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A weak Galerkin finite element method for the Oseen equations

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Abstract In this paper, a weak Galerkin finite element method for the Oseen equations of incompressible fluid flow is proposed and investigated. This method is based on weak gradient and divergence operators which are designed for the finite element discontinuous functions. Moreover, by choosing the usual polynomials of degree $i \geq 1$ for the velocity and polynomials of degree $i-1$ for the pressure and enhancing the polynomials of degree $i-1$ on the interface of a finite element partition for the velocity, this new method has a lot of attractive computational features: more general finite element partitions of arbitrary polygons or polyhedra with certain shape regularity, fewer degrees of freedom and parameter free. Stability and error estimates of optimal order are obtained by defining a weak convection term. Finally, a series

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of numerical experiments are given to show that this method has good stability and accuracy for the Oseen problem.

Keywords Weak Galerkin · Finite element method · The Oseen equations · More general partitions

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1 Introduction

In this paper, we propose a weak Galerkin finite element method for Oseen equations. As an extension of the standard finite elements, the weak Galerkin method substitutes the classical operators (e.g., gradient, divergence, and curl) by weakly defined operators according to integration by parts. The idea of the weak Galerkin method has been introduced and analyzed in [1] for second-order elliptic problems based on local RT or BDM elements, which limited a finite element partition to triangles or tetrahedra. Then, in [4], the weak Galerkin method was extended to allow arbitrary shapes of finite elements in a partition by applying a stabilization idea, which provides a convenient flexibility in mesh generation. A computational process for the weak Galerkin method for second-order elliptic equations with more general finite element partitions has been explained in [5]. In [6], the possibility of an optimal combination of polynomial spaces which minimizes the number of unknowns has been explored and used for several experiments. In [2], by adding stabilization for a flux variable, the weak Galerkin mixed finite element schemes turn out to be applicable for general finite element partitions consisting of arbitrary shapes of polygons or polyhedra. On the base of the weak Galerkin mixed finite element method, in [3] the authors stated the weak Galerkin method for the Stokes equations and proved the L^2 optimal order error estimates for velocity and pressure and the H^1 optimal order error estimates for velocity. Moreover, because the weak Galerkin method inherits the advantages and abandons the weaknesses of discontinuous Galerkin or discontinuous Petrov-Galerkin methods, it has been developed to solve many equations, such as, elliptic interface problem [11], Biharmonic equation [12–14], Helmholtz equation [16, 22], Brinkman equation [18], Darcy-Stokes equation [15, 17], parabolic equation [19–21], etc.

It is well known that the Oseen equations, which are linear equations, show up as an auxiliary problem in many numerical approaches for solving the Navier-Stokes equations. By applying a fixed point iteration for a nonlinear problem, we can see a relationship between the nonlinear problem and the Oseen problem. In addition to the Stokes equations, the Oseen equations possess a convective term and a reactive term. Therefore, compared to the finite element analysis, the most challenges rest in the treatment of the convective term and the reactive term, which have a significant impact on the analysis and numerical computation. The goal of this article is to construct and analyze a stable, parameter-free weak Galerkin finite element scheme for the Oseen equations by using the definition of a weak convection term, which allows the use of finite element partitions with arbitrary shapes of polygons or polyhedra with shape regularity and fewer numbers of unknowns.

This paper is organized as follows. In the next section, we introduce some notation for the Oseen equations and Sobolev spaces. The fundamental definitions and weak Galerkin finite element scheme for the Oseen problem are developed in Section 3. Then, in Section 4, we study the solvability and stability of the weak Galerkin scheme. In Section 5, the H^1 norm error estimates for velocity and the L^2 norm error estimates for both velocity and pressure for the weak Galerkin finite element scheme are derived. In Section 6, numerical results are given to check the stability and accuracy of the present method. Finally, conclusions are drawn in Section 7.

2 Preliminaries

We consider the following stationary Oseen equations in an open bounded domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$, with a Lipschitz continuous boundary $\partial\Omega$:

$$\begin{cases} -\Delta \mathbf{u} + (\mathbf{b} \cdot \nabla)\mathbf{u} + c\mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where the unknowns $\mathbf{u}$ and p, respectively, represent the velocity vector and the pressure, $\mathbf{f} \in [H^{-1}(\Omega)]^d$ is the body force per unit mass, $\mathbf{b}$ is a given convection field, and c is a given scalar function.

For simplicity, we only consider the homogeneous Dirichlet boundary condition. An extension to the nonhomogeneous Dirichlet boundary condition is straightforward.

Our ultimate goal is to study the weak Galerkin finite element method for both the transient and the stationary incompressible Navier-Stokes equations in which $\mathbf{b}$ is substituted by $\mathbf{u}$. By linearizing the incompressible Navier-Stokes equations, e.g., by a semi-implicit iteration, we find it reasonable to study the Oseen problem first. Indeed, for the transient Navier-Stokes equations, $\mathbf{u}$ is the velocity at the current time, $\mathbf{b}$ is the velocity at the previous time and $c = \frac{1}{\Delta t} > 0$ in the Oseen equations; for the stationary case, $\mathbf{u}$ is the velocity at the current iteration step, $\mathbf{b}$ is the velocity at the previous iteration step and $c = 0$ in the Oseen equations.

We introduce the following notation for the variational formulation:

$$\mathbf{X} = [H_0^1(\Omega)]^d, \qquad \mathbf{Y} = [L^2(\Omega)]^d, \qquad Z = \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\}.$$

The Sobolev space $H^k(\Omega)$ is defined in the usual way and, respectively, endowed with the inner product $(\cdot, \cdot)_k$, the norm $\|\cdot\|_k$ and the seminorm $|\cdot|_k$. Especially, when $k = 0$, $H^0(\Omega)$ represents $L^2(\Omega)$. Then the space

$$H(\text{div}, \Omega) = \{\mathbf{v} : \mathbf{v} \in \mathbf{Y}, \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$$

is equipped with the norm $\|\mathbf{v}\|_{H(\text{div},\Omega)} = (\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2)^{\frac{1}{2}}$.

According to integration by parts, the weak formulation of problem (2.1) is given as follows: Find $(\mathbf{u}, p) \in \mathbf{X} \times Z$ such that

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) - d(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{X}', \mathbf{X}} & \forall\, \mathbf{v} \in \mathbf{X}, \\ d(\mathbf{u}, q) = 0 & \forall\, q \in Z, \end{cases} \tag{2.2}$$

where the bilinear forms are defined as

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= (\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{b} \cdot \nabla)\mathbf{u}, \mathbf{v}) + (c\mathbf{u}, \mathbf{v}) \\ &= (\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{b} \cdot \nabla)\mathbf{u}, \mathbf{v}) + \frac{1}{2}((\nabla \cdot \mathbf{b})\mathbf{u}, \mathbf{v}) + (c_0\mathbf{u}, \mathbf{v}) \\ &= (\nabla \mathbf{u}, \nabla \mathbf{v}) + \frac{1}{2}((\mathbf{b} \cdot \nabla)\mathbf{u}, \mathbf{v}) - \frac{1}{2}((\mathbf{b} \cdot \nabla)\mathbf{v}, \mathbf{u}) + (c_0\mathbf{u}, \mathbf{v}) \;\; \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{X}, \\ d(\mathbf{v}, p) &= (\nabla \cdot \mathbf{v}, p) \qquad \forall\, \mathbf{v} \in \mathbf{X}, p \in Z, \end{aligned}$$

and in a general way, we assume that $c(\mathbf{x}) - \frac{1}{2}\nabla \cdot \mathbf{b} := c_0(\mathbf{x}) \geq 0$ for almost all $\mathbf{x} \in \Omega$. In particular, when $\nabla \cdot \mathbf{b} \in L^2(\Omega)$, $\nabla \cdot \mathbf{b} = 0$ almost everywhere in Ω and $c_0(\mathbf{x}) = c(\mathbf{x}) \geq 0$, we have the same weak formulation. Moreover, it is worth mentioning that we denote $((\mathbf{b} \cdot \nabla)\mathbf{u}, \mathbf{v})$ by $(\mathbf{b}\nabla\mathbf{u}, \mathbf{v})$ in the remainder of this article.

As a result, the bilinear form $a(\cdot, \cdot)$ is continuous and coercive on $\mathbf{X} \times \mathbf{X}$, and the bilinear form $d(\cdot, \cdot)$ is continuous and satisfies the inf-sup condition

$$\sup_{\mathbf{v} \in \mathbf{X}} \frac{|\, d(\mathbf{v}, q)\,|}{\|\mathbf{v}\|_1} \geq \beta_1 \|q\| \qquad \forall\, q \in Z.$$

According to the inf-sup condition and Lax-Milgram theorem [7, 23], the existence and uniqueness of a solution to problem (2.1) can be proved.

We also recall the Poincaré-Friedrichs and trace inequalities that are useful in the subsequent analysis [7–10, 23]: There exist constants C_{PF}, C_{T1} and C_{T2}, which depend only on the domain Ω, such that, for all $\mathbf{v} \in \mathbf{X}$,

$$\|\mathbf{v}\|_{L^2(\Omega)} \leq C_{PF} \|\nabla \mathbf{v}\|_{L^2(\Omega)}, \tag{2.3}$$

$$\|\mathbf{v}\|^2_{L^2(\partial\Omega)} \leq C_{T1}(h^{-1}\|\mathbf{v}\|^2_{L^2(\Omega)} + h\|\nabla \mathbf{v}\|^2_{L^2(\Omega)}), \tag{2.4}$$

$$\|\nabla \mathbf{v}\|_{L^2(\partial\Omega)} \leq C_{T2} h^{-1/2} \|\nabla \mathbf{v}\|_{L^2(\Omega)}. \tag{2.5}$$

3 Weak Galerkin discretization

The key to weak Galerkin finite element is to use a weak version to take the place of a strong version in the corresponding variational forms, with the option of adding a stabilization term to enforce a weak continuity of the approximating functions. Discrete weak gradient and divergence operators were introduced in [3], and the rest of the section will review them. Then we discuss the weak Galerkin discretization for the Oseen equations.

