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Unconditional convergence and optimal error estimates of the Euler semi-implicit scheme for a generalized nonlinear Schrödinger equation

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Abstract In this paper, we focus on a linearized backward Euler scheme with a Galerkin finite element approximation for the time-dependent nonlinear Schrödinger equation. By splitting an error estimate into two parts, one from the spatial discretization and the other from the temporal discretization, we obtain unconditionally optimal error estimates of the fully-discrete backward Euler method for a generalized nonlinear Schrödinger equation. Numerical results are provided to support our theoretical analysis and efficiency of this method.

Keywords Unconditional convergence · Optimal error estimate · Backward Euler method · Galerkin finite element method · Time-dependent Schrödinger equation

Mathematics Subject Classification (2010) 65N30

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1 Introduction

Nonlinear Schrödinger equation is a classical partial differential equation that has been applied in modelling the evolution of a wave packet in a nonlinear and dispersive medium. It has been derived in many fields; i.e., nonlinear optics [23, 25], plasma physics [12] and water wave [1, 12, 23, 25]. Moreover, this equation can be used in pattern formulation, where it can model many non-equilibrium pattern forming systems. For example, it can be developed for the optics field as a model for optical pulse propagation in nonlinear fibers [4].

In the past several decades, numerous effort has been devoted to mathematical study of nonlinear Schrödinger equation. There are two main branches of mathematical research for nonlinear Schrödinger equation. One is construction of exact solution, which includes the trial function method [5], Jacobi elliptic function expansion method [8, 30], and G'/G-expansion method [19]. Another important branch is numerical approximation method, including the finite element method, finite difference method and spectral method. Many authors have studied a considerable number of numerical methods for nonlinear Schrödinger equation. For example, Delfour [6], Bao *et al.* [3] and Reichel [26] on the finite difference method, Akrivis [2], Tourigny [28], Sanz-Serna [27], and Zouraris [31] on the finite element method and Feit [10] on the spectral method. In [6], Delfour presented a finite difference method to approximate a Schrödinger equation. The main feature of this method given by Delfour is that it satisfies a discrete analogue of an important conservation law of this equations. In [28], Tourigny obtained optimal H^1 estimates for the fully-implicit backward Euler scheme and Crank-Nicolson scheme for a nonlinear Schrödinger equation by applying a nonlinear stability theory. But these optimal H^1 error estimates required the time step conditions $\Delta t = o(h^{\frac{d}{2}})$ and $\Delta t = o(h^{\frac{d}{4}})$ for the two schemes, respectively, where d represents the dimension of space.

Several time-discrete methods have been widely used to time-dependent nonlinear partial differential equations (PDEs), for example, fully-implicit, semi-implicit, and explicit. The fully-implicit time-discrete Euler method for PDEs must solve nonlinear equations at every time step and needs inner iterations. Compared to the fully-implicit method, the explicit and semi-implicit time-discrete methods have been widely used because they just need to solve a linear system at each time step. However, in order to obtain error estimates of the explicit and semi-implicit methods for PDEs, one usually needs the boundedness of their fully-discrete solution in the L^∞ norm. For this purpose, many authors employ the mathematical induction with an inverse inequality to obtain

$$\left\|U_h^n - R_h u^n\right\|_{L^\infty} \leq C h^{-d/2} \left\|U_h^n - R_h u^n\right\|_{L^2} \leq C h^{-d/2}(\Delta t + h^{r+1}),$$

where U_h^n is a numerical solution, u^n is the exact solution, R_h is a projection operator, r is the degree of the fully discrete Galerkin finite element method, Δt is a time step, and h is a spatial step. However, the above inequality results in a restriction on Δt and h; see [3, 13, 16, 17, 24]. That is, optimal error estimates are obtained under a time-step condition. This condition may result in the use of small time steps. Thus the computational complexity is increased extremely in practice. Recently, a new analytic method is presented in [11, 20–22]. The main approach of these papers is to split an error estimate into a time-discrete error estimate and a spatial-discrete error estimate. The spatial-discrete error is obtained by discretizing the temporal discrete equation, independent of time-step Δt.

In this paper, we apply this approach to study a generalized nonlinear Schrödinger equation by r-order FEMs (finite element methods, $r \geq 1$) and the backward Euler time-discrete method. As the regularity of the time-discrete solution U^n of this Schrödinger equation is obtained, an error estimate of the fully-discrete solution is established without any time step restriction by using the mathematical induction and an inverse inequality:

$$\left\| U_h^n - R_h U^n \right\|_{L^\infty} \leq Ch^{-d/2} \left\| U_h^n - R_h U^n \right\|_{L^2} \leq Ch^{r+1-d/2},$$

where U^n is the numerical solution of the time-discrete Schrödinger equation and R_h is a Ritz operator. Due to the above boundedness of U_h^n in the L^∞ norm, we can obtain the L^2 norm optimal error estimate without any restriction on Δt and h in the traditional way [28]:

$$\left\| u^m - U_h^m \right\|_{L^2} \leq C^{**}(\Delta t + h^{r+1}).$$

The outline of this paper is as follows: In Section 2, a function setting of the Schrödinger equation is introduced, together with some basic assumptions. Moreover, we present a backward Euler FEM for a generalized nonlinear Schrödinger equation. In Section 3, the regularity of a time-discrete numerical solution is obtained. Meanwhile, the boundedness of the time-discrete numerical solution in the L^∞ norm is established. In Section 4, the unconditionally optimal L^2 norm error estimate of the fully-discrete FEM is obtained. In Section 5, we give two numerical experiments to validate our theoretical analysis. Finally, conclusions are drawn in Section 6.

2 Preliminaries

In this section, we focus on a nonlinear Schrödinger equation defined by

$$\begin{cases} iu_t + \Delta u + f(|u|^2)u = 0, & x \in \Omega,\ 0 < t \leq T, \\ u(x, 0) = u_0(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where u is a complex-valued function defined in $\Omega \times [0, T]$ and $\Omega \subset R^2$ is a bounded domain with boundary $\partial\Omega$. Meanwhile, we assume that $f : R \mapsto R$ is a given function belonging to $C^2(R)$.

Let Γ_h be a regular partition of Ω into triangles T_j, $j = 1, 2, ..., M$, in R^2, and $h = \max_{1 \leq j \leq M} \{diam T_j\}$ be the mesh size. If a triangle T_j is on the boundary, we define $\widetilde{T}_j$ as the triangle with one curved side. If T_j is an interior element, we set $\widetilde{T}_j$=T_j.

From the above definition, for a given division Γ_h, we define the finite element spaces

$$V_h = \{v_h \in C(\overline{\Omega}) : v_h|_{T_j} \text{ is a polynomial of degree } r \text{ and } v_h = 0 \ on \ \partial\Omega\},$$
$$S_h = \{v_h \in C(\overline{\Omega}) : v_h|_{\widetilde{T}_j} \text{ is a polynomial of degree } r\},$$

where $\partial\Omega$ is defined by $\widetilde{T}_j \backslash T_j$.

From the above definitions, we can find that V_h is a subspace of $H_0^1(\Omega_h)$ and S_h is a subspace of $H^1(\Omega_h)$. We define $\pounds$:=$\{\pounds v$: $\pounds v = 0$ on $\partial\Omega$; $\pounds v = v$ on T_j $\forall v \in S_h\}$. Moreover, we set $F : C(\overline{\Omega}) \longrightarrow S_h$ to be the Lagrangian operator and define $\Pi_h := \pounds F$. Obviously, Π_h is a projection operator from $C(\overline{\Omega}) \longrightarrow V_h$.

