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High order schemes for the tempered fractional diffusion equations

Can Li[1,2] · Weihua Deng[3]

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Abstract Lévy flight models whose jumps have infinite moments are mathematically used to describe the superdiffusion in complex systems. Exponentially tempering Lévy measure of Lévy flights leads to the tempered stable Lévy processes which combine both the α-stable and Gaussian trends; and the very large jumps are unlikely and all their moments exist. The probability density functions of the tempered stable Lévy processes solve the tempered fractional diffusion equation. This paper focuses on designing the high order difference schemes for the tempered fractional diffusion equation on bounded domain. The high order difference approximations, called the tempered and weighted and shifted Grünwald difference (tempered-WSGD) operators, in space are obtained by using the properties of the tempered fractional calculus and weighting and shifting their first order Grünwald type difference approximations. And the Crank-Nicolson discretization is used in the time direction. The stability and convergence of the presented numerical schemes are established; and the numerical experiments are performed to confirm the theoretical results and testify the effectiveness of the schemes.

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✉ Weihua Deng
dengwh@lzu.edu.cn

Can Li
mathlican@xaut.edu.cn

1 Department of Applied Mathematics, School of Sciences, Xi'an University of Technology, Xi'an, Shaanxi 710054, People's Republic of China

2 Beijing Computational Science Research Center, Beijing 100084, People's Republic of China

3 School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, People's Republic of China

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1 Introduction

The probability density function of Lévy flights [19, 22] has a characteristic function $e^{-D_\alpha |k|^\alpha t}$ $(0 < \alpha < 2)$ of stretched Gaussian form, causing the asymptotic decay as $|x|^{-1-\alpha}$. It produces that the second moment diverges, i.e., $\langle x^2(t)\rangle = \infty$. The divergent second moments may not be feasible for some even non-Brownian physical processes of practical interest which take place in bounded domains and involve observables with finite moments. To overcome the divergence of the variance, many techniques are adopted. By simply discarding the very large jumps, Mantegna and Stanley [20] introduce the truncated Lévy flights and show that the obtained stochastic process ultraslowly converges to a Gaussian. From the point of view of an experimental study, because of the limited time, no expected Gaussian behavior can be observed. Two other modifications to achieve finite second moments are proposed by Sokolov et al. [29], who add a high-order power-law factor, and Chechkin et al. [8], who add a nonlinear friction term. Exponentially tempering the probability of large jumps of Lévy flights, i.e., making the Lévy density decay as $|x|^{-1-\alpha}\mathrm{e}^{-\lambda|x|}$ with $\lambda > 0$, to get finite moments seems to be the most popular one; and the corresponding tempered fractional differential equations are derived [1, 4–6]. In fact, the power-law waiting time can also be exponentially tempered [22]. More detailed descriptions of the applications of tempered fractional differential equations related to finance and geophysical flows can be found in [1, 5] and [22], respectively. This paper focused on providing high order schemes for numerically solving tempered fractional diffusion equations, which involve tempered fractional derivatives. In fact, the tempered fractional integral has a long history. Buschman's earlier work [2] reports the fractional integration with weak singular and exponential kernels; for more detailed discussions, see Srivastava and Buschman's book [15] and the references therein. The definitions of tempered fractional calculus are much similar to the ones of fractional substantial calculus [3]; but they are introduced from completely different physical backgrounds; e.g., fractional substantial calculus is used to characterize the functional distribution of anomalous diffusion [3]. Mathematically the fractional substantial calculus is time-space coupled operator but the tempered fractional calculus is not; numerically the fractional substantial calculus is discretized in the time direction [9], but here the tempered fractional calculus is treated as space operator.

Tempered fractional calculus is the generalization of fractional calculus, or fractional calculus is the special/limiting case of tempered fractional calculus. Some important progresses have been made for numerically solving the fractional partial differential equations (PDEs), e.g., the finite difference methods are used to simulate the space fractional advection diffusion equations [16, 17, 21]. Recently, it seems that more efforts of the researchers are put on the high order schemes and fast algorithms. Based on the Toeplitz-like structure of the matrix corresponding the finite difference methods of fractional PDEs, Wang et al. [33] numerically solve the fractional diffusion equations with the $N \log^2 N$ computational cost. Later, Pang and Sun [26] propose a multigrid method to solve the discretized system of the fractional diffusion equation. By introducing the linear spline approximation, Sousa and Li present a second order discretization for the Riemann-Liouville fractional derivatives, and establish an unconditionally stable weighted finite difference method for the one-dimensional fractional diffusion equation in [30]. Ortigueira [25] gives the "fractional centred derivative" to approximate the Riesz fractional derivative with second order accuracy; and this method is used by Çelik and Duman in [7] to approximate fractional diffusion equation with the Riesz fractional derivative in a finite domain. More recently, by weighting and shifting the Grünwald discretizations, Tian et al. [31] propose a class of second order difference approximations, called WSGD operators, to the Riemann-Liouville fractional derivatives.

So far, there are limited works addressing the finite difference schemes for the tempered fractional diffusion equations. Baeumera and Meerschaert [1] provide finite difference and particle tracking methods for solving the tempered fractional diffusion equation with the second order accuracy. The stability and convergence of the provided schemes are discussed. Cartea and del-Castillo-Negrete [4] derive a general finite difference scheme to numerically solve a Black-Merton-Scholes model with tempered fractional derivatives. Recently, Marom and Momoniat [18] compare the numerical solutions of three kinds of fractional Black-Merton-Scholes equations with tempered fractional derivatives. And the stability and convergence of the presented schemes are not given. To the best of our knowledge, there is no published work to the high order difference schemes for the tempered fractional diffusion equation. In this paper, with the similar method presented in [1, 21], we first propose the first order shifted Grünwald type approximation for the tempered fractional calculus; then motivated by the idea in [31], we design a series of high order schemes, called the tempered-WSGD operators, by weighting and shifting the first order Grünwald type approximations to the tempered fractional calculus. The obtained high order schemes are applied to solve the tempered fractional diffusion equation and the Crank-Nicolson discretization is used in the time direction. The unconditionally numerical stability and convergence are detailedly discussed; and the corresponding numerical experiments are carried out to illustrate the effectiveness of the schemes.

The remainder of the paper is organized as follows. In Section 2, we introduce the definitions of the tempered fractional calculus and derive their first order shifted Grünwald type approximations and the high order discretizations, the tempered-WSGD operators. In Section 3, the tempered fractional diffusion equation is numerically solved by using the tempered-WSGD operators to approximate the space derivative and the Crank-Nicolson discretization to the time derivative; and the numerical stability and convergence are discussed. The effectiveness and convergence orders of the presented schemes are numerically verified in Section 4. And the concluding remarks are given in the last section.

2 Definitions of the tempered fractional calculus and the derivation of the tempered-WSGD operators

We first introduce the definitions of the tempered fractional integral and derivative then focus on deriving their high order discretizations, the tempered-WSGD operators.

2.1 Definitions and Fourier transforms of the tempered fractional calculus

We introduce the definitions of the tempered fractional calculus and perform their Fourier transforms.

Definition 1 ([2, 5]) Let $u(x)$ be piecewise continuous on (a, ∞) (or $(-\infty, b)$ corresponding to the right integral) and integrable on any finite subinterval of $[a, \infty)$ (or $(-\infty, b]$ corresponding to the right integral), $\sigma > 0$, $\lambda \geq 0$. Then

(1) the left Riemann-Liouville tempered fractional integral of order σ is defined to be

$$ {}_aD_x^{-\sigma,\lambda}u(x) = \frac{1}{\Gamma(\sigma)}\int_a^x e^{-\lambda(x-\xi)}(x-\xi)^{\sigma-1}u(\xi)\mathrm{d}\xi; $$

(2) the right Riemann-Liouville tempered fractional integral of order σ is defined to be

$$ {}_xD_b^{-\sigma,\lambda}u(x) = \frac{1}{\Gamma(\sigma)}\int_x^b e^{-\lambda(\xi-x)}(\xi-x)^{\sigma-1}u(\xi)\mathrm{d}\xi. $$

Definition 2 ([12, 14, 27]) For $\alpha \in (n-1, n)$, $n \in \mathbb{N}^+$, let $u(x)$ be $(n-1)$-times continuously differentiable on (a, ∞) (or $(-\infty, b)$ corresponding to the right derivative) and its n-times derivative be integrable on any subinterval of $[a, \infty)$ (or $(-\infty, b]$ corresponding to the right derivative). Then

(1) the left Riemann-Liouville fractional derivative:

$$ {}_aD_x^{\alpha}u(x) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_a^x \frac{u(\xi)}{(x-\xi)^{\alpha-n+1}}\mathrm{d}\xi; $$

(2) the right Riemann-Liouville fractional derivative:

$$ {}_xD_b^{\alpha}u(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_x^b \frac{u(\xi)}{(\xi-x)^{\alpha-n+1}}\mathrm{d}\xi. $$

Definition 3 ([1, 5]) For $\alpha \in (n-1, n)$, $n \in \mathbb{N}^+$, let $u(x)$ be $(n-1)$-times continuously differentiable on (a, ∞) (or $(-\infty, b)$ corresponding to the right derivative) and its n-times derivative be integrable on any subinterval of $[a, \infty)$ (or $(-\infty, b]$ corresponding to the right derivative), $\lambda \geq 0$. Then

(1) the left Riemann-Liouville tempered fractional derivative:

$$ {}_aD_x^{\alpha,\lambda}u(x) = e^{-\lambda x}{}_aD_x^{\alpha}\left(e^{\lambda x}u(x)\right) = \frac{e^{-\lambda x}}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_a^x \frac{e^{\lambda\xi}u(\xi)}{(x-\xi)^{\alpha-n+1}}\mathrm{d}\xi; $$

(2) the right Riemann-Liouville tempered fractional derivative:

$$ {}_xD_b^{\alpha,\lambda}u(x) = e^{\lambda x}{}_xD_b^{\alpha}\left(e^{-\lambda x}u(x)\right) = \frac{(-1)^n e^{\lambda x}}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_x^b \frac{e^{-\lambda\xi}u(\xi)}{(\xi-x)^{\alpha-n+1}}\mathrm{d}\xi. $$

If $\lambda = 0$, then the left and right Riemann-Liouville tempered fractional derivatives ${}_aD_x^{\alpha,\lambda}u(x)$ and ${}_xD_b^{\alpha,\lambda}u(x)$ reduce to the left and right Riemann-Liouville fractional derivatives ${}_aD_x^{\alpha}u(x)$ and ${}_xD_b^{\alpha}u(x)$ defined in Definition 2.

