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A primal-mixed formulation for the strong coupling of quasi-Newtonian fluids with porous media

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Abstract In this work we analyze a primal-mixed finite element method for the coupling of quasi-Newtonian fluids with porous media in 2D and 3D. The flows are governed by a class of nonlinear Stokes and linear Darcy equations, respectively, and the transmission conditions on the interface between the fluid and the porous medium are given by mass conservation, balance of normal forces and the Beavers-Joseph-Saffman law. We apply a primal formulation in the Stokes domain and a mixed formulation in the Darcy formulation. The "*strong coupling*" concept means that the conservation of mass across the interface is introduced as an essential condition in the space where the velocity unknowns live. In this way, under some

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assumptions on the nonlinear kinematic viscosity, a generalization of the Babuška-Brezzi theory is utilized to show the well posedness of the primal-mixed formulation. Then, we introduce a Galerkin scheme in which the discrete conservation of mass is imposed approximately through an orthogonal projector. The unique solvability of this discrete system and its Strang-type error estimate follow from the generalized Babuška-Brezzi theory as well. In particular, the feasible finite element subspaces include Bernadi-Raugel elements for the Stokes flow, and either the Raviart-Thomas elements of lowest order or the Brezzi-Douglas-Marini elements of first order for the Darcy flow, which yield nonconforming and conforming Galerkin schemes, respectively. In turn, piecewise constant functions are employed to approximate in both cases the global pressure field in the Stokes and Darcy domain. Finally, several numerical results illustrating the good performance of both discrete methods and confirming the theoretical rates of convergence, are provided.

Keywords Mixed finite element · Stokes problem · Darcy problem · Quasi-Newtonian fluid · Strong coupling · Non-conforming scheme

Mathematics Subject Classification (2010) 65N15 · 65N30 · 74F10 · 74S05 · 76D07 · 76M10

1 Introduction

The development of suitable numerical methods to solve the Stokes-Darcy and related coupled problems, including porous media with cracks, the incorporation of the Brinkman equation in the model, and linear as well as nonlinear behaviors, has become a very active research area during the last decade (see, e.g., [1, 5–9, 13, 15, 17, 21, 24, 25] and the references therein). In particular, a mixed finite element method for a class of nonlinear Stokes-Darcy coupled problem arising in industrial filtring application and involving a non-Newtonian fluid, is introduced and analized in [7]. Up to the authors' knowledge, this is the first work dealing with the fully-coupled problem for non-Newtonian Stokes and Darcy flows. In fact, the fluid is modeled there by the generalized nonlinear Darcy equation in the porous medium. In addition, the approach in [7] employs the primal method in the Stokes domain and the dual-mixed method in the Darcy region, which means that only the original velocity and pressure unknowns are considered in the fluid, whereas a further unknown (velocity) is added in the porous medium. The corresponding interface conditions are given by the mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law, and since one of them becomes essential, the trace of the Darcy pressure on the interface needs also to be incorporated as an additional Lagrange multiplier. More recently, the model from [7] is recasted in [8] as a reduced matching problem on the interface by using a mortar space approach. As a consequence, a parallel algorithm for the problems in both regions is derived, which allows to solve the coupled problem utilizing existing codes for Stokes and Darcy simulations.

On the other hand, the a priori error analyses of a primal-mixed finite element method for 2D Stokes-Darcy coupled problem, in which primal and mixed formulations are employed in the Stokes and Darcy domains, respectively, were developed in [13] and [19]. This approach allows, on the one hand, to consider the natural unknowns, that is, the velocity vector fields and the pressure field in both domains, and on the other hand, the utilization of different families of finite element subspaces in each media. The model considered in [13] refers to a linearized Stokes equations coupled with a linearized Darcy equations. In addition, since the approach in [13] leads to essential transmission conditions, these are imposed weakly and hence the trace of the porous medium pressure becomes the corresponding Lagrange multiplier. However, in [19], the mass conservation across the interface between both domains was included as an essential condition in the velocity unknowns space, and hence the resulting primal-mixed formulation does not need the trace of the porous media pressure as an additional unknown.

The purpose of the present work is to extend the analysis and results from [19] to the model problem from [7], that is to the coupling of quasi-Newtonian fluids with porous media. To this end, and following a similar approach from [7] (see also [13] and [19]), we apply a primal formulation in the fluid domain while a mixed formulation is applied in the porous medium. In addition, the balance of normal forces and Beavers-Joseph-Saffman law are imposed weakly (exactly as in [13] and [19]), but following the idea introduced in [19], the mass conservation across the interface is imposed as an essential condition in the velocity unknowns space. All these equations yield a nonlinear primal-mixed formulation, whose well-posedness is proved by applying the generalization of the Babuška-Brezzi theory developed in [11] (see also [12]). In addition, since the insertion of the mass conservation as an essential condition in the velocity unknowns space leads to a nonconforming Galerkin scheme, we need to modify the generalized Babuška-Brezzi theory from [11] to be able to show the uniqueness of the discrete scheme and derive the corresponding a priori Strang-type estimate.

The rest of this work is organized as follows: In Section 2 we introduce the model problem and derive the primal-mixed variational formulation, which shows a nonlinear mixed formulation structure. A slight modification of the usual Babuška-Brezzi theory developed in [23] is also given here to analyze the solvability of our continuous formulation. Next, in Section 3 we provide the discrete analogue of the abstract theory developed in [11] (see also [12]), which allows us to establish the solvability and stability of nonconforming Galerkin schemes associated with weak formulations of nonlinear mixed problems. This abstract framework is then applied, under some general assumptions on the finite element subspaces, to prove the well-posedness of the nonconforming discrete scheme associated with our continuous problem. Specific choices of finite element subspaces satisfying these assumptions are also described here. Finally, several numerical results illustrating the performance of the method and confirming the theoretical rates of convergence, are reported in Section 4.

2 The continuous problem

We begin this section by introducing some notations to be used throughout this paper.

2.1 Preliminaries

In what follows, given $d \in \{2, 3\}$, $R^{d\times d}$ denotes the space of tensors (or matrices) $\tau := (\tau_{ij})$ with real entries, and $\mathbb{I}$ is the identity tensor (or identity matrix) of $R^{d\times d}$. Also, in this space we consider the tensorial inner product given by

$$\sigma : \tau := \sum_{i,j=1}^{d} \sigma_{ij}\tau_{ij} \qquad \forall \sigma, \tau \in R^{d\times d},$$

with induced norm

$$|\sigma| := \left\{ \sum_{i,j=1}^{d} \sigma_{ij}^2 \right\}^{1/2} \qquad \forall \sigma \in R^{d\times d}.$$

In turn, given H and Q Hilbert spaces with induced norms $\|\cdot\|_H$ and $\|\cdot\|_Q$, respectively, we endow the product space $H \times Q$ with the product norm $\|\cdot\|_{H\times Q} := \|\cdot\|_H + \|\cdot\|_Q$. In addition, we denote by $\mathbf{H}$ and $\mathbb{H}$ the spaces H^d and $H^{d\times d}$, respectively. Also, If H' denotes the dual space of the Hilbert space H, we let $[\cdot,\cdot]_{H'\times H}$ be the duality pairing between H' and H. Furthermore, we utilize the standard simplified terminology for Sobolev spaces and norms. In particular, given $s \in R$, a domain $U \subseteq R^d$, and an open or closed surface $\Gamma \subseteq R^d$, we consider the Sobolev spaces

$$\mathbf{H}^s(U) := \left[H^s(U)\right]^d \quad \text{and} \quad \mathbf{H}^s(\Gamma) := \left[H^s(\Gamma)\right]^d.$$

However, when $s = 0$ we usually write $L^2(U)$ and $L^2(\Gamma)$ instead of $H^0(U)$ and $H^0(\Gamma)$, respectively, as well as $\mathbf{L}^2(U)$ and $\mathbf{L}^2(\Gamma)$ instead of $\mathbf{H}^0(U)$ and $\mathbf{H}^0(\Gamma)$, respectively. The corresponding norms are denoted by $\|\cdot\|_{s,U}$ and $\|\cdot\|_{s,\Gamma}$ for the respective space on U and Γ, respectively. In addition, given $u, v \in L^2(U)$, $\mathbf{u}, \mathbf{v} \in \mathbf{L}^2(U)$, and $\sigma, \tau \in \mathbb{L}^2(U)$, we set

$$(u, v)_{0,U} := \int_U uv, \quad (\mathbf{u}, \mathbf{v})_{0,U} := \int_U \mathbf{u}\cdot\mathbf{v}$$

and

$$(\sigma, \tau)_{0,U} := \int_U \sigma : \tau.$$

We also need to introduce the space

$$L_0^2(U) := \left\{ u \in L^2(U) : \quad \int_U u = 0 \right\}. \tag{2.1}$$

Further, $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, and between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$ with respect to the $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ inner products, respectively. When Γ is an open surface of R^d and Σ is a closed surface in R^d such that $\Gamma \subseteq \Sigma$, we introduce the extension operator $E_0 : H^{1/2}(\Gamma) \to L^2(\Sigma)$ defined by

$$E_0(\psi) := \begin{cases} \psi \text{ on } \Gamma, \\ 0 \text{ on } \Sigma\backslash\Gamma, \end{cases} \forall \psi \in H^{1/2}(\Gamma),$$

and the space

$$H_{00}^{1/2}(\Gamma) := \left\{ \psi \in H^{1/2}(\Gamma) : \quad E_0(\psi) \in H^{1/2}(\Sigma) \right\},$$

which is endowed with the norm $\|\psi\|_{1/2,00,\Gamma} := \|E_0(\psi)\|_{1/2,\Sigma}$, $\forall \psi \in H_{00}^{1/2}(\Gamma)$. The expression $\langle \cdot, \cdot \rangle_\Gamma$ is also employed in this case to denote the duality pairing between $H_{00}^{1/2}(\Gamma)$ and $H_{00}^{-1/2}(\Gamma)$, where $H_{00}^{-1/2}(\Gamma)$ is the dual space of $H_{00}^{1/2}(\Gamma)$. In particular, note that given $\eta \in H^{-1/2}(\Sigma)$, its restriction to Γ defined by

$$\langle \eta|_\Gamma, \psi \rangle_\Gamma := \langle \eta, E_0(\psi) \rangle_\Sigma \qquad \forall \psi \in H_{00}^{1/2}(\Gamma),$$

is an element of $H_{00}^{-1/2}(\Gamma)$. The corresponding vector versions of $H_{00}^{1/2}(\Gamma)$ and $H_{00}^{-1/2}(\Gamma)$ are denoted by $\mathbf{H}_{00}^{1/2}(\Gamma)$ and $\mathbf{H}_{00}^{-1/2}(\Gamma)$, respectively, and $\langle \cdot, \cdot \rangle_\Gamma$ is also employed to refer to the respective duality pairing.

On the other hand, with div denoting the usual divergence operator, the Hilbert space

$$H(\mathrm{div}; U) := \left\{ \tau \in \mathbf{L}^2(U) : \quad \mathrm{div}\tau \in L^2(U) \right\},$$

is standard in the realm of mixed problems (see [4, 14]). The norm of this space is denoted by $\|\cdot\|_{\mathrm{div},U}$. Moreover, given a nonempty set S of R^d and a nonnegative integer k, we denote by $P_k(S)$ the space of polynomials defined in S with total degree at most k. Also, $\mathbf{P}_k(S)$ denotes the corresponding vector version of $P_k(S)$. Finally, we employ $\mathbf{0}$ to denote a generic null vector, the null functional or the null operator, and we use C with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2.2 The model problem

Let $\Omega \subseteq R^d$ be a Lipschitz polyhedral (polygonal if $d = 2$) domain with boundary $\Gamma := \partial\Omega$ which has been subdivided in two subdomains Ω_S and Ω_D such that $\Omega_S \cap \Omega_D = \emptyset$, $\overline{\Omega} = \overline{\Omega}_S \cup \overline{\Omega}_D$, and $\partial\Omega_S \cap \partial\Omega_D = \Sigma$ is the nonempty polyhedral interface between Ω_S and Ω_D. Also, we let $\Gamma_S := \partial\Omega_S \backslash \overline{\Sigma}$ and $\Gamma_D := \partial\Omega_D \backslash \overline{\Sigma}$. On Σ and on Γ we denote by $\mathbf{n} := (n_1, n_2, ..., n_d)^{\mathrm{t}}$ the unit normal vector which is chosen pointing outward from $\Omega_S \cup \Sigma \cup \Omega_D$ and Ω_S. Note that $\mathbf{n}$ points inward from Σ to Ω_D. In addition, in the 2D case we denote by $\mathbf{t} := (-n_2, n_1)^{\mathrm{t}}$ the fixed unit tangent vector on Σ (see Fig. 1). The model problem we are interested in consists of the movement of an incompressible quasi-Newtonian viscous fluid that occupies the region Ω_S and that flows towards and from the region Ω_D through the interface Σ, where Ω_D is saturated with the same fluid.

More precisely, the governing equations in Ω_S are those of the nonlinear Stokes problem with homogeneous Dirichlet boundary condition on Γ_S, that is:

$$\begin{aligned} -\mathbf{div}\left\{\mu\left(|\nabla \mathbf{u}_S|\right)\nabla \mathbf{u}_S - p_S \mathbb{I}\right\} &= \mathbf{f}_S \ \text{ in } \Omega_S, \\ \mathrm{div}\mathbf{u}_S &= 0 \ \text{ in } \Omega_S, \\ \mathbf{u}_S &= 0 \ \text{ on } \Gamma_S, \end{aligned} \tag{2.2}$$

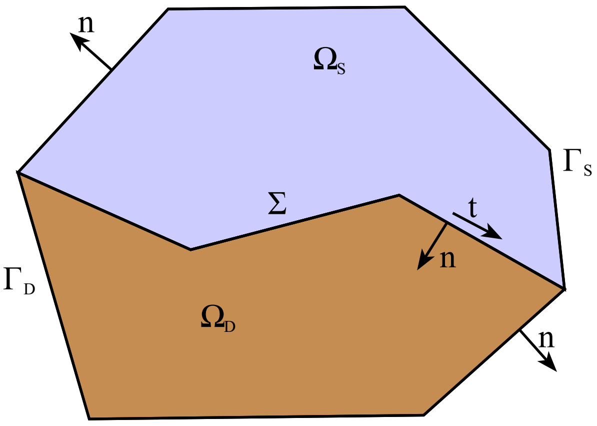

Fig. 1 Layout of the geometry of the coupled problem

where **div** is the usual divergence operator div applied along each row of a tensor, $\mathbf{u}_S$ is the velocity vector field in Ω_S, p_S is the pressure field in Ω_S, $\mu : R^+ \to R^+$ is the nonlinear kinematic viscosity, and $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S)$ is a known volume force. In turn, in Ω_D we consider the linearized Darcy model with Neumann boundary condition on Γ_D:

$$\begin{aligned} \mathbf{K}^{-1}\mathbf{u}_D + \nabla p_D &= 0 \quad \text{in } \Omega_D, \\ \text{div}\mathbf{u}_D &= f_D \quad \text{in } \Omega_D, \\ \mathbf{u}_D \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_D, \end{aligned} \tag{2.3}$$

where $\mathbf{u}_D$ is the velocity vector field in Ω_D, p_D is the pressure field in Ω_D, $f_D \in L^2_0(\Omega_D)$ is a source term, and $\mathbf{K}$ is a symmetric and uniformly positive definite tensor with entries in $L^\infty(\Omega_D)$, which represents the permeability of Ω_D divided by a constant approximation of the viscosity. Finally, the transmission conditions across Σ are given by the conservation of mass, balance of normal forces and Beavers-Joseph-Saffman law:

$$\begin{aligned} \mathbf{u}_S \cdot \mathbf{n} &= \mathbf{u}_D \cdot \mathbf{n} \quad \text{on } \Sigma, \\ \{\mu\,(|\nabla \mathbf{u}_S|)\,\nabla \mathbf{u}_S - p_S \mathbb{I}\}\,\mathbf{n} + \nu\kappa^{-1}\boldsymbol{\pi}_t \mathbf{u}_S &= -p_D \mathbf{n} \quad \text{on } \Sigma, \end{aligned} \tag{2.4}$$

where ν is a constant approximation of the viscosity μ on Σ, $\boldsymbol{\pi}_t \mathbf{w} := \mathbf{w} - (\mathbf{w} \cdot \mathbf{n})\mathbf{n}$ and $\kappa \in L^\infty(\Omega_D)$ is a given coefficient that is bounded from below by a positive constant a.e. on Σ. We remark that the kind of nonlinear Stokes problem given by Eq. 2.2 appears in the modeling of a large class of non-Newtonian fluids (see e.g. [16, 22]). In particular, the Ladyzhenskaya law for fluids with large stresses (see [16]), also known as power law, is given by $\mu(t) = \mu_0 + \mu_1 t^{\beta-2} \quad \forall t \in R^+$, with $\mu_0 \geq 0$, $\mu_1 > 0$ and $\beta > 1$, and the Carreau law for viscoplastic flows (see, e.g. [18] and [22]) reads $\mu(t) = \mu_0 + \mu_1(1+t^2)^{(\beta-2)/2} \quad \forall t \in R^+$,with $\mu_0 \geq 0$, $\mu_1 > 0$ and $\beta \geq 1$. In what follows we let $\mu_{ij} : R^{d\times d} \to R$ be the mapping defined by

$$\mu_{ij}(\sigma) = \mu(|\sigma|)\sigma_{ij} \quad \forall \sigma := (\sigma_{ij}) \in R^{d\times d}. \tag{2.5}$$

Throughout this work we suppose that μ is of class C^1 and that there exist positive constants α_0 and γ_0 such that for all $\sigma, \tau \in R^{d\times d}$

$$|\mu_{ij}(\sigma)| \leq \gamma_0|\sigma|, \qquad \left|\frac{\partial\mu_{ij}}{\partial\sigma_{kl}}(\sigma)\right| \leq \gamma_0, \quad \forall i, j, k, l \in \{1, ..., d\} \tag{2.6}$$

and

$$\sum_{i,j,k,l=1}^{d} \frac{\partial\mu_{ij}}{\partial\sigma_{kl}}(\sigma)\, \tau_{ij}\tau_{kl} \geq \alpha_0|\tau|^2 \,. \tag{2.7}$$

It is easy to check that the Carreau law satisfies Eqs. 2.6 and 2.7 for all $\mu_0 > 0$, and for all $\beta \in [1, 2]$. In particular, with $\beta = 2$ we recover the usual linear Stokes model.

2.3 A primal-mixed formulation

In this section we proceed as in [13] and [19], and introduce a primal-mixed formulation of the coupled problem given by Eqs. 2.2, 2.3 and 2.4. To this end, we consider the spaces

$$\mathbf{H}^1_{\Gamma_S}(\Omega_S) := \left\{\mathbf{v}_S \in \mathbf{H}^1(\Omega_S) : \quad \mathbf{v}_S = \mathbf{0} \quad \text{on } \Gamma_S\right\}$$

and

$$H_{\Gamma_D}(\text{div}; \Omega_D) := \{\mathbf{v}_D \in H(\text{div}; \Omega_D) : \quad \mathbf{v}_D \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_D\}\,.$$

Here, $H(\text{div}; \Omega_D)$ is endowed with the inner product

$$(\mathbf{u}_D, \mathbf{v}_D)_{\text{div},\Omega_D} := (\mathbf{u}_D, \mathbf{v}_D)_{0,\Omega_D} + (\text{div}\mathbf{u}_D, \text{div}\mathbf{v}_D)_{0,\Omega_D} \quad \forall \mathbf{u}_D, \mathbf{v}_D \in H(\text{div}; \Omega_D),$$

and its induced norm $\|\cdot\|^2_{\text{div},\Omega_D} := (\cdot,\cdot)_{\text{div},\Omega_D}$. Next, in order to construct a primal-mixed formulation of Eqs. 2.2, 2.3 and 2.4, we begin by testing the first equation in Eq. 2.2 with $\mathbf{v}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$. In this way, integrating by parts the term $(\mathbf{div}\,\{\mu\,(|\nabla\mathbf{u}_S|)\,\nabla\mathbf{u}_S - p_S\mathbb{I}\}, \mathbf{v}_S)_{0,\Omega_S}$, introducing the Dirichlet boundary condition $\mathbf{u}_S = \mathbf{0}$ on Γ_S, and using that $p_S\mathbb{I} : \nabla\mathbf{v}_S = p_S \,\text{div}\, \mathbf{v}_S$ we obtain

$$(\mu(|\nabla\mathbf{u}_S|)\nabla\mathbf{u}_S, \nabla\mathbf{v}_S)_{0,\Omega_S} - (p_S, \text{div}\mathbf{v}_S)_{0,\Omega_S} - \langle\{\mu(|\nabla\mathbf{u}_S|)\nabla\mathbf{u}_S - p_S\mathbb{I}\}\,\mathbf{n}, \mathbf{v}_S\rangle_\Sigma = (\mathbf{f}_S, \mathbf{v}_S)_{0,\Omega_S}\ ,$$

which, using from Eq. 2.4 that

$$-\{\mu(|\nabla\mathbf{u}_S|)\nabla\mathbf{u}_S - p_S\mathbb{I}\}\,\mathbf{n} = \nu\kappa^{-1}\boldsymbol{\pi}_t\mathbf{u}_S + p_D\mathbf{n} \quad \text{on } \Sigma,$$

yields

$$\begin{aligned}(\mu(|\nabla\mathbf{u}_S|)\nabla\mathbf{u}_S, \nabla\mathbf{v}_S)_{0,\Omega_S} + \left\langle\nu\kappa^{-1}\boldsymbol{\pi}_t\mathbf{u}_S, \boldsymbol{\pi}_t\mathbf{v}_S\right\rangle_\Sigma \\ + \langle\mathbf{v}_S\cdot\mathbf{n}, p_D\rangle_\Sigma - (p_S, \text{div}\mathbf{v}_S)_{0,\Omega_S}\end{aligned} = (\mathbf{f}_S, \mathbf{v}_S)_{0,\Omega_S} \quad \forall \mathbf{v}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S).$$

On the other hand, multiplying the first equation of Eq. 2.3 by $\mathbf{v}_D \in H_{\Gamma_D}(\text{div}; \Omega_D)$, integrating by parts, and using that $-\mathbf{n}$ is the unit normal vector of Σ pointing inward to Ω_D, we arrive at

$$(\mathbf{K}^{-1}\mathbf{u}_D\mathbf{v}_D)_{0,\Omega_D} - \langle\mathbf{v}_D\cdot\mathbf{n}, p_D\rangle_\Sigma - (p_D, \text{div}\mathbf{v}_D)_{0,\Omega_D} = 0 \quad \forall \mathbf{v}_D \in H_{\Gamma_D}(\text{div}; \Omega_D).$$

Hence, adding the last two equations we get

$$\begin{aligned}(\mu(|\nabla \mathbf{u}_S|)\nabla \mathbf{u}_S, \nabla \mathbf{v}_S)_{0,\Omega_S} + \langle \nu\kappa^{-1}\boldsymbol{\pi}_t \mathbf{u}_S, \boldsymbol{\pi}_t \mathbf{v}_S\rangle_\Sigma + (\mathbf{K}^{-1}\mathbf{u}_D, \mathbf{v}_D)_{0,\Omega_D} \\ -(p_S, \operatorname{div}\mathbf{v}_S)_{0,\Omega_S} - (p_D, \operatorname{div}\mathbf{v}_D)_{0,\Omega_D} + \langle(\mathbf{v}_S - \mathbf{v}_D)\cdot \mathbf{n}, p_D\rangle_\Sigma\end{aligned} = (\mathbf{f}_S, \mathbf{v}_S)_{0,\Omega_S}, \tag{2.8}$$

for all $\mathbf{v} := (\mathbf{v}_S, \mathbf{v}_D) \in \mathbf{H}^1_{\Gamma_S}(\Omega_S) \times H_{\Gamma_D}(\operatorname{div}; \Omega_D)$. In turn, from the second equations of Eqs. 2.2 and 2.3, we obtain

$$(q, \operatorname{div}\mathbf{u}_S)_{0,\Omega_S} + (q, \operatorname{div}\mathbf{u}_D)_{0,\Omega_D} = (f_D, q)_{0,\Omega_D} \quad \forall q \in L^2(\Omega). \tag{2.9}$$