For $u, v \in L^2(\Omega)$, we define the L^2 inner product as follows:

$$(u, v) = \int_\Omega u(x)\overline{v(x)}dx,$$

where $\overline{v}$ is the conjugate of v. By an interpolation theory, we obtain

$$\|\Pi_h v - v\|_{L^2} + h\,\|\nabla(\Pi_h v - v)\|_{L^2} \le Ch^{r+1}\,\|v\|_{H^{r+1}}\,, \tag{2.2}$$

where $C > 0$ is a constant. Subsequently, for simplicity, C (with or without a subscript) will denote a positive constant depending only on Ω, which may stand for a different value at its different occurrence.

Assume that R_h: $H_0^1(\Omega) \to V_h$ is a Ritz projection operator defined by

$$(\nabla(v - R_h v), \nabla w) = 0 \quad \forall w \in V_h. \tag{2.3}$$

By the classical finite element theory [7, 29], it holds that

$$\|v - R_h v\|_{L^2} + h\,\|\nabla(v - R_h v)\|_{L^2} \le Ch^s\,\|v\|_{H^s}\,, \quad 1 \le s \le r+1, \quad v \in H_0^1(\Omega) \tag{2.4}$$

and the inverse inequality holds:

$$\|v\|_{L^\infty} \le Ch^{-\frac{d}{2}}\,\|v\|_{L^2}\,, \quad d = 2, 3, \quad v \in V_h. \tag{2.5}$$

Let $\Delta t > 0$ be the time step and $t^n = n\Delta t, n = 0, 1, ..., N$, where $t^N = T$. We denote $u^n = u(x, t_n)$. For a sequence $\{z^n\}_{n=0}^N$, we define

$$D_t z^n = \frac{z^n - z^{n-1}}{\Delta t}, \quad n = 1, 2..., N.$$

With an explicit treatment of the nonlinear term, an Euler semi-implicit scheme is to find $U_h^n \in V_h$ such that

$$i(D_t U_h^n, v) - (\nabla U_h^n, \nabla v) + (f(|U_h^{n-1}|^2)U_h^n, v) = 0, \quad n = 1, 2, ...N, \tag{2.6}$$

for any $v \in V_h$, where $U_h^0 = \Pi_h u_0$.

Meanwhile, we define U^n to be the solution of the following time-discrete system:

$$iD_t U^n + \Delta U^n + f(|U^{n-1}|^2)U^n = 0, \quad n = 1, 2, ...N, \tag{2.7}$$

with the boundary and initial conditions

$$\begin{cases} U^n(x) = 0 & \forall x \in \partial\Omega,\ n = 1, 2, ..., N, \\ U^0(x) = u_0(x) & \forall x \in \Omega. \end{cases} \tag{2.8}$$

The key to our proof in this paper is the following error splitting [11, 20–22]:

$$\left\|U_h^n - u^n\right\| \leq \left\|e^n\right\| + \left\|e_h^n\right\| + \left\|U^n - R_h U^n\right\|$$

for any norm $\|\cdot\|$, where

$$e^n = U^n - u^n, \quad e_h^n = R_h U^n - U_h^n.$$

Below e^n and e_h^n are always defined by the above forms.

Lemma 2.1 *(Discrete Gronwall's inequality* [14, 15, 18]*) Let Δt, B, a_k, b_k, c_k and γ_k, for integers $k \geq 0$, be non-negative numbers such that*

$$a_n + \Delta t \sum_{k=0}^{n} b_k \leq \Delta t \sum_{k=0}^{n} \gamma_k a_k + \Delta t \sum_{k=0}^{n} c_k + B, \quad n \geq 0. \tag{2.9}$$

Suppose that $\Delta t \gamma_k < 1$, and set $\sigma_k = (1 - \Delta t \gamma_k)^{-1}$. Then

$$a_n + \Delta t \sum_{k=0}^{n} b_k \leq exp(\Delta t \sum_{k=0}^{n} \gamma_k \sigma_k)(\Delta t \sum_{k=0}^{n} c_k + B), \quad n \geq 0. \tag{2.10}$$

Throughout this paper we make the following assumption on the prescribed data for problem (2.1), which specifies the regularity of the sultion for our main results.

Assumption (A1): The solution to the initial/boundary value problem (2.1) exists and satisfies

$$\|u_0\|_{H^{r+1}} + \|u\|_{L^\infty((0,T);H^{r+1})} + \|u_t\|_{L^\infty((0,T);H^{r+1})} + \|u_{tt}\|_{L^2((0,T);L^2)} \leq C,$$

where C is a positive constant, which depends only on Ω.

3 Temporal error estimates

In this section, we establish an error bound for $\|u^n - U^n\|_{L^2}$ and the boundedness of the time discrete numerical solution in the L^∞ norm.

We assume that u is the solution of system (2.1). Then we see that

$$iD_t u^n + \Delta u^n + f(|u^{n-1}|^2)u^n = iD_t u^n - iu_t^n + f(|u^{n-1}|^2)u^n - f(|u^n|^2)u^n. \tag{3.1}$$

Let $P^n = iD_t u^n - iu_t^n + f(|u^{n-1}|^2)u^n - f(|u^n|^2)u^n$ and $K = \max_{0 \leq n \leq N} \|u^n\|_{L^\infty} + 1$. Due to the regularity assumption **Assumption(A1)**, it is easy to see that

$$(\sum_{n=1}^{N} \Delta t \left\|P^n\right\|_{L^2}^2)^{\frac{1}{2}} + \Delta t \left\|P^n\right\|_{L^2} \leq C \Delta t \quad \forall v \in V_h, \tag{3.2}$$

where we applying

$$|f(|u^n|^2)| + |f'(\xi)| \leq C_L \tag{3.3}$$

for $|\xi| \leq \|u^n\|_{L^\infty}^2 + \|u^{n-1}\|_{L^\infty}^2 \leq 2K^2$.

Theorem 3.1 *Suppose that (2.1) has a unique solution u satisfying* ***Assumption (A1)****. Then there exists a positive constant* τ *such that when* $\Delta t < \tau_0$*, the time-discrete system defined in Eq. 2.7 has a unique solution* U^n*, n=1,...,N, such that*

$$\left\|U^n\right\|_{H^2} \leq C_0, \tag{3.4}$$

$$\left\|e^n\right\|_{L^2} + \left\|e^n\right\|_{H^1} + (\Delta t)^{\frac{1}{2}} \left\|e^n\right\|_{H^2} \leq C_0 \Delta t. \tag{3.5}$$

Proof System (2.6) is a linear elliptic equation. Following the classical theory of PDEs, we can find that the solution of system (2.6) exists and is unique. From Eqs. 3.1 and 2.7, we see that

$$i \frac{e^1}{\Delta t} + \Delta e^1 + f(|u_0|^2) e^1 = P^1. \tag{3.6}$$

Multiplying Eq. 3.6 by e^1 and integrating the result over Ω, we have

$$i \frac{\|e^1\|_{L^2}^2}{\Delta t} - \left\|\nabla e^1\right\|_{L^2}^2 + (f(|u_0|^2)e^1, e^1) = (P^1, e^1). \tag{3.7}$$

Taking the imaginary parts of the above equation and applying (3.2), we obtain

$$\left\|e^1\right\|_{L^2} \leq \Delta t \left\|P^1\right\|_{L^2} \leq C \Delta t. \tag{3.8}$$