Let $\mathcal{K}_h$ be a regular, quasi-uniform mesh of the domain Ω (see [2] for more details) and the mesh size h be a positive parameter which represents the maximum diameter of the elements in $\mathcal{K}_h$. For each element $K \in \mathcal{K}_h$, K_o and ∂K represent the interior and the boundary of K, respectively. Also, the set of all (d-1)-dimensional edges in

$\mathcal{K}_h$ is denoted by E_h. Moreover, we will write $\mathbf{n}_K$ for the outer unit normal with respect to the cell K. Let $P_i(\mathfrak{H})$ denotes the set of polynomials of degree less than or equal to i on $\mathfrak{H}$; especially, $\mathfrak{H}$ can represent K, K_o and ∂K.

To define discrete weak gradient and divergence operators, we need a weak vector valued function $\mathbf{v} = \{\mathbf{v}_o, \mathbf{v}_b\}$ on each element $K \in \mathcal{K}_h$. These two terms can be, respectively, understood as the values of $\mathbf{v}$ in the interior of K and on the boundary of K.

The discrete weak gradient operator is defined as follows:

Definition 3.1 ([3]) The discrete weak gradient operator denoted by $\nabla_{w,r,K}$ is defined as the unique polynomial $\nabla_{w,r,K}\mathbf{v} \in [P_r(K)]^{d\times d}$ satisfying the following equations:

$$(\nabla_{w,r,K}\mathbf{v}, w)_K = -(\mathbf{v}_o, \nabla \cdot w)_K + \langle \mathbf{v}_b, w \cdot \mathbf{n}\rangle_{\partial K} \qquad \forall\, w \in [P_r(K)]^{d\times d}.$$

The discrete weak divergence operator is defined as follows:

Definition 3.2 ([3]) The discrete weak divergence operator denoted by $\nabla_{w,r,K}\cdot$ is defined as the unique polynomial $(\nabla_{w,r,K} \cdot \mathbf{v}) \in P_r(K)$ satisfying the following equation:

$$(\nabla_{w,r,K} \cdot \mathbf{v}, q)_K = -(\mathbf{v}_o, \nabla q)_K + \langle \mathbf{v}_b \cdot \mathbf{n}, q\rangle_{\partial K} \qquad \forall\, q \in P_r(K).$$

After defining the discrete weak gradient and divergence operators, we introduce the definition of a discrete weak convective term from standard finite elements by analogy.

Definition 3.3 The discrete weak convective term denoted by $\mathbf{b}\nabla_{w,r,K}\mathbf{v}$ is defined as the unique polynomial $\mathbf{b}\nabla_{w,r,K}\mathbf{v} \in [P_r(K)]^d$ satisfying the following equation:

$$(\mathbf{b}\nabla_{w,r,K}\mathbf{v}, \mathbf{w})_K = -(\mathbf{b}\nabla\mathbf{w}, \mathbf{v}_o)_K - (\nabla \cdot \mathbf{b}, \mathbf{v}_o \cdot \mathbf{w})_K + \langle \mathbf{b} \cdot \mathbf{n}, \mathbf{v}_b \cdot \mathbf{w}\rangle_{\partial K} \qquad \forall\, \mathbf{w} \in [P_r(K)]^d.$$

From now on, we can, respectively, introduce the discrete weak Galerkin finite element spaces on a mesh: For the velocity variable,

$$\mathbf{X}_h = \{\mathbf{v} = \{\mathbf{v}_o, \mathbf{v}_b\} : \mathbf{v}_o|_K \in [P_i(K)]^d \text{ for all } K \in \mathcal{K}_h, \mathbf{v}_b|_E \in [P_{i-1}(E)]^d \text{ for all } E \in E_h\}$$

and denote $\mathbf{X}_h^0 = \{\mathbf{v} = \{\mathbf{v}_o, \mathbf{v}_b\} \in \mathbf{X}_h : \mathbf{v}_b = \mathbf{0} \text{ on } \partial\Omega\}$; for the pressure variable,

$$Z_h = \{q \in Z : q|_K \in P_{i-1}(K) \text{ for all } K \in \mathcal{K}_h\}.$$

Moreover, we can see that in the finite element space $\mathbf{X}_h$ the matching discrete weak gradient operator is $\nabla_{w,i-1,K}$ and the matching discrete weak divergence operator is $\nabla_{w,i-1,K}\cdot$; for simplicity of notation, we will drop the subscript $i-1$ and K in the notation for the discrete weak gradient and divergence operators.

We further define two L^2 projection operators: Q_o from $[L^2(K)]^d$ onto $[P_i(K)]^d$ for all $K \in \mathcal{K}_h$; Q_b from $[L^2(E)]^d$ onto $[P_{i-1}(E)]^d$ for all $E \in E_h$. In other words, we can define the L^2 projection operator $Q_h = \{Q_o, Q_b\}$ from $[L^2(K)]^d$ onto $\mathbf{X}_h$.

In order to finish this part, we introduce the bilinear discrete forms as follows:

$$
\begin{aligned}
A(\mathbf{u},\mathbf{v}) &= (\nabla_w \mathbf{u}, \nabla_w \mathbf{v}) + \frac{1}{2}(\mathbf{b}\nabla_w \mathbf{u}, \mathbf{v}_o) - \frac{1}{2}(\mathbf{b}\nabla_w \mathbf{v}, \mathbf{u}_o) + (c_0 \mathbf{u}_o, \mathbf{v}_o) + S(\mathbf{u},\mathbf{v}),\\
S(\mathbf{u},\mathbf{v}) &= \sum_{K\in\mathcal{K}_h} h_K^{-1} \langle Q_b \mathbf{u}_o - \mathbf{u}_b, Q_b \mathbf{v}_o - \mathbf{v}_b \rangle_{\partial K},\\
D(\mathbf{v}, p) &= (\nabla_w \cdot \mathbf{v}, p),
\end{aligned}
$$

where the usual L^2 inner product can be written locally on each element:

$$
(\nabla_w \mathbf{u}, \nabla_w \mathbf{v}) = \sum_{K\in\mathcal{K}_h} (\nabla_w \mathbf{u}, \nabla_w \mathbf{v})_K, \qquad (\nabla_w \cdot \mathbf{v}, p) = \sum_{K\in\mathcal{K}_h} (\nabla_w \cdot \mathbf{v}, p)_K,
$$

$$
(\mathbf{u}_o, \mathbf{v}_o) = \sum_{K\in\mathcal{K}_h} (\mathbf{u}_o, \mathbf{v}_o)_K, \qquad (\mathbf{b}\nabla_w \mathbf{u}, \mathbf{v}_o) = \sum_{K\in\mathcal{K}_h} (\mathbf{b}\nabla_w \mathbf{u}, \mathbf{v}_o)_K.
$$

As a result of all above, the weak Galerkin finite element discrete scheme for the Oseen Eq. 2.1 can be written as follows: Find $\mathbf{u}_h = \{\mathbf{u}_o, \mathbf{u}_b\} \in \mathbf{X}_h^0$ and $p_h \in Z_h$ such that

$$
\begin{cases}
A(\mathbf{u}_h, \mathbf{v}) - D(\mathbf{v}, p_h) = \langle \mathbf{f}, \mathbf{v}_o \rangle_{\mathbf{X}', \mathbf{X}} & \forall\, \mathbf{v} = \{\mathbf{v}_o, \mathbf{v}_b\} \in \mathbf{X}_h^0,\\
D(\mathbf{u}_h, q) = 0 & \forall\, q \in Z_h.
\end{cases} \tag{3.1}
$$

4 Solvability and stability

The solvability and stability of the present weak Galerkin finite element scheme for the Oseen problem is presented as follows. First, we recall several useful definitions and conclusions.

The finite element space $\mathbf{X}_h^0$ is a linear space with a norm given by

$$
|\|\mathbf{v}\||^2 = \sum_{K\in\mathcal{K}_h} \|\nabla_w \mathbf{v}\|_K^2 + \sum_{K\in\mathcal{K}_h} \|c_0^{1/2} \mathbf{v}_o\|_K^2 + \sum_{K\in\mathcal{K}_h} h_K^{-1} \|Q_b \mathbf{v}_o - \mathbf{v}_b\|_{\partial K}^2.
$$

According to the proof in [3], $|\|\cdot\||$ provides a norm in $\mathbf{X}_h^0$ and this norm satisfies

$$
\sum_{K\in\mathcal{K}_h} \|\nabla \mathbf{v}_o\|_K^2 \le C |\|\mathbf{v}\||^2. \tag{4.1}
$$

Lemma 4.1 *For any* $\mathbf{u}, \mathbf{v} \in \mathbf{X}_h^0$, *we have*

$$
\begin{aligned}
|A(\mathbf{u},\mathbf{v})| &\le C |\|\mathbf{u}\|| |\|\mathbf{v}\||,\\
A(\mathbf{u},\mathbf{u}) &= |\|\mathbf{u}\||^2.
\end{aligned}
$$

Proof It is not hard to see that $A(\mathbf{u},\mathbf{u}) = |\|\mathbf{u}\||^2$. Then the result follows from the definition of $|\|\cdot\||$ and the usual Cauchy-Schwarz inequality. □

Lemma 4.2 ([3]) *The projection operators Q_h, $\mathbf{Q}_h$ and $\mathbb{Q}_h$ satisfy the following commutative properties:*

$$\nabla_w(Q_h\mathbf{v}) = \mathbf{Q}_h(\nabla\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathbf{X},$$
$$\nabla_w \cdot (Q_h\mathbf{v}) = \mathbb{Q}_h(\nabla\cdot\mathbf{v}) \qquad \forall\, \mathbf{v} \in H(div, \Omega),$$

where $\mathbf{Q}_h$ and $\mathbb{Q}_h$ represent two local L^2 projections onto $[P_{i-1}(K)]^{d\times d}$ and $P_{i-1}(K)$, respectively.