Now, proceeding as in [19], we introduce the first transmission condition of Eq. 2.4 into the definition of the velocities space $\mathbf{H}$, that is

$$\mathbf{H} := \left\{\mathbf{v} := (\mathbf{v}_S, \mathbf{v}_D) \in \mathbf{H}^1_{\Gamma_S}(\Omega_S) \times H_{\Gamma_D}(\operatorname{div}; \Omega_D) : \quad \mathbf{v}_S\cdot\mathbf{n} = \mathbf{v}_D\cdot\mathbf{n} \quad \text{on } \Sigma\right\}. \tag{2.10}$$

This space is endowed with the usual norm of the product space $\mathbf{H}^1_{\Gamma_S}(\Omega_S) \times H_{\Gamma_D}(\operatorname{div}; \Omega_D)$. Note that, according to the foregoing definition, (2.8) becomes

$$\begin{aligned}(\mu(|\nabla \mathbf{u}_S|)\nabla \mathbf{u}_S, \nabla \mathbf{v}_S)_{0,\Omega_S} + \langle \nu\kappa^{-1}\boldsymbol{\pi}_t \mathbf{u}_S, \boldsymbol{\pi}_t \mathbf{v}_S\rangle_\Sigma \\ + (\mathbf{K}^{-1}\mathbf{u}_D, \mathbf{v}_D)_{0,\Omega_D} - (p_S, \operatorname{div}\mathbf{v}_S)_{0,\Omega_S} - (p_D, \operatorname{div}\mathbf{v}_D)_{0,\Omega_D}\end{aligned} = (\mathbf{f}_S, \mathbf{v}_S)_{0,\Omega_S} \quad \forall\, \mathbf{v} := (\mathbf{v}_S, \mathbf{v}_D) \in \mathbf{H}. \tag{2.11}$$

Then, proceeding as in [13], we find that the resulting weak formulation reduces to a nonlinear system with three unknowns, namely

$$\mathbf{u}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S), \quad \mathbf{u}_D \in H_{\Gamma_D}(\operatorname{div}; \Omega_D) \quad \text{and} \quad p := \begin{cases} p_S & \text{on } \Omega_S \\ p_D & \text{on } \Omega_D \end{cases} \in L^2(\Omega),$$

satisfying Eqs. 2.9 and 2.11. More precisely, our primal-mixed formulation reads: Find $(\mathbf{u}, p) := ((\mathbf{u}_S, \mathbf{u}_D), p) \in \mathbf{H} \times L^2(\Omega)$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= [F, \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} \quad \forall \mathbf{v} \in \mathbf{H}, \\ b(\mathbf{u}, q) &= [G, q]_{\mathbf{Q}'\times\mathbf{Q}} \quad \forall q \in L^2(\Omega), \end{aligned} \tag{2.12}$$

where the semilinear form $a : \mathbf{H} \times \mathbf{H} \to R$, the bilinear form $b : \mathbf{H} \times L^2(\Omega) \to R$, and the functionals $F \in \mathbf{H}'$ and $G \in L^2(\Omega)'$, are defined by

$$a(\mathbf{u}, \mathbf{v}) := (\mu(|\nabla \mathbf{u}_S|)\nabla \mathbf{u}_S, \nabla \mathbf{v}_S)_{0,\Omega_S} + \langle \nu\kappa^{-1}\boldsymbol{\pi}_t \mathbf{u}_S \boldsymbol{\pi}_t \mathbf{v}_S\rangle_\Sigma + (\mathbf{K}^{-1}\mathbf{u}_D \mathbf{v}_D)_{0,\Omega_D} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H},$$

$$b(\mathbf{v}, q) := -(q, \operatorname{div}\mathbf{v}_S)_{0,\Omega_S} - (q, \operatorname{div}\mathbf{v}_D)_{0,\Omega_D} \quad \forall (\mathbf{v}, q) \in \mathbf{H} \times L^2(\Omega),$$

$$[F, \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} := (\mathbf{f}_S, \mathbf{v}_S)_{0,\Omega_S} \quad \forall \mathbf{v} \in \mathbf{H}, \quad \text{and} \quad [G, q]_{\mathbf{Q}'\times\mathbf{Q}} := (f_D, q)_{0,\Omega_D} \quad \forall q \in L^2(\Omega).$$

Now, it is easy to see from Eq. 2.6 that, fixing the first component of a, its second component defines a bounded linear functional. In turn, it is quite clear that b is a bounded bilinear form. Hence, we can introduce the nonlinear operator $\mathbf{A} : \mathbf{H} \to \mathbf{H}'$ and the linear operator $\mathbf{B} : \mathbf{H} \to [L^2(\Omega)]'$ given by

$$[\mathbf{A}(\mathbf{u}), \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} := a(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H},$$

and

$$[\mathbf{B}(\mathbf{v}), q]_{L^2(\Omega)'\times L^2(\Omega)} := b(\mathbf{v}, q) \quad \forall(\mathbf{v}, q) \in \mathbf{H} \times L^2(\Omega),$$

whence the primal-mixed formulation (2.12) can be re-written as: Find $(\mathbf{u}, p) \in \mathbf{H} \times L^2(\Omega)$ such that

$$\begin{aligned} [\mathbf{A}(\mathbf{u}), \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} + [\mathbf{B}(\mathbf{v}), p]_{L^2(\Omega)'\times L^2(\Omega)} &= [F, \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} \ \forall \mathbf{v} \in \mathbf{H}, \\ [\mathbf{B}(\mathbf{u}), q]_{L^2(\Omega)'\times L^2(\Omega)} &= [G, q]_{\mathbf{Q}'\times\mathbf{Q}} \ \forall q \in L^2(\Omega). \end{aligned} \tag{2.13}$$

However, it is easy to show that this system is not unique solvable since, given any solution $(\mathbf{u}, p) := ((\mathbf{u}_S, \mathbf{u}_D), p) \in \mathbf{H} \times L^2(\Omega)$ of Eq. 2.12 (equivalently (2.13)), $(\mathbf{u}, p+c)$ is also a solution for each $c \in R$. In order to overcome this non-uniqueness, we recall the decomposition $L^2(\Omega) = L^2_0(\Omega) \oplus R$, (cf. Eq. 2.1), define $\mathbf{Q} := L^2_0(\Omega)$, and consider the modified primal-mixed formulation: Find $(\mathbf{u}, p) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} [\mathbf{A}(\mathbf{u}), \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} + [\mathbf{B}(\mathbf{v}), p]_{\mathbf{Q}'\times\mathbf{Q}} &= [F, \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} \ \forall \mathbf{v} \in \mathbf{H}, \\ [\mathbf{B}(\mathbf{u}), q]_{\mathbf{Q}'\times\mathbf{Q}} &= [G, q]_{\mathbf{Q}'\times\mathbf{Q}} \ \forall q \in \mathbf{Q}. \end{aligned} \tag{2.14}$$

The following lemma shows the connection between Eqs. 2.13 and 2.14.

Lemma 2.1 *Let* $(\mathbf{u}, p) \in \mathbf{H}\times L^2(\Omega)$ *be a solution of Eq.* 2.13 *and define* $p_0 \in L^2_0(\Omega)$ *by*

$$p_0 := p - \frac{1}{|\Omega|}\int_\Omega p.$$

Then $(\mathbf{u}, p_0) \in \mathbf{H} \times \mathbf{Q}$ *is a solution of Eq.* 2.14. *Conversely, let* $(\mathbf{u}, p_0) \in \mathbf{H} \times \mathbf{Q}$ *be a solution of Eq.* 2.14, *and given* $c \in R$, *define* $p := p_0 + c$. *Then* $(\mathbf{u}, p) \in \mathbf{H} \times L^2(\Omega)$ *is a solution of Eq.* 2.13.

Proof First, let $(\mathbf{u}, p) \in \mathbf{H}\times L^2(\Omega)$ be a solution of Eq. 2.13. We define $p_0 \in L^2_0(\Omega)$ by

$$p_0 := p - c, \quad \text{with} \quad c := \frac{1}{|\Omega|}\int_\Omega p.$$

Then, for any $\mathbf{v} \in \mathbf{H}$ we have, using the first equation in Eq. 2.13,

$$\begin{aligned} [\mathbf{A}(\mathbf{u}), \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} + [\mathbf{B}(\mathbf{v}), p_0]_{\mathbf{Q}'\times\mathbf{Q}} &= [\mathbf{A}(\mathbf{u}), \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} + [\mathbf{B}(\mathbf{v}), p - c]_{L^2(\Omega)'\times L^2(\Omega)} \\ &= [F, \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} - c\,[\mathbf{B}(\mathbf{v}), 1]_{L^2(\Omega)'\times L^2(\Omega)}\,. \end{aligned}$$

Now, since $\mathbf{v}_S \cdot \mathbf{n} = \mathbf{v}_D \cdot \mathbf{n}$ on Σ and $\mathbf{n}$ points inward to Ω_D on Σ, we get

$$[\mathbf{B}(\mathbf{v}), 1]_{L^2(\Omega)'\times L^2(\Omega)} = -\,(1, \mathrm{div}\mathbf{v}_S)_{0,\Omega_S} - (1, \mathrm{div}\mathbf{v}_D)_{0,\Omega_D} = \langle \mathbf{v}_D \cdot \mathbf{n} - \mathbf{v}_S \cdot \mathbf{n}, 1\rangle_\Sigma = 0,$$

which, replaced back into the foregoing equation, gives

$$[\mathbf{A}(\mathbf{u}), \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} + [\mathbf{B}(\mathbf{v}), p_0]_{\mathbf{Q}'\times\mathbf{Q}} = [F, \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} \quad \forall \mathbf{v} \in \mathbf{H}\,,$$

thus showing that the first equation in Eq. 2.14 is satisfied. In turn, the second equation of Eq. 2.14 is clearly satisfied since $\mathbf{Q} \subseteq L^2(\Omega)$.

Conversely, let $(\mathbf{u}, p_0) \in \mathbf{H} \times \mathbf{Q}$ be a solution of Eq. 2.14 and let $c \in R$. Then, defining $p := p_0 + c$ we see from the first equation in Eq. 2.14 that for all $\mathbf{v} \in \mathbf{H}$ there holds

$$\begin{aligned}
&[\mathbf{A}(\mathbf{u}), \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} + [\mathbf{B}(\mathbf{v}), p]_{L^2(\Omega)'\times L^2(\Omega)} \\
&\qquad = [\mathbf{A}(\mathbf{u}), \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} + [\mathbf{B}(\mathbf{v}), p_0]_{\mathbf{Q}'\times\mathbf{Q}} + c \cdot [\mathbf{B}(\mathbf{v}), 1]_{L^2(\Omega)'\times L^2(\Omega)} \\
&\qquad = [\mathbf{A}(\mathbf{u}), \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}} + [\mathbf{B}(\mathbf{v}), p_0]_{\mathbf{Q}'\times\mathbf{Q}} \\
&\qquad = [F, \mathbf{v}]_{\mathbf{H}'\times\mathbf{H}}\,,
\end{aligned}$$

that is the first equation in Eq. 2.13 is satisfied. Now, given $q := q_0 + c \in L^2(\Omega) := L^2_0(\Omega) \oplus R$, with $q_0 \in L^2_0(\Omega)$ and $c \in R$, we deduce, using the second equation in Eq. 2.14 and the identity $G(1) = 0$ (which follows from the fact that $f_{\mathrm{D}} \in L^2_0(\Omega_{\mathrm{D}})$), that

$$[\mathbf{B}(\mathbf{u}), q]_{L^2(\Omega)'\times L^2(\Omega)} = [\mathbf{B}(\mathbf{u}), q_0]_{\mathbf{Q}'\times\mathbf{Q}} + c\cdot[\mathbf{B}(\mathbf{u}), 1]_{L^2(\Omega)'\times L^2(\Omega)} = G(q_0) = G(q),$$

which proves that the second equation in Eq. 2.13 holds. □

According to the previous lemma, throughout the rest of the paper we consider the primal-mixed formulation (2.14).

2.4 An abstract theory for a class of nonlinear mixed problems

Let H and Q be Hilbert spaces with dual spaces H' and Q', and let $A : H \to H'$ be a nonlinear operator, and $B : H \to Q'$ be a linear operator with adjoint $B' : Q \to H'$. Then, given $F \in H'$ and $G \in Q'$, we are interested in the following variational problem: Find $(u, p) \in H \times Q$ such that

$$\begin{aligned}
[A(u), v]_{H'\times H} + [B(v), p]_{Q'\times Q} &= [F, v]_{H'\times H} \quad \forall v \in H, \\
[B(u), q]_{Q'\times Q} &= [G, q]_{Q'\times Q} \quad \forall q \in Q.
\end{aligned} \tag{2.15}$$

In order to analyze the unique solvability of Eq. 2.15, we need to introduce some assumptions on the operators $A : H \to H'$ and $B : H \to Q'$.

(H.1) There exists $\gamma > 0$ such that A is Lipschitz continuous, that is

$$\|A(u) - A(v)\|_{H'} \le \gamma \|u - v\|_H \quad \forall u, v \in H.$$

(H.2) There exists $\alpha > 0$ such that for any $z \in H$, the nonlinear operator $A(z + \cdot)$ is strongly monotone in the null space of the linear operator B, that is

$$\alpha \|u - v\|_H^2 \le [A(z + u) - A(z + v), u - v]_{H'\times H} \quad \forall u, v \in V,$$

where $V := \{v \in H : \quad [B(v), q]_{Q'\times Q} = 0 \quad \forall q \in Q\}$.

(H.3) There exists $\beta > 0$ such that the following continuous inf-sup condition holds

$$\sup_{\substack{v\in H\\ v\neq \mathbf{0}}} \frac{[B(v), q]_{Q'\times Q}}{\|v\|_H} \ge \beta \|q\|_Q \quad \forall q \in Q.$$

We now recall from [14] a result establishing equivalent statements for (H.3).

Lemma 2.2 *The following are equivalent:*

i) *(H.3) is satisfied.*

ii) *B' is an isomorphism from Q onto V°, where*

$$V^\circ := \left\{ F \in H' : \quad [F, v]_{H' \times H} = 0 \quad \forall v \in V \right\}$$

is the polar set of V, and there holds

$$\left\| B'(q) \right\|_{H'} \geq \beta \left\| q \right\|_Q \quad \forall q \in Q.$$

iii) *B is an isomorphism from $V^\perp$ onto Q' and there holds*

$$\left\| B(v) \right\|_{Q'} \geq \beta \left\| v \right\|_H \quad \forall v \in V^\perp.$$

iv) *$B : H \to Q'$ is surjective.*

Proof See [14, Chapter 1, Section 4] for details. □

While the solvability analysis of Eq. 2.15 follows as a particular case of [23, Proposition 2.3], we provide next an alternative proof by adapting the arguments from [12]. Indeed, for each $G \in Q'$, we first set

$$V_G := \left\{ v \in H : \quad [B(v), q]_{Q' \times Q} = [G, q]_{Q' \times Q} \quad \forall q \in Q \right\}.$$

In particular, when $G = \mathbf{0}$, we just write V instead of $V_{\mathbf{0}}$ to denote the null space of the linear operator B. Obviously, since B is linear and bounded, V becomes a closed subspace of H. Then, we associate with Eq. 2.15 the following problem: Find $u \in V_G$ such that

$$[A(u), v]_{H' \times H} = [F, v]_{H' \times H} \quad \forall v \in V. \tag{2.16}$$

The next result establishes the connection between Eqs. 2.15 and 2.16.

Lemma 2.3 *Let $(u, p) \in H \times Q$ be a solution of Eq. 2.15. Then, $u \in V_G$ and u is a solution of Eq. 2.16. Conversely, let $u \in V_G$ be a solution of the problem (2.16). Then, there exists $p \in Q$ such that $(u, p) \in H \times Q$ is a solution of Eq. 2.15.*

Proof Let $(u, p) \in H \times Q$ be a solution of Eq. 2.15. Then, from the second equation in Eq. 2.15 we have that $u \in V_G$, and clearly u is a solution of Eq. 2.16 since $[B(v), p]_{Q' \times Q} = 0 \quad \forall v \in V$. Conversely, let $u \in V_G$ be a solution of Eq. 2.16. It follows that $[B(u), q]_{Q' \times Q} = [G, q]_{Q' \times Q} \quad \forall q \in Q$, which says that the second equation in Eq. 2.15 is satisfied. In turn, from Lemma 2.2 we know that B' is an isomorphism from Q onto V°, and since $F - A(u) \in V^\circ$, we deduce that there exists a unique $p \in Q$ such that $B'(p) = F - A(u)$. In this way, the pair $(u, p) \in H \times Q$ solves (2.15). □

Now, given $G \in Q'$, we know from Lemma 2.2 that there exists a unique $u_G \in V^\perp$ such that $B(u_G) = G$. It follows that for each $u \in V_G$ there holds $u - u_G \in V$, that is

$u = u_\mathbf{0} + u_G$, with $u_\mathbf{0} \in V$ and hence problem (2.16) can be re-stated, equivalently, as: Find $u_\mathbf{0} \in V$ such that

$$[A(u_\mathbf{0} + u_G), v]_{H' \times H} = [F, v]_{H' \times H} \quad \forall v \in V. \tag{2.17}$$

According to the foregoing analysis, we have the following result, which states that problems (2.16) and (2.17) are equivalent.

Lemma 2.4 *Given $u_G \in V_G$, we let $u_\mathbf{0} \in V$ be a solution of Eq. 2.17. Then, $u := u_\mathbf{0} + u_G \in V_G$ is a solution of Eq. 2.16. Conversely, let $u \in V_G$ be a solution of Eq. 2.16. Then, there exist $u_G \in V^\perp$ and $u_\mathbf{0} \in V$ such that $u = u_\mathbf{0} + u_G$, and $u_\mathbf{0} \in V$ is solution of Eq. 2.17.*

The next result establishes the unique solvability of problem (2.17).

Theorem 2.1 *(H.1) and (H.2) imply that problem (2.17) is well posed.*

Proof It follows from a classical result in nonlinear functional analysis (see, e.g. [20, Chapter 3, Section 3]. □

Moreover, we remark from this last result that the solution $u_\mathbf{0} + u_G \in V_G$ of Eq. 2.17 is independent of the election of $u_G \in V^\perp \cap V_G$. In fact, given other $\tilde{u}_G \in V_G$, we let $\tilde{u}_\mathbf{0} \in V$ be the unique solution of

$$\left[A(\tilde{u}_\mathbf{0} + \tilde{u}_G), v\right]_{H' \times H} = [F, v]_{H' \times H} \quad \forall v \in V.$$

Since $\left[A(\tilde{u}_\mathbf{0} + \tilde{u}_G), v\right]_{H' \times H} = \left[A((\tilde{u}_\mathbf{0} + \tilde{u}_G - u_G) + u_G), v\right]_{H' \times H}$ for each $v \in V$,we deduce from Theorem 6.1 with $u_G \in V_G$, that $\tilde{u}_\mathbf{0} + \tilde{u}_G - u_G = u_\mathbf{0}$, whence $\tilde{u}_\mathbf{0} + \tilde{u}_G = u_\mathbf{0} + u_G \in V_G$.

Now, we introduce the main result of this section.

Theorem 2.2 *Assume that (H.1), (H.2) and (H.3) hold. Then, there exists a unique solution $(u, p) \in H \times Q$ of Eq. 2.15. In addition, there exists a constant $C > 0$, depending on the constants α, γ and β provided by (H.1), (H.2) and (H.3), such that*

$$\left\|(u, p)\right\|_{H \times Q} \le C \left\{ \left\|F\right\|_{H'} + \left\|G\right\|_{Q'} + \left\|A(\mathbf{0})\right\|_{H'} \right\}. \tag{2.18}$$

Proof The unique solvability of Eq. 2.15 follows straightforwardly from Lemmas 2.3 and 2.4, and Theorem 2.1. To show the estimate (2.18) we let $u_\mathbf{0} \in V$ and $u_G \in V^\perp \cap V_G$, provided by Lemma 2.4, such that $u = u_\mathbf{0} + u_G$. Then, since B is an isomorphism from $V^\perp$ onto Q' (cf. Lemma 2.2), we get

$$\left\|u_G\right\|_H \le \frac{1}{\beta} \left\|B(u_G)\right\|_{Q'} = \frac{1}{\beta} \left\|G\right\|_{Q'}. \tag{2.19}$$

In turn, from (H.2) and problem (2.17), we have

$$\begin{aligned}\alpha\|u_{\mathbf{0}}\|_H^2 &\le [A(u_{\mathbf{0}}+u_G)-A(u_G),u_{\mathbf{0}}]_{H'\times H}\\ &= [F,u_{\mathbf{0}}]_{H'\times H}+[A(\mathbf{0})-A(u_G),u_{\mathbf{0}}]_{H'\times H}-[A(\mathbf{0}),u_{\mathbf{0}}]_{H'\times H}\,,\end{aligned}$$

which, applying (H.1) and the fact that $F,\ A(\mathbf{0})\in H'$, yields

$$\|u_{\mathbf{0}}\|_H \le \frac{1}{\alpha}\left\{\|F\|_{H'}+\gamma\|u_G\|_H+\|A(\mathbf{0})\|_{H'}\right\}. \tag{2.20}$$

On the other hand, applying (H.3) to $p\in Q$, we get

$$\beta\|p\|_Q \le \sup_{\substack{v\in H\\ v\neq\mathbf{0}}}\frac{[B(v),p]_{Q'\times Q}}{\|v\|_H},$$

whence, using that

$$\begin{aligned}[B(v),p]_{Q'\times Q} &= [F,v]_{H'\times H}-[A(u),v]_{H'\times H}\\ &= [F,v]_{H'\times H}+[A(\mathbf{0})-A(u),v]_{H'\times H}-[A(\mathbf{0}),v]_{H'\times H}\end{aligned}\quad \forall v\in H,$$

and applying (H.1), leads to

$$\|p\|_Q \le \frac{1}{\beta}\left\{\|F\|_{H'}+\gamma\|u\|_H+\|A(\mathbf{0})\|_{H'}\right\}. \tag{2.21}$$

The proof follows by combining Eqs. 2.19 and 2.20 with the inequality $\|u\|_H \le \|u_{\mathbf{0}}\|_H+\|u_G\|_H$, and then replacing the resulting estimate in Eq. 2.21. □

2.5 Analysis of the weak formulation

In this section we show the unique solvability of Eq. 2.14 by checking first that (H.1), (H.2), and (H.3) are satisfied, and then applying Theorem 2.2. We begin our analysis with the characterization of the null space $\mathbf{V}$ of the operator $\mathbf{B}$.

Lemma 2.5 *There holds,*

$$\mathbf{V}=\{\mathbf{v}\in\mathbf{H}:\quad \mathit{div}\,\mathbf{v}_{\mathrm{S}}=0\ \mathit{in}\ \Omega_{\mathrm{S}}\quad \mathit{and}\quad \mathit{div}\,\mathbf{v}_{\mathrm{D}}=0\ \mathit{in}\ \Omega_{\mathrm{D}}\}\,.$$

Proof Given $\mathbf{v}\in\mathbf{V}$, we have

$$-(q,\operatorname{div}\mathbf{v}_{\mathrm{S}})_{0,\Omega_{\mathrm{S}}}-(q,\operatorname{div}\mathbf{v}_{\mathrm{D}})_{0,\Omega_{\mathrm{D}}}=0\quad \forall q\in\mathbf{Q}:=L_0^2(\Omega).$$

In turn, since $\mathbf{v}_{\mathrm{S}}\cdot\mathbf{n}=\mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}$ on Σ, we get

$$0=\langle\mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}-\mathbf{v}_{\mathrm{S}}\cdot\mathbf{n},1\rangle_\Sigma=-(1,\operatorname{div}\mathbf{v}_{\mathrm{D}})_{0,\Omega_{\mathrm{D}}}-(1,\operatorname{div}\mathbf{v}_{\mathrm{S}})_{0,\Omega_{\mathrm{S}}},$$

that is,

$$-(c,\operatorname{div}\mathbf{v}_{\mathrm{D}})_{0,\Omega_{\mathrm{D}}}-(c,\operatorname{div}\mathbf{v}_{\mathrm{S}})_{0,\Omega_{\mathrm{S}}}=0\quad\forall c\in R.$$

Then, the decomposition $L^2(\Omega)=L_0^2(\Omega)\oplus R$ implies that

$$-(q,\operatorname{div}\mathbf{v}_{\mathrm{S}})_{0,\Omega_{\mathrm{S}}}-(q,\operatorname{div}\mathbf{v}_{\mathrm{D}})_{0,\Omega_{\mathrm{D}}}=0\quad\forall q\in L^2(\Omega),$$

which yields div $\mathbf{v}_S = 0$ in Ω_S and div $\mathbf{v}_D = 0$ in Ω_D, thus finishing the proof. □

The continuous inf-sup condition for the operator $\mathbf{B}$ is shown next.