Meanwhile, multiplying Eq. 3.6 by Δe^1 and integrating it over Ω, we get

$$-i \frac{\|\nabla e^1\|_{L^2}^2}{\Delta t} + \left\|\Delta e^1\right\|_{L^2}^2 + (f(|u_0|^2)e^1, \Delta e^1) = (P^1, \Delta e^1), \tag{3.9}$$

which implies

$$\frac{\|\nabla e^1\|_{L^2}^2}{\Delta t} \leq \frac{1}{4} \|\Delta e\|_{L^2}^2 + 2 \left\|P^1\right\|_{L^2}^2 + 2 \left\|f(|u_0|^2)e^1\right\|_{L^2}^2 \tag{3.10}$$

and

$$\left\|\Delta e^1\right\|_{L^2}^2 \leq \frac{1}{4} \left\|\Delta e^1\right\|_{L^2}^2 + 2 \left\|P^1\right\|_{L^2}^2 + 2 \left\|f(|u_0|^2)e^1\right\|_{L^2}^2. \tag{3.11}$$

Summing Eqs. 3.10 and 3.11, we obtain

$$\frac{\|\nabla e^1\|_{L^2}^2}{\Delta t} + \frac{1}{2} \left\|\Delta e^1\right\|_{L^2}^2 \leq 4 \left\|f(|u_0|^2)e^1\right\|_{L^2}^2 + 4 \left\|P^1\right\|_{L^2}^2. \tag{3.12}$$

Thus, applying Eqs. 3.2, 3.3 and 3.8, we see that

$$\left\|\nabla e^1\right\|_{L^2}^2 + \Delta t \left\|\Delta e^1\right\|_{L^2}^2 \leq C_1 \Delta t^2. \tag{3.13}$$

Thanks to the Dirichlet boundary condition, we have

$$(\Delta t)^{\frac{1}{2}} \left\|e^1\right\|_{H^2} \leq C_2 \Delta t. \tag{3.14}$$

Using **Assumption(A1)** and Eq. 3.14, we see that

$$\left\|U^1\right\|_{L^\infty} \le \left\|u^1\right\|_{L^\infty} + \left\|u^1 - U^1\right\|_{L^\infty} \le \left\|u^1\right\|_{L^\infty} + C\left\|e^1\right\|_{H^2} \le K, \tag{3.15}$$

$$\|U^1\|_{H^2} \le \|u^1\|_{H^2} + \|e^1\|_{H^2} \le C_3, \tag{3.16}$$

when $\Delta t \le \tau_1 = \frac{1}{C^2C_2^2}$. Thus, Eqs. 3.4–3.5 hold for m=1.

We assume that Eqs. 3.4 and 3.5 hold for $m \le n-1$. Then

$$\left\|U^m\right\|_{L^\infty} \le \left\|u^m\right\|_{L^\infty} + C\left\|e^m\right\|_{H^2} \le K, \tag{3.17}$$

when $\Delta t \le \tau_2 = \frac{1}{C^2C_0^2}$.

Subtracting Eq. 2.7 from Eq. 3.1 results in the error equation

$$iD_te^n + \Delta e^n + R^n = P^n, \tag{3.18}$$

where

$$R^n = [(f(|u^{n-1}|^2) - f(|U^{n-1}|^2))u^n] + [f(|U^{n-1}|^2)e^n].$$

By the mathematical induction and **Assumption (A1)**, we obtain

$$\begin{aligned}\left\|R^n\right\|_{L^2} &\le \|f'(\xi_1)e^{n-1}(|u^{n-1}| + |U^{n-1}|)u^n\|_{L^2} + \|f(|U^{n-1}|^2)e^n\|_{L^2}\\ &\le C(\left\|e^{n-1}\right\|_{L^2} + \left\|e^n\right\|_{L^2}),\end{aligned} \tag{3.19}$$

where we apply $|f'(\xi_1)| + |f(|U^{n-1}|^2)| \le C_{L1}$ for $|\xi_1| \le \|u^{n-1}\|_{L^\infty}^2 + \|U^{n-1}\|_{L^\infty}^2 \le 2K^2$.

Multiplying Eq. 3.18 by e^n and integrating it over Ω lead to

$$\frac{i}{2\Delta t}(\left\|e^n\right\|_{L^2}^2 - \left\|e^{n-1}\right\|_{L^2}^2 + \left\|e^n - e^{n-1}\right\|_{L^2}^2) - \left\|\nabla e^n\right\|_{L^2}^2 = (P^n, e^n) - (R^n, e^n). \tag{3.20}$$

Taking the imaginary parts of the above equation and applying the Young inequality, we obtain

$$\frac{1}{\Delta t}(\left\|e^n\right\|_{L^2}^2 - \left\|e^{n-1}\right\|_{L^2}^2) \le \frac{1}{2}\left\|P^n\right\|_{L^2}^2 + \frac{1}{2}\left\|R^n\right\|_{L^2}^2 + \left\|e^n\right\|_{L^2}^2. \tag{3.21}$$

Summing inequalities (3.21) up and using Eqs. 3.19, 3.2, and the Young inequality, there exists $\tau_3 > 0$ such that

$$\left\|e^n\right\|_{L^2} \le C_4\Delta t, \tag{3.22}$$

when $\Delta t \le \tau_3$. Moreover, multiplying Eq. 3.18 by D_te^n and integrating it over Ω lead to

$$i\left\|D_te^n\right\|_{L^2}^2 - (\nabla e^n, D_t\nabla e^n) + (R^n, D_te^n) = (P^n, D_te^n). \tag{3.23}$$

Taking the real parts, we obtain

$$\frac{1}{2\Delta t}(\left\|\nabla e^n\right\|_{L^2}^2 - \left\|\nabla e^{n-1}\right\|_{L^2}^2) \le |Re(R^n, D_te^n)| + |Re(P^n, D_te^n)|. \tag{3.24}$$

Multiplying Eq. 3.18 by R^n, we can get

$$i(D_te^n, R^n) - (\nabla e^n, \nabla R^n) + \left\|R^n\right\|_{L^2}^2 = (P^n, R^n). \tag{3.25}$$

Thanks to the above equation, Eqs. 3.19, 3.22 and the Young inequality, we get

$$
\begin{aligned}
|Re(D_t e^n, R^n)| &\le |Im(P^n, R^n)| + |Im(\nabla R^n, \nabla e^n)| \\
&\le \frac{1}{2}\left\|\nabla R^n\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla e^n\right\|_{L^2}^2 + \frac{1}{2}\left\|P^n\right\|_{L^2}^2 + \frac{1}{2}\left\|R^n\right\|_{L^2}^2 \\
&\le C(\left\|\nabla e^n\right\|_{L^2}^2 + \left\|\nabla e^{n-1}\right\|_{L^2}^2) + \frac{1}{2}\left\|P^n\right\|_{L^2}^2 + C\Delta t^2. \quad (3.26)
\end{aligned}
$$

Meanwhile, the second term on the right-hand side of inequality (3.24) can be rewritten as

$$
\begin{aligned}
|Re(P^n, D_t e^n)| &= |Re(iD_t u^n - iu_t^n + f(|u^{n-1}|^2)u^n - f(|u^n|^2)u^n, D_t e^n)| \\
&\le |(D_t u^n - u_t^n, D_t e^n)| + |(f(|u^{n-1}|^2)u^n - f(|u^n|^2)u^n, D_t e^n)|.
\end{aligned} \quad (3.27)
$$

Multiplying Eq. 3.18 by $D_t u^n - u_t^n$ and integrating it over Ω lead to

$$
i(D_t u^n - u_t^n, D_t e^n) - (\nabla e^n, \nabla(D_t u^n - u_t^n)) + (R^n, D_t u^n - u_t^n) = (P^n, D_t u^n - u_t^n). \quad (3.28)
$$