Definition 4 The variants of the left and right Riemann-Liouville tempered fractional derivatives are defined as [1, 5, 23]

$$ {}_a\mathbf{D}_x^{\alpha,\lambda}u(x) = \begin{cases} {}_aD_x^{\alpha,\lambda}u(x) - \lambda^{\alpha}u(x), & 0<\alpha<1, \\ {}_aD_x^{\alpha,\lambda}u(x) - \alpha\lambda^{\alpha-1}\partial_x u(x) - \lambda^{\alpha}u(x), & 1<\alpha<2; \end{cases} \tag{2.1} $$

and

$$ {}_x\mathbf{D}_b^{\alpha,\lambda}u(x) = \begin{cases} {}_xD_b^{\alpha,\lambda}u(x) - \lambda^{\alpha}u(x), & 0<\alpha<1, \\ {}_xD_b^{\alpha,\lambda}u(x) + \alpha\lambda^{\alpha-1}\partial_x u(x) - \lambda^{\alpha}u(x), & 1<\alpha<2, \end{cases} \tag{2.2} $$

where ∂_x denotes the classic first order derivative $\frac{\partial}{\partial x}$.

Remark 1 In the above definitions, the 'a' can be extended to '$-\infty$' and 'b' to '$+\infty$'. In the following analysis, we assume that $u(x)$ is defined on $[a,b]$ and whenever necessary $u(x)$ can be smoothly zero extended to $(-\infty,b)$ or $(a,+\infty)$ or even $(-\infty,+\infty)$. Then ${}_{-\infty}D_x^{-\sigma,\lambda}u(x) = {}_aD_x^{-\sigma,\lambda}u(x)$; ${}_xD_{+\infty}^{-\sigma,\lambda}u(x) = {}_xD_b^{-\sigma,\lambda}u(x)$; ${}_{-\infty}D_x^{\alpha,\lambda}u(x) = {}_aD_x^{\alpha,\lambda}u(x)$; and ${}_xD_{+\infty}^{\alpha,\lambda}u(x) = {}_xD_b^{\alpha,\lambda}u(x)$.

Lemma 1 ([1, 2, 15]) *Let $u(x)$ and its n-times derivative belong to $L^q(\mathbb{R})$, $q \geq 1$. Then the Fourier transforms of the left and right Riemann-Liouville tempered fractional integrals are*

$$ \mathcal{F}({}_{-\infty}D_x^{-\sigma,\lambda}u(x)) = (\lambda + i\omega)^{-\sigma}\hat{u}(\omega); \tag{2.3} $$

and

$$ \mathcal{F}({}_xD_{+\infty}^{-\sigma,\lambda}u(x)) = (\lambda - i\omega)^{-\sigma}\hat{u}(\omega) \tag{2.4} $$

and the Fourier transforms of the left and right Riemann-Liouville tempered fractional derivatives are

$$ \mathcal{F}({}_{-\infty}D_x^{\alpha,\lambda}u(x)) = (\lambda + i\omega)^{\alpha}\hat{u}(\omega); \tag{2.5} $$

and

$$ \mathcal{F}({}_xD_{+\infty}^{\alpha,\lambda}u(x)) = (\lambda - i\omega)^{\alpha}\hat{u}(\omega) \tag{2.6} $$

and the Fourier transforms of the variants of the left and right Riemann-Liouville tempered fractional derivatives give

$$\mathcal{F}(\,_{-\infty}\mathbf{D}_x^{\alpha,\lambda}u(x)) = \begin{cases} (\lambda + i\omega)^{\alpha}\hat{u}(\omega) - \lambda^{\alpha}\hat{u}(\omega), & 0 < \alpha < 1, \\ (\lambda + i\omega)^{\alpha}\hat{u}(\omega) - \alpha i\omega\lambda^{\alpha-1}\hat{u}(\omega) - \lambda^{\alpha}\hat{u}(\omega), & 1 < \alpha < 2; \end{cases} \tag{2.7}$$

and

$$\mathcal{F}(\,_{x}\mathbf{D}_{+\infty}^{\alpha,\lambda}u(x)) = \begin{cases} (\lambda - i\omega)^{\alpha}\hat{u}(\omega) - \lambda^{\alpha}\hat{u}(\omega), & 0 < \alpha < 1, \\ (\lambda - i\omega)^{\alpha}\hat{u}(\omega) + \alpha i\omega\lambda^{\alpha-1}\hat{u}(\omega) - \lambda^{\alpha}\hat{u}(\omega), & 1 < \alpha < 2, \end{cases} \tag{2.8}$$

where the Fourier transform of u is defined by

$$\mathcal{F}(u(x))(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-i\omega x}u(x)\mathrm{d}x,\; i^2 = -1.$$

Remark 2 ([3, 9]) The left and right Riemann-Liouville tempered fractional derivatives can be, respectively, rewritten as

$$_{-\infty}D_x^{\alpha,\lambda}u(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \lambda\right)^n \int_{-\infty}^{x} \frac{e^{-\lambda(x-\xi)}u(\xi)}{(x-\xi)^{\alpha-n+1}}\mathrm{d}\xi;$$

and

$$_{x}D_{+\infty}^{\alpha,\lambda}u(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \lambda\right)^n \int_{x}^{+\infty} \frac{e^{-\lambda(\xi-x)}u(\xi)}{(\xi-x)^{\alpha-n+1}}\mathrm{d}\xi.$$

2.2 Discretizations of the tempered fractional calculus

In this subsection, we derive the Grünwald type discretizations for the tempered fractional calculus. The standard Grünwald discretization generally yields an unstable finite difference scheme when it is used to solve the time dependent fractional PDEs [21]. To remedy this defect, Meerschaert et al. introduce a shifted Grünwald formula. The similar numerical unstability also happens for the time dependent tempered fractional PDEs; so the shift for the Grünwald type discretizations of the tempered fractional derivative is also necessary.

Lemma 2 *Let* $u(x) \in L^1(\mathbb{R})$*,* $_{-\infty}D_x^{\alpha+1,\lambda}u$ *and its Fourier transform belong to* $L^1(\mathbb{R})$*;* $p \in \mathbb{R}, h > 0, \lambda \geq 0$ *and* $\alpha \in (n-1, n), n \in \mathbb{N}^+$*. Defining the shifted Grünwald type difference operator*

$$A_{h,p}^{\alpha,\lambda}u(x) := \frac{1}{h^{\alpha}}\sum_{k=0}^{+\infty} w_k^{(\alpha)}e^{-(k-p)h\lambda}u(x-(k-p)h) - \frac{1}{h^{\alpha}}\left(e^{ph\lambda}(1-e^{-h\lambda})^{\alpha}\right)u(x), \tag{2.9}$$

then

$$A_{h,p}^{\alpha,\lambda}u(x) = \,_{-\infty}D_x^{\alpha,\lambda}u(x) - \lambda^{\alpha}u(x) + O(h), \tag{2.10}$$

where $w_k^{(\alpha)} = (-1)^k\binom{\alpha}{k},\; k \geq 0$ *denotes the normalized Grünwald weights.*

Remark 3 The point $x+(p-\alpha/2)h$ is the superconvergent point of the approximation $A_{h,p}^{\alpha,\lambda}$ to ${}_{-\infty}D_y^{\alpha,\lambda} - \lambda^\alpha$, i.e., $A_{h,p}^{\alpha,\lambda}u(x) = {}_{-\infty}D_y^{\alpha,\lambda}u(y) - \lambda^\alpha u(y) + O(h^2)$ with $y = x + (p-\alpha/2)h$ (the deriving process is similar to the one given in [24]).

Remark 4 Under the assumption given in Lemma 2, for tempered fractional derivatives defined in Eq. 2.1, we have [1]

$$A_{h,p}^{\alpha,\lambda}u(x) = \begin{cases} {}_{-\infty}\mathbf{D}_x^{\alpha,\lambda}u(x) + O(h), & 0 < \alpha < 1, \\ {}_{-\infty}\mathbf{D}_x^{\alpha,\lambda}u(x) + \alpha\lambda^{\alpha-1}\partial_x u(x) + O(h), & 1 < \alpha < 2. \end{cases} \tag{2.11}$$

Proof The proof is similar to the one given in [1]. Taking Fourier transform on both sides of Eq. 2.9, we obtain

$$\begin{aligned} \mathcal{F}[A_{h,p}^{\alpha,\lambda}u](\omega) &= \frac{1}{h^\alpha}\sum_{k=0}^{+\infty} w_k^{(\alpha)} e^{-(k-p)h(\lambda+i\omega)}\hat{u}(\omega) - \frac{1}{h^\alpha}\left(e^{ph\lambda}(1-e^{-h\lambda})^\alpha\right)\hat{u}(\omega) \\ &= e^{ph(\lambda+i\omega)}\left(\frac{1-e^{-h(\lambda+i\omega)}}{h}\right)^\alpha \hat{u}(\omega) - e^{ph\lambda}\left(\frac{1-e^{-h\lambda}}{h}\right)^\alpha \hat{u}(\omega) \\ &= \left[(\lambda+i\omega)^\alpha P_h(\lambda+i\omega) - \lambda^\alpha P_h(\lambda)\right]\hat{u}(\omega), \end{aligned} \tag{2.12}$$

where

$$P_h(z) = e^{phz}\left(\frac{1-e^{-hz}}{hz}\right)^\alpha = 1+(p-\frac{\alpha}{2})hz+O(|z|^2), \text{ with } z = \lambda+i\omega \text{ or } \lambda,\ i^2 = -1. \tag{2.13}$$

Denoting

$$\begin{aligned} \hat{\phi}(\omega,h) &= \mathcal{F}[A_{h,p}^{\alpha,\lambda}u](\omega) - \mathcal{F}[{}_{-\infty}D_x^{\alpha,\lambda}u - \lambda^\alpha u](\omega) \\ &= \left[(\lambda+i\omega)^\alpha\left(P_h(\lambda+i\omega)-1\right) - \lambda^\alpha\left(P_h(\lambda)-1\right)\right]\hat{u}(\omega), \end{aligned}$$

from Eqs. 2.12 and 2.5 there exists

$$|\hat{\phi}(\omega,h)| \le C\left[h|(\lambda+i\omega)|^{\alpha+1} + h|\lambda|^{\alpha+1}\right]|\hat{u}(\omega)|.$$