Lemma 2.6 *There exists a constant* $\beta > 0$ *such that*

$$\sup_{\substack{\mathbf{v}\in\mathbf{H}\\ \mathbf{v}\neq\mathbf{0}}} \frac{[\mathbf{B}(\mathbf{v}), q]_{\mathbf{Q}'\times\mathbf{Q}}}{\|\mathbf{v}\|_{\mathbf{H}}} \geq \beta \|q\|_{\mathbf{Q}} \quad \forall q \in \mathbf{Q}.$$

Proof Let $q \in \mathbf{Q}$. A well-known result (see e.g. [14]) yields the existence of $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ and $C > 0$, independent of $\mathbf{z}$, such that $-\mathrm{div}\mathbf{z} = q$ in Ω and $\|\mathbf{z}\|_{1,\Omega} \leq C\|q\|_{\mathbf{Q}}$. Next, we put $\mathbf{w}_S := \mathbf{z}|_{\Omega_S}$ and $\mathbf{w}_D := \mathbf{z}|_{\Omega_D}$. Then, we observe that $\mathbf{w}_S \cdot \mathbf{n} = \mathbf{w}_D \cdot \mathbf{n}$ on Σ, that is $\mathbf{w} := (\mathbf{w}_S, \mathbf{w}_D) \in \mathbf{H}$. It follows that $[\mathbf{B}(\mathbf{w}), q]_{\mathbf{Q}'\times\mathbf{Q}} = \|q\|^2_{\mathbf{Q}}$ and $\|\mathbf{w}\|_{\mathbf{H}} \leq \|\mathbf{z}\|_{1,\Omega} \leq C\|q\|_{\mathbf{Q}}$, which gives

$$\sup_{\substack{\mathbf{v}\in\mathbf{H}\\ \mathbf{v}\neq\mathbf{0}}} \frac{[\mathbf{B}(\mathbf{v}), q]_{\mathbf{Q}'\times\mathbf{Q}}}{\|\mathbf{v}\|_{\mathbf{H}}} \geq \frac{[\mathbf{B}(\mathbf{w}), q]_{\mathbf{Q}'\times\mathbf{Q}}}{\|\mathbf{w}\|_{\mathbf{H}}} \geq \frac{1}{C}\|q\|_{\mathbf{Q}},$$

and the proof is completed. □

The next lemma shows that the nonlinear operator, induced by the term $(\mu(|\nabla\mathbf{u}_S|)\nabla\mathbf{u}_S, \nabla\mathbf{v}_S)_{0,\Omega_S}$, satisfies (H.1) and (H.2).

Lemma 2.7 *Let* $\mathbf{A}_S : \mathbf{H}^1_{\Gamma_S}(\Omega_S) \to [\mathbf{H}^1_{\Gamma_S}(\Omega_S)]'$ *be the nonlinear operator given by*

$$[\mathbf{A}_S(\mathbf{u}_S), \mathbf{v}_S] := (\mu(|\nabla\mathbf{u}_S|)\nabla\mathbf{u}_S, \nabla\mathbf{v}_S)_{0,\Omega_S} \quad \forall\mathbf{u}_S, \mathbf{v}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S),$$

where $[\cdot,\cdot]$ *denotes the duality pairing between* $\mathbf{H}^1_{\Gamma_S}(\Omega_S)$ *and* $[\mathbf{H}^1_{\Gamma_S}(\Omega_S)]'$. *Then,* $\mathbf{A}_S$ *is Lipschitz continuous, and for each* $\mathbf{z}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$, $\mathbf{A}_S(\mathbf{z}_S + \cdot)$ *is strongly monotone.*

Proof Let $\mathbf{u}_S, \mathbf{v}_S, \mathbf{w}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$. By definition of $\mathbf{A}_S$ we have that

$$[\mathbf{A}_S(\mathbf{u}_S) - \mathbf{A}_S(\mathbf{v}_S), \mathbf{w}_S] = \int_{\Omega_S} (\mu(|\nabla\mathbf{u}_S|)\nabla\mathbf{u}_S - \mu(|\nabla\mathbf{v}_S|)\nabla\mathbf{v}_S) : \nabla\mathbf{w}_S,$$

which, denoting $\sigma := \nabla\mathbf{u}_S$, $\tau := \nabla\mathbf{v}_S$, and $\tilde{\tau} := \nabla\mathbf{w}_S$, becomes

$$\begin{aligned}[\mathbf{A}_S(\mathbf{u}_S) - \mathbf{A}_S(\mathbf{v}_S), \mathbf{w}_S] &= \int_{\Omega_S} (\mu(|\sigma|)\sigma - \mu(|\tau|)\tau) : \tilde{\tau} \\ &= \sum_{i,j=1}^{d} \int_{\Omega_S} (\mu(|\sigma|)\sigma_{ij} - \mu(|\tau|)\tau_{ij})\tilde{\tau}_{ij}.\end{aligned}$$

Next, using Eq. 2.5 and setting $\tilde{\sigma}(m) := m\sigma + (1-m)\tau \quad \forall m \in (0,1)$, we can write for each $i, j \in \{1, ..., d\}$,

$$\begin{aligned}\mu(|\sigma|)\sigma_{ij} - \mu(|\tau|)\tau_{ij} &= \mu_{ij}(\sigma) - \mu_{ij}(\tau) = \int_0^1 \frac{\partial}{\partial m}\mu_{ij}(\tilde{\sigma})dm \\ &= \sum_{k,l=1}^{d} \int_0^1 \frac{\partial \tilde{\sigma}_{kl}}{\partial m}\frac{\partial}{\partial \tilde{\sigma}_{kl}}\mu_{ij}(\tilde{\sigma})dm \\ &= \sum_{k,l=1}^{d} \int_0^1 \frac{\partial}{\partial \tilde{\sigma}_{kl}}\mu_{ij}(\tilde{\sigma})(\sigma_{kl} - \tau_{kl})dm,\end{aligned}$$

which yields

$$[\mathbf{A}_S(\mathbf{u}_S) - \mathbf{A}_S(\mathbf{v}_S), \mathbf{w}_S] = \sum_{i,j,k,l=1}^{d} \int_{\Omega_S} \left(\int_0^1 \frac{\partial}{\partial \tilde{\sigma}_{kl}}\mu_{ij}(\tilde{\sigma})(\sigma_{kl} - \tau_{kl})dm \right) \tilde{\tau}_{ij}.$$

Hence, applying Eq. 2.6 and the Cauchy-Schwarz inequality, we find that

$$\|\mathbf{A}_S(\mathbf{u}_S) - \mathbf{A}_S(\mathbf{v}_S)\|_{\mathbf{H}^1(\Omega_S)'} = \sup_{\substack{\mathbf{w}_S \in \mathbf{H}^1(\Omega_S) \\ \mathbf{w}_S \neq \mathbf{0}}} \frac{[\mathbf{A}_S(\mathbf{u}_S) - \mathbf{A}_S(\mathbf{v}_S), \mathbf{w}_S]}{\|\mathbf{w}_S\|_{1,\Omega_S}} \leq \gamma_0 \|\mathbf{u}_S - \mathbf{v}_S\|_{1,\Omega_S}.$$

Similarly, given $\mathbf{z}_S, \mathbf{u}_S, \mathbf{v}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$, and denoting $\sigma := \nabla \mathbf{z}_S$, $\tau := \nabla \mathbf{u}_S$, $\tilde{\tau} := \nabla \mathbf{v}_S$, and $\hat{\sigma}(m) := m(\sigma + \tau) + (1-m)(\sigma + \tilde{\tau}) \quad \forall m \in (0,1)$, we obtain

$$\begin{aligned}[\mathbf{A}_S(\mathbf{z}_S + \mathbf{u}_S) - \mathbf{A}_S(\mathbf{z}_S + \mathbf{v}_S), \mathbf{u}_S - \mathbf{v}_S] &= \int_{\Omega_S} \{\mu(|\sigma + \tau|)(\sigma + \tau) \\ &\quad -\mu(|\sigma + \tilde{\tau}|)(\sigma + \tilde{\tau})\} : (\tau - \tilde{\tau}) \\ &= \sum_{i,j,k,l=1}^{d} \int_{\Omega_S} \int_0^1 \frac{\partial}{\partial \hat{\sigma}_{kl}}\mu_{ij}(\hat{\sigma})(\tau_{ij} - \tilde{\tau}_{ij}) \\ &\quad \times(\tau_{kl} - \tilde{\tau}_{kl})dm.\end{aligned}$$

In this way, using now Eq. 2.7 and the Friedrich-Poincaré inequality, we get

$$[\mathbf{A}_S(\mathbf{z}_S + \mathbf{u}_S) - \mathbf{A}_S(\mathbf{z}_S + \mathbf{v}_S), \mathbf{u}_S - \mathbf{v}_S] \geq \tilde{\alpha}_0 \|\mathbf{u}_S - \mathbf{v}_S\|^2_{1,\Omega_S},$$

with $\tilde{\alpha}_0 > 0$ depending on α_0 and the constant provided by the aforementioned inequality. □

Note now that the nonlinear operator $\mathbf{A}$ can be written as

$$[\mathbf{A}(\mathbf{u}), \mathbf{v}]_{\mathbf{H}' \times \mathbf{H}} = [\mathbf{A}_S(\mathbf{u}_S), \mathbf{v}_S] + \langle \nu\kappa^{-1}\boldsymbol{\pi}_t\mathbf{u}_S\boldsymbol{\pi}_t\mathbf{v}_S\rangle_\Sigma + (\mathbf{K}^{-1}\mathbf{u}_D\mathbf{v}_D)_{0,\Omega_D} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}. \tag{2.22}$$

The following lemma shows that $\mathbf{A}$ satisfies (H.1) and (H.2).

Lemma 2.8 *Let* $H_{\Gamma_D}(div^0;\Omega_D) := \{\mathbf{v}_D \in H_{\Gamma_D}(div;\Omega_D) : div\mathbf{v}_D = 0\}$. *Then, the nonlinear operator* $\mathbf{A}$ *is Lipschitz continuous in* $\mathbf{H}^1_{\Gamma_S}(\Omega_S) \times H_{\Gamma_D}(div;\Omega_D)$, *and for each* $\mathbf{z} \in \mathbf{H}^1_{\Gamma_S}(\Omega_S) \times H_{\Gamma_D}(div;\Omega_D)$, $\mathbf{A}(\mathbf{z} + \cdot)$ *is strongly monotone in* $\mathbf{H}^1_{\Gamma_S}(\Omega_S) \times H_{\Gamma_D}(div^0;\Omega_D)$.

Proof It follows straightforwardly from the corresponding properties of $\mathbf{A}_S$ (cf. Lemma 2.7) and from the fact that the expressions $\langle \nu\kappa^{-1}\boldsymbol{\pi}_t\mathbf{u}_S, \boldsymbol{\pi}_t\mathbf{v}_S\rangle_\Sigma$ and $\langle \mathbf{K}^{-1}\mathbf{u}_D, \mathbf{v}_D\rangle_{0,\Omega_D}$ induce positive semi-definite, symmetric and uniformly positive definite bilinear forms, respectively. □

The main result of this section is established as follows.

Theorem 2.3 *There exists a unique* $(\mathbf{u}, p) \in \mathbf{H} \times \mathbf{Q}$ *solution of the primal-mixed formulation* (2.14) *and there exists* $C > 0$ *such that*

$$\|(\mathbf{u}, p)\|_{\mathbf{H}\times\mathbf{Q}} \leq C\left\{\|\mathbf{f}_S\|_{0,\Omega_S} + \|f_D\|_{0,\Omega_D}\right\}.$$

Proof It follows from Lemmas 2.5, 2.6, 2.7 and 2.8, and a straightforward application of Theorem 2.2. □

3 The discrete problem

In this section we introduce and analyze a nonconforming Galerkin scheme for the primal-mixed formulation (2.14). We begin with the following discrete abstract analysis.

3.1 A nonconforming discrete scheme

We begin by recalling that the unique solvability of Eq. 2.15 is guaranteed by Theorem 2.2. Now, we let $\tilde{H}$ and $\tilde{Q}$ be two Hilbert spaces with dual spaces $\tilde{H}'$ and $\tilde{Q}'$, respectively, such that $H \subseteq \tilde{H}$ and $Q \subseteq \tilde{Q}$, and we consider finite dimensional subspaces $H_h \subseteq \tilde{H}$ and $Q_h \subseteq \tilde{Q}$. Also, we let $\tilde{A} : \tilde{H} \to \tilde{H}$ be a nonlinear operator, and let $\tilde{B} : \tilde{H} \to \tilde{Q}'$ be a linear operator with adjoint $\tilde{B}' : \tilde{Q} \to \tilde{H}'$. Then, given $\tilde{F} \in \tilde{H}$ and $\tilde{G} \in \tilde{Q}$ we consider the nonconforming discrete scheme of Eq. 2.15: Find $(u_h, p_h) \in H_h \times Q_h$ such that

$$\begin{aligned} \left[\tilde{A}(u_h), v_h\right]_{\tilde{H}'\times\tilde{H}} + \left[\tilde{B}(v_h), p_h\right]_{\tilde{Q}'\times\tilde{Q}} &= \left[\tilde{F}, v_h\right]_{\tilde{H}'\times\tilde{H}} \quad \forall v_h \in H_h, \\ \left[\tilde{B}(u_h), q_h\right]_{\tilde{Q}'\times\tilde{Q}} &= \left[\tilde{G}, q_h\right]_{\tilde{Q}'\times\tilde{Q}} \quad \forall q_h \in Q_h. \end{aligned} \tag{3.1}$$

Note that the nonconformity of Eq. 3.1 is due to the fact that H_h and Q_h are not necessarily contained in H and Q, respectively, and also because $\tilde{A}$ and $\tilde{B}$ dot not necessarily coincide with the operators A and B. Now, given $\tilde{G} \in \tilde{Q}'$, we set

$$V_{\tilde{G},h} := \left\{ v_h \in H_h : \left[\tilde{B}(v_h), q_h\right]_{\tilde{Q}'\times\tilde{Q}} = \left[\tilde{G}, q_h\right]_{\tilde{Q}'\times\tilde{Q}} \quad \forall q_h \in Q_h \right\}.$$

In particular, if $\tilde{G} = \mathbf{0}$, we just write V_h instead of $V_{\mathbf{0},h}$ to denote the discrete kernel of the operator $\tilde{B}$. In order to establish the uniqueness, stability, and corresponding a priori estimate for the discrete scheme (3.1) we need to introduce some hypotheses:

(H.4) There exists a constant $\tilde{\beta} > 0$, independent of h, such that

$$\sup_{\substack{v_h \in H_h \\ v_h \neq \mathbf{0}}} \frac{\left[\tilde{B}(v_h), q_h\right]_{\tilde{Q}'\times\tilde{Q}}}{\|v_h\|_{\tilde{H}}} \geq \tilde{\beta} \|q_h\|_{\tilde{Q}} \quad \forall q_h \in Q_h.$$

(H.5) The operator $\tilde{A}$ is Lispchitz continuous in $\tilde{H}$ with constant $\tilde{\gamma} > 0$, that is

$$\left| \left[\tilde{A}(u) - \tilde{A}(v), w\right]_{\tilde{H}'\times\tilde{H}} \right| \leq \tilde{\gamma} \|u - v\|_{\tilde{H}} \|w\|_{\tilde{H}} \quad \forall\, u, v, w \in \tilde{H}.$$

(H.6) For all $z_h \in H_h$, the operator $\tilde{A}(z_h + \cdot)$ is strongly monotone in V_h with constant $\alpha > 0$ independent of h, that is,

$$\left[\tilde{A}(z_h + u_h) - \tilde{A}(z_h + v_h), u_h - v_h\right]_{\tilde{H}'\times\tilde{H}} \geq \tilde{\alpha} \|u_h - v_h\|_{\tilde{H}}^2 \quad \forall\, u_h, v_h \in V_h.$$

Applying Lemma 2.2 to the present discrete scheme, we deduce from (H.4) that the discrete version of $\tilde{B}$ is an isomorphism from $V_h^{\perp}$ onto Q_h, whence we find that there exists a unique $u_{\tilde{G},h} \in V_h^{\perp}$ such that $\left[\tilde{B}(u_{\tilde{G},h}), q_h\right]_{\tilde{Q}'\times\tilde{Q}} = \left[\tilde{G}, q_h\right]_{\tilde{Q}'\times\tilde{Q}} \quad \forall q_h \in Q_h$. Note that this also says that $u_{\tilde{G},h} \in V_h^{\perp} \cap V_{\tilde{G},h}$. Then, we associate with Eq. 3.1 the discrete problem: Find $u_{\mathbf{0},h} \in V_h$ such that

$$\left[\tilde{A}(u_{\mathbf{0},h} + u_{\tilde{G},h}), v_h\right]_{\tilde{H}'\times\tilde{H}} = \left[\tilde{F}, v_h\right]_{\tilde{H}'\times\tilde{H}} \quad \forall v_h \in V_h, \tag{3.2}$$

which is the discrete analogue of Eq. 2.17. In addition, using similar arguments to those employed in the proof of Lemma 2.3, we can prove the corresponding connection between Eqs. 3.1 and 3.2. Further, similary as in Section 2.4 (cf. Lemma 2.4), we remark that Eq. 3.2 is actually equivalent to the problem: Find $u_h \in V_{\tilde{G},h}$ such that

$$\left[\tilde{A}(u_h), v_h\right]_{\tilde{H}'\times\tilde{H}} = \left[\tilde{F}, v_h\right]_{\tilde{H}'\times\tilde{H}} \quad \forall v_h \in V_h.$$

We now establish the well-posedness of Eq. 3.2.

Lemma 3.1 *Assumptions* (H.5) *and* (H.6) *guarantee the unique solvability of Eq.* 3.2.

Proof It follows from [20, Chapter 3, Theorem 3.3.23]. □

As for the continuous case, we remark here that the solution $u_{\mathbf{0},h} + u_{\tilde{G},h} \in V_{\tilde{G},h}$ of Eq. 3.2 is independent of the choice of $u_{\tilde{G},h} \in V_{\tilde{G},h}$. The well-posedness of Eq. 3.1 is stated now.

Theorem 3.1 *There exists a unique $(u_h, p_h) \in H_h \times Q_h$ solution of Eq. 3.1. In addition, there exists a constant $\tilde{C} > 0$, independent of h, such that*

$$\left\|(u_h, p_h)\right\|_{\tilde{H}\times\tilde{Q}} \leq \tilde{C}\left\{\left\|\tilde{F}\right\|_{\tilde{H}'} + \left\|\tilde{G}\right\|_{\tilde{Q}'} + \left\|\tilde{A}(\mathbf{0})\right\|_{\tilde{H}'}\right\}.$$

Proof The proof follows similarly as for Theorem 2.2. □

We now aim to derive an a priori error estimate for Eq. 2.15 and its discrete scheme (3.1). Hereafter, we let $(u, p) \in H \times Q$ and $(u_h, p_h) \in H_h \times Q_h$ be the unique solutions of the weak formulation (2.15) and the nonconforming Galerkin scheme (3.1), respectively, and let $u_{\tilde{G},h} \in V_{\tilde{G},h}$ and $u_{\mathbf{0},h} \in V_h$, provided by the foregoing analysis, such that $u_h = u_{\tilde{G},h} + u_{\mathbf{0},h}$. The next two preliminary results show partial error estimates for $\left\|u - u_h\right\|_{\tilde{H}}$ and $\left\|p - p_h\right\|_{\tilde{Q}}$, as well as a translation property between the discrete subspaces V_h and H_h.

Lemma 3.2 *Under the assumptions (H.4), (H.5), and (H.6) there hold*

$$\begin{aligned}\left\|u - u_h\right\|_{\tilde{H}} \leq C_1 \Bigg\{ & \inf_{v_h\in V_h}\left\|u - (u_{\tilde{G},h} + v_h)\right\|_{\tilde{H}} + \inf_{q_h\in Q_h}\left\|p - q_h\right\|_{\tilde{Q}} \\ & + \sup_{\substack{w_h\in V_h\\ w_h\neq\mathbf{0}}} \frac{\left[F - \tilde{A}(u) - \tilde{B(p)}, w_h\right]_{\tilde{H}'\times\tilde{H}}}{\left\|w_h\right\|_{\tilde{H}}} \\ & + \sup_{\substack{w_h\in V_h\\ w_h\neq\mathbf{0}}} \frac{\left[\tilde{F} - F, w_h\right]_{\tilde{H}'\times\tilde{H}}}{\left\|w_h\right\|_{\tilde{H}}}\Bigg\},\end{aligned}$$

and

$$\begin{aligned}\left\|p - p_h\right\|_{\tilde{Q}} \leq C_2 \Bigg\{ & \left\|u - u_h\right\|_{\tilde{H}} + \inf_{q_h\in Q_h}\left\|p - q_h\right\|_{\tilde{Q}} \\ & + \sup_{\substack{v_h\in H_h\\ v_h\neq\mathbf{0}}} \frac{\left[F - \tilde{A}(u) - \tilde{B(p)}, v_h\right]_{\tilde{H}'\times\tilde{H}}}{\left\|v_h\right\|_{\tilde{H}}} \\ & + \sup_{\substack{v_h\in H_h\\ v_h\neq\mathbf{0}}} \frac{\left[\tilde{F} - F, v_h\right]_{\tilde{H}'\times\tilde{H}}}{\left\|v_h\right\|_{\tilde{H}}}\Bigg\},\end{aligned}$$

where $C_1 := \frac{1}{\tilde{\alpha}}\max\left\{\tilde{\alpha} + \tilde{\gamma}, \left\|\tilde{B}\right\|_{\tilde{Q}}, 1\right\}$ *and* $C_2 := \frac{1}{\tilde{\beta}}\max\left\{\tilde{\beta} + \left\|\tilde{B}\right\|_{\tilde{Q}}, \tilde{\gamma}, 1\right\}$.