Thanks to the above equation, Eqs. 3.19, 3.22 and the Young inequality, we can get

$$
\begin{aligned}
|(D_t u^n - u_t^n, D_t e^n)| &\le \frac{1}{2}\left\|\nabla e^n\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla(D_t u^n - u_t^n)\right\|_{L^2}^2 + \frac{1}{2}\left\|R^n\right\|_{L^2}^2 \\
&\quad + \frac{1}{2}\left\|P^n\right\|_{L^2}^2 + \left\|(D_t u^n - u_t^n)\right\|_{L^2}^2 \\
&\le \frac{1}{2}\left\|\nabla e^n\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla(D_t u^n - u_t^n)\right\|_{L^2}^2 + \frac{1}{2}\left\|P^n\right\|_{L^2}^2 + C\Delta t^2.
\end{aligned} \quad (3.29)
$$

Multiplying Eq. 3.18 by $f(|u^{n-1}|^2)u^n - f(|u^n|^2)u^n$ and integrating it over Ω lead to

$$
\begin{aligned}
& i(D_t e^n, f(|u^{n-1}|^2)u^n - f(|u^n|^2)u^n) + (\Delta e^n, f(|u^{n-1}|^2)u^n - f(|u^n|^2)u^n) \\
& + (R^n, f(|u^{n-1}|^2)u^n - f(|u^n|^2)u^n) = (P^n, f(|u^{n-1}|^2)u^n - f(|u^n|^2)u^n).
\end{aligned}
$$

Thanks to the above equation, Eqs. 3.19, 3.22, **Assumption(A1)**, Eq. 3.3, and the Young inequality, we can get the following inequality:

$$
\begin{aligned}
& |(D_t e^n, f(|u^{n-1}|^2)u^n - f(|u^n|^2)u^n)| \\
\le\ & \frac{1}{2}\left\|\nabla e^n\right\|_{L^2}^2 + \frac{1}{2}\left\|f(|u^{n-1}|^2)u^n - f(|u^n|^2)u^n\right\|_{H^1}^2 + \frac{1}{2}\left\|R^n\right\|_{L^2}^2 \\
& + \frac{1}{2}\left\|P^n\right\|_{L^2}^2 + \left\|f(|u^{n-1}|^2)u^n - f(|u^n|^2)u^n\right\|_{L^2}^2 \\
=\ & \frac{1}{2}\left\|\nabla e^n\right\|_{L^2}^2 + \frac{1}{2}\|f'(\xi)(u^{n-1} - u^n)(|u^{n-1}| + |u^n|)u^n\|_{H^1}^2 + \frac{1}{2}\left\|R^n\right\|_{L^2}^2 \\
& + \frac{1}{2}\left\|P^n\right\|_{L^2}^2 + \|f'(\xi)(u^{n-1} - u^n)(|u^{n-1}| + |u^n|)u^n\|_{L^2}^2 \\
\le\ & \frac{1}{2}\left\|\nabla e^n\right\|_{L^2}^2 + \frac{1}{2}\left\|P^n\right\|_{L^2}^2 + C\Delta t^2.
\end{aligned} \quad (3.30)
$$

From Eqs. 3.24, 3.26, 3.27, 3.29, and 3.30, we can obtain the following result:

$$\|\nabla e^n\|_{L^2}^2 - \left\|\nabla e^{n-1}\right\|_{L^2}^2 \leq C\Delta t(\|\nabla e^n\|_{L^2}^2 + \left\|\nabla e^{n-1}\right\|_{L^2}^2) + 3\Delta t\,\|P^n\|_{L^2}^2$$
$$+C\Delta t^3 + \Delta t\,\|D_t u^n - u_t^n\|_{H^1}^2 .$$

Summing the above inequalities up and using the Gronwall inequality, Eq. 3.2, there exists $\tau_4 > 0$ such that

$$\|\nabla e^n\|_{L^2} \leq C_5 \Delta t, \tag{3.31}$$

when $\Delta t \leq \tau_4$. Multiplying Eq. 3.18 by Δe^n and integrating it over Ω lead to

$$-i\frac{\|\nabla e^n\|_{L^2}^2}{\Delta t} + i\frac{(\nabla e^{n-1}, \nabla e^n)}{\Delta t} + \|\Delta e^n\|_{L^2}^2 + (R^n, \Delta e^n) = (P^n, \Delta e^n).$$

Taking the real parts and applying the Young inequality, we obtain

$$\|\Delta e^n\|_{L^2}^2 \leq \frac{1}{2\Delta t}(\left\|\nabla e^{n-1}\right\|_{L^2}^2 + \|\nabla e^n\|_{L^2}^2) + \|P^n\|_{L^2}^2 + \|R^n\|_{L^2}^2 + \frac{1}{2}\|\Delta e^n\|_{L^2}^2 . \tag{3.32}$$

By Eqs. 3.2, 3.18, 3.21, and 3.30, we obtain

$$(\Delta t)^{\frac{1}{2}}\|e^n\|_{H^2} \leq C_6 \Delta t. \tag{3.33}$$

Using Eq. 3.32 and **Assumption(A1)**, we see that

$$\|U^n\|_{L^\infty} \leq \|u^n\|_{L^\infty} + C\,\|e^n\|_{H^2} \leq K, \tag{3.34}$$
$$\|U^n\|_{H^2} \leq \|u^n\|_{H^2} + \|e^n\|_{H^2} \leq C_7, \tag{3.35}$$

when $\Delta t \leq \tau_5 = \frac{1}{C^2 C_6^2}$. With $\tau_0 = min\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$ and $C_0 = C + \sum_{i=1}^7 C_i$ the proof of Theorem 3.1 can be completed. □

4 The fully-discrete finite element solution

In this section, we study the error $\|U^n - U_h^n\|_{L^2}$ of the Galerkin finite element for the time-discrete system (2.7)–(2.8).

Lemma 4.1 *Suppose that the time-discrete system* (2.7)–(2.8) *has a unique solution* U^n. *Then*

$$\|R_h U^n\|_{L^\infty} \leq M. \tag{4.1}$$

Proof Thanks to Eqs. 2.4, 2.5, 3.4, and 3.34, we can obtain the following result:

$$\begin{aligned}\|R_h U^n\|_{L^\infty} &= \|U^n - R_h U^n\|_{L^\infty} + \|U^n\|_{L^\infty}\\ &\leq K + Ch^{-\frac{d}{2}}\|U^n - R_h U^n\|_{L^2}\\ &\leq Ch^{-\frac{d}{2}}h^2\|U^n\|_{H^2} + K\\ &\leq M.\end{aligned}$$

The proof of Lemma 4.1 is complete. □

The variational form of the time-discrete system (2.7) can be defined by

$$i(D_tU^n, v) - (\nabla U^n, \nabla v) + (f(|U^{n-1}|^2)U^n, v) = 0 \quad \forall v \in H_0^1. \tag{4.2}$$

Subtracting Eq. 2.6 from Eq. 4.2, we can obtain the following equation:

$$i(D_t(U^n - U_h^n), v) - (\nabla U^n - \nabla U_h^n, \nabla v) + (f(|U^{n-1}|^2)U^n - f(|U_h^{n-1}|^2)U_h^n, v) = 0 \tag{4.3}$$

for any $v \in V_h$.