With the condition $\mathcal{F}[{}_{-\infty}D_x^{\alpha+1,\lambda}u](k) \in L^1(\mathbb{R})$, and using the Riemann-Lebesgue Lemma, it yields

$$\begin{aligned} |A_{h,p}^{\alpha,\lambda}u(x) - {}_{-\infty}D_x^{\alpha,\lambda}u(x) + \lambda^\alpha u(x)| &= |\phi| \le \frac{1}{2\pi}\int_{\mathbb{R}}|\hat{\phi}(\omega,h)|d\omega \\ &\le C\|\mathcal{F}[{}_{-\infty}D_x^{\alpha+1,\lambda}u + \lambda^{\alpha+1}u(x)](\omega)\|_{L^1}h = O(h), \end{aligned}$$

where the property of the Fourier transforms for the left Riemann-Liouville tempered fractional derivatives (2.5) is used. □

Lemma 3 *Let $u(x) \in L^1(\mathbb{R})$, ${}_xD_{+\infty}^{\alpha+1,\lambda}u$ and its Fourier transform belong to $L^1(\mathbb{R})$; $p \in \mathbb{R}$, $h > 0$, $\lambda \ge 0$ and $\alpha \in (n-1,n), n \in \mathbb{N}^+$. Define the tempered shifted*

Grünwald type difference operator

$$B_{h,p}^{\alpha,\lambda}u(x) := \frac{1}{h^\alpha}\sum_{k=0}^{+\infty} w_k^{(\alpha)} e^{-(k-p)h\lambda}u(x+(k-p)h) - \frac{1}{h^\alpha}\left(e^{ph\lambda}(1-e^{-h\lambda})^\alpha\right)u(x). \tag{2.14}$$

Then

$$B_{h,p}^{\alpha,\lambda}u(x) = {}_xD_{+\infty}^{\alpha,\lambda}u(x) - \lambda^\alpha u(x) + O(h). \tag{2.15}$$

Remark 5 The point $x-(p-\alpha/2)h$ is the superconvergent point of the approximation $B_{h,p}^{\alpha,\lambda}$ to ${}_yD_\infty^{\alpha,\lambda} - \lambda^\alpha$, i.e., $B_{h,p}^{\alpha,\lambda}u(x) = {}_yD_\infty^{\alpha,\lambda}u(y) - \lambda^\alpha u(y) + O(h^2)$ with $y = x-(p-\alpha/2)h$ (the deriving process is similar to the one given in [24]).

Remark 6 Under the assumption given in Lemma 3, for tempered fractional derivatives defined in Eq. 2.2, we have

$$B_{h,q}^{\alpha,\lambda}u(x) = \begin{cases} {}_x\mathbf{D}_{+\infty}^{\alpha,\lambda}u(x) + O(h), & 0<\alpha<1, \\ {}_x\mathbf{D}_{+\infty}^{\alpha,\lambda}u(x) - \alpha\lambda^{\alpha-1}\partial_x u(x) + O(h), & 1<\alpha<2. \end{cases} \tag{2.16}$$

Proof Taking Fourier transform on both sides of Eq. 2.14, we obtain

$$\begin{aligned}\mathcal{F}[B_{h,p}^{\alpha,\lambda}u](\omega) &= \frac{1}{h^\alpha}\sum_{k=0}^{+\infty} w_k^{(\alpha)} e^{-(k-p)h(\lambda-i\omega)}\hat{u}(\omega) - \frac{1}{h^\alpha}\left(e^{ph\lambda}(1-e^{-h\lambda})^\alpha\right)\hat{u}(\omega) \\ &= e^{ph(\lambda-i\omega)}\left(\frac{1-e^{-h(\lambda-i\omega)}}{h}\right)^\alpha \hat{u}(\omega) - e^{ph\lambda}\left(\frac{1-e^{-h\lambda}}{h}\right)^\alpha \hat{u}(\omega) \\ &= \left[(\lambda-i\omega)^\alpha P_h(\lambda-i\omega) - \lambda^\alpha P_h(\lambda)\right]\hat{u}(\omega),\end{aligned}$$

where $P_h(z)$ is defined by Eq. 2.13 with $z = \lambda - i\omega$ or λ. Denoting $\hat{\phi}(\omega,h) = \mathcal{F}[B_{h,p}^{\alpha,\lambda}u](\omega) - \mathcal{F}[{}_xD_\infty^{\alpha,\lambda}u - \lambda^\alpha u](\omega)$, then with the similar method used in the proof of Lemma 2, and using the Fourier transform of the right Riemann-Liouville tempered fractional derivative (2.6), we obtain

$$\begin{aligned}|B_{h,p}^{\alpha,\lambda}u(x) - {}_xD_{+\infty}^{\alpha,\lambda}u(x) + \lambda^\alpha u(x)| &= |\phi| \le \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{\phi}(\omega,h)|d\omega \\ &\le C\|\mathcal{F}[{}_xD_{+\infty}^{\alpha+1,\lambda}u + \lambda^{\alpha+1}u(x)](\omega)\|_{L^1}h = O(h).\end{aligned}$$

□

The approximation accuracy of the classic difference operator can be improved by adding the band of discretization stencils [10]. And then the computational cost increases accordingly. However, because of the nonlocal property of the fractional operator, even for the first order discretizations, the stencil covers the whole interval. Without introducing new computational cost, we can improve the approximation accuracy of the discretized fractional operators by modifying the Grünwald type weights. The improved discretized tempered fractional operators are called tempered weighted and shifted Grünwald difference (tempered-WSGD) operators.

Theorem 1 *Let* $u(x) \in L^1(\mathbb{R})$, ${}_{-\infty}D_x^{\alpha+\ell,\lambda}u$ *and its Fourier transform belong to* $L^1(\mathbb{R})$*; and define the left tempered-WSGD operator by*

$$ {}_L\mathcal{D}_{h,p_1,p_2,\ldots,p_m}^{\alpha,\gamma_1,\gamma_2,\ldots,\gamma_m}u(x) = \sum_{j=1}^{m}\gamma_j A_{h,p_j}^{\alpha,\lambda}u(x), \tag{2.17}$$

where p_j *and* γ_j *are determined by Eqs. 2.21–2.24. Then, for any integer* $m \geq \ell$*, there exists*

$$ {}_L\mathcal{D}_{h,p_1,p_2,\ldots,p_m}^{\alpha,\gamma_1,\gamma_2,\ldots,\gamma_m}u(x) = {}_{-\infty}D_x^{\alpha,\lambda}u(x) - \lambda^\alpha u(x) + O(h^\ell), \tag{2.18}$$

uniformly for $x \in \mathbb{R}$.

Let $u(x) \in L^1(\mathbb{R})$, ${}_xD_\infty^{\alpha+\ell,\lambda}u$ *and its Fourier transform belong to* $L^1(\mathbb{R})$*; and define the right tempered-WSGD operator by*

$$ {}_R\mathcal{D}_{h,p_1,p_2,\ldots,p_m}^{\alpha,\gamma_1,\gamma_2,\ldots,\gamma_m}u(x) = \sum_{j=1}^{m}\gamma_j B_{h,p_j}^{\alpha,\lambda}u(x), \tag{2.19}$$

where p_j *and* γ_j *are determined by Eqs. 2.21–2.24. Then, for any integer* $m \geq \ell$*, there is*

$$ {}_R\mathcal{D}_{h,p_1,p_2,\ldots,p_m}^{\alpha,\gamma_1,\gamma_2,\ldots,\gamma_m}u(x) = {}_xD_{+\infty}^{\alpha,\lambda}u(x) - \lambda^\alpha u(x) + O(h^\ell), \tag{2.20}$$

uniformly for $x \in \mathbb{R}$.

For $\ell = 2$, p_j, γ_j *are real numbers and satisfy the linear system*

$$\begin{cases} \sum_{j=1}^m \gamma_j = 1, \\ \sum_{j=1}^m \gamma_j \left[p_j - \frac{\alpha}{2}\right] = 0. \end{cases} \tag{2.21}$$

For $\ell = 3$, p_j, γ_j *are real numbers and satisfy*

$$\begin{cases} \sum_{j=1}^m \gamma_j = 1, \\ \sum_{j=1}^m \gamma_j \left[p_j - \frac{\alpha}{2}\right] = 0, \\ \sum_{j=1}^m \gamma_j \left[\frac{p_j^2}{2} - \frac{\alpha p_j}{2} + \frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8}\right] = 0. \end{cases} \tag{2.22}$$

For $\ell = 4$, p_j, γ_j *are real numbers and the following hold*

$$\begin{cases} \sum_{j=1}^m \gamma_j = 1, \\ \sum_{j=1}^m \gamma_j \left[p_j - \frac{\alpha}{2}\right] = 0, \\ \sum_{j=1}^m \gamma_j \left[\frac{p_j^2}{2} - \frac{\alpha p_j}{2} + \frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8}\right] = 0, \\ \sum_{j=1}^m \gamma_j \left[\frac{p_j^3}{6} - \frac{\alpha p_j^2}{4} + \left(\frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8}\right)p_j - \frac{\alpha}{24} - \frac{\alpha(\alpha-1)}{12} - \frac{\alpha(\alpha-1)(\alpha-2)}{48}\right] = 0. \end{cases} \tag{2.23}$$

For $\ell = 5$, p_j, γ_j are real numbers and the following hold

$$\begin{cases} \sum_{j=1}^m \gamma_j = 1, \\ \sum_{j=1}^m \gamma_j \left[p_j - \frac{\alpha}{2}\right] = 0, \\ \sum_{j=1}^m \gamma_j \left[\frac{p_j^2}{2} - \frac{\alpha p_j}{2} + \frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8}\right] = 0, \\ \sum_{j=1}^m \gamma_j \left[\frac{p_j^3}{6} - \frac{\alpha p_j^2}{4} + \left(\frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8}\right) p_j - \frac{\alpha}{24} - \frac{\alpha(\alpha-1)}{12} - \frac{\alpha(\alpha-1)(\alpha-2)}{48}\right] = 0, \\ \sum_{j=1}^m \gamma_j \left[\frac{p_j^4}{24} - \frac{\alpha p_j^3}{4} + \frac{1}{2}\left(\frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8}\right) p_j^2 + \left(-\frac{\alpha}{24} - \frac{\alpha(\alpha-1)}{12} - \frac{\alpha(\alpha-1)(\alpha-2)}{48}\right) p_j \right. \\ \left. + \frac{\alpha}{120} + \frac{5\alpha(\alpha-1)}{144} + \frac{\alpha(\alpha-1)(\alpha-2)}{48} + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{384}\right] = 0. \end{cases} \tag{2.24}$$