Lemma 4.3 *When $c_0 \in L^\infty(\Omega)$, there exists a positive constant β_2, independent of h, such that*

$$\sup_{\mathbf{v}\in\mathbf{X}_h^0} \frac{|\ D(\mathbf{v}, q)\ |}{|\|\mathbf{v}\||} \geq \beta_2\|q\| \qquad \forall\, q \in Z_h.$$

Proof First, we claim that the following holds true

$$|\|\mathbf{v}\|| \leq C\|\hat{\mathbf{v}}\|_1. \tag{4.2}$$

In order to proof the aforementioned inequality (4.2), we set $\mathbf{v} = Q_h\hat{\mathbf{v}} \in \mathbf{X}_h$ and use Lemma 4.2 and Hŏlder inequality to obtain

$$\begin{aligned}\sum_{K\in\mathcal{K}_h}\|\nabla_w\mathbf{v}\|_K^2 + \sum_{K\in\mathcal{K}_h}\|c_0^{1/2}\mathbf{v}_o\|_K^2 &= \sum_{K\in\mathcal{K}_h}\|\nabla_w(Q_h\hat{\mathbf{v}})\|_K^2 + \sum_{K\in\mathcal{K}_h}\|c_0^{1/2}Q_o\hat{\mathbf{v}}\|_K^2 \\ &= \sum_{K\in\mathcal{K}_h}\|\mathbf{Q}_h(\nabla\hat{\mathbf{v}})\|_K^2 + \sum_{K\in\mathcal{K}_h}\|c_0^{1/2}Q_o\hat{\mathbf{v}}\|_K^2 \\ &\leq C\|\nabla\hat{\mathbf{v}}\|^2 + \|c_0\|_\infty\|Q_o\hat{\mathbf{v}}\|^2 \leq C\|\nabla\hat{\mathbf{v}}\|^2.\end{aligned} \tag{4.3}$$

According to [3], we obtain

$$\sum_{K\in\mathcal{K}_h} h_K^{-1}\|Q_b\mathbf{v}_o - \mathbf{v}_b\|_{\partial K}^2 \leq C\|\nabla\hat{\mathbf{v}}\|^2. \tag{4.4}$$

Combining the estimate (4.3) with (4.4) yields the desired inequality (4.2).

Then, for any given $q \in Z_h \subset Z$, it is well known [7–9] that there exists a vector-valued function $\hat{\mathbf{v}} \in \mathbf{X}$ such that

$$\frac{(\nabla\cdot\hat{\mathbf{v}}, q)}{\|\hat{\mathbf{v}}\|_1} \geq C\|q\|, \tag{4.5}$$

where $C > 0$ is a constant depending only on the domain Ω. And it follows from Lemma 4.2 and the definition of $\mathbb{Q}_h$ that

$$D(\mathbf{v}, q) = (\nabla_w\cdot(Q_h\hat{\mathbf{v}}), q) = (\mathbb{Q}_h(\nabla\cdot\hat{\mathbf{v}}), q) = (\nabla\cdot\hat{\mathbf{v}}, q). \tag{4.6}$$

Using Eqs. 4.2, 4.5 and 4.6, we have

$$\frac{|\ D(\mathbf{v}, q)\ |}{|\|\mathbf{v}\||} \geq \frac{|\ (\nabla\cdot\hat{\mathbf{v}}, q)\ |}{C\|\hat{\mathbf{v}}\|_1} \geq \beta_2\|q\|.$$

This completes the proof of the lemma. □

From Lemma 4.1 and Lemma 4.3, the following solvability holds true for the weak Galerkin finite element scheme (3.1).

Theorem 4.4 *The weak Galerkin finite element scheme* (3.1) *has a unique solution.*

5 Error estimates

In this section, we concentrate on an error analysis for the present method for the Oseen equations and obtain optimal error estimates. To establish the error estimates, the following result will be used:

Lemma 5.1 ([3]) *Let* $(\mathbf{u}, p) \in [H^{r+1}(\Omega)]^d \times H^r(\Omega)$ *with* $1 \le r \le i$ *and* $\mathcal{K}_h$ *be a finite element partition of* Ω *satisfying the shape regularity assumption as specified in* [2]. *Then, for* $0 \le s \le 1$, *we have*

$$\sum_{K\in\mathcal{K}_h} h_K^{2s}\|\mathbf{u}-Q_o\mathbf{u}\|_{K,s}^2 \le Ch^{2(r+1)}\|\mathbf{u}\|_{r+1}^2,$$

$$\sum_{K\in\mathcal{K}_h} h_K^{2s}\|\nabla\mathbf{u}-\mathbf{Q}_h(\nabla\mathbf{u})\|_{K,s}^2 \le Ch^{2r}\|\mathbf{u}\|_{r+1}^2,$$

$$\sum_{K\in\mathcal{K}_h} h_K^{2s}\|p-\mathbb{Q}_h p\|_{K,s}^2 \le Ch^{2r}\|p\|_r^2.$$

Lemma 5.2 *Let* $(\mathbf{u}, p) \in [H^1(\Omega)]^d \times L^2(\Omega)$ *satisfy the first equation of* Eq. 2.1 *in the bounded domain* $\Omega \subseteq \mathbb{R}^d$ *with a Lipschitz continuous boundary and be sufficiently smooth. Moreover,* $c_0(\mathbf{x}) \ge 0$ *and* $\mathbf{b} \in [L^\infty(\Omega)]^d, c, c_0 \in L^\infty(\Omega)$. *Let* $Q_h\mathbf{u} = \{Q_o\mathbf{u}, Q_b\mathbf{u}\}$ *and* $\mathbb{Q}_h p$ *be the* L^2 *projection of* $(\mathbf{u}, p)$ *into the finite element space* $(\mathbf{X}_h, Z_h)$. *Then the following equation holds true:*

$$\begin{aligned}
&(\nabla_w(Q_h\mathbf{u}), \nabla_w\mathbf{v}) - (\nabla_w\cdot\mathbf{v}, \mathbb{Q}_h p) + \frac{1}{2}(\mathbf{b}\nabla_w(Q_h\mathbf{u}), \mathbf{v}_o) - \frac{1}{2}(\mathbf{b}\nabla_w\mathbf{v}, Q_o\mathbf{u})\\
&+ (c_0 Q_o\mathbf{u}, \mathbf{v}_o) = \langle \mathbf{f}, \mathbf{v}_o\rangle_{\mathbf{X}',\mathbf{X}} + l_{\mathbf{u}}(\mathbf{v}) - \theta_p(\mathbf{v}) - r^1_{\mathbf{u},\mathbf{b}}(\mathbf{v}) - r^2_{\mathbf{u},\mathbf{b}}(\mathbf{v})\\
&- \frac{1}{2}(\nabla\cdot\mathbf{b}, (Q_o\mathbf{u}-\mathbf{u})\cdot\mathbf{v}_o) - (\mathbf{b}\nabla\mathbf{v}_o, Q_o\mathbf{u}-\mathbf{u}) - (c_0(\mathbf{u}-Q_o\mathbf{u}), \mathbf{v}_o)
\end{aligned}$$

for all $\mathbf{v} \in \mathbf{X}_h^0$, *where* $l_{\mathbf{u}}(\mathbf{v})$, $\theta_p(\mathbf{v})$, $r^1_{\mathbf{u},\mathbf{b}}(\mathbf{v})$ *and* $r^2_{\mathbf{u},\mathbf{b}}(\mathbf{v})$ *are linear functions on* $\mathbf{X}_h^0$ *defined by*

$$l_{\mathbf{u}}(\mathbf{v}) = \sum_{K\in\mathcal{K}_h}\langle \mathbf{v}_o-\mathbf{v}_b, \nabla\mathbf{u}\cdot\mathbf{n}-\mathbf{Q}_h(\nabla\mathbf{u})\cdot\mathbf{n}\rangle_{\partial K},$$

$$\theta_p(\mathbf{v}) = \sum_{K\in\mathcal{K}_h}\langle \mathbf{v}_o-\mathbf{v}_b, (p-\mathbb{Q}_h p)\mathbf{n}\rangle_{\partial K},$$

$$r^1_{\mathbf{u},\mathbf{b}}(\mathbf{v}) = \frac{1}{2}\sum_{K\in\mathcal{K}_h}\langle \mathbf{b}\cdot\mathbf{n}, (\mathbf{u}-Q_b\mathbf{u})\cdot(\mathbf{v}_o-\mathbf{v}_b)\rangle_{\partial K},$$

$$r^2_{\mathbf{u},\mathbf{b}}(\mathbf{v}) = \frac{1}{2}\sum_{K\in\mathcal{K}_h}\langle \mathbf{b}\cdot\mathbf{n}, (\mathbf{u}-Q_o\mathbf{u})\cdot(\mathbf{v}_o-\mathbf{v}_b)\rangle_{\partial K}.$$