Thanks to Eq. 2.3, the above formulation can be rewritten as

$$\begin{aligned} i(D_te_h^n, v) \; - \; (\nabla e_h^n, \nabla v) + (f(|U^{n-1}|^2)U^n \\ - \; f(|U_h^{n-1}|^2)U_h^n, v) = -i(D_t(U^n - R_hU^n), v). \end{aligned} \tag{4.4}$$

Theorem 4.1 *Assume that the unique solution u of Eq. 2.1 satisfies* ***Assumption(A1)****. Then the fully discrete system* (2.6) *has a unique solution* U_h^m*, m=1,2,...,N, and there exists* $\tau' > 0$ *such that, when* $\Delta t \le \tau'$*,* $h \le h'$

$$\|e_h^n\|_{L^2} \le C^* h^2, \tag{4.5}$$

$$\|e_h^n\|_{H^1} \le C^* h. \tag{4.6}$$

Proof As in [28], the existence and uniqueness of a solution to the fully-discrete system (2.6) can be shown. Next, we prove the error estimates Eqs. 4.5 and 4.6 by using the mathematical induction. Assuming $U_h^0 = \Pi_h u_0$ and using (2.2) and **Assumption (A1)**, it is easy to see that

$$\|u_0 - \Pi_h u_0\|_{L^2} \le Ch^2 \|u_0\|_{H^2} \le Ch^2.$$

With above inequality, Eq. 2.4, and **Assumption(A1)**, we obtain

$$\begin{aligned} \left\|e_h^0\right\|_{L^2} &= \left\|R_hU^0 - U_h^0\right\|_{L^2} = \|R_hu_0 - u_0 + u_0 - \Pi_h u_0\|_{L^2} \\ &\le \|R_hu_0 - u_0\|_{L^2} + \|u_0 - \Pi_hu_0\|_{L^2} \le Ch^2 \|u_0\|_{H^2} \le Ch^2. \end{aligned} \tag{4.7}$$

Defining $K_1 = \max\limits_{0 \le n \le N} \|R_hU^n\|_{L^\infty} + 1$ and using Eqs. 4.1, 2.5 and 4.7, we have the following result:

$$\begin{aligned} \left\|U_h^0\right\|_{L^\infty} &\le \left\|R_hU^0\right\|_{L^\infty} + \left\|R_hU^0 - U_h^0\right\|_{L^\infty} \\ &\le M + Ch^{-\frac{d}{2}} \left\|R_hU^0 - U_h^0\right\|_{L^2} \\ &\le C^2h^{-\frac{d}{2}}h^2 + M \\ &\le K_1, \end{aligned} \tag{4.8}$$

when $h \le h_1 = C^{-\frac{4}{4-d}}$.

First, we study the error estimates at the initial time step. From Eq. 4.3, we can obtain the following formulation:

$$\frac{i}{\Delta t}(e_h^1, v) - (\nabla e_h^1, \nabla v) + (f(|U^0|^2)U^1 - f(|U_h^0|^2)U_h^1, v)$$
$$= \frac{i}{\Delta t}(U^0 - U_h^0, v) - \frac{i}{\Delta t}(U^1 - R_h U^1, v). \tag{4.9}$$

Let $v = e_h^1$ in Eq. 4.9, and then it follows that

$$\frac{i}{\Delta t}\left\|e_h^1\right\|_{L^2}^2 - \left\|\nabla e_h^1\right\|_{L^2}^2 + (f(|U^0|^2)U^1 - f(|U_h^0|^2)U_h^1, e_h^1)$$
$$= \frac{i}{\Delta t}(U^0 - U_h^0, e_h^1) - \frac{i}{\Delta t}(U^1 - R_h U^1, e_h^1). \tag{4.10}$$

Taking the imaginary parts of the above equation and applying Eqs. 2.2, 2.4, 3.3 and 4.8, **Assumption(A1)**, and the Young inequality, we can obtain

$$\begin{aligned}
\left\|e_h^1\right\|_{L^2}^2 &= -Im((f(|U^0|^2) - f(|U_h^0|^2))U^1, e_h^1)\Delta t \\
&\quad -Im(f(|U_h^0|^2)(U^1 - R_h U^1), e_h^1)\Delta t \\
&\quad -Im(f(|U_h^0|^2)(R_h U^1 - U_h^1), e_h^1)\Delta t + Re(U^0 - U_h^0, e_h^1) \\
&\quad -Re(U^1 - R_h U^1, e_h^1) \le \frac{1}{2}\left\|(f(|U^0|^2) - f(|U_h^0|^2))U^1\right\|_{L^2}^2 \\
&\quad +\frac{1}{2}\left\|f(|U_h^0|^2)(U^1 - R_h U^1)\right\|_{L^2}^2 + \|f(|U_h^0|^2)e_h^1\|_{L^2}^2\Delta t \\
&\quad +\left\|U^0 - U_h^0\right\|_{L^2}^2 + \left\|U^1 - R_h U^1\right\|_{L^2}^2 + (\Delta t + \frac{1}{2})\left\|e_h^1\right\|_{L^2}^2 \\
&\le \frac{1}{2}\|f'(\xi_3)(U^0 - U_h^0)(|U^0| + |U_h^0|)U^1\|_{L^2}^2 \\
&\quad +\frac{1}{2}\|f(|U_h^0|^2)(U^1 - R_h U^1)\|_{L^2}^2 + \|U^0 - U_h^0\|_{L^2}^2 + \|U^1 - R_h U^1\|_{L^2}^2 \\
&\quad +(\frac{1}{2} + (1 + C_k^2)\Delta t)\|e_h^1\|_{L^2}^2 \le ((1 + C_{L2}^2)\Delta t + \frac{1}{2})\left\|e_h^1\right\|_{L^2}^2 + Ch^4,
\end{aligned} \tag{4.11}$$

where we using $f(|U_h^0|^2)+f'(\xi_2) \le C_{L2}$ for $|\xi_2| \le \|U^0\|_{L^\infty}^2+\|U_h^0\|_{L^\infty}^2 \le K^2+K_1^2$. So, we can get

$$\|e_h^1\|_{L^2} \le C_7 h^2, \tag{4.12}$$

when $\Delta t \le \tau_6 = \frac{1}{2(1+C_{L3}^2)}$. Thus, Eq. 4.5 holds for n=1.

From Eqs. 4.1, 2.5, and 4.11, we can get the following result:

$$\left\|U_h^1\right\|_{L^\infty} \le \left\|R_h U^1\right\|_{L^\infty} + \left\|e_h^1\right\|_{L^\infty} \le M + CC_7 h^{-\frac{d}{2}} h^2 \le K_1, \tag{4.13}$$

when $h \le h_1 = (CC_7)^{-\frac{2}{4-d}}$.

We assume that Eq. 4.5 holds for $m \leq n-1$. Similar to the derivation of the above inequality, we get

$$\left\|U_h^m\right\|_{L^\infty} \leq \|R_h U^m\|_{L^\infty} + \|e_h^m\|_{L^\infty} \leq M + CC^* h^{-\frac{d}{2}} h^2 \leq K_1, \tag{4.14}$$

when $h \leq h_2 = (CC^*)^{-\frac{2}{4-d}}$.