Proof The standard Fourier transforms are again used here. Performing the Fourier transform on the left hand of Eq. 2.17, we obtain

$$\begin{aligned} \mathcal{F}\left[\,{}_L\mathcal{D}_{h,p_1,p_2,\ldots,p_m}^{\alpha,\gamma_1,\gamma_2,\ldots,\gamma_m} u(x)\right](\omega) &= \sum_{j=1}^m \gamma_j \left(\frac{1}{h^\alpha}\sum_{k=0}^\infty w_k^{(\alpha)} e^{-(k-p_j)h(\lambda+i\omega)}\hat{u}(\omega)\right. \\ &\qquad \left. -\frac{1}{h^\alpha}\left(e^{p_j h\lambda}(1-e^{-h\lambda})^\alpha\right)\hat{u}(\omega)\right) \\ &= \sum_{j=1}^m \left[(\lambda+i\omega)^\alpha P_{h,j}(\lambda+i\omega) - \lambda^\alpha P_{h,j}(\lambda)\right]\hat{u}(\omega)\gamma_j, \end{aligned} \tag{2.25}$$

where $P_{h,j}(z) = e^{p_j hz}\left(\frac{1-e^{-hz}}{hz}\right)^\alpha$, $z = \lambda + i\omega$ or λ, $i = \sqrt{-1}$. By a simple Taylor's expansion, we get

$$\begin{aligned} e^{p_j hz}\left(\frac{1-e^{-hz}}{hz}\right)^\alpha &= 1 + \left[p_j - \frac{\alpha}{2}\right]hz + \left[\frac{p_j^2}{2} - \frac{\alpha p_j}{2} + \frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8}\right](hz)^2 \\ &\quad + \left[\frac{p_j^3}{6} - \frac{\alpha p_j^2}{4} + \left(\frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8}\right)p_j - \frac{\alpha}{24} - \frac{\alpha(\alpha-1)}{12} - \frac{\alpha(\alpha-1)(\alpha-2)}{48}\right](hz)^3 \\ &\quad + \left[\frac{p_j^4}{24} - \frac{\alpha p_j^3}{4} + \frac{1}{2}\left(\frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8}\right)p_j^2 + \left(-\frac{\alpha}{24} - \frac{\alpha(\alpha-1)}{12} - \frac{\alpha(\alpha-1)(\alpha-2)}{48}\right)p_j \right. \\ &\quad \left. + \frac{\alpha}{120} + \frac{5\alpha(\alpha-1)}{144} + \frac{\alpha(\alpha-1)(\alpha-2)}{48} + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{384}\right](hz)^4 \\ &\quad + O(|zh|^5). \end{aligned} \tag{2.26}$$

Denoting $\mathcal{F}[{}_L\mathcal{D}_{h,p_1,p_2,\ldots,p_m}^{\alpha,\gamma_1,\gamma_2,\ldots,\gamma_m} u(x)](\omega) = \mathcal{F}[{}_{-\infty}D_x^{\alpha,\lambda}u - \lambda^\alpha u](\omega) + \hat{\phi}(\omega, h)$, in view of Eqs. 2.26, 2.5, and 2.21–2.24, we have

$$|\hat{\phi}(\omega, h)| \le Ch^l\left[|\lambda + i\omega|^{\alpha+\ell} + |\lambda|^{\alpha+\ell}\right]|\hat{u}(\omega)|. \tag{2.27}$$

Due to $\mathcal{F}[\,_{-\infty}D_x^{\alpha+\ell,\lambda}u](\omega) \in L^1(\mathbb{R})$, there exists

$$| \,_L\mathcal{D}_{h,p_1,p_2,\ldots,p_m}^{\alpha,\gamma_1,\gamma_2,\ldots,\gamma_m} u - \,_{-\infty}D_x^{\alpha,\lambda}u + \lambda^\alpha u| = |\phi| \le \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{\phi}(\omega,h)|d\omega$$
$$\le C\|\mathcal{F}[_{-\infty}D_x^{\alpha+\ell,\lambda}u + \lambda^{\alpha+\ell}u](\omega)\|_{L^1}h^\ell = O(h^\ell).$$

By the similar arguments we can prove (2.20). □

Remark 7 Under the assumptions given by Theorem 1, for the tempered fractional derivatives defined in Eqs. 2.1 and 2.2, we deduce that

$$\,_L\mathcal{D}_{h,p_1,p_2,\ldots,p_m}^{\alpha,\gamma_1,\gamma_2,\ldots,\gamma_m} u(x) = \begin{cases} \,_{-\infty}\mathbf{D}_x^{\alpha,\lambda}u(x) + O(h^\ell), & 0 < \alpha < 1, \\ \,_{-\infty}\mathbf{D}_x^{\alpha,\lambda}u(x) + \alpha\lambda^{\alpha-1}\partial_x u(x) + O(h^\ell), & 1 < \alpha < 2; \end{cases} \tag{2.28}$$

and

$$\,_R\mathcal{D}_{h,p_1,p_2,\ldots,p_m}^{\alpha,\gamma_1,\gamma_2,\ldots,\gamma_m} u(x) = \begin{cases} \,_x\mathbf{D}_{+\infty}^{\alpha,\lambda}u(x) + O(h^\ell), & 0 < \alpha < 1, \\ \,_x\mathbf{D}_{+\infty}^{\alpha,\lambda}u(x) - \alpha\lambda^{\alpha-1}\partial_x u(x) + O(h^\ell), & 1 < \alpha < 2. \end{cases} \tag{2.29}$$

Remark 8 If $u(x) \in L^1(\mathbb{R})$, $\,_{-\infty}D_x^{\alpha+1,\lambda}u$ and its Fourier transform belong to $L^1(\mathbb{R})$; $p \in \mathbb{R}, h > 0, \lambda \ge 0$ and $\alpha \in (n-1,n), n \in \mathbb{N}^+$. Defining the shifted Grünwald type difference operator

$$\tilde{A}_{h,p}^{\alpha,\lambda}u(x) := \frac{1}{h^\alpha}\sum_{k=0}^{+\infty} w_k^{(\alpha)} e^{-(k-p)h\lambda} u(x-(k-p)h), \tag{2.30}$$

then

$$\tilde{A}_{h,p}^{\alpha,\lambda}u(x) = \,_{-\infty}D_x^{\alpha,\lambda}u(x) + O(h), \tag{2.31}$$

where $w_k^{(\alpha)} = (-1)^k\binom{\alpha}{k}$, $k \ge 0$ denotes the normalized Grünwald weights.

If $u(x) \in L^1(\mathbb{R})$, $\,_xD_{+\infty}^{\alpha+1,\lambda}u$ and its Fourier transform belong to $L^1(\mathbb{R})$; $p \in \mathbb{R}$, $h > 0$, $\lambda \ge 0$ and $\alpha \in (n-1,n), n \in \mathbb{N}^+$. Define the shifted Grünwald type difference operator

$$\tilde{B}_{h,p}^{\alpha,\lambda}u(x) := \frac{1}{h^\alpha}\sum_{k=0}^{+\infty} w_k^{(\alpha)} e^{-(k-p)h\lambda} u(x+(k-p)h). \tag{2.32}$$

Then

$$\tilde{B}_{h,p}^{\alpha,\lambda}u(x) = \,_xD_{+\infty}^{\alpha,\lambda}u(x) + O(h). \tag{2.33}$$

Moreover, if $u(x) \in L^1(\mathbb{R})$, $\,_{-\infty}D_x^{\alpha+\ell,\lambda}u$ and its Fourier transform belong to $L^1(\mathbb{R})$; and define the left tempered-WSGD operator by

$$\,_L\tilde{\mathcal{D}}_{h,p_1,p_2,\ldots,p_m}^{\alpha,\gamma_1,\gamma_2,\ldots,\gamma_m} u(x) = \sum_{j=1}^{m} \gamma_j \tilde{A}_{h,p_j}^{\alpha,\lambda}u(x), \tag{2.34}$$

where p_j and γ_j are determined by Eq. 2.21–2.24. Then, for any integer $m \ge \ell$, there exists

$$\,_L\tilde{\mathcal{D}}_{h,p_1,p_2,\ldots,p_m}^{\alpha,\gamma_1,\gamma_2,\ldots,\gamma_m} u(x) = \,_{-\infty}D_x^{\alpha,\lambda}u(x) + O(h^\ell), \tag{2.35}$$

uniformly for $x \in \mathbb{R}$.

If $u(x) \in L^1(\mathbb{R})$, ${}_xD_{\infty}^{\alpha+\ell,\lambda}u$ and its Fourier transform belong to $L^1(\mathbb{R})$; and define the right tempered-WSGD operator by

$$
{}_R\tilde{\mathcal{D}}_{h,p_1,p_2,\ldots,p_m}^{\alpha,\gamma_1,\gamma_2,\ldots,\gamma_m}u(x) = \sum_{j=1}^{m}\gamma_j \tilde{B}_{h,p_j}^{\alpha,\lambda}u(x), \tag{2.36}
$$

where p_j and γ_j are determined by Eq. 2.21–2.24. Then, for any integer $m \geq \ell$, there is

$$
{}_R\tilde{\mathcal{D}}_{h,p_1,p_2,\ldots,p_m}^{\alpha,\gamma_1,\gamma_2,\ldots,\gamma_m}u(x) = {}_xD_{+\infty}^{\alpha,\lambda}u(x) + O(h^{\ell}), \tag{2.37}
$$

uniformly for $x \in \mathbb{R}$.

Remark 9 To get the discretizations, including the first and high orders, of the left and right Riemann-Liouville tempered fractional integrals of order $\sigma > 0$: ${}_{-\infty}D_x^{-\sigma,\lambda}u(x)$ and ${}_xD_{\infty}^{-\sigma,\lambda}u(x)$, just use $-\sigma$ to replace α existing in the corresponding discretizations of the left and right Riemann-Liouville tempered fractional derivatives of order $\alpha > 0$: ${}_{-\infty}D_x^{\alpha,\lambda}u(x)$ and ${}_xD_{+\infty}^{\alpha,\lambda}u(x)$.