Proof We first estimate $\|u - u_h\|_{\tilde{H}}$. Given $v_h \in V_h$, we have from the triangle inequality

$$\|u-u_h\|_{\tilde{H}} = \|u-(u_{\tilde{G},h}+u_{\mathbf{0},h})\|_{\tilde{H}} \leq \|u-(u_{\tilde{G},h}+v_h)\|_{\tilde{H}} + \|u_{\mathbf{0},h}-v_h\|_{\tilde{H}}. \quad (3.3)$$

Now, applying (H.6) with $z_h = u_{\tilde{G},h}$, we deduce that

$$\begin{aligned}\tilde{\alpha}\|u_{\mathbf{0},h}-v_h\|_{\tilde{H}}^2 &\leq \left[\tilde{A}(u_{\tilde{G},h}+u_{\mathbf{0},h}) - \tilde{A}(u_{\tilde{G},h}+v_h), u_{\mathbf{0},h}-v_h\right]_{H'\times H} \\ &= \left[\tilde{A}(u_h), u_{\mathbf{0},h}-v_h\right]_{\tilde{H}'\times\tilde{H}} - \left[\tilde{A}(u_{\tilde{G},h}+v_h), u_{\mathbf{0},h}-v_h\right]_{\tilde{H}'\times\tilde{H}}.\end{aligned}$$

Then, using that

$$\left[\tilde{B}(u_{\mathbf{0},h}-v_h), q_h\right]_{\tilde{Q}'\times\tilde{Q}} = 0 \quad \forall q_h \in Q_h\,,$$

and that

$$\left[\tilde{A}(u_h), u_{\mathbf{0},h}-v_h\right]_{\tilde{H}'\times\tilde{H}} = \left[\tilde{F}, u_{\mathbf{0},h}-v_h\right]_{\tilde{H}'\times\tilde{H}}\,,$$

we find, after adding and substracting appropiate terms, that

$$\begin{aligned}\tilde{\alpha}\|u_{\mathbf{0},h}-v_h\|_{\tilde{H}}^2 &\leq \left[\tilde{F}, u_{\mathbf{0},h}-v_h\right]_{\tilde{H}'\times\tilde{H}} - \left[\tilde{B}(u_{\mathbf{0},h}-v_h), q_h\right]_{\tilde{Q}'\times\tilde{Q}} \\ &\quad - \left[\tilde{A}(u_{\tilde{G},h}+v_h), u_{\mathbf{0},h}-v_h\right]_{\tilde{H}'\times\tilde{H}} \\ &= \left[F-\tilde{A}(u)-\tilde{B(p)}, u_{\mathbf{0},h}-v_h\right]_{\tilde{H}'\times\tilde{H}} \\ &\quad + \left[\tilde{F}-F, u_{\mathbf{0},h}-v_h\right]_{\tilde{H}'\times\tilde{H}} \\ &\quad + \left[\tilde{B}(u_{\mathbf{0},h}-v_h), p-q_h\right]_{\tilde{Q}'\times\tilde{Q}} \\ &\quad + \left[\tilde{A}(u)-\tilde{A}(u_{\tilde{G},h}+v_h), u_{\mathbf{0},h}-v_h\right]_{\tilde{H}'\times\tilde{H}},\end{aligned}$$

which, applying the boundedness provided by the duality parings and the assumption (H.5), dividing by $\tilde{\alpha}\|u_{\mathbf{0},h}-v_h\|_{\tilde{H}}$, and then combining the resulting inequality with Eq. 3.3, implies that for each $(v_h, q_h) \in V_h \times Q_h$ there holds

$$\begin{aligned}\|u-u_h\|_{\tilde{H}} \leq \frac{1}{\tilde{\alpha}}\Bigg\{&(\tilde{\alpha}+\tilde{\gamma})\,\|u-(u_{\tilde{G},h}+v_h)\|_{\tilde{H}} + \|\tilde{B}\|\|p-q_h\|_{\tilde{Q}} \\ &+ \sup_{\substack{w_h\in V_h\\ w_h\neq\mathbf{0}}} \frac{\left[F-\tilde{A}(u)-\tilde{B}(p), w_h\right]\tilde{H}'\times\tilde{H}}{\|w_h\|_{\tilde{H}}} \\ &+ \sup_{\substack{w_h\in V_h\\ w_h\neq\mathbf{0}}} \frac{\left[\tilde{F}-F, w_h\right]\tilde{H}'\times\tilde{H}}{\|w_h\|_{\tilde{H}}}\Bigg\}. \quad (3.4)\end{aligned}$$

On the other hand, applying (H.4) we obtain for each $q_h \in Q_h$

$$\tilde{\beta} \| p_h - q_h \|_{\tilde{Q}} \leq \sup_{\substack{v_h \in H_h \\ v_h \neq \mathbf{0}}} \frac{\left[\tilde{B}(v_h), p_h - q_h \right]_{\tilde{Q}' \times \tilde{Q}}}{\| v_h \|_{\tilde{H}}}, \tag{3.5}$$

and according to the first equation of Eq. 3.1, we can write

$$\begin{aligned} \left[\tilde{B}(v_h), p_h - q_h \right]_{\tilde{Q}' \times \tilde{Q}} &= \left[\tilde{B}(v_h), p_h \right]_{\tilde{Q}' \times \tilde{Q}} - \left[\tilde{B}(v_h), q_h \right]_{\tilde{Q}' \times \tilde{Q}} \\ &= \left[\tilde{F}, v_h \right]_{\tilde{H}' \times \tilde{H}} - \left[\tilde{A}(u_h), v_h \right]_{\tilde{H}' \times \tilde{H}} - \left[\tilde{B}(v_h), q_h \right]_{\tilde{Q}' \times \tilde{Q}} \\ &= \left[\tilde{F} - F, v_h \right]_{\tilde{H}' \times \tilde{H}} + \left[F - \tilde{A}(u) - \tilde{B}'(p), v_h \right]_{\tilde{H}' \times \tilde{H}} \\ &\quad + \left[\tilde{A}(u) - \tilde{A}(u_h), v_h \right]_{\tilde{H}' \times \tilde{H}} + \left[\tilde{B}(v_h), p - q_h \right]_{\tilde{Q}' \times \tilde{Q}}, \end{aligned}$$

that is, for each $(v_h, q_h) \in H_h \times Q_h$ there holds

$$\begin{aligned} \left[\tilde{B}(v_h), p_h - q_h \right]_{\tilde{Q}' \times \tilde{Q}} = \left[\tilde{F} - F, v_h \right]_{\tilde{H}' \times \tilde{H}} + \left[F - \tilde{A}(u) - \tilde{B}'(p), v_h \right]_{\tilde{H}' \times \tilde{H}} \\ + \left[\tilde{A}(u) - \tilde{A}(u_h), v_h \right]_{\tilde{H}' \times \tilde{H}} + \left[\tilde{B}(v_h), p - q_h \right]_{\tilde{Q}' \times \tilde{Q}}. \end{aligned}$$

Replacing the foregoing identity back into Eq. 3.5, and applying (H.5) and the boundedness of $\tilde{B}$, we arrive at

$$\begin{aligned} \| p_h - q_h \|_{\tilde{Q}} \leq \frac{1}{\tilde{\beta}} \Bigg\{ & \tilde{\gamma} \| u - u_h \|_{\tilde{H}} + \| \tilde{B} \|_{\tilde{Q}'} \| p - q_h \|_{\tilde{Q}} \\ & + \sup_{\substack{w_h \in H_h \\ w_h \neq \mathbf{0}}} \frac{\left[F - \tilde{A}(u) - \tilde{B}'(p), w_h \right]_{\tilde{H}' \times \tilde{H}}}{\| w_h \|_{\tilde{H}}} \\ & + \sup_{\substack{w_h \in H_h \\ w_h \neq \mathbf{0}}} \frac{\left[\tilde{F} - F, w_h \right]_{\tilde{H}' \times \tilde{H}}}{\| w_h \|_{\tilde{H}}} \Bigg\}. \end{aligned}$$

Hence, applying the triangle inequality we conclude that

$$\begin{aligned}\|p - p_h\|_{\tilde{Q}} &\leq \|p - q_h\|_{\tilde{Q}} + \|p_h - q_h\|_{\tilde{Q}} \\ &\leq \frac{1}{\tilde{\beta}} \Bigg\{ \tilde{\gamma} \|u - u_h\|_{\tilde{H}} + \left(\tilde{\beta} + \|\tilde{B}\|_{\tilde{Q}'}\right) \|p - q_h\|_{\tilde{Q}} \\ &\quad + \sup_{\substack{w_h \in H_h \\ w_h \neq \mathbf{0}}} \frac{\left[F - \tilde{A}(u) - \tilde{B}'(p), w_h\right]_{\tilde{H}' \times \tilde{H}}}{\|w_h\|_{\tilde{H}}} \\ &\quad + \sup_{\substack{w_h \in H_h \\ w_h \neq \mathbf{0}}} \frac{\left[\tilde{F} - F, w_h\right]_{\tilde{H}' \times \tilde{H}}}{\|w_h\|_{\tilde{H}}} \Bigg\}. \end{aligned} \tag{3.6}$$

Finally, the result follows applying infimum on V_h and Q_h in Eq. 3.4, and also taking infimum on Q_h in the Eq. 3.6. □

It remains to estimate $\inf_{v_h \in V_h} \|u - (u_{\tilde{G},h} + v_h)\|_{\tilde{H}}$, which is provided by the following lemma.

Lemma 3.3 *There holds*

$$\begin{aligned}\inf_{v_h \in V_h} \|u - (u_{\tilde{G},h} + v_h)\|_{\tilde{H}} \leq C \Bigg\{ &\inf_{v_h \in H_h} \|u - v_h\|_{\tilde{H}} \\ &+ \sup_{\substack{q_h \in Q_h \\ q_h \neq \mathbf{0}}} \frac{\left[G - \tilde{B}(u), q_h\right]_{\tilde{Q}' \times \tilde{Q}}}{\|q_h\|_{\tilde{Q}}} \\ &+ \sup_{\substack{q_h \in Q_h \\ q_h \neq \mathbf{0}}} \frac{\left[\tilde{G} - G, q_h\right]_{\tilde{Q}' \times \tilde{Q}}}{\|q_h\|_{\tilde{Q}}} \Bigg\}, \end{aligned}$$

with $C := \frac{1}{\tilde{\beta}} \max\left\{\tilde{\beta} + \|\tilde{B}\|_{\tilde{Q}'}, 1\right\}$.

Proof Given $\hat{v}_h \in H_h$, we know from (H.4) that there exists a unique $w_h \in V_h^{\perp} \cap H_h$ such that

$$\left[\tilde{B}(w_h), q_h\right]_{\tilde{Q}' \times \tilde{Q}} = \left[\tilde{B}(u_{\tilde{G},h} - \hat{v}_h), q_h\right]_{\tilde{Q}' \times \tilde{Q}} \quad \forall q_h \in Q_h, \tag{3.7}$$

and there holds

$$\begin{aligned}
\|w_h\|_{\tilde{H}} &\leq \frac{1}{\tilde{\beta}} \sup_{\substack{q_h \in Q_h \\ q_h \neq \mathbf{0}}} \frac{\left[\tilde{B}(w_h), q_h\right]_{\tilde{Q}' \times \tilde{Q}}}{\|q_h\|_{\tilde{Q}}} = \frac{1}{\beta} \sup_{\substack{q_h \in Q_h \\ q_h \neq \mathbf{0}}} \frac{[\tilde{B}(u_{\tilde{G},h} - \hat{v}_h), q_h]_{\tilde{Q} \times \tilde{Q}}}{\|q_h\|_{\tilde{Q}}} \\
&= \frac{1}{\tilde{\beta}} \sup_{\substack{q_h \in Q_h \\ q_h \neq \mathbf{0}}} \frac{\left[\tilde{B}(u - \hat{v}_h) - \tilde{B}(u - u_{\tilde{G},h}), q_h\right]_{\tilde{Q}' \times \tilde{Q}}}{\|q_h\|_{\tilde{Q}}} \\
&\leq \frac{\|\tilde{B}\|_{\tilde{Q}'}}{\tilde{\beta}} \|u - \hat{v}_h\|_{\tilde{Q}} \\
&\quad + \frac{1}{\tilde{\beta}} \left\{ \sup_{\substack{q_h \in Q_h \\ q_h \neq \mathbf{0}}} \frac{\left[G - \tilde{B}(u), q_h\right]_{\tilde{Q}' \times \tilde{Q}}}{\|q_h\|_{\tilde{Q}}} + \sup_{\substack{q_h \in Q_h \\ q_h \neq \mathbf{0}}} \frac{\left[\tilde{G} - G, q_h\right]_{\tilde{Q}' \times \tilde{Q}}}{\|q_h\|_{\tilde{Q}}} \right\},
\end{aligned}$$

where the foregoing expressions have arised after adding and substracting $\tilde{B}(u)$ and G, and realizing that $[\tilde{B}(u_{\tilde{G},h}), q_h]_{\tilde{Q}' \times \tilde{Q}} = [\tilde{G}, q_h]_{\tilde{Q}' \times \tilde{Q}} \quad \forall q_h \in Q_h$. Then, noting from Eq. 3.7 that $\hat{v}_h + w_h - u_{\tilde{G},h} \in V_h$, we find that

$$\begin{aligned}
\inf_{v_h \in V_h} \|u - (u_{\tilde{G},h} + v_h)\|_{\tilde{H}} &\leq \|u - (u_{\tilde{G},h} + \hat{v}_h + w_h - u_{\tilde{G},h})\|_{\tilde{H}} \leq \|u - \hat{v}_h\|_{\tilde{H}} + \|w_h\|_{\tilde{H}} \\
&\leq \left(1 + \frac{\|\tilde{B}\|_{\tilde{Q}'}}{\tilde{\beta}}\right) \|u - \hat{v}_h\|_{\tilde{H}} \\
&\quad + \frac{1}{\tilde{\beta}} \left\{ \sup_{\substack{q_h \in Q_h \\ q_h \neq \mathbf{0}}} \frac{\left[G - \tilde{B}(u), q_h\right]_{\tilde{Q}' \times \tilde{Q}}}{\|q_h\|_{\tilde{Q}}} \right. \\
&\quad \left. + \sup_{\substack{q_h \in Q_h \\ q_h \neq \mathbf{0}}} \frac{\left[\tilde{G} - G, q_h\right]_{\tilde{Q}' \times \tilde{Q}}}{\|q_h\|_{\tilde{Q}}} \right\},
\end{aligned}$$

which, taking infimum on $\hat{v}_h \in H_h$, yields the required inequality and completes the proof. □

The main result of this section is established as follows.

Theorem 3.2 *Under the assumptions (H.4), (H.5), and (H.6), the nonconforming discrete scheme* (3.1) *is stable, and there holds the Strang-type error estimate*

$$\begin{aligned}\left\|(u-u_h, p-p_h)\right\|_{\tilde{H}\times\tilde{Q}} \leq C \Bigg\{ &\inf_{v_h\in H_h} \left\|u-v_h\right\|_{\tilde{H}} + \inf_{q_h\in Q_h} \left\|p-q_h\right\|_{\tilde{Q}} \\ &+ \sup_{\substack{w_h\in H_h\\ w_h\neq\mathbf{0}}} \frac{\left[F-\tilde{A}(u)-\tilde{B}'(p), w_h\right]_{\tilde{H}'\times\tilde{H}}}{\left\|w_h\right\|_{\tilde{H}}} + \sup_{\substack{w_h\in H_h\\ w_h\neq\mathbf{0}}} \frac{\left[\tilde{F}-F, w_h\right]_{\tilde{H}'\times\tilde{H}}}{\left\|w_h\right\|_{\tilde{H}}} \\ &+ \sup_{\substack{q_h\in Q_h\\ q_h\neq\mathbf{0}}} \frac{\left[G-\tilde{B}(u), q_h\right]_{\tilde{Q}'\times\tilde{Q}}}{\left\|q_h\right\|_{\tilde{Q}}} + \sup_{\substack{q_h\in Q_h\\ q_h\neq\mathbf{0}}} \frac{\left[\tilde{G}-G, q_h\right]_{\tilde{Q}'\times\tilde{Q}}}{\left\|q_h\right\|_{\tilde{Q}}} \Bigg\}.\end{aligned}$$

Proof The proof follows from a straightforward application of Lemmas 3.2 and 3.3. □

It is important to observe from Theorem 3.2 that if $H_h \subseteq H$, then

$$\sup_{\substack{v_h\in H_h\\ v_h\neq\mathbf{0}}} \frac{\left[F-\tilde{A}(u)-\tilde{B}'(p), v_h\right]_{\tilde{H}'\times\tilde{H}}}{\left\|v_h\right\|_{\tilde{H}}} = 0 \quad \text{and} \quad \sup_{\substack{v_h\in H_h\\ v_h\neq\mathbf{0}}} \frac{\left[\tilde{F}-F, v_h\right]_{\tilde{H}'\times\tilde{H}}}{\left\|v_h\right\|_{\tilde{H}}} = 0.$$

Similarly, if $Q_h \subseteq Q$, then

$$\sup_{\substack{q_h\in Q_h\\ q_h\neq\mathbf{0}}} \frac{\left[G-\tilde{B}(u), q_h\right]_{\tilde{Q}'\times\tilde{Q}}}{\left\|q_h\right\|_{\tilde{Q}}} = 0 \quad \text{and} \quad \sup_{\substack{q_h\in Q_h\\ q_h\neq\mathbf{0}}} \frac{\left[\tilde{G}-G, q_h\right]_{\tilde{Q}'\times\tilde{Q}}}{\left\|q_h\right\|_{\tilde{Q}}} = 0.$$

Therefore, when $H_h \subseteq H$ and $Q_h \subseteq Q$, the a priori error bound provided by Theorem 3.2 becomes the usual Cea error estimate. In other words, the last four terms in that estimate constitute the consistency error for the case in which H_h and Q_h are not subspaces of H and Q, respectively.

3.2 Analysis of the Galerkin scheme

Let $\mathcal{T}_{\mathrm{S}}$ and $\mathcal{T}_{\mathrm{D}}$ be separate shape-regular families of triangulations, that is, satisfying the minimum angle condition, of $\overline{\Omega}_{\mathrm{S}}$ and $\overline{\Omega}_{\mathrm{D}}$, respectively, by triangles (or tetrahedra) T of diameter h_T, assume that the vertices of $\mathcal{T}_{\mathrm{S}}$ and $\mathcal{T}_{\mathrm{D}}$ coincide on the interface Σ, and let $\mathcal{T}_h := \mathcal{T}_{\mathrm{S}} \cup \mathcal{T}_{\mathrm{D}}$, where $h := \max\{h_{\mathrm{S}}, h_{\mathrm{D}}\}$, $h_{\mathrm{S}} := \max\{h_T : T \in \mathcal{T}_{\mathrm{S}}\}$, and $h_{\mathrm{D}} := \max\{h_T : T \in \mathcal{T}_{\mathrm{D}}\}$. Since the triangulations $\mathcal{T}_{\mathrm{S}}$ and $\mathcal{T}_{\mathrm{D}}$ coincide on Σ, we let Σ_h be the set of edges/faces inherited from $\mathcal{T}_{\mathrm{S}}$ and $\mathcal{T}_{\mathrm{D}}$. Then, we let $\mathbf{H}_{\mathrm{S},h}$, $\mathbf{H}_{\mathrm{D},h}$ and

$\mathbf{Q}_h$ be discrete finite dimensional subspaces of $\mathbf{H}^1_{\Gamma_S}(\Omega_S)$, $H_{\Gamma_D}(\text{div};\Omega_D)$ and $L^2(\Omega)$, respectively, and we set

$$\mathbf{Q}_{h,0} := \mathbf{Q}_h \cap L^2_0(\Omega). \tag{3.8}$$

In addition, we denote by $\Phi_{S,h}$ and $\Phi_{D,h}$ the subspaces of the normal components on Σ from $\mathbf{H}_{S,h}$ and $\mathbf{H}_{D,h}$, respectively, that is,

$$\Phi_{S,h} := \left\{\mathbf{v}_{S,h}\cdot\mathbf{n}|_\Sigma : \mathbf{v}_{S,h}\in\mathbf{H}_{S,h}\right\} \quad \text{and} \quad \Phi_{D,h} := \left\{\mathbf{v}_{D,h}\cdot\mathbf{n}|_\Sigma : \mathbf{v}_{D,h}\in\mathbf{H}_{D,h}\right\}.$$

Then, if $\Pi_h : L^2(\Sigma)\to\Phi_{D,h}$ denotes the orthogonal projector, and $\tilde{\mathbf{H}}_h := \mathbf{H}_{S,h}\times\mathbf{H}_{D,h}$, we introduce the finite element subspace

$$\mathbf{H}_h := \left\{\mathbf{v}_h := (\mathbf{v}_{S,h},\mathbf{v}_{D,h})\in\tilde{\mathbf{H}}_h : \Pi_h(\mathbf{v}_{S,h}\cdot\mathbf{n}-\mathbf{v}_{D,h}\cdot\mathbf{n}) = 0 \quad \text{on } \Sigma\right\}. \tag{3.9}$$

From this definition we observe that the discrete subspace $\mathbf{H}_h$ is not contained in $\mathbf{H}$, but the space $\tilde{\mathbf{H}} := \mathbf{H}^1_{\Gamma_S}(\Omega_S)\times H_{\Gamma_D}(\text{div};\Omega_D)$ contains both $\mathbf{H}_h$ and $\mathbf{H}$. Also, we observe that $\mathbf{A}:\tilde{\mathbf{H}}\to\tilde{\mathbf{H}}'$ is a well-defined nonlinear operator, $\mathbf{B}:\tilde{\mathbf{H}}\to\mathbf{Q}'$ is a well-defined linear and bounded operator, and the extension of F to $\tilde{\mathbf{H}}$ belongs to $\tilde{\mathbf{H}}'$. Then, we now introduce the nonconforming Galerkin scheme: Find $(\mathbf{u}_h, p_h)\in\mathbf{H}_h\times\mathbf{Q}_{h,0}$ such that

$$\begin{aligned} [\mathbf{A}(\mathbf{u}_h),\mathbf{v}_h]_{\mathbf{H}'\times\mathbf{H}} + [\mathbf{B}(\mathbf{v}_h), p_h]_{\mathbf{Q}'\times\mathbf{Q}} &= [F,\mathbf{v}_h]_{\mathbf{H}'\times\mathbf{H}} \quad \forall \mathbf{v}_h\in\mathbf{H}_h,\\ [\mathbf{B}(\mathbf{u}_h), q_h]_{\mathbf{Q}'\times\mathbf{Q}} &= [G, q_h]_{\mathbf{Q}'\times\mathbf{Q}} \quad \forall q_h\in\mathbf{Q}_{h,0}. \end{aligned} \tag{3.10}$$

The nonconformity of this discrete scheme refers to the fact that $\mathbf{H}_h$ is not contained in $\mathbf{H}$. We note from the definition of the finite element subspace $\mathbf{H}_h$ that $\Pi_h(\mathbf{v}_{S,h}\cdot\mathbf{n}-\mathbf{v}_{D,h}\cdot\mathbf{n}) = 0$ on Σ, for all $\mathbf{v}_h\in\mathbf{H}_h$, which is equivalent to saying that $\Pi_h(\mathbf{v}_{S,h}\cdot\mathbf{n}) - \mathbf{v}_{D,h}\cdot\mathbf{n} = 0$ on Σ, for all $\mathbf{v}_h\in\mathbf{H}_h$. Then, since $\Pi_h : L^2(\Sigma)\to\Phi_{D,h}$ is the orthogonal projector, the discrete scheme (3.10) becomes conforming if only if the discrete normal components on Σ from $\mathbf{H}_{S,h}$ are contained in the discrete normal components on Σ from $\mathbf{H}_{D,h}$, i.e., if only if $\Phi_{S,h}\subseteq\Phi_{D,h}$.

In what follows we need to consider some hypotheses concerning the subspaces involved in the discrete formulation (3.10), the linear operator $\mathbf{B}$, and the existence of a stable lifting operator from $\Phi_{D,h}$ onto $\mathbf{H}_{D,h}$. The set of assumptions is as follows.

(H.7) there holds $P_0(\Sigma_h)\subseteq\Phi_{D,h}$, where $P_0(\Sigma_h)$ is the space of piecewise constant functions defined on Σ_h.

(H.8) there exists $\tilde{\beta}>0$, independent of h, such that

$$\sup_{\substack{\mathbf{v}_h\in\mathbf{H}_h\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{[\mathbf{B}(\mathbf{v}_h), q_h]_{\mathbf{Q}'\times\mathbf{Q}}}{\|\mathbf{v}_h\|_{\mathbf{H}}} \geq \tilde{\beta}\|q_h\|_{\mathbf{Q}} \quad \forall q_h\in\mathbf{Q}_{h,0}.$$

(H.9) $\text{div}\mathbf{H}_{D,h}$ is contained in the restriction of the discrete subspace $\mathbf{Q}_h$ to Ω_D.

(H.10) there exists an operator $\mathbf{L}_h : \Phi_{D,h}\to\mathbf{H}_{D,h}$, satisfying the following properties:

a) there exists a constant $C>0$, independent of h, such that

$$\|\mathbf{L}_h(\phi_{D,h})\|_{\text{div},\Omega_D} \leq C\|\phi_{D,h}\|_{-1/2,00,\Sigma} \quad \forall \phi_{D,h}\in\Phi_{D,h}.$$

b) for all $\phi_{\mathrm{D},h} \in \Phi_{\mathrm{D},h}$ there holds

$$\mathbf{L}_h(\phi_{\mathrm{D},h}) \cdot \mathbf{n} = \phi_{\mathrm{D},h} \quad \text{on } \Sigma.$$

We say in this case that $\mathbf{L}_h$ is a stable discrete lifting of $\Phi_{\mathrm{D},h}$.