Proof From [3], we obtain

$$(\nabla_w(Q_h\mathbf{u}), \nabla_w\mathbf{v}) = (\nabla\mathbf{u}, \nabla\mathbf{v}_o) - \sum_{K\in\mathcal{K}_h}\langle\mathbf{v}_o - \mathbf{v}_b, \mathbf{Q}_h(\nabla\mathbf{u})\cdot\mathbf{n}\rangle_{\partial K},$$

$$(\nabla_w\cdot\mathbf{v}, \mathbb{Q}_h p) = -(\mathbf{v}_o, \nabla p) + \sum_{K\in\mathcal{K}_h}\langle\mathbf{v}_o - \mathbf{v}_b, (p - \mathbb{Q}_h p)\mathbf{n}\rangle_{\partial K}.$$

Then, according to Definition 3.3 and the fact $\sum_{K\in\mathcal{K}_h}\langle\mathbf{b}\cdot\mathbf{n}, \mathbf{u}_b\cdot\mathbf{v}_b\rangle_{\partial K} = 0$, we have

$$(\mathbf{b}\nabla_{w,r,K}\mathbf{u}, \mathbf{v}_o) = -(\mathbf{b}\nabla\mathbf{v}_o, \mathbf{u}_o) - (\nabla\cdot\mathbf{b}, \mathbf{u}_o\cdot\mathbf{v}_o) + \sum_{K\in\mathcal{K}_h}\langle\mathbf{b}\cdot\mathbf{n}, \mathbf{u}_b\cdot(\mathbf{v}_o - \mathbf{v}_b)\rangle_{\partial K} \tag{5.1}$$

and

$$\begin{aligned}(\mathbf{b}\nabla_{w,r,K}\mathbf{v}, \mathbf{u}_o) &= -(\mathbf{b}\nabla\mathbf{u}_o, \mathbf{v}_o) - (\nabla\cdot\mathbf{b}, \mathbf{u}_o\cdot\mathbf{v}_o) + \sum_{K\in\mathcal{K}_h}\langle\mathbf{b}\cdot\mathbf{n}, \mathbf{v}_b\cdot\mathbf{u}_o\rangle_{\partial K}\\ &= (\mathbf{b}\nabla\mathbf{v}_o, \mathbf{u}_o) - \sum_{K\in\mathcal{K}_h}\langle\mathbf{b}\cdot\mathbf{n}, (\mathbf{v}_o - \mathbf{v}_b)\cdot\mathbf{u}_o\rangle_{\partial K}.\end{aligned} \tag{5.2}$$

By using Eqs. 5.1 and 5.2, we obtain

$$(\mathbf{b}\nabla_w(Q_h\mathbf{u}), \mathbf{v}_o) = -(\mathbf{b}\nabla\mathbf{v}_o, Q_o\mathbf{u}) - (\nabla\cdot\mathbf{b}, Q_o\mathbf{u}\cdot\mathbf{v}_o) + \sum_{K\in\mathcal{K}_h}\langle\mathbf{b}\cdot\mathbf{n}, Q_b\mathbf{u}\cdot(\mathbf{v}_o - \mathbf{v}_b)\rangle_{\partial K}, \tag{5.3}$$

$$(\mathbf{b}\nabla_w\mathbf{v}, Q_o\mathbf{u}) = (\mathbf{b}\nabla\mathbf{v}_o, Q_o\mathbf{u}) - \sum_{K\in\mathcal{K}_h}\langle\mathbf{b}\cdot\mathbf{n}, (\mathbf{v}_o - \mathbf{v}_b)\cdot Q_o\mathbf{u}\rangle_{\partial K}, \tag{5.4}$$

$$\begin{aligned}&(\mathbf{b}\nabla_w(Q_h\mathbf{u}), \mathbf{v}_o) + \tfrac{1}{2}(\nabla\cdot\mathbf{b}, Q_o\mathbf{u}\cdot\mathbf{v}_o) - \tfrac{1}{2}\sum_{K\in\mathcal{K}_h}\langle\mathbf{b}\cdot\mathbf{n}, (Q_b\mathbf{u} - Q_o\mathbf{u})\cdot(\mathbf{v}_o - \mathbf{v}_b)\rangle_{\partial K}\\ &= \tfrac{1}{2}(\mathbf{b}\nabla_w(Q_h\mathbf{u}), \mathbf{v}_o) - \tfrac{1}{2}(\mathbf{b}\nabla\mathbf{v}_o, Q_o\mathbf{u}) + \tfrac{1}{2}\sum_{K\in\mathcal{K}_h}\langle\mathbf{b}\cdot\mathbf{n}, Q_o\mathbf{u}\cdot(\mathbf{v}_o - \mathbf{v}_b)\rangle_{\partial K}\\ &= \tfrac{1}{2}(\mathbf{b}\nabla_w(Q_h\mathbf{u}), \mathbf{v}_o) - \tfrac{1}{2}(\mathbf{b}\nabla_w\mathbf{v}, Q_o\mathbf{u}).\end{aligned} \tag{5.5}$$

Now, testing the first equation of Eq. 2.1 by function $\mathbf{v}_o$ in $\mathbf{v} = \{\mathbf{v}_o, \mathbf{v}_b\} \in \mathbf{X}_h^0$ and combining with Eqs. 5.3–5.5 and the fact that $\sum_{K\in\mathcal{K}_h}\langle\mathbf{v}_b, \nabla\mathbf{u}\cdot\mathbf{n}\rangle_{\partial K} = 0$, $\sum_{K\in\mathcal{K}_h}\langle\mathbf{v}_b, p\mathbf{n}\rangle_{\partial K} = 0$, and $\sum_{K\in\mathcal{K}_h}\langle\mathbf{b}\cdot\mathbf{n}, \mathbf{u}\cdot\mathbf{v}_b\rangle_{\partial K} = 0$, we obtain

$$\begin{aligned}-(\Delta\mathbf{u}, \mathbf{v}_o) &= (\nabla\mathbf{u}, \nabla\mathbf{v}_o) - \sum_{K\in\mathcal{K}_h}\langle\mathbf{v}_o - \mathbf{v}_b, \nabla\mathbf{u}\cdot\mathbf{n}\rangle_{\partial K},\\ (\nabla p, \mathbf{v}_o) &= -(\nabla_w\cdot\mathbf{v}, \mathbb{Q}_h p) + \sum_{K\in\mathcal{K}_h}\langle\mathbf{v}_o - \mathbf{v}_b, (p - \mathbb{Q}_h p)\mathbf{n}\rangle_{\partial K},\\ (c\mathbf{u}, \mathbf{v}_o) - \frac{1}{2}(\nabla\cdot\mathbf{b}, \mathbf{u}\cdot\mathbf{v}_o) &= (c_0\mathbf{u}, \mathbf{v}_o) = (c_0(\mathbf{u} - Q_o\mathbf{u}), \mathbf{v}_o) + (c_0 Q_o\mathbf{u}, \mathbf{v}_o),\\ (\mathbf{b}\nabla\mathbf{u}, \mathbf{v}_o) + \frac{1}{2}(\nabla\cdot\mathbf{b}, \mathbf{u}\cdot\mathbf{v}_o) &= -(\mathbf{b}\nabla\mathbf{v}_o, \mathbf{u}) - \frac{1}{2}(\nabla\cdot\mathbf{b}, \mathbf{u}\cdot\mathbf{v}_o) + \sum_{K\in\mathcal{K}_h}\langle\mathbf{b}\cdot\mathbf{n}, \mathbf{u}\cdot(\mathbf{v}_o - \mathbf{v}_b)\rangle_{\partial K}\\ &= \frac{1}{2}(\mathbf{b}\nabla_w(Q_h\mathbf{u}), \mathbf{v}_o) - \frac{1}{2}(\mathbf{b}\nabla_w\mathbf{v}, Q_o\mathbf{u})\\ &\quad+\frac{1}{2}(\nabla\cdot\mathbf{b}, (Q_o\mathbf{u} - \mathbf{u})\cdot\mathbf{v}_o) + \frac{1}{2}\sum_{K\in\mathcal{K}_h}\langle\mathbf{b}\cdot\mathbf{n}, (\mathbf{u} - Q_b\mathbf{u})\cdot(\mathbf{v}_o - \mathbf{v}_b)\rangle_{\partial K}\\ &\quad+(\mathbf{b}\nabla\mathbf{v}_o, Q_o\mathbf{u} - \mathbf{u}) + \frac{1}{2}\sum_{K\in\mathcal{K}_h}\langle\mathbf{b}\cdot\mathbf{n}, (\mathbf{u} - Q_o\mathbf{u})\cdot(\mathbf{v}_o - \mathbf{v}_b)\rangle_{\partial K}.\end{aligned}$$

By combining the above formulations, we complete the proof of the lemma. □

Lemma 5.3 ([3]) *Let* $1 \le r \le i$ *and* $(\mathbf{u}, p) \in [H^{r+1}(\Omega)]^d \times H^r(\Omega)$. *Assume that the finite element partition* $\mathcal{K}_h$ *is shape regular* [2]. *Then the following estimates hold true for all* $\mathbf{v} \in \mathbf{X}_h$*:*

$$| S(Q_h\mathbf{u}, \mathbf{v}) | \le Ch^r \|\mathbf{u}\|_{r+1} |||\mathbf{v}|||,$$
$$| l_{\mathbf{u}}(\mathbf{v}) | \le Ch^r \|\mathbf{u}\|_{r+1} |||\mathbf{v}|||,$$
$$| \theta_p(\mathbf{v}) | \le Ch^r \|p\|_r |||\mathbf{v}|||,$$

where $l_{\mathbf{u}}(\mathbf{v})$ *and* $\theta_p(\mathbf{v})$ *are defined as in Lemma 5.2.*

Theorem 5.4 *Assume that* $(\mathbf{u}, p) \in [H_0^1(\Omega) \cap H^{i+1}(\Omega)]^d \times (L_0^2(\Omega) \cap H^i(\Omega))$ *and* $(\mathbf{u}_h, p_h) \in \mathbf{X}_h^0 \times Z_h$ *are the solutions of* Eqs. 2.2 *and* 3.1*, respectively. Then it holds*

$$|||Q_h\mathbf{u} - \mathbf{u}_h||| + \|\mathbb{Q}_h p - p_h\| \le Ch^i(\|\mathbf{u}\|_{i+1} + \|p\|_i)$$

where $c_0(\mathbf{x}) \ge 0$ *and* $\mathbf{b} \in [L^\infty(\Omega)]^d$, $c, c_0 \in L^\infty(\Omega)$.