Substituting $v = e_h^n$ in Eq. 4.4, we derive at

$$\frac{i}{2\Delta t}(\|e_h^n\|_{L^2}^2 - \|e_h^{n-1}\|_{L^2}^2 + \|e_h^n - e_h^{n-1}\|_{L^2}^2) - \|\nabla e_h^n\|_{L^2}^2 + (f(|U^{n-1}|^2)U^n - f(|U_h^{n-1}|^2)U_h^n, e_h^n) = -i(D_t(U^n - R_h U^n), e_h^n). \tag{4.15}$$

Taking the imaginary parts of the above equation and using Eqs. 2.4, 3.4, 3.34, 3.35 and 4.14, and the Young inequality, we get

$$\begin{aligned}
\frac{\|e_h^n\|_{L^2}^2 - \left\|e_h^{n-1}\right\|_{L^2}^2}{2\Delta t} &\leq |(f(|U^{n-1}|^2)U^n - f(|U_h^{n-1}|^2)U_h^n, e_h^n)| \\
&\quad + |(D_t(U^n - R_h U^n), e_h^n)| \\
&= |(f'(\xi_4)(U^{n-1} - R_h U^{n-1} \\
&\quad + R_h U^{n-1} - U_h^{n-1})(|U^{n-1}| + |U_h^{n-1}|)U^n \\
&\quad + f(|U_h^{n-1}|^2)(U^n - R_h U^n + R_h U^n - U_h^n), e_h^n)| \\
&\quad + |(D_t(U^n - R_h U^n), e_h^n)| \leq C(\left\|e_h^n\right\|_{L^2}^2 + \left\|e_h^{n-1}\right\|_{L^2}^2) \\
&\quad + Ch^4 + |(D_t(U^n - R_h U^n), e_h^n)|,
\end{aligned} \tag{4.16}$$

where we applying $|f'(\xi_3)| + |f(|U_h^{n-1}|^2)| \leq C_{L3}$ for $|\xi_3| \leq \|U_h^{n-1}\|_{L^\infty}^2 + \|U^{n-1}\|_{L^\infty}^2 \leq K_1^2 + K^2$.

With Eq. 2.4 and the Young inequality, we see that

$$\sum_{m=1}^{n} \Delta t|(D_t(U^m - R_h U^m), e_h^m)| \leq Ch^4 \sum_{m=1}^{n} \Delta t\|D_t U^m\|_{H^2}^2 + \frac{1}{2}\sum_{m=1}^{n} \Delta t\|e_h^m\|_{L^2}^2. \tag{4.17}$$

Multiplying Eq. 3.18 by Δe^n and integrating it over Ω lead to

$$-\frac{i}{2\Delta t}(\|\nabla e^n\|_{L^2}^2 - \|\nabla e^{n-1}\|_{L^2}^2 + \|\nabla e^n - \nabla e^{n-1}\|_{L^2}^2) + \|\Delta e^n\|_{L^2}^2 + (R^n, \Delta e^n) = (P^n, \Delta e^n) \tag{4.18}$$

Taking real parts of above inequality and applying Gronwall inequality, we have

$$\frac{\Delta t}{2}\|\Delta e^n\|_{L^2}^2 \leq \Delta t\|R^n\|_{L^2}^2 + \Delta t\|P^n\|_{L^2}^2. \tag{4.19}$$

Summing inequalities (4.19) up and using Eqs. 3.19, 3.22, and 3.2 lead to

$$\sum_{m=1}^{n} \Delta t \|\Delta e^m\|_{L^2}^2 \le 2\sum_{m=1}^{n} \Delta t \|R^m\|_{L^2}^2 + 2\sum_{m=1}^{n} \Delta t \|P^m\|_{L^2}^2 \le C\Delta t^2. \tag{4.20}$$

Thus, we have

$$\sum_{m=1}^{n} \Delta t \|\Delta D_t e^m\|_{L^2}^2 \le C\Delta t^{-2} \sum_{m=1}^{n} \Delta t \|\Delta e^m\|_{L^2}^2 \le C \tag{4.21}$$

From above inequality and **Assumption(A1)**, we can get

$$\sum_{m=1}^{n} \Delta t \|D_t U^m\|_{H^2} \le \sum_{m=1}^{n} \Delta t \|D_t u^m\|_{H^2}^2 + \sum_{m=1}^{n} \Delta t \|\Delta D_t e^m\|_{L^2}^2 \le C. \tag{4.22}$$

Summing inequalities (4.16) up and using Eqs. 4.17, 4.22, and 4.7, we can obtain

$$\|e_h^n\|_{L^2}^2 \le \|e_h^0\|_{L^2}^2 + C\Delta t \sum_{m=0}^{n} \|e_h^m\|_{L^2}^2 + \sum_{m=1}^{n} \Delta t |(D_t(U^m - R_h U^m), e_h^m)| + Ch^4$$

$$\le C\Delta t \sum_{m=0}^{n} \|e_h^m\|_{L^2}^2 + Ch^4. \tag{4.23}$$

By Gronwall inequality, there exists $\tau_7 > 0$ such that

$$\|e_h^n\|_{L^2} \le C_8 h^2, \tag{4.24}$$

when $\Delta t \le \tau_7$.

Due to the Δt-independent property of estimate (4.5), we can obtain the H^1 error estimate by using an inverse inequality:

$$\|e_h^n\|_{H^1} \le Ch^{-1} \|e_h^n\|_{L^2} \le C_9 h. \tag{4.25}$$

Thus, choosing $\tau' = min(\tau_0, \tau_6, \tau_7)$, $h' = min(h_1, h_2)$ and $C^* = C + \sum_{i=7}^{9} C_i$, we finish the proof of Eqs. 4.5–4.6. □

Theorem 4.2 *Assume that the unique solution u and the initial datum u_0 of system* (2.1) *satisfy* ***Assumption(A1)****. Then the finite element system* (2.6) *has a unique solution U_h^m, $m = 1, 2, ..., N$, and there exists $\tau'' > 0$, $h' > 0$, such that, when $\Delta t \le \tau''$, $h \le h'$,*

$$\|u^m - U_h^m\|_{L^2} \le C^{**}(\Delta t + h^{r+1}), \tag{4.26}$$

where C^{**} is a positive constant independent of Δt and h.

Proof First, we study the optimal estimate as r=1. Thanks to Eqs. 2.4 and 3.35, Theorem 3.1, and Theorem 4.1, we have

$$\|u^m - U_h^m\|_{L^2} \le C_{10}(\Delta t + h^2),$$

when $\Delta t < \tau'$, $h \le h'$.

Next, we derive at the optimal L^2 error estimates as $r > 1$.
By Eqs. 2.2, 2.4 and **Assumption(A1)**, we can get

$$\begin{aligned}\|R_h u_0 - U_h^0\|_{L^2} &\le \|R_h u_0 - u_0\|_{L^2} + \|u_0 - U_h^0\|_{L^2}\\ &\le Ch^{r+1}\|u_0\|_{H^{r+1}} \le Ch^{r+1}\end{aligned} \tag{4.27}$$

Assuming that u is the solution of system (2.1), then we can get

$$i(u_t^n, v) - (\nabla u^n, \nabla v) + (f(|u^n|^2)u^n, v) = 0 \tag{4.28}$$

for any $v \in V_h$. Subtracting Eq. 4.28 from Eq. 2.6 and using Eq. 2.3, we derive at

$$\begin{aligned}&i(D_t(R_h u^n - U_h^n), v) - (\nabla(R_h u^n - U_h^n), \nabla v) + (f(|u^n|^2)u^n - f(|U_h^{n-1}|^2)U_h^n, v)\\ &= i(D_t R_h u^n - u_t^n, v).\end{aligned} \tag{4.29}$$

We assume that $\sigma_h^n = R_h u^n - U_h^n$ and $v = \sigma_h^n$; by Eq. 4.29, we see that

$$\begin{aligned}&\frac{i}{2\Delta t}(\|\sigma_h^n\|_{L^2}^2 - \left\|\sigma_h^{n-1}\right\|_{L^2}^2 + \left\|\sigma_h^n - \sigma_h^{n-1}\right\|_{L^2}^2) - \|\nabla\sigma_h^n\|_{L^2}^2 + (f(|u^n|^2)u^n\\ &-f(|U_h^{n-1}|^2)U_h^n, \sigma_h^n) = i(D_t R_h u^n - u_t^n, \sigma_h^n).\end{aligned} \tag{4.30}$$