It is easy to prove that (H.7) and a classical duality argument imply the following approximation property of the projector Π_h:

$$\|\xi - \Pi_h(\xi)\|_{-1/2,00,\Sigma} \leq C h^{1/2} \|\xi\|_{0,\Sigma} \quad \forall \xi \in L^2(\Sigma)\,. \tag{3.11}$$

Moreover, employing Sobolev interpolation estimates we find that (see, e.g. [10, Proof of Lemma 4.8])

$$\|\xi - \Pi_h(\xi)\|_{0,\Sigma} \leq C\, h^{1/2}\, \|\xi\|_{1/2,\Sigma} \quad \forall \xi \in H^{1/2}(\Sigma)\,. \tag{3.12}$$

We now establish the first result of this section.

Lemma 3.4 *Let* $\mathbf{v}_h := (\mathbf{v}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{D},h}) \in \mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{H}_h : \quad [\mathbf{B}(\mathbf{v}_h), q_h]_{\mathbf{Q}'\times\mathbf{Q}} = 0 \quad \forall q_h \in \mathbf{Q}_{h,0}\}$. *Then,* $div\mathbf{v}_{\mathrm{D},h} = 0$ *on* Ω_{D}.

Proof By definition of the linear operator $\mathbf{B}$ we get

$$-\left(q_h, \mathrm{div}\mathbf{v}_{\mathrm{S},h}\right)_{0,\Omega_{\mathrm{S}}} - \left(q_h, \mathrm{div}\mathbf{v}_{\mathrm{D},h}\right)_{0,\Omega_{\mathrm{D}}} = 0 \quad \forall q_h \in \mathbf{Q}_{h,0}.$$

Also, (H.7) and the orthogonalilty condition satisfied by Π_h imply

$$\begin{aligned} 0 &= \left\langle \Pi_h(\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n} - \mathbf{v}_{\mathrm{D},h}\cdot\mathbf{n}), 1\right\rangle_\Sigma = \left\langle \mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n} - \mathbf{v}_{\mathrm{D},h}\cdot\mathbf{n}, 1\right\rangle_\Sigma \\ &= \left(1, \mathrm{div}\mathbf{v}_{\mathrm{S},h}\right)_{0,\Omega_{\mathrm{S}}} + \left(1, \mathrm{div}\mathbf{v}_{\mathrm{D},h}\right)_{0,\Omega_{\mathrm{D}}}, \end{aligned}$$

which, together with the decomposition $\mathbf{Q}_h = \mathbf{Q}_{h,0} \oplus R$, yield

$$-\left(q_h, \mathrm{div}\mathbf{v}_{\mathrm{S},h}\right)_{0,\Omega_{\mathrm{S}}} - \left(q_h, \mathrm{div}\mathbf{v}_{\mathrm{D},h}\right)_{0,\Omega_{\mathrm{D}}} = 0 \quad \forall q_h \in \mathbf{Q}_h.$$

In particular, $\left(q_h, \mathrm{div}\mathbf{v}_{\mathrm{D},h}\right)_{0,\Omega_{\mathrm{D}}} = 0$,for all q_h belonging to the restriction of $\mathbf{Q}_h$ to Ω_{D}, and hence (H.9) and the foregoing identity give $\mathrm{div}\mathbf{v}_{\mathrm{D},h} = 0$ on Ω_{D}. □

The next result establishes the well-posedness of our discrete scheme (3.10).

Lemma 3.5 *There exists a unique solution* $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times \mathbf{Q}_{h,0}$ *of the nonconforming discrete scheme* (3.10). *In addition, there exists* $C > 0$, *independent of h, such that*

$$\|(\mathbf{u}_h, p_h)\|_{\mathbf{H}\times\mathbf{Q}} \leq C \left\{ \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|f_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \right\}.$$

Proof We first recall from Lemma 2.8 that the nonlinear operator $\mathbf{A}$ is Lipschitz continuous in $\tilde{\mathbf{H}}$. Also, it is clear from Lemma 3.4 that $\mathbf{V}_h \subseteq \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times H_{\Gamma_{\mathrm{D}}}(\mathrm{div}^0; \Omega_{\mathrm{D}})$. Then, given $\mathbf{z}_h \in \mathbf{H}_h$, we know from Lemma 2.8 that the nonlinear operator $\mathbf{A}(\mathbf{z}_h + \cdot)$ is strongly monotone in $\mathbf{V}_h$, and hence the nonlinear operator $\mathbf{A}$ satisfies (H.5) and (H.6) (cf. Section 3.1). Therefore, noting also that (H.4) follows from (H.8), the proof becomes a straightforward application of Theorem 3.1. □

We now show the a priori error estimate for the primal-mixed formulation (2.14) and the Galerkin scheme (3.10).

Lemma 3.6 *Let* $(\mathbf{u}, p) \in \mathbf{H} \times \mathbf{Q}$ *and* $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times \mathbf{Q}_{h,0}$ *be the unique solutions, guaranteed by Theorem 2.2 and Lemma 3.5 of the continuous problem* (2.14) *and its nonconforming discrete scheme* (3.10)*, respectively. Then there exists* $C > 0$*, independent of h, such that*

$$\left\|(\mathbf{u}-\mathbf{u}_h, p-p_h)\right\|_{\mathbf{H}\times\mathbf{Q}} \leq C\left\{\inf_{\mathbf{v}_h\in\mathbf{H}_h}\left\|\mathbf{u}-\mathbf{v}_h\right\|_{\mathbf{H}} + \inf_{q_h\in\mathbf{Q}_{h,0}}\left\|p-q_h\right\|_{\mathbf{Q}} + h^{1/2}\left\|p_{\mathrm{D}}-\Pi_h(p_{\mathrm{D}})\right\|_{0,\Sigma}\right\}.$$

Proof Applying Theorem 3.2 we have the estimate

$$\left\|(\mathbf{u}-\mathbf{u}_h, p-p_h)\right\|_{\mathbf{H}\times\mathbf{Q}} \leq C_1\left\{\inf_{\mathbf{v}_h\in\mathbf{H}_h}\left\|\mathbf{u}-\mathbf{v}_h\right\|_{\mathbf{H}} + \inf_{q_h\in\mathbf{Q}_{h,0}}\left\|p-q_h\right\|_{\mathbf{Q}} + \sup_{\substack{\mathbf{v}_h\in\mathbf{H}_h\\ \mathbf{v}_h\neq\mathbf{0}}}\frac{\left[F-\mathbf{A}(\mathbf{u})-\mathbf{B}'(p),\mathbf{v}_h\right]_{\mathbf{H}'\times\mathbf{H}}}{\left\|\mathbf{v}_h\right\|_{\mathbf{H}}}\right\}, \quad (3.13)$$

where $C_1 > 0$ is a constant independent of h. Now, we just need to bound the consistency term on the right hand side of the above inequality. To this end, we proceed as in [19] and let $\mathcal{P}_0 : L^2(\Sigma) \to P_0(\Sigma_h)$ be the orthogonal projector and $\mathcal{P}_0 : \mathbf{L}^2(\Sigma) \to \mathbf{P}_0(\Sigma_h)$ its vector version. Recalling Eq. 2.8, we note that $p_{\mathrm{D}} \in H^1(\Omega_{\mathrm{D}})$. Then the consistency error term in Eq. 3.13 yields

$$\left[F-\mathbf{A}(\mathbf{u})-\mathbf{B}'(p),\mathbf{v}_h\right]_{\mathbf{H}'\times\mathbf{H}} = \left\langle(\mathbf{v}_{\mathrm{S},h}-\mathbf{v}_{\mathrm{D},h})\cdot\mathbf{n}, p_{\mathrm{D}}\right\rangle_{\Sigma} \quad \forall\mathbf{v}_h\in\mathbf{H}_h. \quad (3.14)$$

Now, given $\mathbf{v}_h \in \mathbf{H}_h$, we first observe that

$$\begin{aligned}\left\langle(\mathbf{v}_{\mathrm{S},h}-\mathbf{v}_{\mathrm{D},h})\cdot\mathbf{n}, p_{\mathrm{D}}\right\rangle_{\Sigma} &= \left\langle(\mathbf{v}_{\mathrm{S},h}-\mathbf{v}_{\mathrm{D},h})\cdot\mathbf{n}, p_{\mathrm{D}}\right\rangle_{\Sigma} - \left\langle\Pi_h(\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n}-\mathbf{v}_{\mathrm{D},h}\cdot\mathbf{n}), p_{\mathrm{D}}\right\rangle_{\Sigma}\\ &= \left\langle(\mathbf{v}_{\mathrm{S},h}-\mathbf{v}_{\mathrm{D},h})\cdot\mathbf{n}, p_{\mathrm{D}}\right\rangle_{\Sigma} - \left\langle\Pi_h(\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n})-\mathbf{v}_{\mathrm{D},h}\cdot\mathbf{n}, p_{\mathrm{D}}\right\rangle_{\Sigma}\\ &= \left\langle\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n}, p_{\mathrm{D}}\right\rangle_{\Sigma} - \left\langle\Pi_h(\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n}), p_{\mathrm{D}}\right\rangle_{\Sigma}\\ &= \left\langle\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n}, p_{\mathrm{D}}-\Pi_h(p_{\mathrm{D}})\right\rangle_{\Sigma}.\end{aligned}$$

Further, from (H.7) we find that for all $v \in H^{1/2}(\Sigma)$ there holds

$$\begin{aligned}\left\langle\mathcal{P}_0(\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n}), v-\Pi_h(v)\right\rangle_{\Sigma} &= \left\langle\mathcal{P}_0(\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n}), v\right\rangle_{\Sigma} - \left\langle\mathcal{P}_0(\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n}), \Pi_h(v)\right\rangle_{\Sigma}\\ &= \left\langle\mathcal{P}_0(\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n}), v\right\rangle_{\Sigma} - \left\langle\Pi_h\left(\mathcal{P}_0(\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n})\right), v\right\rangle_{\Sigma}\\ &= 0,\end{aligned}$$

that is, $\left\langle\mathcal{P}_0(\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n}), v-\Pi_h(v)\right\rangle_{\Sigma} = 0 \quad \forall v \in H^{1/2}(\Sigma)$. Then, taking in particular $v = p_{\mathrm{D}}|_{\Sigma} \in H^{1/2}(\Sigma)$, we obtain from the foregoing identity

$$\begin{aligned}\left\langle(\mathbf{v}_{\mathrm{S},h}-\mathbf{v}_{\mathrm{D},h})\cdot\mathbf{n}, p_{\mathrm{D}}\right\rangle_{\Sigma} &= \left\langle\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n}, p_{\mathrm{D}}-\Pi_h(p_{\mathrm{D}})\right\rangle_{\Sigma} - \left\langle\mathcal{P}_0(\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n}), p_{\mathrm{D}}-\Pi_h(p_{\mathrm{D}})\right\rangle_{\Sigma}\\ &= \left\langle\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n}-\mathcal{P}_0(\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n}), p_{\mathrm{D}}-\Pi_h(p_{\mathrm{D}})\right\rangle_{\Sigma}.\end{aligned}$$

In turn, since $\mathcal{P}_0(\mathbf{v}_{S,h}) \cdot \mathbf{n} \in P_0(\Sigma_h)$, we deduce that

$$\left\langle \mathcal{P}_0(\mathbf{v}_{S,h} \cdot \mathbf{n}) - \mathcal{P}_0(\mathbf{v}_{S,h}) \cdot \mathbf{n}, v - \Pi_h(v) \right\rangle_\Sigma = 0 \quad \forall v \in H^{1/2}(\Sigma)\,,$$

whence

$$\left\langle (\mathbf{v}_{S,h} - \mathbf{v}_{D,h}) \cdot \mathbf{n}, p_D \right\rangle_\Sigma = \left\langle \mathbf{v}_{S,h} \cdot \mathbf{n} - \mathcal{P}_0(\mathbf{v}_{S,h}) \cdot \mathbf{n}, p_D - \Pi_h(p_D) \right\rangle_\Sigma .$$

Then, from the normal trace theorem in $\mathbf{H}^1(\Omega_S)$, using a well known approximation estimate for piecewise constant functions and the trace theorem in $\mathbf{H}^1(\Omega_S)$, we deduce that

$$\begin{aligned}
\left\langle (\mathbf{v}_{S,h} - \mathbf{v}_{D,h}) \cdot \mathbf{n}, p_D \right\rangle_\Sigma &\leq \left\| \mathbf{v}_{S,h} \cdot \mathbf{n} - \mathcal{P}_0(\mathbf{v}_{S,h}) \cdot \mathbf{n} \right\|_{0,\Sigma} \left\| p_D - \Pi_h(p_D) \right\|_{0,\Sigma} \\
&\leq C h^{1/2} \left\| \mathbf{v}_{S,h} \right\|_{1/2,00,\Sigma} \left\| p_D - \Pi_h(p_D) \right\|_{0,\Sigma} \\
&\leq \tilde{C} h^{1/2} \left\| \mathbf{v}_{S,h} \right\|_{1,\Omega_S} \left\| p_D - \Pi_h(p_D) \right\|_{0,\Sigma},
\end{aligned}$$

that is,

$$\left\langle (\mathbf{v}_{S,h} - \mathbf{v}_{D,h}) \cdot \mathbf{n}, p_D \right\rangle_\Sigma \leq \tilde{C} h^{1/2} \left\| \mathbf{v}_{S,h} \right\|_{1,\Omega_S} \left\| p_D - \Pi_h(p_D) \right\|_{0,\Sigma},$$

with $\tilde{C} > 0$ a constant independent of h. Thus, dividing the previous inequality by $\left\| \mathbf{v}_{S,h} \right\|_{1,\Omega_S}$, noting that $\left\| \mathbf{v}_{S,h} \right\|_{1,\Omega_S} \leq \left\| \mathbf{v}_h \right\|_{\mathbf{H}}$, and taking supremum on $\mathbf{H}_h$, we conclude that

$$\sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{\left\langle (\mathbf{v}_{S,h} - \mathbf{v}_{D,h}) \cdot \mathbf{n}, p_D \right\rangle_\Sigma}{\left\| \mathbf{v}_h \right\|_{\mathbf{H}}} \leq \tilde{C} h^{1/2} \left\| p_D - \Pi_h(p_D) \right\|_{0,\Sigma}.$$

The result follows by combining the previous inequality with Eq. 3.13 after replacing Eq. 3.14 back into Eq. 3.13. □

The next result establishes an approximation property of the discrete space $\mathbf{H}_h$.

Lemma 3.7 *There exists $C > 0$, independent of h, such that for each $\mathbf{v} := (\mathbf{v}_S, \mathbf{v}_D) \in \mathbf{H}$ there holds*

$$\begin{aligned}
\inf_{\mathbf{v}_h \in \mathbf{H}_h} \left\| \mathbf{v} - \mathbf{v}_h \right\|_{\mathbf{H}} \leq C \Bigg\{ &\inf_{\mathbf{v}_{S,h} \in \mathbf{H}_{S,h}} \left\| \mathbf{v}_S - \mathbf{v}_{S,h} \right\|_{1,\Omega_S} + \inf_{\mathbf{v}_{D,h} \in \mathbf{H}_{D,h}} \left\| \mathbf{v}_D - \mathbf{v}_{D,h} \right\|_{div,\Omega_D} \\
&+ h^{1/2} \left\| \mathbf{v}_S \cdot \mathbf{n} - \Pi_h(\mathbf{v}_S \cdot \mathbf{n}) \right\|_{0,\Sigma} \Bigg\},
\end{aligned}$$

with $C > 0$ a constant independent of h.

Proof This proof is provided in [19, Proposition 4.1]. In what follows we describe the main aspects of it. Let $\Pi_{S,h} : \mathbf{H}^1_{\Gamma_S}(\Omega_S) \to \mathbf{H}_{S,h}$ and $\Pi_{D,h} : H_{\Gamma_D}(\text{div}; \Omega_D) \to \mathbf{H}_{D,h}$ be the orthogonal projectors with respect to the inner products $\mathbf{L}^2(\Omega_S)$ and $\mathbf{L}^2(\Omega_D)$, respectively. Then, given $\mathbf{v} := (\mathbf{v}_S, \mathbf{v}_D) \in \mathbf{H}$, we set

$$\mathbf{v}_{S,h} := \Pi_{S,h}(\mathbf{v}_S) \quad \text{and} \quad \mathbf{v}_{D,h} := \Pi_{D,h}(\mathbf{v}_D) - \mathbf{L}_h \left(\Pi_{D,h}(\mathbf{v}_D) \cdot \mathbf{n} - \Pi_h(\Pi_{S,h}(\mathbf{v}_S) \cdot \mathbf{n}) \right),$$

where $\mathbf{L}_h : \Phi_{D,h} \to \mathbf{H}_{D,h}$ is the stable discrete lifting defined in (H.10). It follows precisely from (H.10) that

$$\begin{aligned}\mathbf{v}_{D,h} \cdot \mathbf{n} &= \Pi_{D,h}(\mathbf{v}_D) \cdot \mathbf{n} - \mathbf{L}_h \big(\Pi_{D,h}(\mathbf{v}_D) \cdot \mathbf{n} - \Pi_h(\Pi_{S,h}(\mathbf{v}_S) \cdot \mathbf{n})\big) \cdot \mathbf{n} \\ &= \Pi_h(\Pi_{S,h}(\mathbf{v}_S) \cdot \mathbf{n}) = \Pi_h(\mathbf{v}_{S,h} \cdot \mathbf{n}) \quad \text{on } \Sigma,\end{aligned}$$

which shows that the pair $\mathbf{v}_h := (\mathbf{v}_{S,h}, \mathbf{v}_{D,h})$ belongs to $\mathbf{H}_h$. Next, the triangle inequality and (H.10) again imply that

$$\begin{aligned}\|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{H}} &= \|\mathbf{v}_S - \mathbf{v}_{S,h}\|_{1,\Omega_S} + \|\mathbf{v}_D - \mathbf{v}_{D,h}\|_{\mathrm{div},\Omega_D} \\ &\le \|\mathbf{v}_{S,h} - \Pi_{S,h}(\mathbf{v}_S)\|_{1,\Omega_S} + \|\mathbf{v}_D - \Pi_{D,h}(\mathbf{v}_D)\|_{\mathrm{div},\Omega_D} \\ &\quad + \|\mathbf{L}_h(\Pi_{D,h}(\mathbf{v}_D) \cdot \mathbf{n} - \Pi_h(\Pi_{S,h}(\mathbf{v}_S) \cdot \mathbf{n}))\|_{\mathrm{div},\Omega_D} \\ &\le \|\mathbf{v}_{S,h} - \Pi_{S,h}(\mathbf{v}_S)\|_{1,\Omega_S} + \|\mathbf{v}_D - \Pi_{D,h}(\mathbf{v}_D)\|_{\mathrm{div},\Omega_D} \\ &\quad + C\|\Pi_{D,h}(\mathbf{v}_D) \cdot \mathbf{n} - \Pi_h(\Pi_{S,h}(\mathbf{v}_S) \cdot \mathbf{n})\|_{-1/2,00,\Sigma}.\end{aligned}$$

Now, since $\mathbf{v}_S \cdot \mathbf{n} = \mathbf{v}_D \cdot \mathbf{n}$ on Σ, using the normal trace theorem in $H(\mathrm{div};\Omega_D)$ we get

$$\begin{aligned}&\|\Pi_{D,h}(\mathbf{v}_D) \cdot \mathbf{n} - \Pi_h(\Pi_{S,h}(\mathbf{v}_S) \cdot \mathbf{n})\|_{-1/2,00,\Sigma} \\ &\le \|\mathbf{v}_D \cdot \mathbf{n} - \Pi_{D,h}(\mathbf{v}_D) \cdot \mathbf{n}\|_{-1/2,00,\Sigma} + \|\mathbf{v}_S \cdot \mathbf{n} - \Pi_h(\mathbf{v}_{S,h} \cdot \mathbf{n})\|_{-1/2,00,\Sigma} \\ &\le C\|\mathbf{v}_D - \Pi_{D,h}(\mathbf{v}_D)\|_{\mathrm{div},\Omega_D} + \|\mathbf{v}_S \cdot \mathbf{n} - \Pi_h(\mathbf{v}_{S,h} \cdot \mathbf{n})\|_{-1/2,00,\Sigma},\end{aligned}$$

whence, adding and substracting appropiate terms, employing the estimate (3.11) twice, and applying the trace theorem in $\mathbf{H}^1(\Omega_S)$, we find that

$$\begin{aligned}\|\mathbf{v}_S \cdot \mathbf{n} - \Pi_h(\Pi_{S,h}(\mathbf{v}_S) \cdot \mathbf{n})\|_{-1/2,00,\Sigma} &\le \|(\mathrm{I} - \Pi_h)(\mathbf{v}_S \cdot \mathbf{n} - \Pi_h(\mathbf{v}_S \cdot \mathbf{n}))\|_{-1/2,00,\Sigma} \\ &\quad + \|(\mathrm{I} - \Pi_h)(\mathbf{v}_S \cdot \mathbf{n} - \mathbf{v}_{S,h} \cdot \mathbf{n})\|_{-1/2,00,\Sigma} + \|\mathbf{v}_S \cdot \mathbf{n} - \mathbf{v}_{S,h} \cdot \mathbf{n}\|_{-1/2,00,\Sigma} \\ &\le Ch^{1/2}\left\{\|\mathbf{v}_S \cdot \mathbf{n} - \Pi_h(\mathbf{v}_S \cdot \mathbf{n})\|_{0,\Sigma} + \|\mathbf{v}_S \cdot \mathbf{n} - \mathbf{v}_{S,h} \cdot \mathbf{n}\|_{0,\Sigma}\right\} \\ &\quad + \|\mathbf{v}_S \cdot \mathbf{n} - \mathbf{v}_{S,h} \cdot \mathbf{n}\|_{-1/2,00,\Sigma} \\ &\le \tilde{C}\left\{\|\mathbf{v}_S - \mathbf{v}_{S,h}\|_{1,\Omega_S} + h^{1/2}\|\mathbf{v}_S - \Pi_h(\mathbf{v}_S \cdot \mathbf{n})\|_{0,\Sigma}\right\},\end{aligned}$$

which completes the proof. □

We now summarize the unique solvability and the Strang-type a priori error estimate for the nonconforming discrete scheme (3.10) in the following theorem.

Theorem 3.3 *There exists a unique* $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times \mathbf{Q}_{h,0}$ *solution of Eq.* 3.10*, and there holds*

$$\begin{aligned}\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{\mathbf{H}\times\mathbf{Q}} \le C\Bigg\{&\inf_{\mathbf{v}_{S,h}\in\mathbf{H}_{S,h}} \|\mathbf{u}_S - \mathbf{v}_{S,h}\|_{1,\Omega_S} \\ &+ \inf_{\mathbf{v}_{D,h}\in\mathbf{H}_{D,h}} \|\mathbf{u}_D - \mathbf{v}_{D,h}\|_{div,\Omega_D} + \inf_{q_h\in\mathbf{Q}_{h,0}} \|p - q_h\|_{\mathbf{Q}} \\ &+ h^{1/2}\Big(\|p_D - \Pi_h(p_D)\|_{0,\Sigma} + \|\mathbf{u}_S \cdot \mathbf{n} - \Pi_h(\mathbf{u}_S \cdot \mathbf{n})\|_{0,\Sigma}\Big)\Bigg\}, \qquad (3.15)\end{aligned}$$

where $C > 0$ is a constant independent of h.