Proof First, since $(\mathbf{u}, p)$ satisfies the first equation of Eq. 2.1, we derive from Lemma 5.2

$$\begin{aligned}(\nabla_w(Q_h\mathbf{u}), \nabla_w\mathbf{v}) - (\nabla_w \cdot \mathbf{v}, \mathbb{Q}_h p) + \tfrac{1}{2}(\mathbf{b}\nabla_w(Q_h\mathbf{u}), \mathbf{v}_o) - \tfrac{1}{2}(\mathbf{b}\nabla_w\mathbf{v}, Q_o\mathbf{u})\\ +(c_0 Q_o\mathbf{u}, \mathbf{v}_o) = \langle \mathbf{f}, \mathbf{v}_o\rangle_{\mathbf{X}',\mathbf{X}} + l_{\mathbf{u}}(\mathbf{v}) - \theta_p(\mathbf{v}) - r^1_{\mathbf{u},\mathbf{b}}(\mathbf{v}) - r^2_{\mathbf{u},\mathbf{b}}(\mathbf{v})\\ -\tfrac{1}{2}(\nabla \cdot \mathbf{b}, (Q_o\mathbf{u} - \mathbf{u}) \cdot \mathbf{v}_o) - (\mathbf{b}\nabla\mathbf{v}_o, Q_o\mathbf{u} - \mathbf{u}) - (c_0(\mathbf{u} - Q_o\mathbf{u}), \mathbf{v}_o).\end{aligned}$$

Adding $S(Q_h\mathbf{u}, \mathbf{v})$ to both sides of the above equation gives

$$\begin{aligned}A(Q_h\mathbf{u}, \mathbf{v}) - D(\mathbf{v}, \mathbb{Q}_h p) = \langle \mathbf{f}, \mathbf{v}_o\rangle_{\mathbf{X}',\mathbf{X}} + l_{\mathbf{u}}(\mathbf{v}) - \theta_p(\mathbf{v}) + S(Q_h\mathbf{u}, \mathbf{v}) - r^1_{\mathbf{u},\mathbf{b}}(\mathbf{v})\\ - r^2_{\mathbf{u},\mathbf{b}}(\mathbf{v}) - \frac{1}{2}(\nabla \cdot \mathbf{b}, (Q_o\mathbf{u} - \mathbf{u}) \cdot \mathbf{v}_o) - (\mathbf{b}\nabla\mathbf{v}_o, Q_o\mathbf{u} - \mathbf{u}) - (c_0(\mathbf{u} - Q_o\mathbf{u}), \mathbf{v}_o).\end{aligned} \tag{5.6}$$

Next, testing the second equation of Eq. 2.1 by $q \in Z_h$ and using Lemma 4.2, we obtain

$$(\nabla \cdot \mathbf{u}, q) = (\mathbb{Q}_h(\nabla \cdot \mathbf{u}), q) = (\nabla_w \cdot Q_h\mathbf{u}, q) = 0$$

i.e.,

$$D(Q_h\mathbf{u}, q) = 0. \tag{5.7}$$

Subtracting Eq. 3.1 from Eqs. 5.6, 5.7, respectively, yields the following equation:

$$\begin{aligned}A(\varepsilon_h, \mathbf{v}) - D(\mathbf{v}, \eta_h) + D(\varepsilon_h, q) = l_{\mathbf{u}}(\mathbf{v}) - \theta_p(\mathbf{v}) + S(Q_h\mathbf{u}, \mathbf{v}) - r^1_{\mathbf{u},\mathbf{b}}(\mathbf{v})\\ - r^2_{\mathbf{u},\mathbf{b}}(\mathbf{v}) - \frac{1}{2}(\nabla \cdot \mathbf{b}, (Q_o\mathbf{u} - \mathbf{u}) \cdot \mathbf{v}_o) - (\mathbf{b}\nabla\mathbf{v}_o, Q_o\mathbf{u} - \mathbf{u}) - (c_0(\mathbf{u} - Q_o\mathbf{u}), \mathbf{v}_o),\end{aligned} \tag{5.8}$$

where $\varepsilon_h = Q_h\mathbf{u} - \mathbf{u}_h$ and $\eta_h = Q_h p - p_h$.

By letting $(\mathbf{v}, q) = (\varepsilon_h, \eta_h)$ in Eq. 5.8, we see that

$$\begin{aligned}|||\varepsilon_h|||^2 = l_{\mathbf{u}}(\varepsilon_h) - \theta_p(\varepsilon_h) + S(Q_h\mathbf{u}, \varepsilon_h) - r^1_{\mathbf{u},\mathbf{b}}(\varepsilon_h) - r^2_{\mathbf{u},\mathbf{b}}(\varepsilon_h)\\ - \frac{1}{2}(\nabla \cdot \mathbf{b}, (Q_o\mathbf{u} - \mathbf{u}) \cdot \varepsilon_o) - (\mathbf{b}\nabla\varepsilon_o, Q_o\mathbf{u} - \mathbf{u}) - (c_0(\mathbf{u} - Q_o\mathbf{u}), \varepsilon_o).\end{aligned}$$

Clearly, by using the trace inequality (2.4), (2.5), Lemma 5.1 and (4.1), we obtain

$$
\begin{aligned}
&| r^1_{\mathbf{u},\mathbf{b}}(\varepsilon_h) |=| \frac{1}{2}\sum_{K\in\mathcal{K}_h} \langle \mathbf{b}\cdot\mathbf{n}, (\mathbf{u}-Q_b\mathbf{u})\cdot(\varepsilon_o-\varepsilon_b)\rangle_{\partial K} | \\
&\le \frac{1}{2} | \sum_{K\in\mathcal{K}_h} \langle \mathbf{b}\cdot\mathbf{n}, (\mathbf{u}-Q_b\mathbf{u})\cdot(\varepsilon_o-Q_b\varepsilon_o)\rangle_{\partial K} | + \frac{1}{2} | \sum_{K\in\mathcal{K}_h} \langle \mathbf{b}\cdot\mathbf{n}, (\mathbf{u}-Q_b\mathbf{u})\cdot(Q_b\varepsilon_o-\varepsilon_b)\rangle_{\partial K} | \\
&\le \frac{1}{2}\left(\sum_{K\in\mathcal{K}_h}\|\mathbf{b}\|_{L^\infty(\partial K)}\right)\left(\sum_{K\in\mathcal{K}_h}\|\varepsilon_o-Q_b\varepsilon_o\|^2_{\partial K}\right)^{1/2}\left(\sum_{K\in\mathcal{K}_h}\|\mathbf{u}-Q_b\mathbf{u}\|^2_{\partial K}\right)^{1/2} \\
&\quad + \frac{1}{2}\left(\sum_{K\in\mathcal{K}_h}\|\mathbf{b}\|_{L^\infty(\partial K)}\right)\left(\sum_{K\in\mathcal{K}_h}h^{-1}\|Q_b\varepsilon_o-\varepsilon_b\|^2_{\partial K}\right)^{1/2}\left(\sum_{K\in\mathcal{K}_h}h\|\mathbf{u}-Q_b\mathbf{u}\|^2_{\partial K}\right)^{1/2} \\
&\le C\|\mathbf{b}\|_\infty\left[\left(\sum_{K\in\mathcal{K}_h}h\|\nabla\varepsilon_o\|^2_{\partial K}\right)^{1/2}+\left(\sum_{K\in\mathcal{K}_h}h^{-1}\|Q_b\varepsilon_o-\varepsilon_b\|^2_{\partial K}\right)^{1/2}\right]\left(\sum_{K\in\mathcal{K}_h}h\|\mathbf{u}-Q_b\mathbf{u}\|^2_{\partial K}\right)^{1/2} \\
&\le Ch^{i+1}|\|\varepsilon_h\||\|\mathbf{u}\|_{i+1}.
\end{aligned}
$$

Similarly,

$$| r^2_{\mathbf{u},\mathbf{b}}(\varepsilon_h) |\le Ch^{i+1}|\|\varepsilon_h\||\|\mathbf{u}\|_{i+1}.$$

Furthermore, we can derive

$$
\begin{aligned}
&| -\frac{1}{2}(\nabla\cdot\mathbf{b}, (Q_o\mathbf{u}-\mathbf{u})\cdot\varepsilon_o) - (\mathbf{b}\nabla\varepsilon_o, Q_o\mathbf{u}-\mathbf{u}) - (c_0(\mathbf{u}-Q_o\mathbf{u}), \varepsilon_o) | \\
&\le \|\nabla\cdot\mathbf{b}\|_\infty\|Q_o\mathbf{u}-\mathbf{u}\|\|\varepsilon_o\| + \|\mathbf{b}\|_\infty\|Q_o\mathbf{u}-\mathbf{u}\|\|\nabla\varepsilon_o\| + \|c_0\|_\infty\|Q_o\mathbf{u}-\mathbf{u}\|\|\varepsilon_o\| \\
&\le Ch^{i+1}|\|\varepsilon_h\||\|\mathbf{u}\|_{i+1}.
\end{aligned}
$$