Taking the imaginary parts of the above equality and using Eqs. 4.14, 2.4, **Assumption(A1)**, and the Young inequality, we obtain

$$\begin{aligned}&\frac{\|\sigma_h^n\|_{L^2}^2 - \left\|\sigma_h^{n-1}\right\|_{L^2}^2}{\Delta t}\\ &\le 2|(f(|u^n|^2)u^n - f(|U_h^{n-1}|^2)u^n, \sigma_h^n)| + 2|(f(|U_h^{n-1}|^2))(u^n - U_h^n), \sigma_h^n)|\\ &+2|(D_t R_h u^n - u_t^n, \sigma_h^n)|\\ &\le 2|(f'(\xi_5)(u^n - U_h^{n-1})(|u^n| + |U_h^{n-1}|)u^n, \sigma_h^n)| + 2|(f(|U_h^{n-1}|^2)(u^n-\\ &R_h u^n), \sigma_h^n)| + 2|(f(|U_h^{n-1}|^2)(R_h u^n - U_h^n), \sigma_h^n)| + 2|(D_t R_h u^n - u_t^n, \sigma_h^n)|\\ &\le 2|(f'(\xi_5)(|u_t(\xi_6)|)\Delta t(|u^n| + |U_h^{n-1}|)u^n, \sigma_h^n)| + 2|(f'(\xi_5)(|u^{n-1} - R_h u^{n-1}|)\\ &(|u^n| + |U_h^{n-1}|)u^n, \sigma_h^n)| + 2|(f'(\xi_5)(|R_h u^{n-1} - U_h^{n-1}|)(|u^n| + |U_h^{n-1}|)u^n,\\ &\sigma_h^n)| + 2|(f(|U_h^{n-1}|^2)(u^n - R_h u^n), \sigma_h^n)| + 2|(f(|U_h^{n-1}|^2)(R_h u^n - U_h^n),\\ &\sigma_h^n)| + 2|(D_t R_h u^n - u_t^n, \sigma_h^n)|\\ &\le C(\|\sigma_h^n\|_{L^2}^2 + \left\|\sigma_h^{n-1}\right\|_{L^2}^2) + Ch^{2(r+1)} + C\Delta t^2 + \|D_t R_h u^n - u_t^n\|_{L^2}^2\end{aligned} \tag{4.31}$$

where we applying $|f'(\xi_4)| + |f(|U_h^{n-1}|^2)| \le C_{L4}$ for $|\xi_4| \le \|U_h^{n-1}\|_{L^\infty}^2 + \|u^n\|_{L^\infty}^2 \le K^2 + K_1^2$.

Applying Eqs. 2.4, 3.2 and **Assumption(A1)**, we can get the following result:

$$\begin{aligned}\sum_{m=1}^{n} \Delta t \left\| D_t R_h u^m - u_t^m \right\|_{L^2}^2 &\leq \sum_{m=1}^{n} \Delta t \left\| D_t (R_h u^m - u^m) \right\|_{L^2}^2 \\ &\quad + \sum_{m=1}^{n} \Delta t \left\| D_t u^m - u_t^m \right\|_{L^2}^2 \\ &\leq C \Delta t h^{2(r+1)} \sum_{m=1}^{n} \| D_t u^m \|_{H^{r+1}}^2 + C \Delta t^2 \\ &\leq C h^{2(r+1)} + C \Delta t^2. \end{aligned} \tag{4.32}$$

Summing inequalities (4.31) up and using Eqs. 4.27, 4.32, and **Assumption(A1)**, we can obtain the following result:

$$\begin{aligned}\left\|\sigma_h^n\right\|_{L^2}^2 \leq \|\sigma_h^0\|_{L^2}^2 + C\Delta t \sum_{m=0}^{n} \|\sigma_h^m\|_{L^2}^2 + Ch^{2(r+1)} + C\Delta t^2 + \Delta t \sum_{m=1}^{n} \| D_t R_h u^m \\ - u_t^m \|_{L^2}^2 \leq C\Delta t \sum_{m=0}^{n} \|\sigma_h^m\|_{L^2}^2 + C\Delta t^2 + Ch^{2(r+1)}. \end{aligned} \tag{4.33}$$

Using Gronwall inequality,

$$\|\sigma_h^n\|_{L^2} \leq C_{11}(\Delta t + h^{r+1}), \tag{4.34}$$

when $\Delta t \leq \tau_8$ and $h \leq h'$. Thanks to Eq. 2.4 and **Assumption(A1)**, we have

$$\begin{aligned}\left\|u^n - U_h^n\right\|_{L^2} &\leq \left\|u^n - R_h u^n\right\|_{L^2} + \left\|R_h u^n - U_h^n\right\|_{L^2} \\ &\leq Ch^{r+1} + C_{11}(\Delta t + h^{r+1}). \end{aligned} \tag{4.35}$$

With $\tau'' = min(\tau', \tau_8)$, $C^{**} = max(C, C_{11})$ and $h \leq h'$, we complete the proof of Theorem 4.2. □

Remark. In the above proof, if $r = 1$, it is easy to show (4.26) (the L^2-norm optimal error estimate). However, as the Galerkin FEM order r is bigger than 1, we cannot obtain the optimal error estimates by using only Theorems 3.1 and 4.1.

5 Numerical experiments

In this section, we present two numerical examples to validate the theoretical analysis in the previous sections. All numerical results are performed by free software FreeFEM++[9].

Table 1 L^2 error estimates of the linear FEM with $h^2 = \Delta t$ (Example 5.1)

$\|\|u(\cdot, t_n) - U_h^n\|\|_{L^2}$								
	t=0.5	order	t=1	order	t=1.5	order	t=2	order
$1/h = 5$	1.45815E-02		3.01352E-02		7.18496E-02		1.88144E-01	
$1/h = 10$	4.32468E-03	1.7535	8.40798E-03	1.8416	2.00191E-02	1.8436	6.54744E-02	1.5228
$1/h = 20$	1.31054E-03	1.7224	2.22639E-03	1.9171	5.14973E-03	1.9588	1.80891E-02	1.8558
$1/h = 40$	3.52571E-04	1.8942	5.64739E-04	1.9791	1.30656E-03	1.9787	4.59479E-03	1.9770

Example 5.1 Let $f(s) = s$, and we can get the cubic Schrödinger equation

$$\begin{cases} iu_t + \Delta u + |u|^2 u = g, \ x \in \Omega, \ 0 < t \leq T, \\ u(x, 0) = u_0(x), \qquad x \in \Omega, \\ u = 0, \qquad x \in \partial\Omega, \end{cases} \tag{5.1}$$

where $\Omega = [0, 1] \times [0, 1]$. Moreover, the exact solution u of the above system is given as follows:

$$u = 5e^{it}(1 + 2t^2)(1 - x)(1 - y)\sin(x)\sin(y)$$

and the right-hand side g is given by the exact solution u and system (5.1).

Next, we solve system (5.1) by the semi-implicit backward Euler method with a linear finite element approximation and a quadratic finite element approximation, respectively. To check the optimal convergence rate in the L^2 norm, we pick $\Delta t = h^2$ for the linear finite element approximation and $\Delta t = h^3$ for the quadratic finite element approximation, respectively. We choose $t = 0.5$, 1, 1.5, and 2 to present our numerical results. From Tables 1–2, we can see that the results completely agree with the theoretical analysis above.