Proof The proof follows from a straightforward application of Lemmas 3.5, 3.6 and 3.7. □

3.3 Particular choices of finite element subspaces

In this section we specify concrete 2D examples of finite element subspaces of $\mathbf{H}^1_{\Gamma_S}(\Omega_S)$, $H_{\Gamma_D}(\text{div}, \Omega_D)$ and $L^2_0(\Omega)$ satisfying (H.7)-(H.10). Given $T \in \mathcal{T}_S$, we first define the local Bernardi-Raugel space (see [2]), denoted by $BR(T)$, as

$$BR(T) := \mathbf{P}_1(T) \oplus \text{span}\{\eta_2\eta_3\mathbf{n}_1, \eta_1\eta_3\mathbf{n}_2, \eta_1\eta_2\mathbf{n}_3\}, \tag{3.16}$$

where η_1, η_2 and η_3 are the barycentric coordinates of the triangle T, and $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are the three unit normal components to the opposite sides of its corresponding vertices, which point outwards on ∂T. In turn, given $T \in \mathcal{T}_D$, we let $RT(T)$ be the local Raviart-Thomas space of lowest order, that is

$$RT_0(T) := \mathbf{P}_0(T) \oplus P_0(T)\mathbf{x}. \tag{3.17}$$

where $\mathbf{x}$ denotes a generic vector of R^2. Also, we consider the local Brezzi-Douglas-Marini space of order one, which is given by

$$BDM_1(T) := \mathbf{P}_1(T). \tag{3.18}$$

In what follows, we describe two different examples of finite element subspaces for the Stokes and Darcy domains in terms of the local spaces defined in Eqs. 3.16, 3.17 and 3.18, with their corresponding finite element subspaces approximating the pressure field in Ω.

3.3.1 Bernadi-Raugel + Raviart-Thomas

The subspaces $\mathbf{H}_{S,h}$, $\mathbf{H}_{D,h}$, $\mathbf{H}_h$ (cf. (3.9)), and $\mathbf{Q}_{h,0}$ of $\mathbf{H}^1_{\Gamma_S}(\Omega_S)$, $H_{\Gamma_D}(\text{div}; \Omega_D)$, $\tilde{\mathbf{H}}$, and $L^2_0(\Omega)$, respectively, are defined as

$$\begin{aligned}
\mathbf{H}_{S,h} &:= \left\{\mathbf{v}_{S,h} \in [C(\overline{\Omega}_S)]^2 : \mathbf{v}_{S,h}|_T \in BR(T) \quad \forall T \in \mathcal{T}_S\right\} \cap \mathbf{H}^1_{\Gamma_S}(\Omega_S), \\
\mathbf{H}_{D,h} &:= \left\{\mathbf{v}_{D,h} \in H(\text{div}; \Omega_D) : \mathbf{v}_{D,h}|_T \in RT_0(T) \quad \forall T \in \mathcal{T}_D\right\} \cap H_{\Gamma_D}(\text{div}; \Omega_D), \\
\mathbf{H}_h &:= \left\{\mathbf{v}_h := (\mathbf{v}_{S,h}, \mathbf{v}_{D,h}) \in \mathbf{H}_{S,h} \times \mathbf{H}_{D,h} : \Pi_h(\mathbf{v}_{S,h} \cdot \mathbf{n} - \mathbf{v}_{D,h} \cdot \mathbf{n}) = 0 \text{ on } \Sigma\right\},
\end{aligned} \tag{3.19}$$

and

$$\mathbf{Q}_{h,0} := \left\{q_h \in L^2(\Omega) : q_h|_T \in P_0(T) \quad \forall T \in \mathcal{T}_h\right\} \cap L^2_0(\Omega). \tag{3.20}$$

From these particular choices of finite element subspaces, and taking into account the definition of the local spaces BR and RT (cf. (3.16) and (3.17), respectively), we observe that the discrete space $\Phi_{S,h}$ becomes the continuous piecewise quadratic functions while the discrete space $\Phi_{D,h}$ becomes the piecewise constant functions. Note that the discrete space $\Phi_{S,h}$ is not contained in $\Phi_{D,h}$, which means that the discrete scheme (3.10) is nonconforming in this case. In turn, it is clear that (H.7) and

(H.9) are satisfied. In addition, (H.10) has been shown in the 2D case (see [19]) without any requeriment on the meshes for both the Raviart-Thomas subspace of lowest order (cf. (3.17)) and the Brezzi-Douglas-Marini subspaces for any nonnegative integer $l \geq 1$. Finally, in order to verify (H.8) we proceed similarly as in [13]. To this end, we let $\Pi_{\mathrm{S}} : \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \to \mathbf{H}_{\mathrm{S},h}$ be the Bernadi-Raugel interpolation operator (cf. [2, 14]), which is linear and bounded with respect to the $\mathbf{H}^1(\Omega_{\mathrm{S}})$-norm. More precisely, given $\mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, this interpolation operator is characterized by the following identities:

$$\int_e \Pi_{\mathrm{S}}(\mathbf{v}_{\mathrm{S}}) \cdot \mathbf{n}_e = \int_e \mathbf{v}_{\mathrm{S}} \cdot \mathbf{n}_e, \quad \text{for each edge } e \text{ of } \mathcal{T}_{\mathrm{S}}, \tag{3.21}$$

and

$$\Pi_{\mathrm{S}}(\mathbf{v}_{\mathrm{S}}(a)) = \mathbb{I}_h(\mathbf{v}_{\mathrm{S}}(a)) \quad \text{for each node } a \text{ of } \mathcal{T}_{\mathrm{S}},$$

where $\mathbb{I}_h$ is the Clément regularization operator defined in [14, Appendix A, A.3]. Note that, as a consequence of Eq. 3.21, there holds

$$\int_{\Omega_{\mathrm{S}}} q_h \operatorname{div} \Pi_{\mathrm{S}}(\mathbf{v}_{\mathrm{S}}) = \int_{\Omega_{\mathrm{S}}} q_h \operatorname{div} \mathbf{v}_{\mathrm{S}} \quad \forall q_h \text{ in the restriction of } \mathbf{Q}_h \text{ to } \Omega_{\mathrm{S}}. \tag{3.22}$$

Equivalently, if $\mathcal{P}_{\mathrm{S}}$ denotes the $L^2(\Omega_{\mathrm{S}})-$orthogonal projection onto the restriction of $\mathbf{Q}_h$ to Ω_{S}, then the relation (3.22) can be rewritten as

$$\mathcal{P}_{\mathrm{S}}(\operatorname{div}(\Pi_{\mathrm{S}}(\mathbf{v}_{\mathrm{S}}))) = \mathcal{P}_{\mathrm{S}}(\operatorname{div} \mathbf{v}_{\mathrm{S}}) \quad \forall \mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}). \tag{3.23}$$

In turn, we let $\Pi_{\mathrm{D}} : \mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) \to \mathbf{H}_{\mathrm{D},h}$ be the Raviart-Thomas interpolation operator of lowest order, which, given $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})$, is characterized by:

$$\int_e \Pi_{\mathrm{D}}(\mathbf{v}_{\mathrm{D}}) \cdot \mathbf{n}_e = \int_e \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n}_e, \quad \text{for each edge } e \text{ of } \mathcal{T}_{\mathrm{D}}. \tag{3.24}$$

Similarly as for Π_{S}, we find that Eq. 3.24 yields

$$\int_{\Omega_{\mathrm{D}}} q_h \operatorname{div} \Pi_{\mathrm{D}}(\mathbf{v}_{\mathrm{D}}) = \int_{\Omega_{\mathrm{D}}} q_h \operatorname{div} \mathbf{v}_{\mathrm{D}} \quad \forall q_h \text{ in the restriction of } \mathbf{Q}_h \text{ to } \Omega_{\mathrm{D}}. \tag{3.25}$$

Equivalently, if $\mathcal{P}_{\mathrm{D}}$ denotes the $L^2(\Omega_{\mathrm{D}})-$orthogonal projection onto the restriction of $\mathbf{Q}_h$ to Ω_{D}, then the relation (3.25) can be rewritten as

$$\operatorname{div}(\Pi_{\mathrm{D}}(\mathbf{v}_{\mathrm{D}})) = \mathcal{P}_{\mathrm{D}}(\operatorname{div} \mathbf{v}_{\mathrm{D}}) \quad \forall \mathbf{v}_{\mathrm{D}} \in \mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}). \tag{3.26}$$

In addition, we know that the Raviart-Thomas interpolation operator Π_{D} satisfies the following approximation property: For any $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}^1(\Omega_{\mathrm{D}})$, there exists $C > 0$, independent of h, such that

$$\left\|\mathbf{v}_{\mathrm{D}} - \Pi_{\mathrm{D}}(\mathbf{v}_{\mathrm{D}})\right\|_{0,\Omega_{\mathrm{D}}} \leq C h_{\mathrm{D}} \left\|\mathbf{v}_{\mathrm{D}}\right\|_{1,\Omega_{\mathrm{D}}}. \tag{3.27}$$

The next result shows that (H.8) also holds.

Lemma 3.8 *There exists $\beta_1 > 0$, independent of h, such that*

$$\sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{[\mathbf{B}(\mathbf{v}_h), q_h]_{\mathbf{Q}' \times \mathbf{Q}}}{\left\|\mathbf{v}_h\right\|_{\mathbf{H}}} \geq \beta_1 \left\|q_h\right\|_{\mathbf{Q}} \quad \forall q_h \in \mathbf{Q}_{h,0}.$$

Proof Given $q_h \in \mathbf{Q}_{h,0}$, a well-known result (see, e.g. [14]) implies the existence of $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ such that $-\mathrm{div}\mathbf{z} = q_h$ in Ω and $\|\mathbf{z}\|_{1,\Omega} \le C\|q_h\|_{0,\Omega}$. We define

$$\mathbf{w}_{\mathrm{S},h} := \Pi_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}}) \in \mathbf{H}_{\mathrm{S},h} \quad \text{and} \quad \mathbf{w}_{\mathrm{D},h} := \Pi_{\mathrm{D}}(\mathbf{w}_{\mathrm{D}}) \in \mathbf{H}_{\mathrm{D},h},$$

where $\mathbf{w}_{\mathrm{S}} := \mathbf{z}|_{\Omega_{\mathrm{S}}}$ and $\mathbf{w}_{\mathrm{D}} := \mathbf{w}|_{\Omega_{\mathrm{D}}}$. It is clear that $\mathbf{w} := (\mathbf{w}_{\mathrm{S}}, \mathbf{w}_{\mathrm{D}})$ belongs to $\mathbf{H}$. This fact together with Eqs. 3.21 and 3.24 yield

$$\int_e \mathbf{w}_{\mathrm{S},h} \cdot \mathbf{n}_e = \int_e \mathbf{w}_{\mathrm{S}} \cdot \mathbf{n}_e = \int_e \mathbf{w}_{\mathrm{D}} \cdot \mathbf{n}_e = \int_e \mathbf{w}_{\mathrm{D},h} \cdot \mathbf{n}_e \quad \forall e \in \Sigma_h. \tag{3.28}$$

Now, since $\Pi_h : L^2(\Sigma) \to \Phi_{\mathrm{D},h}$ is the orthogonal projector and $\Phi_{\mathrm{D},h}$ becomes the piecewise constant functions, we obtain that

$$\int_e \{\xi - \Pi_h(\xi)\} = 0 \quad \forall \xi \in L^2(\Sigma), \quad \forall e \text{ edge of } \Sigma.$$

Then Eq. 3.28 and the foregoing identity applied to $\xi = \mathbf{w}_{\mathrm{S},h} \cdot \mathbf{n} \in L^2(\Sigma)$ imply that

$$\int_e \Pi_h(\mathbf{w}_{\mathrm{S},h} \cdot \mathbf{n}) = \int_e \mathbf{w}_{\mathrm{S},h} \cdot \mathbf{n} = \int_e \mathbf{w}_{\mathrm{D},h} \cdot \mathbf{n} \quad \forall e \text{ edge of } \Sigma,$$

and combining this last relation with the fact that $\Pi_h(\mathbf{w}_{\mathrm{S},h} \cdot \mathbf{n}) - \mathbf{w}_{\mathrm{D},h} \cdot \mathbf{n} \in P_0(\Sigma_h)$, we deduce that $\Pi_h(\mathbf{w}_{\mathrm{S},h} \cdot \mathbf{n}) = \mathbf{w}_{\mathrm{D},h} \cdot \mathbf{n}$ on Σ, that is the pair $\mathbf{w}_h := (\mathbf{w}_{\mathrm{S},h}, \mathbf{w}_{\mathrm{D},h})$ belongs to $\mathbf{H}_h$. Further, Eq. 3.23 yields

$$\mathcal{P}_{\mathrm{S}}\left(\mathrm{div}\mathbf{w}_{\mathrm{S},h}\right) = \mathcal{P}_{\mathrm{S}}\left(\mathrm{div}\mathbf{w}_{\mathrm{S}}\right) = \mathcal{P}_{\mathrm{S}}(-q_h) = -q_h \quad \text{in } \Omega_{\mathrm{S}},$$

and Eq. 3.26 implies that

$$\mathrm{div}\mathbf{w}_{\mathrm{D},h} = \mathcal{P}_{\mathrm{D}}\left(\mathrm{div}\mathbf{w}_{\mathrm{D}}\right) = \mathcal{P}_{\mathrm{D}}(-q_h) = -q_h \quad \text{in } \Omega_{\mathrm{D}}.$$

It follows that

$$[\mathbf{B}(\mathbf{w}_h), q_h]_{\mathbf{Q}' \times \mathbf{Q}} = \|q_h\|_{\mathbf{Q}}^2. \tag{3.29}$$

On the other hand, since the operator Π_{S} is bounded, there holds

$$\|\mathbf{w}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \le C\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \le C\|\mathbf{z}\|_{1,\Omega} \le c_1\|q_h\|_{0,\Omega},$$

and applying Eq. 3.27 we have that

$$\begin{aligned}\|\mathbf{w}_{\mathrm{D},h}\|_{\mathrm{div},\Omega_{\mathrm{D}}} &= \|\mathbf{w}_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}} + \|\mathrm{div}\mathbf{w}_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}} \\ &\le Ch\|\mathbf{w}_{\mathrm{D}}\|_{1,\Omega_{\mathrm{D}}} + \|\mathbf{w}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|q_h\|_{0,\Omega} \le c_2\|q_h\|_{0,\Omega},\end{aligned}$$

where we used here, from the previous estimate, that $\|\mathbf{w}_{\mathrm{D}}\|_{1,\Omega_{\mathrm{D}}} \le \|\mathbf{z}\|_{1,\Omega} \le C\|q_h\|_{0,\Omega}$. Therefore, we have that $\|\mathbf{w}_h\|_{\mathbf{H}} \le c_3\|q_h\|_{0,\Omega}$, and using Eq. 3.29 we conclude that

$$\sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{[\mathbf{B}(\mathbf{v}_h), q_h]_{\mathbf{Q}' \times \mathbf{Q}}}{\|\mathbf{v}_h\|_{\mathbf{H}}} \ge \frac{[\mathbf{B}(\mathbf{w}_h), q_h]_{\mathbf{Q}' \times \mathbf{Q}}}{\|\mathbf{w}_h\|_{\mathbf{H}}} \ge \frac{1}{c_3}\|q_h\|_{0,\Omega},$$

with $c_3 > 0$ a constant independent of h. □

Finally, we recall from [14] (see also [2]) an approximation property for the Bernadi-Raugel interpolation operator Π_S, that is: for each $\mathbf{v}_S \in \mathbf{H}^2(\Omega_S)$, there exists $C > 0$, independent of h_S, such that

$$\|\mathbf{v}_S - \Pi_S(\mathbf{v}_S)\|_{1,\Omega_S} \leq C h_S \|\mathbf{v}_S\|_{2,\Omega_S}. \tag{3.30}$$

We are now in a position to establish the main result of this section.

Theorem 3.4 *Let $\mathbf{H}_h$ and $\mathbf{Q}_{h,0}$ be the finite element subspaces defined by Eqs. 3.19 and 3.20, respectively. Then the nonconforming discrete scheme (3.10) has a unique solution $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times \mathbf{Q}_{h,0}$ and there exists $c_1 > 0$, independent of h, such that*

$$\|(\mathbf{u}_h, p_h)\|_{\mathbf{H}\times\mathbf{Q}} \leq c_1 \left\{ \|F_h\|_{\mathbf{H}'} + \|G_h\|_{\mathbf{Q}'} \right\},$$

where $F_h := F|_{\mathbf{H}_h}$ and $G_h := G|_{\mathbf{Q}_{h,0}}$. In addition, assume that the unique solution $(\mathbf{u}, p) \in \mathbf{H} \times \mathbf{Q}$ of the primal-mixed formulation (2.14) is such that $\mathbf{u}_S \in \mathbf{H}^2(\Omega_S)$, $\mathbf{u}_S \cdot \mathbf{n}|_\Sigma \in H^{1/2}(\Sigma)$, $\mathbf{u}_D \in \mathbf{H}^1(\Omega_D)$, $div\, \mathbf{u}_D \in H^1(\Omega_D)$, and $p \in H^1(\Omega)$. Then there exists $c_2 > 0$, independent of h, such that

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{\mathbf{H}\times\mathbf{Q}} \leq c_2 \left\{ h_S |\mathbf{u}_S|_{2,\Omega_S} + h_D \left(|\mathbf{u}_D|_{1,\Omega_D} + |div \mathbf{u}_D|_{1,\Omega_D} \right) \right.$$
$$\left. + h|p|_{1,\Omega} + h \|\mathbf{u}_S \cdot \mathbf{n}\|_{1/2,\Sigma} \right\}.$$

Proof The proof follows from a straightforward application of Theorem 3.3 and the approximation properties of the subspaces and projectors involved. In particular, Eq. 3.12 allows to estimate the expressions $\|p_D - \Pi_h(p_D)\|_{0,\Sigma}$ and $\|\mathbf{u}_S \cdot \mathbf{n} - \Pi_h(\mathbf{u}_S \cdot \mathbf{n})\|_{0,\Sigma}$ in Eq. 3.15. □

3.3.2 Bernadi-Raugel + Brezzi-Douglas-Marini

The specific subspaces $\mathbf{H}_{S,h}$, $\mathbf{H}_{D,h}$, $\mathbf{H}_h$ (cf. (3.9)), and $\mathbf{Q}_{h,0}$ of $\mathbf{H}^1_{\Gamma_S}(\Omega_S)$, $H_{\Gamma_D}(\text{div}; \Omega_D)$, $\tilde{\mathbf{H}}$, and $L^2_0(\Omega)$, respectively, are

$$\mathbf{H}_{S,h} := \left\{ \mathbf{v}_{S,h} \in \left[C(\overline{\Omega}_S)\right]^2 : \mathbf{v}_{S,h}|_T \in BR(T) \quad \forall T \in \mathcal{T}_S \right\} \cap \mathbf{H}^1_{\Gamma_S}(\Omega_S),$$
$$\mathbf{H}_{D,h} := \left\{ \mathbf{v}_{D,h} \in H(\text{div}; \Omega_D) : \mathbf{v}_{D,h}|_T \in BDM_1(T) \quad \forall T \in \mathcal{T}_D \right\} \cap H_{\Gamma_D}(\text{div}; \Omega_D),$$
$$\mathbf{H}_h := \left\{ \mathbf{v}_h := (\mathbf{v}_{S,h}, \mathbf{v}_{D,h}) \in \mathbf{H}_{S,h} \times \mathbf{H}_{D,h} : \Pi_h(\mathbf{v}_{S,h} \cdot \mathbf{n} - \mathbf{v}_{D,h} \cdot \mathbf{n}) = 0 \text{ on } \Sigma \right\}, \tag{3.31}$$

and

$$\mathbf{Q}_{h,0} := \left\{ q_h \in L^2(\Omega) : q_h|_T \in P_0(T) \quad \forall T \in \mathcal{T}_h \right\} \cap L^2_0(\Omega). \tag{3.32}$$

We observe that the discrete space $\Phi_{S,h}$ is formed by continuous piecewise quadratic functions while the discrete space $\Phi_{D,h}$ becomes the piecewise linear functions. Therefore, the discrete mixed formulation (3.10) is nonconforming as well. In turn, (H.7) holds because $P_0(\Sigma_h) \subseteq P_1(\Sigma_h) = \Phi_{D,h}$. Further, it is clear that (H.9) is satisfied. Also, we know from [19, Appendix] that (H.10) is satisfied in the

2D case with no requeriment on the meshes around Σ for the Raviart-Thomas subspace of lowest order (cf. (3.17)) and for the Brezzi-Douglas-Marini subspace of any nonnegative integer order.

On the other hand, in order to prove the discrete inf-sup condition for the linear operator $\mathbf{B}$ (cf. (H.8)), we introduce the BDM interpolation operator $\Pi_{D,h}$: $\mathbf{H}^1_{\Gamma_D}(\Omega_D) \to \mathbf{H}_{D,h}$ (cf. [3]) which, given $\mathbf{v}_D \in \mathbf{H}^1_{\Gamma_D}(\Omega_D)$, is characterized by the following indentity:

$$\int_e (\mathbf{v}_D - \Pi_{D,h}(\mathbf{v}_D)) \cdot \mathbf{n}_e \, p = 0 \quad \forall p \in P_1(e) \quad \forall e \text{ edge of } \mathcal{T}_D. \tag{3.33}$$

Moreover, if we denote by $\mathcal{P}_D$ the $L^2(\Omega_D)$-orthogonal onto the restriction of $\mathbf{Q}_h$ to Ω_D, (3.33) implies that

$$\operatorname{div}\Pi_{D,h}(\mathbf{v}_D) = \mathcal{P}_D(\operatorname{div}\mathbf{v}_D) \quad \forall \mathbf{v}_D \in \mathbf{H}^1_{\Gamma_D}(\Omega_D). \tag{3.34}$$

We now recall from [4] an approximation property of the interpolation operator $\Pi_{D,h}$: there exists $C > 0$, independent of h, such that for each $\mathbf{v}_D \in \mathbf{H}^1(\Omega_D)$ there holds

$$\left\| \mathbf{v}_D - \Pi_{D,h}(\mathbf{v}_D) \right\|_{0,\Omega_D} \le C h_D \left\| \mathbf{v}_D \right\|_{1,\Omega_D}. \tag{3.35}$$

In addition, we recall from [19, Appendix] the following result summarizing the properties of a stable lifting.

Lemma 3.9 *There exists an operator* $\mathbf{L}_h : \Phi_{D,h} \to \mathbf{H}_{D,h}$ *with the properties indicated in* (H.10) *(cf. Section* 3.2). In addition, there holds

$$\operatorname{div}\mathbf{L}(\phi_h) = \frac{1}{|\Sigma|} \int_\Sigma \phi_h \quad \forall \phi_h \in \Phi_{D,h}. \tag{3.36}$$

Proof See [19, Appendix]. □

The hypothesis (H.8) is proved next.