Then, combining the above inequalities with Lemma 5.3, we have

$$|\|\varepsilon_h\||^2 \le Ch^i(\|\mathbf{u}\|_{i+1}+\|p\|_i)|\|\varepsilon_h\||,$$

which implies the first part of the desired estimate. To estimate $\|\eta_h\|$, from Lemmas 4.3, 5.3 and (5.8), we have

$$\|\eta_h\| \le \beta_2^{-1}\sup_{\mathbf{v}\in\mathbf{X}_h^0}\frac{| D(\mathbf{v},\eta_h) |}{|\|\mathbf{v}\||} \le Ch^i(\|\mathbf{u}\|_{i+1}+\|p\|_i),$$

which yields the desired estimate. □

In order to derive the L^2 error estimate for the velocity, we consider the following dual problem: Find $(\Phi,\Psi)\in\mathbf{X}\times Z$ such that

$$
\begin{cases}
-\Delta\Phi-(\mathbf{b}\cdot\nabla)\Phi-(\nabla\cdot b)\Phi+c\Phi-\nabla\Psi = Q_o\mathbf{u}-\mathbf{u}_o & \text{in } \Omega, \\
\nabla\cdot\Phi = 0 & \text{in } \Omega, \\
\Phi = \mathbf{0} & \text{on } \partial\Omega.
\end{cases}
\tag{5.9}
$$

Assume that the dual problem satisfies $(\Phi,\Psi)\in[H^2(\Omega)]^d\times H^1(\Omega)$ and the convexity of the domain Ω. Then the following regularity property holds:

$$\|\Phi\|_2+\|\Psi\|_1\le C\|\varepsilon_o\|. \tag{5.10}$$

Theorem 5.5 *Assume that* $(\mathbf{u}, p) \in [H_0^1(\Omega) \cap H^{i+1}(\Omega)]^d \times (L_0^2(\Omega) \cap H^i(\Omega))$ *and* $(\mathbf{u}_h, p_h) \in \mathbf{X}_h^0 \times Z_h$ *are the solutions of* Eqs. 2.2 *and* 3.1, *respectively. Then it holds*

$$\|Q_o\mathbf{u} - \mathbf{u}_o\| \le Ch^{i+1}(\|\mathbf{u}\|_{i+1} + \|p\|_i)$$

when $c_0(\mathbf{x}) \ge 0$ *and* $\mathbf{b} \in [L^\infty(\Omega)]^d$, $c, c_0 \in L^\infty(\Omega)$.

Proof Copying the derivation of Lemma 5.2 and adding $S(\mathbf{v}, Q_h\Phi)$ to both sides give

$$\begin{aligned} &A(\mathbf{v}, Q_h\Phi) + D(\mathbf{v}, \mathbb{Q}_h\Psi) = (\varepsilon_o, \mathbf{v}_o) + l_\Phi(\mathbf{v}) + \theta_\Psi(\mathbf{v}) + S(\mathbf{v}, Q_h\Phi) + r^1_{\Phi,\mathbf{b}}(\mathbf{v}) \\ &+ r^2_{\Phi,\mathbf{b}}(\mathbf{v}) + \frac{1}{2}(\nabla\cdot\mathbf{b}, (Q_o\Phi - \Phi)\cdot\mathbf{v}_o) + (\mathbf{b}\nabla\mathbf{v}_o, Q_o\Phi - \Phi) - (c_0(\Phi - Q_o\Phi), \mathbf{v}_o), \end{aligned} \tag{5.11}$$

and thanks to Eq. 5.7, we see that

$$D(Q_h\Phi, q) = 0.$$

By letting $(\mathbf{v}, q) = (\varepsilon_h, \eta_h) = (Q_h\mathbf{u} - \mathbf{u}_h, Q_hp - p_h)$ in the above equations, we obtain

$$\begin{aligned} &A(\varepsilon_h, Q_h\Phi) + D(\varepsilon_h, Q_h\Psi) - D(Q_h\Phi, \eta_h) = (\varepsilon_o, \varepsilon_o) + l_\Phi(\varepsilon_h) + \theta_\Psi(\varepsilon_h) + S(\varepsilon_h, Q_h\Phi) \\ &+ r^1_{\Phi,\mathbf{b}}(\varepsilon_h) + r^2_{\Phi,\mathbf{b}}(\varepsilon_h) + \frac{1}{2}(\nabla\cdot\mathbf{b}, (Q_o\Phi - \Phi)\cdot\varepsilon_o) + (\mathbf{b}\nabla\varepsilon_o, Q_o\Phi - \Phi) - (c_0(\Phi - Q_o\Phi), \varepsilon_o). \end{aligned} \tag{5.12}$$

In Eq. 5.8, we substitute $\mathbf{v}$ by $Q_h\Phi$ and q by $Q_h\Psi$

$$\begin{aligned} &A(\varepsilon_h, Q_h\Phi) - D(Q_h\Phi, \eta_h) + D(\varepsilon_h, Q_h\Psi) = l_\mathbf{u}(Q_h\Phi) - \theta_p(Q_h\Phi) + S(Q_h\mathbf{u}, Q_h\Phi) - r^1_{\mathbf{u},\mathbf{b}}(Q_h\Phi) \\ &- r^2_{\mathbf{u},\mathbf{b}}(Q_h\Phi) - \frac{1}{2}(\nabla\cdot\mathbf{b}, (Q_o\mathbf{u} - \mathbf{u})\cdot Q_o\Phi) - (\mathbf{b}\nabla Q_o\Phi, Q_o\mathbf{u} - \mathbf{u}) - (c_0(\mathbf{u} - Q_o\mathbf{u}), Q_o\Phi). \end{aligned} \tag{5.13}$$

Subtracting Eq. 5.12 from Eq. 5.13 yields

$$\begin{aligned} &\|\varepsilon_o\|^2 = -l_\Phi(\varepsilon_h) - \theta_\Psi(\varepsilon_h) - S(\varepsilon_h, Q_h\Phi) - r^1_{\Phi,\mathbf{b}}(\varepsilon_h) - r^2_{\Phi,\mathbf{b}}(\varepsilon_h) - \frac{1}{2}(\nabla\cdot\mathbf{b}, (Q_o\Phi - \Phi)\cdot\varepsilon_o) \\ &- (\mathbf{b}\nabla\varepsilon_o, Q_o\Phi - \Phi) + (c_0(\Phi - Q_o\Phi), \varepsilon_o) + l_\mathbf{u}(Q_h\Phi) - \theta_p(Q_h\Phi) + S(Q_h\mathbf{u}, Q_h\Phi) - r^1_{\mathbf{u},\mathbf{b}}(Q_h\Phi) \\ &- r^2_{\mathbf{u},\mathbf{b}}(Q_h\Phi) - \frac{1}{2}(\nabla\cdot\mathbf{b}, (Q_o\mathbf{u} - \mathbf{u})\cdot Q_o\Phi) - (\mathbf{b}\nabla Q_o\Phi, Q_o\mathbf{u} - \mathbf{u}) - (c_0(\mathbf{u} - Q_o\mathbf{u}), Q_o\Phi). \end{aligned} \tag{5.14}$$

It is clear from Lemma 5.3, Theorem 5.4 and (5.10) that

$$\begin{aligned} &| -l_\Phi(\varepsilon_h) - \theta_\Psi(\varepsilon_h) - S(\varepsilon_h, Q_h\Phi) | \le Ch\|\Phi\|_2 |\|\varepsilon_h\|| \le Ch|\|\varepsilon_h\||\|\varepsilon_o\|, \\ &| -r^1_{\Phi,\mathbf{b}}(\varepsilon_h) - r^2_{\Phi,\mathbf{b}}(\varepsilon_h) - \frac{1}{2}(\nabla\cdot\mathbf{b}, (Q_o\Phi - \Phi)\cdot\varepsilon_o) - (\mathbf{b}\nabla\varepsilon_o, Q_o\Phi - \Phi) + (c_0(\Phi - Q_o\Phi), \varepsilon_o) | \\ &\le Ch\|\Phi\|_2 |\|\varepsilon_h\|| \le Ch|\|\varepsilon_h\||\|\varepsilon_o\|. \end{aligned}$$

According to the L^2-error estimates in [3], we have

$$| l_\mathbf{u}(Q_h\Phi) - \theta_p(Q_h\Phi) + S(Q_h\mathbf{u}, Q_h\Phi) | \le Ch^{i+1}(\|\mathbf{u}\|_{i+1} + \|p\|_i)\|\varepsilon_o\|.$$