Considering a well-defined function $u(x)$ on the bounded interval $[a, b]$, the function $u(x)$ can be zero extended for $x < a$ or $x > b$. Then the α-th order left and right Riemann-Liouville tempered fractional derivatives of $u(x)$ at point x can be approximated by the tempered-WSGD operators

$$
\begin{aligned}
{}_aD_x^{\alpha,\lambda}u(x) - \lambda^{\alpha}u(x) &= \sum_{j=1}^{m}\frac{\gamma_j}{h^{\alpha}}\left(\sum_{k=0}^{[\frac{x-a}{h}]+p_j} w_k^{(\alpha)}e^{-(k-p_j)h\lambda}u(x-(k-p_j)h) - \left(e^{p_jh\lambda}(1-e^{-h\lambda})^{\alpha}\right)u(x)\right)\\
&\quad + O(h^{\ell});\\
{}_xD_b^{\alpha,\lambda}u(x) - \lambda^{\alpha}u(x) &= \sum_{j=1}^{m}\frac{\gamma_j}{h^{\alpha}}\left(\sum_{k=0}^{[\frac{b-x}{h}]+p_j} w_k^{(\alpha)}e^{-(k-p_j)h\lambda}u(x+(k-p_j)h) - \left(e^{p_jh\lambda}(1-e^{-h\lambda})^{\alpha}\right)u(x)\right)\\
&\quad + O(h^{\ell}),
\end{aligned} \tag{2.38}
$$

and the σ-th order left and right Riemann-Liouville tempered fractional integrals of $u(x)$ at point x can be approximated by the tempered-WSGD operators

$$
\begin{aligned}
{}_aD_x^{-\sigma,\lambda}u(x) &= \sum_{j=1}^{m}\gamma_j h^{\sigma}\left(\sum_{k=0}^{[\frac{x-a}{h}]+p_j} w_k^{(-\sigma)}e^{-(k-p_j)h\lambda}u(x-(k-p_j)h)\right) + O(h^{\ell}),\\
{}_xD_b^{-\sigma,\lambda}u(x) &= \sum_{j=1}^{m}\gamma_j h^{\sigma}\left(\sum_{k=0}^{[\frac{b-x}{h}]+p_j} w_k^{(-\sigma)}e^{-(k-p_j)h\lambda}u(x+(k-p_j)h)\right) + O(h^{\ell}),
\end{aligned} \tag{2.39}
$$

where the weight parameters γ_j are determined by the above linear algebraic systems

given in Theorem 1.

Remark 10 The parameters $[(x-a)/h]+p_j$ are the numbers of the points located on the right/left hand of the point x used for evaluating the α-th (or σ-th) order left/right Riemann-Liouville tempered fractional derivatives (or integrals) at x; thus, when employing the discretizations (2.38) (or Eq. 2.39) for approximating non-periodic boundary problems on bounded interval, p_j should be chosen satisfying $|p_j| \leq 1$ to ensure that the nodes at which the values of u are needed in Eq. 2.38 (or Eq. 2.39) are within the bounded interval; otherwise, we need to use another methodology to discretize the tempered fractional derivative when x is close to the right/left boundary just like classic ones [10].

It is easy to check that any one of the linear systems (2.21–2.24) with $m = \ell$ has an unique solution. And for $m > l$, using the knowledge of linear algebra, we know that the system (2.21–2.24) has infinitely many solutions. As we have discussed in Theorem 1, in principle the arbitrarily high order difference approximations can be obtained. For computational purposes, we are more interested in the schemes with $|p_j| \leq 1$. And for the easy of presentation but without loss of the generality, in the following sections, we focus on the second order difference approximations ($l = 2$) of Eq. 2.38 with three to be determined weights γ_j, $j = 1, 2, 3$ ($m = 3$), i.e.,

$$\begin{aligned} {}_L\mathcal{D}_{h,1,0,-1}^{\alpha,\gamma_1,\gamma_2,\gamma_3}u(x) := & \frac{\gamma_1}{h^\alpha}\sum_{k=0}^{[\frac{x-a}{h}]+1} w_k^{(\alpha)} e^{-(k-1)h\lambda} u(x-(k-1)h) + \frac{\gamma_2}{h^\alpha}\sum_{k=0}^{[\frac{x-a}{h}]} w_k^{(\alpha)} e^{-kh\lambda} u(x-kh) \\ & + \frac{\gamma_3}{h^\alpha}\sum_{k=0}^{[\frac{x-a}{h}]-1} w_k^{(\alpha)} e^{-(k+1)h\lambda} u(x-(k+1)h) \\ & - \frac{1}{h^\alpha}\left((\gamma_1 e^{h\lambda} + \gamma_2 + \gamma_3 e^{-h\lambda})(1-e^{-h\lambda})^\alpha\right) u(x); \end{aligned} \tag{2.40}$$

and

$$\begin{aligned} {}_R\mathcal{D}_{h,1,0,-1}^{\alpha,\gamma_1,\gamma_2,\gamma_3}u(x) := & \frac{\gamma_1}{h^\alpha}\sum_{k=0}^{[\frac{b-x}{h}]+1} w_k^{(\alpha)} e^{-(k-1)h\lambda} u(x+(k-1)h) \\ & + \frac{\gamma_2}{h^\alpha}\sum_{k=0}^{[\frac{b-x}{h}]} w_k^{(\alpha)} e^{-kh\lambda} u(x+kh) + \frac{\gamma_3}{h^\alpha}\sum_{k=0}^{[\frac{b-x}{h}]-1} w_k^{(\alpha)} e^{-(k+1)h\lambda} u(x+(k+1)h) \\ & - \frac{1}{h^\alpha}\left((\gamma_1 e^{h\lambda} + \gamma_2 + \gamma_3 e^{-h\lambda})(1-e^{-h\lambda})^\alpha\right) u(x), \end{aligned} \tag{2.41}$$

where the parameters γ_j, $j = 1, 2, 3$, satisfy the following linear system

$$\begin{cases} \gamma_1 + \gamma_2 + \gamma_3 = 1, \\ \gamma_1 - \gamma_3 = \frac{\alpha}{2}. \end{cases} \tag{2.42}$$

The system (2.42) has infinitely many solutions. With the help of the knowledge of linear algebra, the solutions of the system of linear algebraic Eq. 2.42 can be collected by the following three sets

$$\mathcal{S}_1^{\alpha}(\gamma_1,\gamma_2,\gamma_3)=\left\{\gamma_1 \text{ is given}, \gamma_2=\frac{2+\alpha}{2}-2\gamma_1,\ \gamma_3=\gamma_1-\frac{\alpha}{2}\right\}; \tag{2.43}$$

or

$$\mathcal{S}_2^{\alpha}(\gamma_1,\gamma_2,\gamma_3)=\left\{\gamma_1=\frac{2+\alpha}{4}-\frac{\gamma_2}{2},\ \gamma_2 \text{ is given},\ \gamma_3=\frac{2-\alpha}{4}-\frac{\gamma_2}{2}\right\}; \tag{2.44}$$

or

$$\mathcal{S}_3^{\alpha}(\gamma_1,\gamma_2,\gamma_3)=\left\{\gamma_1=\frac{\alpha}{2}+\gamma_3,\ \gamma_2=\frac{2-\alpha}{2}-2\gamma_3,\ \gamma_3 \text{ is given}\right\}. \tag{2.45}$$

The parameter values presented in the sets $\mathcal{S}_j^{\alpha}$, $j=1,2,3$ produce infinite number of second order approximations for the Riemann-Liouville tempered fractional derivative. Particularly, if taking $\lambda=0$ and $\gamma_j=0$ in $\mathcal{S}_j^{\alpha}$, $j=1,2,3$, they recover the second order approximations presented in [31] for the Riemann-Liouville fractional derivative. After rearranging the weights $w_k^{(\alpha)}$, the Riemann-Liouville tempered fractional derivatives at point x_j are approximated as

$$\begin{aligned}
{}_aD_x^{\alpha,\lambda}u(x_j)-\alpha\lambda^{\alpha}u(x_j) &= \frac{1}{h^{\alpha}}\sum_{k=0}^{j+1}g_k^{(\alpha)}u(x_{j-k+1})-\frac{1}{h^{\alpha}}\left((\gamma_1e^{h\lambda}+\gamma_2+\gamma_3e^{-h\lambda})(1-e^{-h\lambda})^{\alpha}\right)u(x_j)\\
&\quad+O(h^2),\\
{}_xD_b^{\alpha,\lambda}u(x_j)-\alpha\lambda^{\alpha}u(x_j) &= \frac{1}{h^{\alpha}}\sum_{k=0}^{N-j+1}g_k^{(\alpha)}u(x_{j+k-1})-\frac{1}{h^{\alpha}}\left((\gamma_1e^{h\lambda}+\gamma_2+\gamma_3e^{-h\lambda})(1-e^{-h\lambda})^{\alpha}\right)u(x_j)\\
&\quad+O(h^2),
\end{aligned} \tag{2.46}$$

where the weights are given as

$$\begin{aligned}
g_0^{(\alpha)}&=\gamma_1w_0^{(\alpha)}e^{h\lambda},\ g_1^{(\alpha)}=\gamma_1w_1^{(\alpha)}+\gamma_2w_0^{(\alpha)},\\
g_k^{(\alpha)}&=\left(\gamma_1w_k^{(\alpha)}+\gamma_2w_{k-1}^{(\alpha)}+\gamma_3w_{k-2}^{(\alpha)}\right)e^{-(k-1)h\lambda},\ k\ge 2.
\end{aligned} \tag{2.47}$$

Remark 11 Similarly, for the Riemann-Liouville tempered fractional derivatives defined in Definition 3, we have the second order difference approximations,

$$\begin{aligned}
{}_L\tilde{\mathcal{D}}_{h,1,0,-1}^{\alpha,\gamma_1,\gamma_2,\gamma_3}u(x) &= \frac{\gamma_1}{h^{\alpha}}\sum_{k=0}^{[\frac{x-a}{h}]+1}w_k^{(\alpha)}e^{-(k-1)h\lambda}u(x-(k-1)h)\\
&\quad+\frac{\gamma_2}{h^{\alpha}}\sum_{k=0}^{[\frac{x-a}{h}]}w_k^{(\alpha)}e^{-kh\lambda}u(x-kh)\\
&\quad+\frac{\gamma_3}{h^{\alpha}}\sum_{k=0}^{[\frac{x-a}{h}]-1}w_k^{(\alpha)}e^{-(k+1)h\lambda}u(x-(k+1)h);
\end{aligned} \tag{2.48}$$

and

$$
\begin{aligned}
{}_R\tilde{\mathcal{D}}^{\alpha,\gamma_1,\gamma_2,\gamma_3}_{h,1,0,-1}u(x) = & \frac{\gamma_1}{h^\alpha}\sum_{k=0}^{[\frac{b-x}{h}]+1} w_k^{(\alpha)}e^{-(k-1)h\lambda}u(x+(k-1)h) \\
& +\frac{\gamma_2}{h^\alpha}\sum_{k=0}^{[\frac{b-x}{h}]} w_k^{(\alpha)}e^{-kh\lambda}u(x+kh) \\
& +\frac{\gamma_3}{h^\alpha}\sum_{k=0}^{[\frac{b-x}{h}]-1} w_k^{(\alpha)}e^{-(k+1)h\lambda}u(x+(k+1)h). \quad (2.49)
\end{aligned}
$$

After rearranging the weights $w_k^{(\alpha)}$, the Riemann-Liouville tempered fractional derivatives at point x_j are approximated as

$$
\begin{aligned}
{}_aD_x^{\alpha,\lambda}u(x_j) &= \frac{1}{h^\alpha}\sum_{k=0}^{j+1} g_k^{(\alpha)}u(x_{j-k+1}) + O(h^2), \\
{}_xD_b^{\alpha,\lambda}u(x_j) &= \frac{1}{h^\alpha}\sum_{k=0}^{N-j+1} g_k^{(\alpha)}u(x_{j+k-1}) + O(h^2), \quad (2.50)
\end{aligned}
$$

where $g_k^{(\alpha)}$ is given in Eq. 2.47.