Lemma 3.10 *There exists* $\beta_2 > 0$, *independent of* h, *such that*

$$\sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{[\mathbf{B}(\mathbf{v}_h), q_h]_{\mathbf{Q}' \times \mathbf{Q}}}{\left\| \mathbf{v}_h \right\|_{\mathbf{H}}} \ge \beta_2 \left\| q_h \right\|_{\mathbf{Q}} \quad \forall q_h \in \mathbf{Q}_{h,0}.$$

Proof Let $q_h \in L^2_0(\Omega)$. We know that there exists $\mathbf{z} \in \mathbf{H}^1_0(\Omega)$ such that

$$-\operatorname{div}\mathbf{z} = q_h \quad \text{in } \Omega \text{ and } \quad \left\| \mathbf{z} \right\|_{1,\Omega} \le C \left\| q_h \right\|_{0,\Omega}. \tag{3.37}$$

We let $\mathbf{w}_S := \mathbf{z}|_{\Omega_S}$, $\mathbf{w}_D := \mathbf{z}|_{\Omega_D}$, and then we define

$$\mathbf{w}_{S,h} := \Pi_S(\mathbf{w}_S) \in \mathbf{H}_{S,h} \quad \text{and} \quad \mathbf{w}_{D,h} := \Pi_{D,h}(\mathbf{w}_D) + \mathbf{L}_h\left(\Pi_h \mathbf{w}_{S,h} \cdot \mathbf{n} - \Pi_{D,h}(\mathbf{w}_D) \cdot \mathbf{n}\right) \in \mathbf{H}_{D,h}.$$

It is clear that $\mathbf{w} := (\mathbf{w}_S, \mathbf{w}_D) \in \mathbf{H}$, and (H.10) implies that the pair $\mathbf{w}_h := (\mathbf{w}_{S,h}, \mathbf{w}_{D,h})$ belongs to $\mathbf{H}_h$. In addition, Eqs. 3.23 and 3.34 yield

$$\mathcal{P}_S\left(\operatorname{div}\mathbf{w}_{S,h}\right) = \mathcal{P}_S\left(\operatorname{div}\Pi_S(\mathbf{w}_S)\right) = \mathcal{P}_S\left(\operatorname{div}\mathbf{w}_S\right) = \mathcal{P}_S(-q_h) = -q_h \quad \text{in } \Omega_S,$$

and

$$\mathrm{div}\Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}}) = \mathcal{P}_{\mathrm{D}}(\mathrm{div}\mathbf{w}_{\mathrm{D}}) = \mathcal{P}_{\mathrm{D}}(-q_h) = -q_h \quad \text{in } \Omega_{\mathrm{D}}.$$

Next,

$$[\mathbf{B}(\mathbf{w}_h), q_h]_{\mathbf{Q}'\times\mathbf{Q}} = \|q_h\|_{0,\Omega}^2 - \left(q_h, \mathrm{div}\mathbf{L}_h\left(\Pi_h \mathbf{w}_{\mathrm{S},h}\cdot\mathbf{n} - \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\cdot\mathbf{n}\right)\right)_{0,\Omega_{\mathrm{D}}}. \tag{3.38}$$

Moreover, from Eq. 3.36 (cf. Lemma 3.9) we get

$$\mathrm{div}\mathbf{L}_h\left(\Pi_h \mathbf{w}_{\mathrm{S},h}\cdot\mathbf{n} - \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\cdot\mathbf{n}\right) = \frac{1}{|\Sigma|}\int_\Sigma \left\{\Pi_h \mathbf{w}_{\mathrm{S},h}\cdot\mathbf{n} - \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\cdot\mathbf{n}\right\},$$

whence, using Eqs. 3.21, 3.33 and the fact that $\mathbf{w} := (\mathbf{w}_{\mathrm{S}}, \mathbf{w}_{\mathrm{D}})$ belongs to $\mathbf{H}$, we find for each e edge of Σ that

$$\int_e \Pi_h \mathbf{w}_{\mathrm{S},h}\cdot\mathbf{n} = \int_e \mathbf{w}_{\mathrm{S},h}\cdot\mathbf{n} = \int_e \mathbf{w}_{\mathrm{S}}\cdot\mathbf{n} = \int_e \mathbf{w}_{\mathrm{D}}\cdot\mathbf{n} = \int_e \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\cdot\mathbf{n},$$

which proves that

$$\mathrm{div}\mathbf{L}_h\left(\Pi_h \mathbf{w}_{\mathrm{S},h}\cdot\mathbf{n} - \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\cdot\mathbf{n}\right) = 0.$$

The foregoing relation and Eq. 3.38 lead to

$$[\mathbf{B}(\mathbf{w}_h), q_h]_{\mathbf{Q}'\times\mathbf{Q}} = \|q_h\|_{0,\Omega}^2. \tag{3.39}$$

On the other hand, the boundedness of the interpolation operator Π_{S} and Eq. 3.37 imply that

$$\|\mathbf{w}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \le C\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \le C\|\mathbf{z}\|_{1,\Omega} \le c_1\|q_h\|_{0,\Omega}. \tag{3.40}$$

In turn, since $\mathrm{div}\mathbf{w}_{\mathrm{D}} = \mathrm{div}\Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}}) = -q_h$ we have that

$$\|\mathbf{w}_{\mathrm{D}} - \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\|_{\mathrm{div},\Omega_{\mathrm{D}}} = \|\mathbf{w}_{\mathrm{D}} - \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\|_{0,\Omega_{\mathrm{D}}},$$

so that the above relation, the uniform boundedness of $\mathbf{L}_h$ (cf. (H.10)), Eqs. 3.35 and 3.37 lead to

$$\begin{aligned}
\|\mathbf{w}_{\mathrm{D},h}\|_{\mathrm{div},\Omega} \le\ & \|\mathbf{w}_{\mathrm{D}} - \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\|_{\mathrm{div},\Omega_{\mathrm{D}}} + \|\mathbf{w}_{\mathrm{D}}\|_{\mathrm{div},\Omega_{\mathrm{D}}} \\
& + \|\mathbf{L}_h\left(\Pi_h\Pi_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})\cdot\mathbf{n} - \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\cdot\mathbf{n}\right)\|_{\mathrm{div},\Omega_{\mathrm{D}}} \\
\le\ & \|\mathbf{w}_{\mathrm{D}} - \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\|_{0,\Omega_{\mathrm{D}}} + \|\mathbf{w}_{\mathrm{D}}\|_{1,\Omega_{\mathrm{D}}} + \tilde{C}\|\Pi_h\Pi_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})\cdot\mathbf{n} \\
& - \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\cdot\mathbf{n}\|_{-1/2,00,\Sigma} \\
\le\ & Ch_{\mathrm{D}}\|\mathbf{w}_{\mathrm{D}}\|_{1,\Omega} + \|\mathbf{w}_{\mathrm{D}}\|_{1,\Omega_{\mathrm{D}}} + \tilde{C}\|\Pi_h\Pi_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})\cdot\mathbf{n} \\
& - \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\cdot\mathbf{n}\|_{-1/2,00,\Sigma} \\
\le\ & Ch_{\mathrm{D}}\|\mathbf{z}\|_{1,\Omega} + \|\mathbf{z}\|_{1,\Omega} + \tilde{C}\|\Pi_h\Pi_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})\cdot\mathbf{n} \\
& - \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\cdot\mathbf{n}\|_{-1/2,00,\Sigma} \\
\le\ & c_2\|q_h\|_{\mathbf{Q}} + \tilde{C}\|\Pi_h\Pi_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})\cdot\mathbf{n} - \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\cdot\mathbf{n}\|_{-1/2,00,\Sigma},
\end{aligned}$$

that is

$$\|\mathbf{w}_{\mathrm{D},h}\|_{\mathrm{div},\Omega_{\mathrm{D}}} \le c_2\|q_h\|_{\mathbf{Q}} + \tilde{C}\|\Pi_h\Pi_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})\cdot\mathbf{n} - \Pi_{\mathrm{D},h}(\mathbf{w}_{\mathrm{D}})\cdot\mathbf{n}\|_{-1/2,00,\Sigma}. \tag{3.41}$$

Now, the trace theorems on $\mathbf{H}^1(\Omega_S)$ and on $H(\text{div}; \Omega_D)$, the boundedness of Π_h and Π_S, and the estimates (3.35) and (3.37) imply that the second term on the right hand side of Eq. 3.41 can be bounded as follows

$$\begin{aligned}
&\big\| \Pi_h \Pi_S(\mathbf{w}_S) \cdot \mathbf{n} - \Pi_{D,h}(\mathbf{w}_D) \cdot \mathbf{n} \big\|_{-1/2,00,\Sigma} \\
&\quad \leq C \big\| \Pi_h \Pi_S(\mathbf{w}_S) \cdot \mathbf{n} \big\|_{0,\Sigma} + \big\| \Pi_{D,h}(\mathbf{w}_D) \cdot \mathbf{n} \big\|_{-1/2,00,\Sigma} \\
&\quad \leq C \big\| \Pi_S(\mathbf{w}_S) \big\|_{0,\Sigma} + C_2 \big\| \Pi_{D,h}(\mathbf{w}_D) \big\|_{\text{div},\Omega_D} \\
&\quad \leq C_1 \big\| \Pi_S(\mathbf{w}_S) \big\|_{1,\Omega_S} + C_2 \big\| \Pi_{D,h}(\mathbf{w}_D) \big\|_{\text{div},\Omega_D} \\
&\quad \leq \tilde{C}_1 \big\| \mathbf{w}_S \big\|_{1,\Omega} + C_2 \big\| \mathbf{w}_D - \Pi_{D,h}(\mathbf{w}_D) \big\|_{0,\Omega_D} + C_2 \big\| \mathbf{w}_D \big\|_{\text{div},\Omega_D} \\
&\quad \leq \tilde{C}_1 \big\| \mathbf{w}_S \big\|_{1,\Omega} + \tilde{C}_2 h_D \big\| \mathbf{w}_D \big\|_{1,\Omega_D} + C_2 \big\| \mathbf{w}_D \big\|_{1,\Omega_D} \\
&\quad \leq C_3 \big\| \mathbf{z} \big\|_{1,\Omega} \leq c_3 \big\| q_h \big\|_{0,\Omega},
\end{aligned}$$

i.e.,

$$\big\| \Pi_h \Pi_S(\mathbf{w}_S) \cdot \mathbf{n} - \Pi_D(\mathbf{w}_D) \cdot \mathbf{n} \big\|_{-1/2,00,\Sigma} \leq c_3 \big\| q_h \big\|_{0,\Omega}.$$

Replacing this last inequality back into Eq. 3.41 and combining the resulting estimate with Eq. 3.40 we can deduce that

$$\big\| \mathbf{w}_h \big\|_{\mathbf{H}} \leq \big\| \mathbf{w}_{S,h} \big\|_{1,\Omega_S} + \big\| \mathbf{w}_{D,h} \big\|_{\text{div},\Omega} \leq c_4 \big\| q_h \big\|_{0,\Omega}. \tag{3.42}$$

Thus, from Eqs. 3.39 and 3.42 we conclude that

$$\sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{[\mathbf{B}(\mathbf{v}_h), q_h]_{\mathbf{Q}' \times \mathbf{Q}}}{\big\| \mathbf{v}_h \big\|_{\mathbf{H}}} \geq \frac{[\mathbf{B}(\mathbf{w}_h), q_h]_{\mathbf{Q}' \times \mathbf{Q}}}{\big\| \mathbf{w}_h \big\|_{\mathbf{H}}} \geq \frac{1}{c_4} \big\| q_h \big\|_{\mathbf{Q}},$$

with $c_4 > 0$ a constant independent of h. □

Then, by applying again Theorem 3.3 and the approximation properties of the subspaces and projectors involved, we arrive at the following main result.

Theorem 3.5 *Let $\mathbf{H}_h$ and $\mathbf{Q}_{h,0}$ be the finite element subspaces defined by Eqs. 3.31 and 3.32, respectively. Then, the nonconforming Galerkin scheme (3.10) has a unique solution $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times \mathbf{Q}_{h,0}$, and there exists $c_3 > 0$, independent of h, such that*

$$\big\| (\mathbf{u}_h, p_h) \big\|_{\mathbf{H} \times \mathbf{Q}} \leq c_3 \left\{ \big\| F_h \big\|_{\mathbf{H}'} + \big\| G_h \big\|_{\mathbf{Q}'} \right\},$$

where $F_h := F|_{\mathbf{H}_h}$ and $G_h := G|_{\mathbf{Q}_{h,0}}$. In addition, assume that the unique solution $(\mathbf{u}, p) \in \mathbf{H} \times \mathbf{Q}$ of the primal-mixed formulation (2.14) is such that $\mathbf{u}_S \in \mathbf{H}^2(\Omega_S)$,

$\mathbf{u}_S \cdot \mathbf{n}|_\Sigma \in H^{1/2}(\Sigma)$, $\mathbf{u}_D \in \mathbf{H}^1(\Omega_D)$, *div* $\mathbf{u}_D \in H^1(\Omega_D)$, *and* $p \in H^1(\Omega)$. *Then, there exists* $c_4 > 0$, *independent of* h, *such that*

$$\|(\mathbf{u}-\mathbf{u}_h, p-p_h)\|_{\mathbf{H}\times\mathbf{Q}} \le c_4 \Big\{ h_S \|\mathbf{u}_S\|_{2,\Omega} + h_D \Big(|\mathbf{u}_D|_{1,\Omega_D} + |div \mathbf{u}_D|_{1,\Omega_D} \Big) + h|p|_{1,\Omega} + h\|\mathbf{u}_S\cdot\mathbf{n}\|_{1/2,\Sigma} \Big\}.$$

3.3.3 A general approach

Irrespective of the previous analysis in Sections 3.3.1 and 3.3.2, we remark that the results in [19] can be extended to the present situation in such a way that (H.8) is simplified as follows:

(H.11) there exists $\tilde{\beta} > 0$, independent of h, such that

$$\sup_{\substack{\mathbf{v}_h\in\bar{\mathbf{H}}_h \\ \mathbf{v}\neq\mathbf{0}}} \frac{[\mathbf{B}(\mathbf{v}_h), q_h]_{\mathbf{Q}'\times\mathbf{Q}}}{\|\mathbf{v}_h\|_{\mathbf{H}}} \ge \tilde{\beta}\|q_h\|_{\mathbf{Q}} \quad \forall q_h \in \bar{\mathbf{Q}}_h,$$

where

$$\bar{\mathbf{H}}_h := \Big[\mathbf{H}_{S,h} \cap \mathbf{H}_0^1(\Omega_S)\Big] \times \big[\mathbf{H}_{D,h} \cap H_0(\mathrm{div};\Omega_D)\big]$$

and

$$\bar{\mathbf{Q}}_h := \Big\{ q_h \in \mathbf{Q}_h : \int_{\Omega_S} q_h = 0, \quad \int_{\Omega_D} q_h = 0 \Big\}.$$

Indeed, it was shown in [19] that one can combine either the RT-element or the BDM-element of order k, with any stable FEM for Stokes of the same order, to obtain a global (conforming as in Table 1 or nonconforming as in Table 2) coupled scheme of order of convergence k. In particular, when the BR elements are employed in the fluid, the corresponding face bubbles do not need to be considered on the faces lying on Σ, which yields a conforming scheme (see [19, Proposition 3.1] for the respective proof). Note also that, in spite of the foregoing modification, the associated approximation property remains unaltered.

Table 1 Coupling of Stokes elements with BDM elements

Stokes	Velocity	Press	Darcy	Vel	Press	Order
MINI	P_1+bubbles	P_1^{cont}	BDM_1	P_1	P_0	h
Taylor-Hood, $k \ge 2$	P_k	P_{k-1}^{cont}	BDM_k	P_k	P_{k-1}	h^k
Conf Crouzeix-Raviart	P_2+bubbles	P_1	BDM_2	P_2	P_1	h^2
Bernardi-Raugel	P_1+face bubbles	P_0	BDM_1	P_1	P_0	h

The superscript $^{\text{cont}}$ refers to the demand of contnuity for the discrete pressure space. The bubbles are used for velocities in the MINI and conformal Crouzeix-Raviart elements: an internal $\mathbf{P}_{d+1}(T)$ bubble is added to the velocity space on each element. For the Bernardi-Raugel element, face bubbles are included on all internal faces, but no bubbles are added on faces lying on Σ. When these bubbles (not needed for stability) are added, the method stops being a particular case of this class

Table 2 Coupling of Stokes elements with BDM and RT elements and their order of convergence

Stokes	Velocity	Press	Darcy	Vel	Press	Order
MINI	$\mathbf{P}_1$+bubbles	$\mathbf{P}_1^{\text{cont}}$	RT_0	RT_0	$\mathbf{P}_0$	h
Taylor-Hood, $k \geq 2$	$\mathbf{P}_k$	$\mathbf{P}_{k-1}^{\text{cont}}$	RT_{k-1}	RT_{k-1}	$\mathbf{P}_{k-1}$	h^k
Bernardi-Raugel	$\mathbf{P}_1$+face bubbles	$\mathbf{P}_0$	RT_0	RT_0	$\mathbf{P}_0$	h
$\mathbf{P}_2$-iso-$\mathbf{P}_1$	$\mathbf{P}_1(\mathcal{T}_{\mathrm{S}}^{h/2})$	$\mathbf{P}_1^{\text{cont}}$	BDM_1	$\mathbf{P}_1$	$\mathbf{P}_0$	h

The superscript $^{\text{cont}}$ refers to the demand of continuity for the discrete pressure space. The bubbles are used for velocities in the MINI element. The triangulation $\mathcal{T}_{\mathrm{S}}^{h/2}$ is a one level refinement of $\mathcal{T}_{\mathrm{S}}^{h}$ and $\mathbf{P}_1(\mathcal{T}_{\mathrm{S}}^{h/2})$ is the space of piecewise linear functions with respect to $\mathcal{T}_{\mathrm{S}}^{h/2}$. For the Bernardi-Raugel element, face bubbles are only included on the internal faces. Adding them to faces on Σ does not change the convergence order. In that case Bernardi-Raugel can be coupled with BDM_1 as well

4 Numerical results

In this section we present numerical examples in 2D illustrating the good performance of the discrete scheme (3.10) on a set of uniform triangulations of the domains Ω_{S} and Ω_{D}. We begin by introducing additional notations. In what follows, N stands for the number of degree of freedom defining the corresponding finite element subspaces $\mathbf{H}_h$ and $\mathbf{Q}_{h,0}$. Then, given the unique solutions $(\mathbf{u}, p) := ((\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}), p) \in \mathbf{H} \times \mathbf{Q}$ and $(\mathbf{u}_h, p_h) := ((\mathbf{u}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}), p_h) \in \mathbf{H}_h \times \mathbf{Q}_{h,0}$ of the primal-mixed formulation (2.14) and the discrete scheme (3.10), respectively, the corresponding individual and global errors are denoted by

$$\mathrm{e}(\mathbf{u}_{\mathrm{S}}) := \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}}, \quad \mathrm{e}(\mathbf{u}_{\mathrm{D}}) := \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div},\Omega_{\mathrm{D}}}, \quad \text{and} \quad \mathrm{e}(p) := \|p - p_h\|_{0,\Omega},$$

and

$$\mathrm{e}(\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, p) := \left\{ \mathrm{e}(\mathbf{u}_{\mathrm{S}})^2 + \mathrm{e}(\mathbf{u}_{\mathrm{D}})^2 + \mathrm{e}(p)^2 \right\}^{1/2}.$$

Also, we let $r(\mathbf{u}_{\mathrm{S}})$, $r(\mathbf{u}_{\mathrm{D}})$ and $r(p)$ be the experimental rates of convergence given by

$$r(\mathbf{u}_{\mathrm{S}}) := \frac{\log(\mathrm{e}(\mathbf{u}_{\mathrm{S}})/\mathrm{e}'(\mathbf{u}_{\mathrm{S}}))}{\log(h/h')}, \quad r(\mathbf{u}_{\mathrm{D}}) := \frac{\log(\mathrm{e}(\mathbf{u}_{\mathrm{D}})/\mathrm{e}'(\mathbf{u}_{\mathrm{D}}))}{\log(h/h')}$$

and

$$r(p) := \frac{\log(\mathrm{e}(p)/\mathrm{e}'(p))}{\log(h/h')},$$

where h and h' denote two consecutive meshsizes with errors e and e$'$, respectively. Further, we let $r(\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, p)$ be the experimental rate for the total error defined by

$$r(\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, p) := \frac{\log(\mathrm{e}(\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, p)/\mathrm{e}'(\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, p))}{\log(h/h')}$$

In the following two sections we present several numerical examples for the nonconforming and conforming versions of the discrete scheme (3.10). For both cases, we choose $\kappa = 1$, $\mathbf{K} = \mathbb{I}$, and consider the nonlinear function $\mu : R^+ \to R^+$ given by a particular case of the Carreau law for viscoplastic flows, that is

$$\mu(t) := \mu_0 + \mu_1 (1 + t^2)^{(\beta-2)}/2 \quad \forall t \in R^+,$$

with $\mu_0 = \mu_1 = 0.5$ and $\beta = 1.5$. It is easy to check in this case that the assumptions (2.6) and (2.7) are satisfied with

$$\gamma_0 = \mu_0 + \mu_1 \left\{ \frac{|\beta - 2|}{2} + 1 \right\} \quad \text{and} \quad \alpha_0 = \mu_0.$$

4.1 A nonconforming case

Here we consider the pair of finite element subspaces $\mathbf{H}_h$ and $\mathbf{Q}_{h,0}$ given in Section 3.3.1 (cf. (3.19), (3.20)), which yields a nonconforming discrete scheme (3.10). In what follows we set

$$\mathbf{curl}\, q := \left(\frac{\partial q}{\partial x_2}, -\frac{\partial q}{\partial x_1} \right)^{\mathtt{t}}.$$

In Example 1 we set the regions $\Omega_S := (-1, 1)^2 \backslash [0, 1)^2$ and $\Omega_D := (0, 1)^2$ of R^2, and choose the data $\mathbf{f}_S$ and f_D so that the exact solution is given by the smooth functions

$$\mathbf{u}_S(\mathbf{x}) := \mathbf{curl} \left(3(x_1^2 + x_2^2)^{13/3} (x_1^2 - 1)^2 (x_2^2 - 1)^2 \right),$$

and

$$p(\mathbf{x}) := \begin{cases} -\frac{\pi}{4} \cos\left(\frac{\pi x_1}{2}\right) \left\{ x_2 + \frac{1}{2} - 2\cos^2\left[\frac{\pi}{2}\left(x_2 + \frac{1}{2}\right)\right] \right\} & \text{on } \Omega_S \\ (x_1 - 1)^2 \sin^3(2\pi(x_2 + 0.5)) & \text{on } \Omega_D. \end{cases}$$

Next, in Example 2 we consider the regions $\Omega_S := (-1, 1)^2 \setminus (-1, 0]^2$ and $\Omega_D := (-1, 0)^2$ of R^2, and choose the data $\mathbf{f}_S$ and f_D so that the exact solutions is given by

$$\mathbf{u}_S(\mathbf{x}) := \mathbf{curl} \left(3(x_1^2 + x_2^2)^{2/3} (x_1^2 - 1)^2 (x_2^2 - 1)^2 \right),$$

and

$$p(\mathbf{x}) := \begin{cases} \exp(x_1 + x_2) x_1 x_2 & \text{on } \Omega_S \\ (x_1 + 1)^2 \sin^3(2\pi(x_2 + 0.5)) & \text{on } \Omega_D. \end{cases}$$

Note that in this example $\mathbf{u}_S$ becomes singular at the origin.