Moreover, by using the trace inequality (2.4), (2.5) and Lemma 5.1, we obtain

$$
\begin{aligned}
&\mid r^1_{\mathbf{u},\mathbf{b}}(Q_h\Phi)\mid=\mid \frac{1}{2}\sum_{K\in\mathcal{K}_h}\langle \mathbf{b}\cdot\mathbf{n},(\mathbf{u}-Q_b\mathbf{u})\cdot(Q_o\Phi-Q_b\Phi)\rangle_{\partial K}\mid\\
&\le \frac{1}{2}\mid \sum_{K\in\mathcal{K}_h}\langle \mathbf{b}\cdot\mathbf{n},(\mathbf{u}-Q_b\mathbf{u})\cdot(Q_o\Phi-\Phi)\rangle_{\partial K}\mid+\frac{1}{2}\mid \sum_{K\in\mathcal{K}_h}\langle \mathbf{b}\cdot\mathbf{n},(\mathbf{u}-Q_b\mathbf{u})\cdot(Q_b\Phi-\Phi)\rangle_{\partial K}\mid\\
&\le \frac{1}{2}\left(\sum_{K\in\mathcal{K}_h}\|\mathbf{b}\|_{L^\infty(\partial K)}\right)\left(\sum_{K\in\mathcal{K}_h}\|Q_o\Phi-\Phi\|^2_{\partial K}\right)^{1/2}\left(\sum_{K\in\mathcal{K}_h}\|\mathbf{u}-Q_b\mathbf{u}\|^2_{\partial K}\right)^{1/2}\\
&\quad+\frac{1}{2}\left(\sum_{K\in\mathcal{K}_h}\|\mathbf{b}\|_{L^\infty(\partial K)}\right)\left(\sum_{K\in\mathcal{K}_h}\|Q_b\Phi-\Phi)\|^2_{\partial K}\right)^{1/2}\left(\sum_{K\in\mathcal{K}_h}\|\mathbf{u}-Q_b\mathbf{u}\|^2_{\partial K}\right)^{1/2}\\
&\le C\|\mathbf{b}\|_\infty\left(\sum_{K\in\mathcal{K}_h}h\|\nabla\Phi\|^2_{\partial K}\right)^{1/2}\left(\sum_{K\in\mathcal{K}_h}h\|\mathbf{u}-Q_b\mathbf{u}\|^2_{\partial K}\right)^{1/2}\le Ch^{i+1}\|\mathbf{u}\|_{i+1}\|\varepsilon_o\|.
\end{aligned}
$$

Similarly,

$$
\mid r^2_{\mathbf{u},\mathbf{b}}(Q_h\Phi)\mid\le Ch^{i+1}\|\mathbf{u}\|_{i+1}\|\varepsilon_o\|.
$$

Now, it is not hard to derive

$$
\begin{aligned}
&\mid -\frac{1}{2}(\nabla\cdot\mathbf{b},(Q_o\mathbf{u}-\mathbf{u})\cdot Q_o\Phi)-(\mathbf{b}\nabla Q_o\Phi,Q_o\mathbf{u}-\mathbf{u})-(c_0(\mathbf{u}-Q_o\mathbf{u}),Q_o\Phi)\mid\\
&\le \|\nabla\cdot\mathbf{b}\|_\infty\|Q_o\mathbf{u}-\mathbf{u}\|\|Q_o\Phi\|+\|\mathbf{b}\|_\infty\|\nabla Q_o\Phi\|\|Q_o\mathbf{u}-\mathbf{u}\|+\|c_0\|_\infty\|\mathbf{u}-Q_o\mathbf{u}\|\|Q_o\Phi\|\\
&\le Ch^{i+1}(\|\nabla\cdot\mathbf{b}\|_\infty+\|\mathbf{b}\|_\infty+\|c_0\|_\infty)\|\Phi\|_2\|\mathbf{u}\|_{i+1}\le Ch^{i+1}\|\varepsilon_o\|\|\mathbf{u}\|_{i+1}.
\end{aligned}
$$

Finally, substituting the above inequalities into Eq. 5.14 gives

$$
\|\varepsilon_o\|^2\le Ch^{i+1}(\|\mathbf{u}\|_{i+1}+\|p\|_i)\|\varepsilon_o\|+Ch|\|\varepsilon_h\||\|\varepsilon_o\|,
$$

which, together with Theorem 5.4, derives the desired result. □

6 Numerical experiments

In this section, we will give several numerical experiments to examine the stability and efficiency of the present method for the Oseen equations and prove that this new method is designed on finite element partitions consisting of arbitrary shapes of polygons or polyhedra which are shape regularity. For all the following examples, we choose $i = 1$ for the weak Galerkin finite element method, i.e., P_1 for the velocity in the interior of a finite element partition, P_0 for the velocity on the interface of the finite element partition, and P_0 for the pressure. The computational domain Ω can be designed as $[0, 1] \times [0, 1]$ and the boundary data and the source term are chosen according to the exact solution. In addition, in each refined triangular mesh, each grid cell is divided into four similar cells by connecting the edge midpoints; therefore, a mesh width h_L in grid level L is twice as long as that in grid level $L + 1$.

Table 1 Error results with Dirichlet data being approximated by the usual nodal point interpolation

h	$\|\|\|\varepsilon_h\|\|\|$	$\|\varepsilon_o\|$	$\|Q_h p - p_h\|$	$\|\varepsilon_b\|$	$\|\nabla_w \mathbf{u}_h - \nabla \mathbf{u}\|$
1/8	2.9220e+00	1.8638e-01	3.3487e-01	5.9264e-02	1.7512e+00
1/16	1.4579e+00	4.6442e-02	1.5352e-01	1.5813e-02	8.9289e-01
1/32	7.2850e-01	1.1594e-02	7.3901e-02	4.0259e-03	4.4881e-01
1/64	3.6419e-01	2.8975e-03	3.6517e-02	1.0110e-03	2.2471e-01
1/128	1.8209e-01	7.2430e-04	1.8198e-02	2.5297e-04	1.1239e-01
Rate	1.0011	2.0019	1.0505	1.9680	0.9905

Moreover, the norms in the following table can be introduced [5] to explain the error estimate results further:

$$\text{Edge based } L^2 \text{ norm} : \|\varepsilon_b\| = \left(\sum_{E \in E_h} h_K \int_E |\varepsilon_b|^2 ds \right)^{1/2},$$

$$\|\nabla_w \mathbf{u}_h - \nabla \mathbf{u}\| = \left(\sum_{K \in \mathcal{K}_h} \int_K |\nabla_w \mathbf{u}_h - \nabla \mathbf{u}|^2 dx \right)^{1/2},$$

where h_K stands for the size of the element K that takes E as an edge.

Example 6.1 The exact solution is given by

$$\begin{cases} u_1 = \cos(2\pi x)\sin(2\pi y), \\ u_2 = -\sin(2\pi x)\cos(2\pi y), \\ p = 0, \end{cases}$$

the convection field $\mathbf{b} = (1, 1)$, and $c = 1$. Then, when the boundary datum $\mathbf{u} = \mathbf{g}$ is approximated by different schemes (nodal interpolation schemes and L^2 projection), we can see the effect of treating the Dirichlet boundary data and the common optimal order of convergence for the weak Galerkin method in Tables 1 and 2. In each table, we list the errors and convergence rates between the exact solution and the approximate solution, which illustrate the validity of the theoretical analysis.

Table 2 Error results with Dirichlet data being approximated by L^2 Projection

h	$\|\|\|\varepsilon_h\|\|\|$	$\|\varepsilon_o\|$	$\|Q_h p - p_h\|$	$\|\varepsilon_b\|$	$\|\nabla_w \mathbf{u}_h - \nabla \mathbf{u}\|$
1/8	2.9203e+00	1.8011e-01	3.6176e-01	6.4293e-02	1.7480e+00
1/16	1.4577e+00	4.4762e-02	1.5882e-01	1.7557e-02	8.9245e-01
1/32	7.2847e-01	1.1165e-02	7.4691e-02	4.5099e-03	4.4875e-01
1/64	3.6419e-01	2.7895e-03	3.6621e-02	1.1360e-03	2.2470e-01
1/128	1.8209e-01	6.9727e-04	1.8211e-02	2.8455e-04	1.1239e-01
Rate	1.00085	2.003225	1.0781	1.9550	0.9898

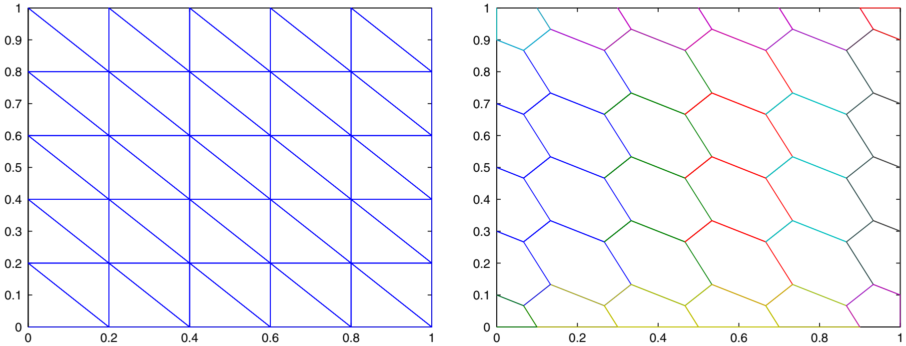

Fig. 1 triangle mesh (*left*) and hexagon mesh (*right*)

Example 6.2 The exact solution is given by

$$\begin{cases} u_1 = 2x^2(x-1)^2 y(y-1)(2y-1), \\ u_2 = -2y^2(y-1)^2 x(x-1)(2x-1), \\ p = x^3 + y^3 - 0.5, \end{cases}$$

the convection field

$$\mathbf{b} = \begin{pmatrix} \sin(x)\sin(y) \\ \cos(x)\cos(y) \end{pmatrix},$$

and $c = 100$ (the choice of c corresponds to a length of the time step of 0.01 in the nonstationary Navier-Stokes equations). Then, because of the properties of polynomials, for the boundary datum $\mathbf{u} = \mathbf{g}$ approximated by a nodal interpolation scheme or the L^2 projection, we have the same errors and convergence rates.

Moreover, since this new method has an attractive computational features: more general finite element partitions of arbitrary polygons or polyhedra with certain shape regularity as shown in Fig. 1, we compare the numerical results in triangular elements with those in hexagon elements in Tables 3 and 4, which also prove the theoretical analysis.