Lemma 4 *The weights appeared in Eq.* 2.47 *with* $1<\alpha<2$ *satisfy*

(1). $w_0^{(\alpha)}=1,\ w_1^{(\alpha)}=-\alpha<0,\ w_k^{(\alpha)}=\left(1-\frac{1+\alpha}{k}\right)w_{k-1}^{(\alpha)}\ (k\geq 1);\ 1\geq w_2^{(\alpha)}\geq w_3^{(\alpha)}\geq\ldots\geq 0,\ \sum_{k=0}^{\infty}w_k^{(\alpha)}=0,\ \sum_{k=0}^{m}w_k^{(\alpha)}<0\ (m\geq 1)$;

(2). *For* $h>0,\ \lambda\geq 0$, *if* γ_2,γ_1 *and* γ_3 *are chosen in set* $\mathcal{S}_2^\alpha(\gamma_1,\gamma_2,\gamma_3)$ *with* $\frac{(\alpha-4)(\alpha^2+3\alpha+2)+24}{2(\alpha^2+3\alpha+2)}\leq\gamma_2\leq\min\left\{\frac{(\alpha-2)(\alpha^2+3\alpha+4)+16}{2(\alpha^2+3\alpha+4)},\frac{(\alpha-6)(\alpha^2+3\alpha+2)+48}{2(\alpha^2+3\alpha+2)}\right\}$, *or set* $\mathcal{S}_1^\alpha(\gamma_1,\gamma_2,\gamma_3)$ *with* $\max\left\{\frac{2(\alpha^2+3\alpha-4)}{\alpha^2+3\alpha+2},\frac{\alpha^2+3\alpha}{\alpha^2+3\alpha+4}\right\}\leq\gamma_1\leq\frac{3(\alpha^2+3\alpha-2)}{2(\alpha^2+3\alpha+2)}$, *or set* $\mathcal{S}_3^\alpha(\gamma_1,\gamma_2,\gamma_3)$ *with* $\max\left\{\frac{(2-\alpha)(\alpha^2+\alpha-8)}{\alpha^2+3\alpha+2},\frac{(1-\alpha)(\alpha^2+2\alpha)}{2(\alpha^2+3\alpha+4)}\right\}\leq\gamma_3\leq\frac{(2-\alpha)(\alpha^2+2\alpha-3)}{2(\alpha^2+3\alpha+2)}$, *then there exist*

$$
g_1^{(\alpha)}\leq 0,\ g_2^{(\alpha)}+g_0^{(\alpha)}\geq 0,\ g_k^{(\alpha)}\geq 0\ (k\geq 3). \quad (2.51)
$$

Proof For the proof of the first part of this lemma, one can see [21, 27]. For the second part of this lemma, we only prove the conclusion for γ_1,γ_2 and γ_3 selected in set $\mathcal{S}_1^\alpha(\gamma_1,\gamma_2,\gamma_3)$. The conclusions for γ_1,γ_2 and γ_3 selected in sets $\mathcal{S}_2^\alpha(\gamma_1,\gamma_2,\gamma_3)$ and $\mathcal{S}_3^\alpha(\gamma_1,\gamma_2,\gamma_3)$ can be proved in a similar manner. According to Eq. 2.42, we deduce that

$$\begin{aligned} g_1^{(\alpha)} &= -\alpha\gamma_1 + \gamma_2 \\ &= -(2+\alpha)\gamma_1 + \frac{2+\alpha}{2}. \end{aligned} \tag{2.52}$$

Obviously, $\gamma_1 \geq \frac{1}{2}$ implies $g_1^{(\alpha)} \leq 0$. Noting (2.42), we see that

$$\begin{aligned} g_2^{(\alpha)} + g_0^{(\alpha)} &= \left(\frac{\alpha^2-\alpha}{2}\gamma_1 - \alpha\gamma_2 + \gamma_3\right)e^{-h\lambda} + \gamma_1 e^{h\lambda} \\ &\geq \left(\frac{\alpha^2-\alpha+2}{2}\gamma_1 - \alpha\gamma_2 + \gamma_3\right)e^{-h\lambda} \\ &= \left(\frac{\alpha^2+3\alpha+4}{2}\gamma_1 - \frac{\alpha^2+3\alpha}{2}\right)e^{-h\lambda} \geq 0, \end{aligned} \tag{2.53}$$

if $\gamma_1 \geq \frac{\alpha^2+3\alpha}{\alpha^2+3\alpha+4}$. In view of Eq. 2.42, by a straightforward calculation, we obtain

$$\begin{aligned} g_3^{(\alpha)} &= \left(\frac{(2-\alpha)(\alpha-1)\alpha}{6}\gamma_1 + \frac{\alpha^2-\alpha}{2}\gamma_2 - \alpha\gamma_3\right)e^{-2h\lambda} \\ &= \left(-\frac{\alpha(\alpha^2+3\alpha+2)}{6}\gamma_1 + \frac{\alpha(\alpha^2+3\alpha-2)}{4}\right)e^{-2h\lambda} \geq 0, \end{aligned} \tag{2.54}$$

if $\gamma_1 \leq \frac{3(\alpha^2+3\alpha-2)}{2(\alpha^2+3\alpha+2)}$. More generally, for $k \geq 4$, using the recurrence relation of $w_k^{(\alpha)}$, we have

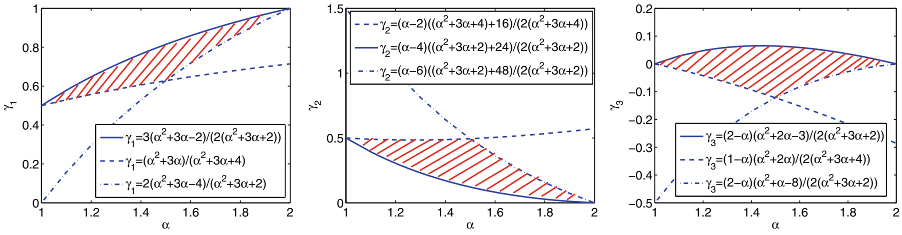
$$\begin{aligned} g_k^{(\alpha)} &= \left(\gamma_1 w_k^{(\alpha)} + \gamma_2 w_{k-1}^{(\alpha)} + \gamma_3 w_{k-2}^{(\alpha)}\right)e^{-(k-1)h\lambda} \\ &= \left(\frac{(k-1-\alpha)(k-2-\alpha)}{k(k-1)}\gamma_1 + \frac{k-2-\alpha}{k-1}\gamma_2 + \gamma_3\right)w_{k-2}^{(\alpha)}e^{-(k-1)h\lambda} \\ &= \left(\frac{\alpha^2+3\alpha+2}{k(k-1)}\gamma_1 + \frac{-\alpha^2-3\alpha+2k-4}{2(k-1)}\right)w_{k-2}^{(\alpha)}e^{-(k-1)h\lambda} \geq 0, \end{aligned} \tag{2.55}$$

if $\gamma_1 \geq \frac{k(\alpha^2+3\alpha+4-2k)}{2(\alpha^2+3\alpha+2)}$. It is easy to check that the bound $\frac{k(\alpha^2+3\alpha+4-2k)}{2(\alpha^2+3\alpha+2)}$ is decreasing with respect to the variable k $(k \geq 2)$ for $1 < \alpha < 2$. Combining the above formulas, we obtain the desired bounds of γ_1. □

Remark 12 The bounds of γ_1, γ_2 and γ_3 are illustrated in Fig. 1. For the Riemann-Liouville fractional calculus (i.e., $\lambda = 0$), the restrictions for γ_1, γ_2 and γ_3 given in Lemma 4 can be relaxed when using the generating function method [31] to prove the numerical stability of time dependent fractional PDEs. And if the parameters $\gamma_j,\ p_j,\ j = 1, 2, 3$, in Eqs. 2.17 and 2.18 are taken as

$$p_1 = 1,\ p_2 = 0,\ p_3 = -1,\ \gamma_1 = \frac{3\alpha^2+5\alpha}{24},\ \gamma_2 = \frac{12-3\alpha^2+\alpha}{12},\ \gamma_3 = \frac{3\alpha^2-7\alpha}{24}, \tag{2.56}$$

then the corresponding tempered-WSGD operators have third order accuracy. It is easy to check that the parameters do not fall in the domains described in Lemma 4.



(a) The bounds of γ_1 in set $\mathcal{S}_1^\alpha$. (b) The bounds of γ_2 in set $\mathcal{S}_2^\alpha$. (c) The bounds of γ_3 in set $\mathcal{S}_3^\alpha$.

Fig. 1 The bounds of γ_1, γ_2, and γ_3 described in Lemma 4

Using the difference formulae (2.56) to approximate the tempered fractional derivatives for fractional diffusion equations seems not to be stable. In the next section, we select the stability ones to solve the time dependent tempered fractional PDEs.