In [20], Tourigny showed the optimal L^2 error estimate with the condition of $\Delta t = o(h^{2/d})$. However, in Theorem 4.2, the L^2 optimal error estimate without any condition is obtained. For checking the unconditional convergence, we discuss (5.1) with different Δt on gradually refined meshes at $t = 1.0$. From Fig. 1, we can see that for a fixed Δt, the L^2-error of the linear FEM and quadratic FEM asymptotically converges to a small constant as $1/h$ increases. Obviously, it shows no restriction on Δt and h.

Table 2 L^2 error estimates of the quadratic FEM with $h^3 = \Delta t$ (Example 5.1)

$\|u(\cdot, t_n) - U_h^n\|_{L^2}$								
	t=0.5	order	t=1	order	t=1.5	order	t=2	order
$1/h = 5$	2.26241E-04		4.54483E-04		1.70451E-03		8.67418E-03	
$1/h = 10$	1.78403E-05	3.6646	2.80334E-05	4.0190	1.69899E-04	3.3266	1.03279E-03	3.0702
$1/h = 20$	1.72314E-06	3.3720	1.94346E-06	3.8504	1.80592E-05	3.2339	1.21222E-04	3.0908
$1/h = 40$	2.13961E-07	3.0096	2.05939E-07	3.2383	2.05455E-06	3.1358	1.50013E-05	3.0145

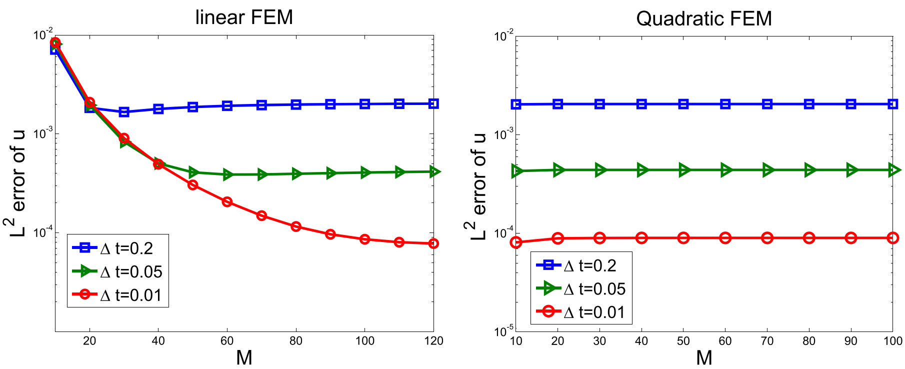

Fig. 1 L^2-norm errors of the linear and quadratic FEMs (Example 5.1)

Example 5.2 Let $f(s) = -s + s^2$, and we can get the cubic-quintic Schrödinger equation

$$\begin{cases} iu_t + \Delta u - |u|^2 u + |u|^4 u = g, & x \in \Omega, \ 0 < t \leq T, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{5.2}$$

where $\Omega = [0, 1] \times [0, 1]$. In addition, the exact solution u of the above system is given as follows:

$$u = e^{it+(x+y)/2}(1 + 3t^2)x(1 - x)y(1 - y).$$

The right-hand side g is given by system (5.2) and the exact solution u. We solve system (5.2) with the Euler semi-implicit scheme by applying the linear and quadratic FEMs. Similarly, to verify our theoretical analysis, we choose $\Delta t = h^2$ for the linear FEM and $\Delta t = h^3$ for the quadratic FEM, respectively. From Table 3 and 4, we can see that the L^2 error estimates of linear FEM are proportional to h^2 and the L^2 error estimates of quadratic FEM are proportional to h^3. Meanwhile, we choose different mesh scales $1/h$=10,20,...,100 with different $\Delta t = 0.2, 0.05, 0.01$ at $t = 1.0$. Also, from Fig. 2, we see that the errors converge to a constant as $\frac{h}{\Delta t} \to 0$, which shows

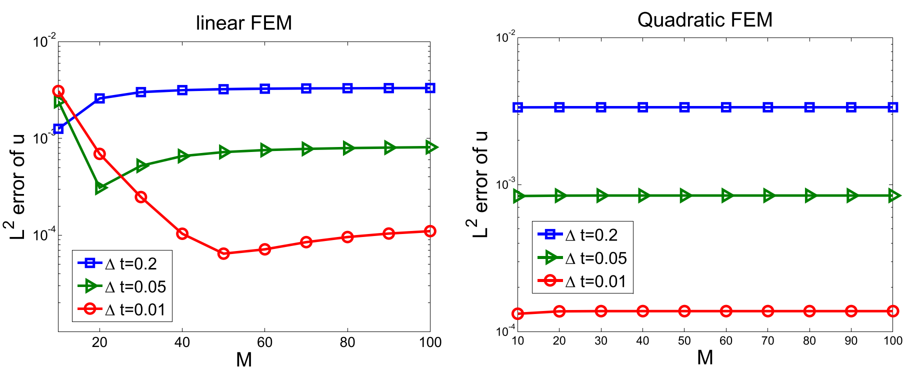

Fig. 2 L^2-norm errors of the linear and quadratic FEMs (Example 5.2)

Table 3 L^2 error estimates of the linear FEM with $h^2 = \Delta t$ (Example 5.2)

$\|u(\cdot,t_n) - U_h^n\|_{L^2}$								
	t=0.5	order	t=1	order	t=1.5	order	t=2	order
$1/h = 5$	5.1901E-03		1.1201E-02		2.2399E-02		4.9182E-02	
$1/h = 10$	1.4603E-03	1.8124	3.0493E-03	1.8770	5.8612E-03	1.9342	1.4945E-02	1.7185
$1/h = 20$	4.2169E-04	1.7920	8.1699E-04	1.9001	1.4209E-03	2.0444	4.0147E-03	1.8963
$1/h = 40$	1.1064E-04	1.9303	2.0785E-04	1.9748	3.4029E-04	2.0619	1.0369E-03	1.9530

Table 4 L^2 error estimates of the quadratic FEM with $h^3 = \Delta t$ (Example 5.2)

$\|u(\cdot,t_n) - U_h^n\|_{L^2}$								
	t=0.5	order	t=1	order	t=1.5	order	t=2	order
$1/h = 5$	1.1249E-04		2.4069E-04		4.6009E-04		1.5751E-03	
$1/h = 10$	9.7713E-06	3.5251	2.2870E-05	3.3956	3.4378E-05	3.7423	1.5102E-04	3.3826
$1/h = 20$	1.0696E-06	3.1914	2.7265E-06	3.0684	3.1570E-06	3.4449	1.6416E-05	3.2623
$1/h = 40$	1.3297E-07	3.0079	3.4950E-07	2.9636	3.6936E-07	3.0954	1.9160E-06	3.0989

the unconditional convergence by using the Euler semi-implicit FEMs for solving the Schrödinger equation.

6 Conclusions

In this paper, we obtain the optimal error estimates of an Euler semi-implicit method for a generalized nonlinear Schrödinger equation without any time step restriction. This method is based on a splitting of an error into a time error and a spatial error. As the regularity of the solution U^n of the time-discrete formulation is obtained, the solution of the fully-discrete Euler method in the L^∞ norm can be bounded by

$$\left\|U_h^n\right\|_{L^\infty} \le \left\|R_h U^n\right\|_{L^\infty} + \left\|U_h^n - R_h U^n\right\|_{L^\infty} \le K_1.$$

Applying the above inequality, the optimal error estimate in the L^2 norm of the fully-discrete scheme can be obtained as follow:

$$\left\|u^m - U_h^m\right\|_{L^2} \le C^{**}(\Delta t + h^{r+1}).$$

This optimal error estimate has no restriction on the time and spatial steps. The analytic approach in this paper can be extended similarly to other PDEs.

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