Table 3 Error results for triangular mesh

h	$\|\|\|\varepsilon_h\|\|\|$	$\|\varepsilon_o\|$	$\|Q_h p - p_h\|$	$\|\varepsilon_b\|$	$\|\nabla_w \mathbf{u}_h - \nabla \mathbf{u}\|$
1/8	1.0345e-01	5.5221e-03	1.1784e-01	2.9341e-03	3.5902e-02
1/16	5.8998e-02	1.8377e-03	3.9930e-02	1.0957e-03	2.4431e-02
1/32	3.0770e-02	5.0574e-04	1.1308e-02	3.1871e-04	1.3649e-02
1/64	1.5568e-02	1.2999e-04	3.0168e-03	8.3605e-05	7.0526e-03
1/128	7.8085e-03	3.2746e-05	8.0920e-04	2.1202e-05	3.5586e-03
Rate	0.9319	1.8494	1.7966	1.7782	0.8337

Table 4 Error results for hexagon mesh

h	$\|\|\|\varepsilon_h\|\|\|$	$\|\varepsilon_o\|$	$\|Q_h p - p_h\|$	$\|\varepsilon_b\|$	$\|\nabla_w \mathbf{u}_h - \nabla \mathbf{u}\|$
1/8	1.0627e-01	5.9165e-03	1.5030e-01	5.2274e-03	2.7770e-02
1/16	6.3929e-02	2.2098e-03	6.0358e-02	2.1433e-03	1.5106e-02
1/32	3.4446e-02	6.5742e-04	2.1956e-02	6.6916e-04	6.8099e-03
1/64	1.7692e-02	1.7535e-04	8.7760e-03	1.8199e-04	2.9440e-03
1/128	8.9313e-03	4.4864e-05	4.0092e-03	4.6906e-05	1.3228e-03
Rate	0.89316	1.7608	1.3071	1.7001	1.0980

Example 6.3 In this test, we choose the two-dimensional analytical solution of the incompressible Navier-Stokes equations and take $\mathbf{b} = \mathbf{u}$, $c = 0$; then, with our choice of $\mathbf{b}$ and c, we have that $\nabla \cdot \mathbf{b} = 0$ and $c_0 = c = 0$. The exact solution is given by

$$\begin{cases} u_1 = x^3 + x^2 y + x^2 - 3xy^2 - 2xy + x, \\ u_2 = -3x^2 y - xy^2 - 2xy + y^3 + y^2 - y, \\ p = x^3 y^2 + xy + x + y - \frac{4}{3}. \end{cases}$$

The errors and the order of convergence are presented in Table 5. Again, the computational results agree with the theoretical results.

Example 6.4 In the previous examples, we always consider the situation $\nabla \cdot \mathbf{b} = 0$. Now, we choose the convection field

$$\mathbf{b} = \begin{pmatrix} x \\ y \end{pmatrix},$$

and $c = xy + 1$, which satisfy that $\nabla \cdot \mathbf{b} \neq 0$ and $c_0(\mathbf{x}) \geq 0$. Moreover, we choose the exact solution

$$\begin{cases} u_1 = 10x^2(x-1)^2 y(y-1)(2y-1), \\ u_2 = -10y^2(y-1)^2 x(x-1)(2x-1), \\ p = 10(2x-1)(2y-1). \end{cases}$$

Table 5 Error results with Dirichlet data being approximated by the usual nodal point interpolation or L^2 projection

h	$\|\|\|\varepsilon_h\|\|\|$	$\|\varepsilon_o\|$	$\|Q_h p - p_h\|$	$\|\varepsilon_b\|$	$\|\nabla_w \mathbf{u}_h - \nabla \mathbf{u}\|$
1/8	4.7954e-01	2.9656e-02	3.9168e-01	2.9724e-02	5.3178e-01
1/16	2.4274e-01	7.7408e-03	1.9180e-01	8.4815e-03	2.7041e-01
1/32	1.2195e-01	1.9902e-03	9.4275e-02	2.2786e-03	1.3596e-01
1/64	6.1068e-02	5.0759e-04	4.6736e-02	5.9806e-04	6.8094e-02
1/128	3.0547e-02	1.2904e-04	2.3282e-02	1.5556e-04	3.4062e-02
Rate	0.99312	1.9611	1.0181	1.8945	0.99117

Table 6 Error results with Dirichlet data being approximated by the usual nodal point interpolation or L^2 projection

h	$\|\|\|\varepsilon_h\|\|\|$	$\|\varepsilon_o\|$	$\|Q_h p - p_h\|$	$\|\varepsilon_b\|$	$\|\nabla_w \mathbf{u}_h - \nabla \mathbf{u}\|$
1/8	1.0691e+00	7.3870e-02	4.5737e-01	5.1657e-02	4.9260e-01
1/16	5.4982e-01	1.9916e-02	1.6234e-01	1.6097e-02	2.7913e-01
1/32	2.7804e-01	5.1307e-03	5.1243e-02	4.3478e-03	1.4581e-01
1/64	1.3957e-01	1.2963e-03	1.5302e-02	1.1158e-03	7.3979e-02
1/128	6.9876e-02	3.2518e-04	4.4850e-03	2.8133e-04	3.7162e-02
Rate	0.9839	1.9569	1.6680	1.8801	0.9321

The results are reported in Table 6 for the Dirchlet boundary data approximated by the usual nodal point interpolation/L^2 projection, which show optimal rates of convergence in all norms for the present method.

7 Conclusions

In this paper, we have extended the weak Galerkin finite element method proposed for the Stokes system to the Oseen equations on arbitrary polygons or polyhedra with certain shape regularity. A stability analysis and error estimates have been performed and the numerical examples have shown that this method is stable and efficient for the Oseen equations.

Future work will be focused on the extension of the present method to the stationary and nonstationary Navier-Stokes equations both from a numerical and a theoretical standpoint. Furthermore, from a practical point of view, numerical methods will be developed for more general problems with different boundary conditions.

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References

1. Wang, J., Ye, X.: A weak Galerkin finite element method for second-order elliptic problems. J. Comp. Appl Math. **241**, 103–115 (2013)
2. Wang, J., Ye, X.: A weak Galerkin mixed finite element method for second-order elliptic problems. Math. Comp. **83**, 2101–2126 (2014)
3. Wang, J., Ye, X.: A weak Galerkin finite element method for the Stokes equations. Adv. Comput. Math. **42**, 155–174 (2016)
4. Mu, L., Wang, J., Ye, X.: Weak Galerkin finite element methods on Polytopal Meshes. Int. J. Numer. Anal Mod. **12**, 31–53 (2015)
5. Mu, L., Wang, J., Wang, Y., Ye, X.: A computational study of the weak Galerkin method for second-order elliptic equations. Numer. Algor. **63**, 753–777 (2013)

6. Mu, L., Wang, J., Ye, X.: A weak Galerkin finite element method with polynomial reduction. J. Comp. Appl. Math. **285**, 45–58 (2015)
7. Girault, V., Raviart, P.: Finite Element Methods for Navier-Stokes Equations. Springer Series in Computational Mathematics. Springer, Berlin (1986)
8. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)
9. Temam, R.: Navier-Stokes Equations, Theory and Numerical Analysis. Reprint of the 1984 Edition. AMS Chelsea Publishing, Providence (2001)
10. Chen, Z.: Finite Element Methods and Their Applications. Scientific Computation. Springer, Berlin (2005)
11. Mu, L., Wang, J., Wei, G., Ye, X., Zhao, S.: Weak Galerkin methods for second order elliptic interface problems. J. Comp. Phys. **250**, 106–125 (2013)
12. Wang, C., Wang, J.: An efficient numerical scheme for the biharmonic equation by weak Galerkin finite element methods on polygonal or polyhedral meshes. Comput. Math Appl. **68**, 2314–2330 (2014)
13. Wang, C., Wang, J.: A hybridized weak Galerkin finite element method for the Biharmonic equation. Int. J. Numer. Anal. Mod. **12**, 302–317 (2015)
14. Mu, L., Wang, J., Ye, X.: Weak Galerkin finite element method for the Biharmonic equation on polytopal meshes. Numer. Meth. Part. D E **30**, 1003–1029 (2014)
15. Chen, W., Wang, F., Wang, Y.: Weak Galerkin method for the coupled Darcy-Stokes flow. IMA. J. Numer. Anal. **36**, 897–921 (2016)
16. Mu, L., Wang, J., Ye, X.: A new weak Galerkin finite element method for the Helmholtz equation. IMA J. Numer. Anal. **35**, 1228–1255 (2015)
17. Lin, G., Liu, J., Mu, L., Ye, X.: Weak Galerkin finite element methods for Darcy flow Anisotropy and heterogeneity. J. Comp. Phys. **276**, 422–437 (2014)
18. Mu, L., Wang, J., Ye, X.: A stable numerical algorithm for the Brinkman equations by weak Galerkin finite elemnt methods. J. Comp. Phys. **273**, 327–342 (2014)
19. Gao, F., Mu, L.: On error estimate for weak Galerkin finite element methods for parabolic problems. J. Comput. Math. **32**, 195–204 (2014)
20. Gao, F., Wang, X.: A modified weak Galerkin finite element method for a class of parabolic problems. J. Comp. Appl. Math. **271**, 1–19 (2014)
21. Li, Q., Wang, J.: Weak Galerkin finite element methods for parabolic equations. Numer. Meth. Part. D E **29**, 2004–2024 (2013)
22. Mu, L., Wang, J., Ye, X., Zhao, S.: A numerical study on the weak Galerkin method for the Helmholtz equation. Commun. Comput. Phys. **15**, 1461–1479 (2014)
23. Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods. Springer Series in Computational Mathematics. Springer, New York (1991)