3 Numerical schemes for the tempered fractional diffusion equation

In this section, we apply the second order approximations of the Riemann-Liouville tempered fractional derivative presented in Eq. 2.46 to the following tempered fractional diffusion equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = (l\ {}_a\mathbf{D}_x^{\alpha,\lambda} + r\ {}_x\mathbf{D}_b^{\alpha,\lambda})u(x,t) + s(x,t), & (x,t) \in (a,b) \times [0,T], \\ u(x,0) = u_0(x), & x \in (a,b), \\ u(a,t) = \phi_l(t), \quad u(b,t) = \phi_r(t), & t \in [0,T], \end{cases} \tag{3.1}$$

where $u = u(x,t)$ is the concentration of a solute at a point x at time t; $s(x,t)$ is the source term; and the weighting factors l, r usually control the bias of the diffusion. The diffusion coefficients l and r are nonnegative constants with $l + r = 1$. And if $l \neq 0$, then $\phi_l(t) \equiv 0$; if $r \neq 0$, then $\phi_r(t) \equiv 0$. Next we discretize (3.1) by the second order tempered-WSGD operators given in Eq. 2.46. In the following numerical analysis, we assume that Eq. 3.1 has an unique and sufficiently smooth solution.

Table 1 Numerical errors and orders of accuracy for ${}_0D_x^{\alpha,\lambda}(e^{-\lambda x}x^{2+\alpha}) = \frac{\Gamma(3+\alpha)}{2}e^{-\lambda x}x^2$ computed by the tempered-WSGD operators (2.48) for different λ in the interval [0, 1] with fixed $\alpha = 1.6$ and the $(\gamma_1, \gamma_2, \gamma_3)$ are selected in set $\mathcal{S}_3^\alpha(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_3 = 0.001$

	$\lambda = 0$		$\lambda = 1$		$\lambda = 10$	
h	$\|e\|_h$-error	order	$\|e\|_h$-error	order	$\|e\|_h$-error	order
1/10	3.63e−03		2.49e−03		4.57e−04	
1/20	9.02e−04	2.01	6.15e−04	2.02	1.22e−04	1.90
1/40	2.25e−04	2.00	1.53e−04	2.01	3.04e−05	2.01
1/80	5.62e−05	2.00	3.82e−05	2.00	7.54e−06	2.01

Table 2 Numerical errors and orders of accuracy for ${}_xD_1^{\alpha,\lambda}(e^{\lambda x}(1-x)^{2+\alpha}) = \frac{\Gamma(3+\alpha)}{2}e^{\lambda x}(1-x)^2$ computed by the tempered-WSGD operators (2.49) for different λ in the interval [0, 1] with fixed $\alpha = 1.6$ and the $(\gamma_1, \gamma_2, \gamma_3)$ are selected in set $\mathcal{S}_3^\alpha(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_3 = 0$

	$\lambda = 0$		$\lambda = 1$		$\lambda = 10$	
h	$\|e\|_h$-error	order	$\|e\|_h$-error	order	$\|e\|_h$-error	order
1/10	3.63e−03		6.75e−03		1.01e+01	
1/20	9.02e−04	2.01	1.67e−03	2.01	2.69e+00	1.90
1/40	2.25e−04	2.00	4.16e−04	2.01	6.69e−01	2.01
1/80	5.62e−05	2.00	1.04e−05	2.00	1.66e−01	2.01

3.1 Crank-Nicolson-tempered-WSGD schemes

To derive the numerical schemes for problem (3.1), we first introduce some notations used later. The spatial interval $[a, b]$ is divided into N_x parts by the uniform mesh with the space step $h = (b-a)/N_x$ and the temporal interval is partitioned into N_t parts using the grid-points $t_n = n\tau$, where the equidistant temporal step gives $\tau = T/N_t$. And the set of grid points are denoted by $x_j = a + jh$ and $t_n = n\tau$ for $1 \le j \le N_x$ and $0 \le n \le N_t$. Denoting $t_{n+1/2} = (t_n + t_{n+1})/2$ and setting $u_j^n = u(x_j, t_n)$, $u_j^{n+1/2} = (u_j^n + u_j^{n+1})/2$, $s_j^n = s(x_j, t_n)$, we get the following Crank-Nicolson time discretization for Eq. 3.1 at mesh point (x_j, t_n):

$$\frac{u_j^{n+1} - u_j^n}{\tau} - \left(l({}_a\mathbf{D}_x^{\alpha,\lambda}u)_j^{n+1/2} + r({}_x\mathbf{D}_b^{\alpha,\lambda}u)_j^{n+1/2}\right) = s_j^{n+1/2} + O(\tau^2).$$

Using the tempered-WSGD operators ${}_L\mathcal{D}_{h,1,0,-1}^{\alpha,\gamma_1,\gamma_2,\gamma_3}u(x,t)$ and ${}_R\mathcal{D}_{h,1,0,-1}^{\alpha,\gamma_1,\gamma_2,\gamma_3}u(x,t)$ to approximate the space Riemann-Liouville tempered fractional derivatives

Table 3 Numerical errors and orders of accuracy for ${}_0D_x^{-\sigma,\lambda}(e^{-\lambda x}x^{1+\sigma}) = \frac{\Gamma(2+\sigma)}{\Gamma(2+2\sigma)}e^{-\lambda x}x^{1+2\sigma}$ computed by the tempered-WSGD operators (replacing α by $-\sigma$ in Eq. 2.48) for different λ in the interval [0, 1] with fixed $\sigma = 0.6$ and the $(\gamma_1, \gamma_2, \gamma_3)$ are selected in set $\mathcal{S}_3^\sigma(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_3 = 0.04$

	$\lambda = 0$		$\lambda = 2$		$\lambda = 5$	
h	$\|e\|_h$-error	order	$\|e\|_h$-error	order	$\|e\|_h$-error	order
1/10	3.48e−03		2.43e−03		1.68e−03	
1/20	8.88e−04	1.97	6.71e−04	1.86	5.30e−04	1.66
1/40	2.25e−04	1.98	1.77e−04	1.92	1.49e−04	1.83
1/80	5.69e−05	1.98	4.56e−05	1.96	3.98e−05	1.91

${}_a\mathbf{D}_x^{\alpha,\lambda}u(x,t)$ and ${}_x\mathbf{D}_b^{\alpha,\lambda}u(x,t)$, respectively, yields

$$\frac{u_j^{n+1}-u_j^n}{\tau}-\left(l\,{}_L\mathcal{D}_{h,1,0,-1}^{\alpha,\gamma_1,\gamma_2,\gamma_3}u_j^{n+1/2}+r\,{}_R\mathcal{D}_{h,1,0,-1}^{\alpha,\gamma_1,\gamma_2,\gamma_3}u_j^{n+1/2}\right)+\alpha\lambda^{\alpha-1}(l-r)\,\delta_x u_j^{n+1/2}$$
$$=s_j^{n+1/2}+O(\tau^2+h^2), \tag{3.2}$$

where $\delta_x u_j^n=(u_{j+1}^n-u_{j-1}^n)/2h$. Rearranging the above discretization (3.2) leads to

$$u_j^{n+1}-l\,\tau\,{}_L\mathcal{D}_{h,1,0,-1}^{\alpha,\gamma_1,\gamma_2,\gamma_3}u_j^{n+1/2}-r\,\tau\,{}_R\mathcal{D}_{h,1,0,-1}^{\alpha,\gamma_1,\gamma_2,\gamma_3}u_j^{n+1/2}+\tau\alpha\lambda^{\alpha-1}(l-r)\,\delta_x u_j^{n+1/2}$$
$$=u_j^n+\tau s_j^{n+1/2}+O(\tau^3+\tau h^2). \tag{3.3}$$

From Eq. 2.46, we can recast (3.3) as

$$u_j^{n+1}-\frac{l\tau}{h^\alpha}\sum_{k=0}^{j+1}g_k^{(\alpha)}u_{j-k+1}^{n+1/2}-\frac{r\tau}{h^\alpha}\sum_{k=0}^{N_x-j+1}g_k^{(\alpha)}u_{j+k-1}^{n+1/2}+\tau\alpha\lambda^{\alpha-1}(l-r)\,\delta_x u_j^{n+1/2}$$
$$+\frac{\tau(l+r)}{h^\alpha}\left((\gamma_1e^{h\lambda}+\gamma_2+\gamma_3e^{-h\lambda})(1-e^{-h\lambda})^\alpha\right)u_j^{n+1/2}=u_j^n+\tau s_j^{n+1/2}+O(\tau^3+\tau h^2). \tag{3.4}$$

Denoting U_j^n as the numerical approximation of u_j^n and omitting the local truncation errors, we get the Crank-Nicolson-tempered-WSGD scheme of Eq. 3.1 being given by

$$U_j^{n+1}-\frac{l\,\tau}{h^\alpha}\sum_{k=0}^{j+1}g_k^{(\alpha)}U_{j-k+1}^{n+1/2}-\frac{r\,\tau}{h^\alpha}\sum_{k=0}^{N_x-j+1}g_k^{(\alpha)}U_{j+k-1}^{n+1/2}+\tau\alpha\lambda^{\alpha-1}(l-r)\,\frac{U_{j+1}^{n+1/2}-U_{j-1}^{n+1/2}}{2h}$$
$$+\frac{\tau(l+r)}{h^\alpha}\left((\gamma_1e^{h\lambda}+\gamma_2+\gamma_3e^{-h\lambda})(1-e^{-h\lambda})^\alpha\right)U_j^{n+1/2}=U_j^n+\tau s_j^{n+1/2}. \tag{3.5}$$