The numerical results shown below were obtained using a MATLAB code. In Tables 3 and 4 we summarize the convergence history of the discrete primal-mixed scheme (3.10) as applied to Examples 1 and 2, for sequences of quasi-uniform triangulations of the domains. We observe in Table 3, looking at the corresponding experimental rates of convergence, that the $O(h)$ predicted by Theorem 3.4 is attained by all the unknowns in Example 1. In addition, we notice that the dominant error is given by $\mathrm{e}(\mathbf{u}_D)$. The behavior of the experimental rates of convergence can

Table 3 Example 1, convergence history

h	N	$\mathtt{e}(\mathbf{u}_S)$	$r(\mathbf{u}_S)$	$\mathtt{e}(\mathbf{u}_D)$	$r(\mathbf{u}_D)$	$\mathtt{e}(p)$	$r(p)$	$\mathtt{e}(\mathbf{u}_S, \mathbf{u}_D, p)$	$r(\mathbf{u}_S, \mathbf{u}_D, p)$
1/17	12828	3.989E−00	–	5.612E−00	–	2.806E−01	–	6.891E−00	– (5)
1/19	16114	3.509E−00	1.153	5.040E−00	0.967	2.741E−01	0.209	6.147E−00	1.027 (5)
1/21	19882	3.178E−00	0.990	4.491E−00	1.152	2.573E−01	0.635	5.508E−00	1.097 (5)
1/25	28121	2.718E−00	0.896	3.787E−00	0.978	2.003E−01	1.436	4.666E−00	0.951 (5)
1/35	54222	1.915E−00	1.040	2.763E−00	0.937	1.331E−01	1.216	3.365E−00	0.971 (5)
1/45	91170	1.482E−00	1.022	2.072E−00	1.145	1.007E−01	1.109	2.550E−00	1.104 (5)
1/55	135720	1.201E−00	1.049	1.721E−00	0.925	8.112E−02	1.077	2.100E−00	0.966 (5)
1/65	190019	1.017E−00	0.991	1.461E−00	0.982	7.188E−02	0.724	1.782E−00	0.984 (5)
1/75	254402	8.851E−01	0.974	1.244E−00	1.123	6.147E−02	1.093	1.528E−00	1.073 (5)
1/85	325129	7.754E−01	1.057	1.101E−00	0.973	5.173E−02	1.378	1.348E−00	1.001 (5)
1/95	403178	6.953E−01	0.981	9.951E−01	0.913	4.445E−02	1.364	1.215E−00	0.936 (5)
1/105	493751	6.296E−01	0.991	9.021E−01	0.980	4.114E−02	0.773	1.101E−00	0.984 (5)
1/115	592931	5.691E−01	1.111	8.196E−01	1.054	3.650E−02	1.315	9.985E−01	1.073 (5)
1/125	705036	5.246E−01	0.976	7.469E−01	1.113	3.416E−02	0.796	9.134E−01	1.068 (5)

be also checked from Fig. 2, where we display the mesh size h and the errors $\mathtt{e}(\mathbf{u}_S)$, $\mathtt{e}(\mathbf{u}_D)$ and $\mathtt{e}(p)$ versus the degrees of freedom N. In particular, we realize there that $\mathtt{e}(p)$ is quite below the other individual errors and that, in spite of its convergence slower than $O(h)$ at the beginning, it rapidly stabilizes around that order later on. Concerning Example 2, we note on the contrary in Table 4 that $r(\mathbf{u}_S)$ lies around

Table 4 Example 2, convergence history

h	N	$\mathtt{e}(\mathbf{u}_S)$	$r(\mathbf{u}_S)$	$\mathtt{e}(\mathbf{u}_D)$	$r(\mathbf{u}_D)$	$\mathtt{e}(p)$	$r(p)$	$\mathtt{e}(\mathbf{u}_S, \mathbf{u}_D, p)$	$r(\mathbf{u}_S, \mathbf{u}_D, p)$
1/17	12853	2.856E−00	–	5.560E−00	–	5.160E−01	–	6.272E−00	– (5)
1/19	16108	2.671E−00	0.602	5.059E−00	0.848	4.950E−01	0.372	5.743E−00	0.793 (5)
1/21	19671	2.577E−00	0.359	4.596E−00	0.959	4.854E−01	0.196	5.292E−00	0.817 (5)
1/25	28444	2.313E−00	0.622	3.761E−00	1.151	4.365E−01	0.609	4.437E−00	1.011 (5)
1/35	54513	2.144E−00	0.225	2.774E−00	0.904	5.668E−01	–	3.552E−00	0.661 (5)
1/45	91225	1.767E−00	0.769	2.090E−00	1.127	3.214E−01	2.257	2.756E−00	1.010 (5)
1/55	136347	1.704E−00	0.182	1.724E−00	0.960	3.269E−01	–	2.446E−00	0.595 (5)
1/65	190171	1.597E−00	0.388	1.463E−00	0.982	3.012E−01	0.489	2.187E−00	0.670 (5)
1/75	254577	1.493E−00	0.469	1.257E−00	1.063	3.224E−01	–	1.978E−00	0.701 (5)
1/85	324355	1.427E−00	0.360	1.109E−00	0.997	2.650E−01	1.567	1.827E−00	0.635 (4)
1/95	403975	1.457E−00	–	9.954E−01	0.973	3.119E−01	–	1.792E−00	0.173 (4)
1/105	496126	1.359E−00	0.698	9.007E−01	0.998	2.800E−01	1.079	1.654E−00	0.800 (4)
1/115	595622	1.331E−00	0.229	8.211E−01	1.018	2.895E−01	–	1.590E−00	0.432 (4)
1/125	707479	1.262E−00	0.634	7.532E−01	1.034	2.328E−01	2.614	1.488E−00	0.795 (4)

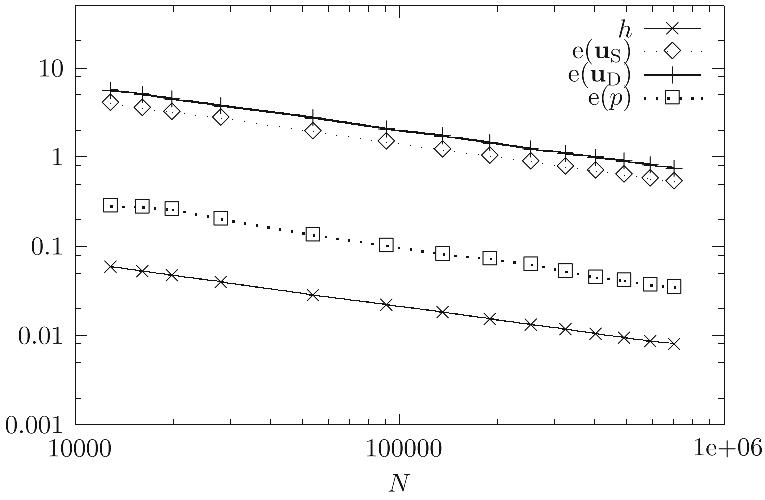

Fig. 2 Example 1, h and errors versus degree of freedom N

$1/2$ whereas $r(p)$ shows large oscillations, which is certainly due to the singular behaviour of the corresponding exact solution. However, $r(\mathbf{u}_\mathrm{D})$ does not seem to be affected by the lack of regularity of $\mathbf{u}_\mathrm{S}$ since it behaves always as $O(h)$. The foregoing facts are also observed in Fig. 3, where we display the mesh size h and the errors $\mathrm{e}(\mathbf{u}_\mathrm{S})$, $\mathrm{e}(\mathbf{u}_\mathrm{D})$ and $\mathrm{e}(p)$ versus the degrees of freedom N. This example is certainly very suitable to explore in the future the application of an adaptive algorithm based on a posteriori error estimates. Indeed, one would expect that by means of this procedure the optimal rates of convergence would be recovered by all the unknowns. On the other hand, in Figs. 4, 5, 6, and 7, we show some components of the approximate

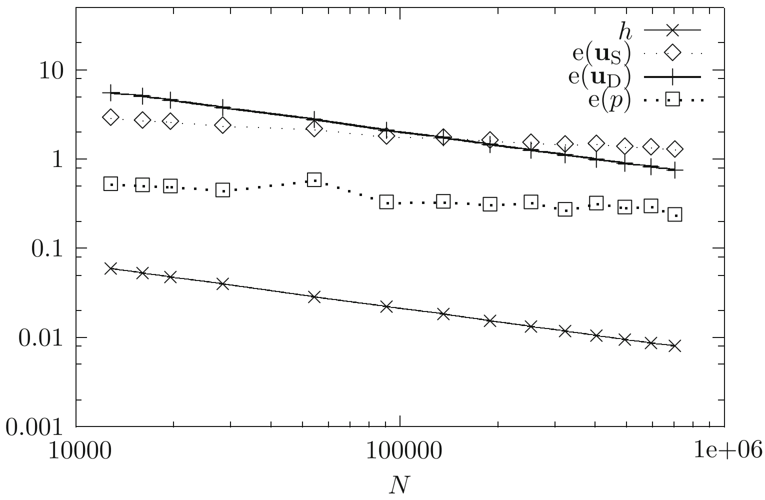

Fig. 3 Example 2, h and errors versus degree of freedom N

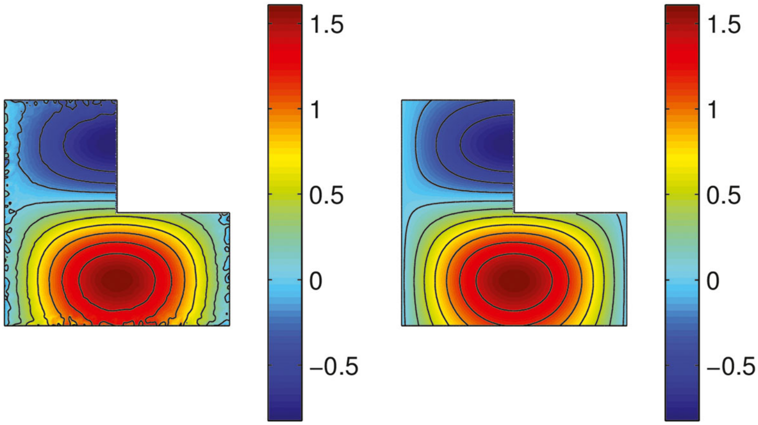

Fig. 4 Example 1, Stokes pressure with $N = 54222$

(left) and exact (right) solutions. We notice from Figs. 4 and 5 that the piecewise constant functions approximate quite well the pressure in the Darcy domain Ω_D and the interior of the Stokes region Ω_S, whereas this approximation deteriorates a bit near by $\partial\Omega_S \backslash \Sigma$ (Fig. 4 and 5). In turn, in Figs. 6 and 7 we see that the Bernardi-Raugel subspace provides a quite good approximation of the velocity in the Stokes domain Ω_S.

4.2 A conforming case

We now consider the pair of finite element subspaces $\mathbf{H}_h$ and $\mathbf{Q}_{h,0}$ given in Section 3.3.2 (cf. (3.31), (3.32)), but with the modification explained at the end of Section 3.3.3 so that the resulting scheme (3.10) becomes conforming. Then, for the Example 3 we set the regions $\Omega_S := (-1, 1) \times (-1, 0)$ and $\Omega_D := (-1, 1) \times (0, 1)$

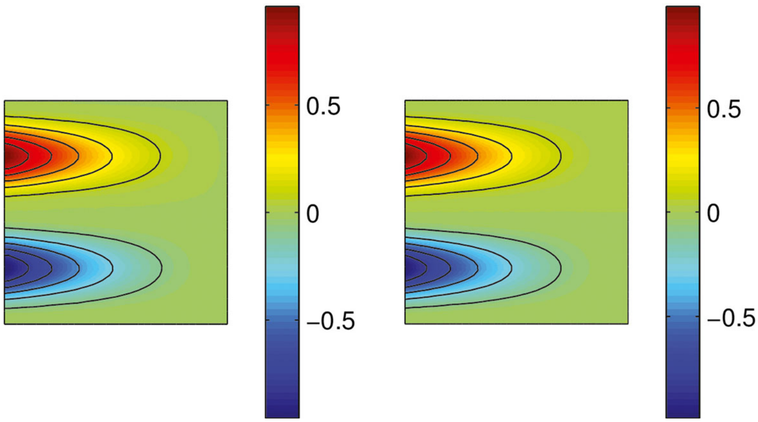

Fig. 5 Example 1, Darcy pressure with $N = 54222$

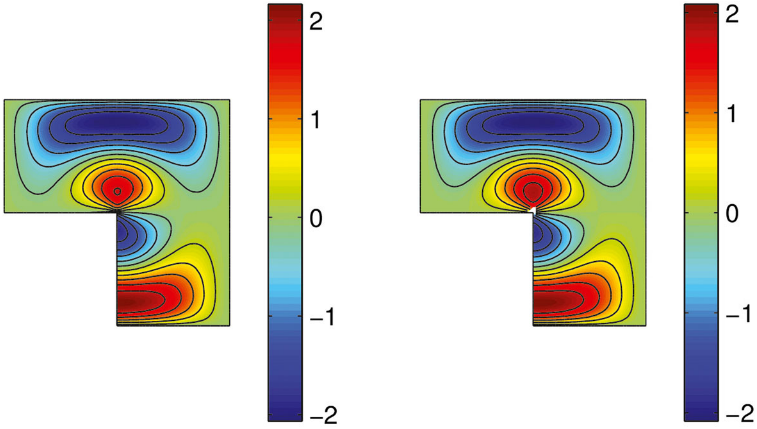

Fig. 6 Example 2, first component of the Stokes velocity with $N = 54513$

of R^2, and choose the data $\mathbf{f}_S$ and f_D so that the exact solution is given by the smooth functions

$$\mathbf{u}_S(\mathbf{x}) := \mathbf{curl}\left(\sin(\pi x_2 + \pi/4)\sin^2(2\pi x_1)(1 + x_2)^2\right),$$

and

$$p(\mathbf{x}) := \begin{cases} \exp(x_1 + x_2)x_1 x_2 & \text{on } \Omega_S \\ 3\pi\left(1 - x_2 - \dfrac{1}{\pi}\sin(\pi x_2)\right)\sin^2(\pi x_1)\cos(\pi x_1) & \text{on } \Omega_D\,. \end{cases}$$

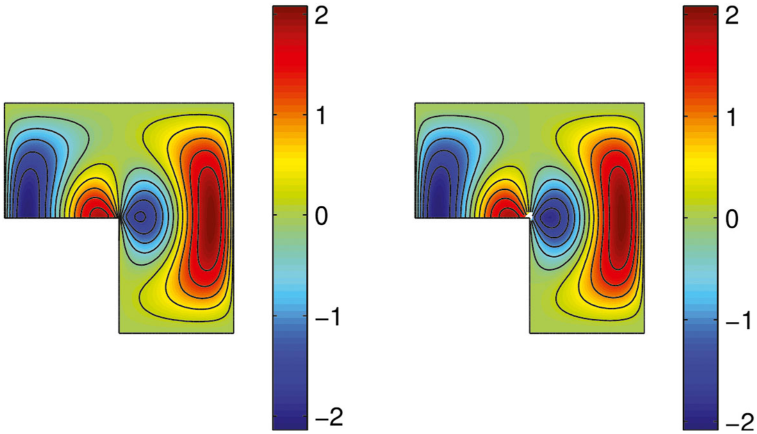

Fig. 7 Example 2, second component of the Stokes velocity with $N = 54513$

Table 5 Example 3, convergence history

h	N	$e(\mathbf{u}_S)$	$r(\mathbf{u}_S)$	$e(\mathbf{u}_D)$	$r(\mathbf{u}_D)$	$e(p)$	$r(p)$	$e(\mathbf{u}_S, \mathbf{u}_D, p)$	$r(\mathbf{u}_S, \mathbf{u}_D, p)$
1/32	7923	6.372E−00	–	1.260E+01	–	6.352E−01	–	1.413E+01	– (6)
1/64	31203	3.188E−00	0.999	6.319E−00	0.996	2.411E−01	1.398	7.082E−00	0.997 (5)
1/96	69843	2.124E−00	1.002	4.215E−00	0.999	1.472E−01	1.216	4.722E−00	0.999 (5)
1/128	123843	1.592E−00	1.001	3.162E−00	0.999	1.062E−01	1.136	3.542E−00	1.000 (5)
1/160	193203	1.274E−00	1.001	2.530E−00	1.000	8.292E−02	1.107	2.833E−00	1.000 (5)
1/192	277923	1.061E−00	1.001	2.108E−00	1.000	6.813E−02	1.077	2.361E−00	1.000 (5)
1/224	378003	9.094E−01	1.001	1.807E−00	1.000	5.776E−02	1.071	2.024E−00	1.000 (5)
1/256	493443	7.956E−01	1.001	1.581E−00	1.000	5.017E−02	1.056	1.771E−00	1.000 (5)
1/288	624243	7.072E−01	1.001	1.406E−00	1.000	4.433E−02	1.050	1.574E−00	1.000 (5)
1/320	770403	6.364E−01	1.001	1.265E−00	1.000	3.973E−02	1.041	1.417E−00	1.000 (5)
1/352	931923	5.785E−01	1.001	1.150E−00	1.000	3.597E−02	1.042	1.288E−00	1.000 (5)
1/384	1108803	5.303E−01	1.001	1.054E−00	1.000	3.288E−02	1.034	1.180E−00	1.000 (4)

The numerical results shown below were also obtained using a MATLAB code. In Table 5 we summarize the convergence history of the discrete primal-mixed scheme (3.10) as applied to Example 3, for sequences of quasi-uniform triangulations of the domains. Similarly as for Example 1, we observe there, looking at the corresponding experimental rates of convergence, that the order $O(h)$ predicted by Theorem 3.5 is attained by all the unknowns. In addition, the individual errors $e(\mathbf{u}_S)$ and $e(\mathbf{u}_D)$ are the dominant ones in this example. This fact is even more clear in Fig. 8 where

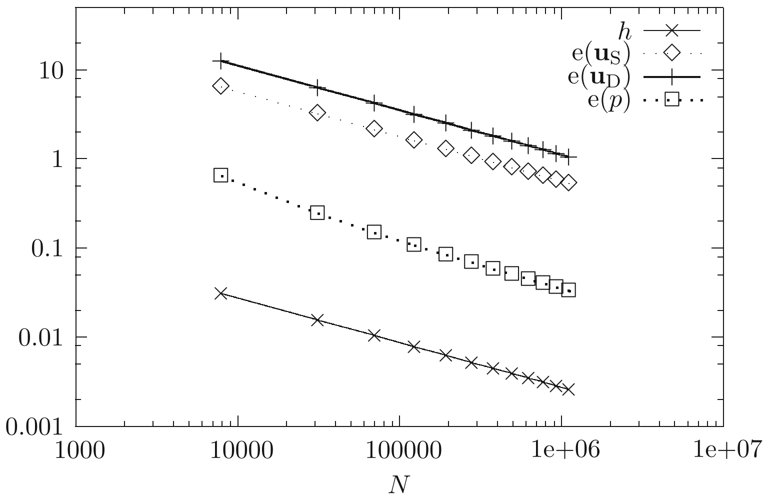

Fig. 8 Example 3, h and errors versus degree of freedom N

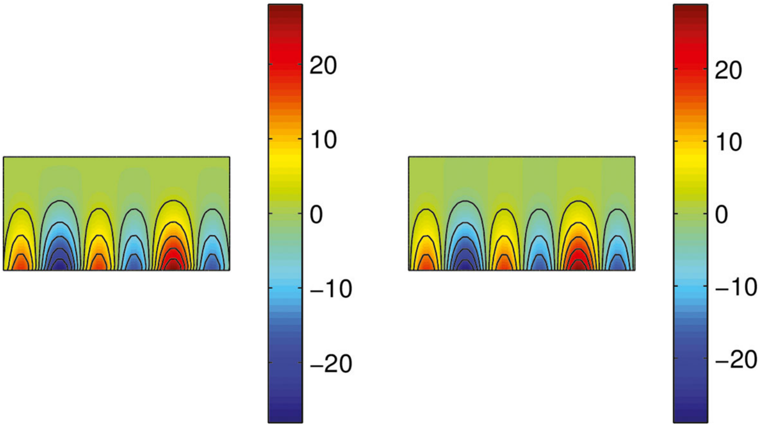

Fig. 9 Example 3, first component of the Darcy velocity with $N = 31203$

one sees that $e(\mathbf{u}_S)$ and $e(\mathbf{u}_D)$ are quite above $e(p)$. Moreover, we observe there that $e(p)$ seems to converge a bit faster than $O(h)$ at the beginning but then it rapidly stabilizes around that order. Finally, in Figs. 9 and 10 we show some components of the approximate (left) and exact (right) solutions for this example. In particular, we remark that the Raviart-Thomas subspace reconstructs quite accurately the velocity in the porous medium Ω_D.

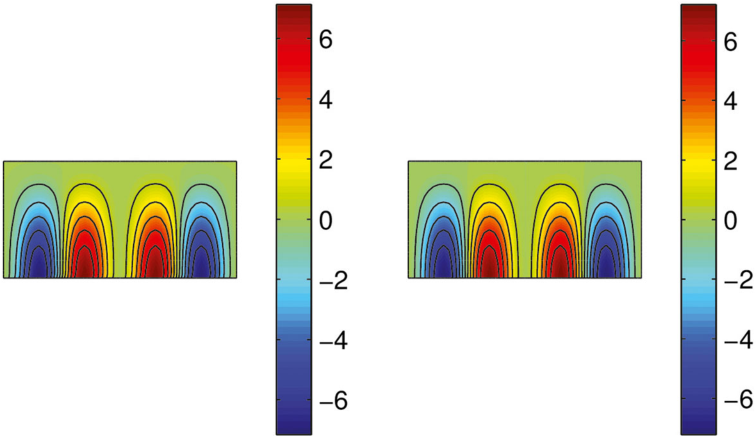

Fig. 10 Example 3, second component of the Darcy velocity with $N = 31203$

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References

1. Bernardi, C., Hecht, F., Pironneau, O.: Coupling Darcy and Stokes equations for porous media with cracks. ESAIM: Mathematical Modelling and Numerical Analysis **39**(1), 7–35 (2005)
2. Bernardi, C., Raugel, G.: Analysis of some finite elements for the Stokes problem. Math. Comput. **44**(169), 71–79 (1985)
3. Brezzi, F., Douglas, J., Marini, L.D.: Two families of mixed finite elements for second order elliptic problems. Numer. Math. **47**(2), 217–235 (1985)
4. Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods. Springer (1991)
5. Discacciati, M.: Domain Decomposition Methods for the Coupling of Surface and Groundwater Flows. Ph.D. Thesis, Ecole Polytechnique Fédérale de Lausanne (2004)
6. Discacciati, M., Miglio, E., Quarteroni, A.: Mathematical and numerical models for coupling surface and groundwater flows. Appl. Numer. Math. **43**(1-2), 57–74 (2002)
7. Ervin, V.J., Jenkins, E.W., Sun, S.: Coupled generalized nonlinear Stokes flow with flow through a porous medium. SIAM J. Numer. Anal. **47**(2), 929–952 (2009)
8. Ervin, V.J., Jenkins, E.W., Sun, S.: Coupling nonlinear Stokes and Darcy flow using mortar finite elements. Appl. Numer. Math. **61**(11), 1198–1222 (2011)
9. Galvis, J., Sarkis, M.: Non-matching mortar discretization analysis for the coupling Stokes-Darcy equations. Electron. Trans. Numer. Anal. **26**, 350–384 (2007)
10. Gatica, G.N.: A Simple Introduction to the Mixed Finite Element Method. Theory and Applications. SpringerBriefs in Mathematics, Springer, Cham (2014)
11. Gatica, G.N.: Solvability and Galerkin approximations of a class of nonlinear operator equations. Zeitschrift fur Analysis und ihre Anwendungen **21**(3), 761–781 (2002)
12. Gatica, G.N., Heuer, N., Meddahi, S.: On the numerical analysis of nonlinear twofold saddle point problems. IMA J. Numer. Anal. **23**(2), 301–330 (2003)
13. Gatica, G.N., Meddahi, S., Oyarzúa, R.: A conforming mixed finite-element method for the coupling of fluid flow with porous media flow. IMA J. Numer. Anal. **29**(1), 86–108 (2009)
14. Girault, V., Raviart, P.-A.: Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms. Springer (1986)
15. Karper, T., Mardal, K.-A., Winther, R.: Unified finite element discretizations of coupled Darcy-Stokes flow. Numerical Methods for Partial Differential Equations **25**(2), 311–326 (2009)
16. Ladyzhenskaya, O.: New equations for the description of the viscous incompressible fluids and solvability in the large for the boundary value problems of them. In: Boundary Value Problems of Mathematical Physics V, Providence, RI: AMS (1970)
17. Layton, W.J., Schieweck, F., Yotov, I.: Coupling fluid flow with porous media flow. SIAM J. Numer. Anal. **40**(6), 2195–2218 (2003)
18. Loula, A.F.D., Guerreiro, J.N.C.: Finite element analysis of nonlinear creeping flows. Comput. Methods Appl. Mech. Eng. **79**(1), 87–109 (1990)
19. Márquez, A., Meddahi, S., Sayas, F.-J.: Strong coupling of finite element methods for the Stokes-Darcy problem. IMA J. Numer. Anal. **35**(2), 969–988 (2015)
20. Necas, J.: Introduction to the Theory of Nonlinear Elliptic Equations. Wiley (1986)
21. Riviere, B., Yotov, I.: Locally conservative coupling of Stokes and Darcy flows. SIAM J. Numer. Anal. **42**(5), 1959–1977 (2005)
22. Sandri, D.: Sur l'approximation numérique des écoulements quasi-Newtoniens dont la viscosité suit la loi puissance ou la loi de Carreau. ESAIM: Mathematical Modelling and Numerical Analysis **27**(2), 131–155 (1993)

23. Scheurer, B.: Existence et approximation de points selles pour certains problemes non lineaires. RAIRO Analyse Numerique **11**(4), 369–400 (1977)
24. Urquiza, J.M., NDri, D., Garon, A., Delfour, M.C.: Coupling Stokes and Darcy equations. Appl. Numer. Math. **58**(5), 525–538 (2008)
25. Xie, X., Xu, J., Xue, G.: Uniformly stable finite element methods for Darcy-Stokes-Brinkman models. J. Comput. Math. **26**(3), 437–455 (2008)