If we introduce the matrix form of the grid functions $U^n=\left(U_1^n,U_2^n,\cdots,U_{N_x-1}^n\right)^{\mathrm{T}}$, then the numerical scheme (3.5) can be rewritten as

$$\left(I-\frac{\tau}{2h^\alpha}(l\,A+r\,A^{\mathrm{T}})-\frac{\tau\alpha\lambda^{\alpha-1}(r-l)}{4h}B\right)U^{n+1}=\left(I+\frac{\tau}{2h^\alpha}(l\,A+r\,A^{\mathrm{T}})+\frac{\tau\alpha\lambda^{\alpha-1}(r-l)}{4h}B\right)U^n+\tau F^{n+1/2}, \tag{3.6}$$

where the matrix $A=\left(a_{m,j}\right)_{N_x-1,N_x-1}$ with the entries

$$a_{m,j}=\begin{cases}0, & j>m+1,\\ g_0^{(\alpha)}, & j=m+1,\\ g_1^{(\alpha)}-(l+r)(\gamma_1e^{h\lambda}+\gamma_2+\gamma_3e^{-h\lambda})(1-e^{-h\lambda})^\alpha, & j=m,\\ g_2^{(\alpha)}, & j=m-1,\\ g_{m-j+1}^{(\alpha)}, & j\le m-2,\end{cases} \tag{3.7}$$

and $B = tridiag\{-1, 0, 1\}$, is a skew-symmetric tri-diagonal matrix of $N_x - 1$-square. The term $F^{n+1/2}$ gives

$$F^{n+1/2} = \begin{pmatrix} s_1^{n+1/2} \\ s_2^{n+1/2} \\ \vdots \\ s_{N_x-2}^{n+1/2} \\ s_{N_x-1}^{n+1/2} \end{pmatrix} + \frac{U_0^{n+1/2}}{2h^\alpha} \begin{pmatrix} l\ g_2^{(\alpha)} + r\ g_0^{(\alpha)} \\ l\ g_3^{(\alpha)} \\ \vdots \\ l\ g_{N_x-1}^{(\alpha)} \\ l\ g_{N_x}^{(\alpha)} \end{pmatrix} + \frac{U_{N_x}^{n+1/2}}{2h^\alpha} \begin{pmatrix} r\ g_{N_x}^{(\alpha)} \\ r\ g_{N_x-1}^{(\alpha)} \\ \vdots \\ r\ g_3^{(\alpha)} \\ l\ g_0^{(\alpha)} + r\ g_2^{(\alpha)} \end{pmatrix}$$
$$+ \frac{\alpha\lambda^{\alpha-1}(r-l)}{4h} \begin{pmatrix} U_0^{n+1/2} \\ 0 \\ \vdots \\ 0 \\ -U_{N_x}^{n+1/2} \end{pmatrix}.$$

3.2 Stability and convergence

Now we discuss the numerical stability and convergence for the Crank-Nicolson-tempered-WSGD schemes (3.5). We explore the properties of the eigenvalues of the iterative matrix of Eq. 3.5 on the grid points $\{x_j = a + jh,\ h = (b-a)/N_x,\ j = 1, 2, \ldots, N_x - 1\}$. If the real parts of the eigenvalues are negative, then the schemes are stable. First, we introduce several lemmas.

Lemma 5 ([28]) *A real matrix A of order n is positive definite if and only if its symmetric part $H = \frac{A+A^T}{2}$ is positive definite; H is positive definite if and only if the eigenvalues of H are positive.*

Lemma 6 ([28]) *If $A \in \mathbb{C}^{n\times n}$, let $H = \frac{A+A^*}{2}$ be the hermitian part of A, A^* the conjugate transpose of A, then for any eigenvalue μ of A, there exists*

$$\mu_{\min}(H) \le \mathrm{Re}(\mu(A)) \le \mu_{\max}(H),$$

Table 4 Numerical errors and orders of accuracy for ${}_xD_1^{-\sigma,\lambda}(e^{\lambda x}(1-x)^{1+\sigma}) = \frac{\Gamma(2+\sigma)}{\Gamma(2+2\sigma)} e^{\lambda x}(1-x)^{1+2\sigma}$ computed by the tempered-WSGD operators (replacing α by $-\sigma$ in Eq. 2.49) for different λ in the interval [0, 1] with fixed $\sigma = 0.6$ and the $(\gamma_1, \gamma_2, \gamma_3)$ are selected in set $\mathcal{S}_3^\sigma(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_3 = -0.01$

	$\lambda = 0$		$\lambda = 2$		$\lambda = 5$	
h	$\|e\|_h$-error	order	$\|e\|_h$-error	order	$\|e\|_h$-error	order
1/10	4.35e−03		2.22e−02		3.05e−01	
1/20	1.11e−04	1.96	6.16e−03	1.85	9.71e−02	1.66
1/40	2.84e−04	1.97	1.63e−03	1.92	2.75e−02	1.82
1/80	7.18e−05	1.98	4.22e−04	1.95	7.37e−03	1.90

Table 5 Numerical errors and orders of accuracy for ${}_0D_x^{\alpha,\lambda}(e^{-\lambda x}x^{2+\alpha}) - \lambda^\alpha(e^{-\lambda x}x^{2+\alpha})$ computed by the tempered-WSGD operators (2.40) for different λ in the interval [0, 1] and the $(\gamma_1, \gamma_2, \gamma_3)$ are selected in set $\mathcal{S}_3^\alpha(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_3 = 0.02$

		$\lambda = 0$		$\lambda = 1$		$\lambda = 10$	
α	h	$\|e\|_h$-error	order	$\|e\|_h$-error	order	$\|e\|_h$-error	order
	1/10	3.56e−03		4.84e−03		6.64e−04	
	1/20	8.91e−04	2.00	1.15e−03	2.08	2.27e−04	1.55
$\alpha = 0.5$	1/40	2.23e−04	2.00	2.85e−04	2.01	6.15e−05	1.88
	1/80	5.57e−05	2.00	7.11e−05	2.00	1.47e−05	2.06
	1/10	4.53e−03		2.54e−03		1.19e−04	
$\alpha = 1.5$	1/20	1.12e−03	2.02	6.28e−04	2.02	3.80e−05	1.65
	1/40	2.79e−04	2.01	1.56e−04	2.01	1.05e−05	1.86
	1/80	6.96e−05	2.00	3.90e−05	2.00	2.72e−06	1.94

where $\mathrm{Re}(\mu(\mathrm{A}))$ *represents the real part of* μ*, and* $\mu_{\min}(H)$ *and* $\mu_{\max}(H)$ *are the minimum and maximum of the eigenvalues of* H*.*

Theorem 2 *Let the martrices* $A = (a_{m,j})_{N_x-1,N_x-1}$, $A^T = (a_{j,m})_{N_x-1,N_x-1}$ *be given in numerical scheme* (3.6)*. If* γ_1, γ_2 *and* γ_3 *are chosen in set* $\mathcal{S}_1^\alpha(\gamma_1, \gamma_2, \gamma_3)$ *with* $\max\left\{\frac{2(\alpha^2+3\alpha-4)}{\alpha^2+3\alpha+2}, \frac{\alpha^2+3\alpha}{\alpha^2+3\alpha+4}\right\} \le \gamma_1 \le \frac{3(\alpha^2+3\alpha-2)}{2(\alpha^2+3\alpha+2)}$*, or set* $\mathcal{S}_2^\alpha(\gamma_1, \gamma_2, \gamma_3)$ *with* $\frac{(\alpha-4)(\alpha^2+3\alpha+2)+24}{2(\alpha^2+3\alpha+2)} \le \gamma_2 \le \min\left\{\frac{(\alpha-2)(\alpha^2+3\alpha+4)+16}{2(\alpha^2+3\alpha+4)}, \frac{(\alpha-6)(\alpha^2+3\alpha+2)+48}{2(\alpha^2+3\alpha+2)}\right\}$*, or set* $\mathcal{S}_3^\alpha(\gamma_1, \gamma_2, \gamma_3)$ *with* $\max\left\{\frac{(2-\alpha)(\alpha^2+\alpha-8)}{\alpha^2+3\alpha+2}, \frac{(1-\alpha)(\alpha^2+2\alpha)}{2(\alpha^2+3\alpha+4)}\right\} \le \gamma_3 \le \frac{(2-\alpha)(\alpha^2+2\alpha-3)}{2(\alpha^2+3\alpha+2)}$.

Table 6 Numerical errors and orders of accuracy for ${}_xD_1^{\alpha,\lambda}(e^{\lambda x}(1-x)^{2+\alpha}) - \lambda^\alpha(e^{\lambda x}(1-x)^{2+\alpha})$ computed by the tempered-WSGD operators (2.41) for different λ in the interval [0, 1] and the $(\gamma_1, \gamma_2, \gamma_3)$ are selected in set $\mathcal{S}_3^\alpha(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_3 = -0.02$

		$\lambda = 0$		$\lambda = 1$		$\lambda = 10$	
α	h	$\|e\|_h$-error	order	$\|e\|_h$-error	order	$\|e\|_h$-error	order
	1/10	2.45e−03		8.62e−03		1.30e+01	
$\alpha = 0.5$	1/20	6.19e−04	1.98	2.15e−03	2.01	3.97e+00	1.71
	1/40	1.56e−04	1.99	5.39e−04	1.99	8.51e−01	2.22
	1/80	3.91e−05	2.00	1.35e−04	1.99	2.23e−01	1.94
	1/10	3.51e−03		5.38e−03		2.40e+00	
$\alpha = 1.5$	1/20	8.65e−04	2.02	1.32e−03	2.03	7.14e−01	1.75
	1/40	2.15e−04	2.01	3.28e−04	2.01	1.88e−01	1.93
	1/80	5.38e−05	2.00	8.19e−05	2.00	4.75e−02	1.98

Then the matrix $Q = \frac{A+A^T}{2}$ *is diagonally dominant for* $1 < \alpha < 2$ *and all the eigenvalues of* Q *are negative.*

Proof Denote $Q = \frac{A+A^T}{2} = \left(q_{m,j}\right)_{N_x-1,N_x-1}$ with the entries

$$q_{m,j} = \begin{cases} \frac{1}{2} g_{j-m+1}^{(\alpha)}, & j > m+1, \\ \frac{1}{2}(g_0^{(\alpha)} + g_2^{(\alpha)}), & j = m+1, \\ g_1^{(\alpha)} - (l+r)(\gamma_1 e^{h\lambda} + \gamma_2 + \gamma_3 e^{-h\lambda})(1-e^{-h\lambda})^\alpha, & j = m, \\ \frac{1}{2}(g_2^{(\alpha)} + g_0^{(\alpha)}), & j = m-1, \\ \frac{1}{2} g_{m-j+1}^{(\alpha)}, & j \le m-2. \end{cases} \tag{3.8}$$

With the help of the following binomial formula

$$\sum_{m=0}^{+\infty} w_m^{(\alpha)} e^{-mh\lambda} = (1-e^{-h\lambda})^\alpha,$$

we have

$$\begin{aligned} \sum_{m=0}^{+\infty} g_m^{(\alpha)} &= \gamma_1 w_0^{(\alpha)} e^{h\lambda} + \gamma_1 w_1^{(\alpha)} + \gamma_2 w_0^{(\alpha)} + \sum_{m=2}^{+\infty} \left(\gamma_1 w_m^{(\alpha)} + \gamma_2 w_{m-1}^{(\alpha)} + \gamma_3 w_{m-2}^{(\alpha)}\right) e^{-(m-1)h\lambda} \\ &= (\gamma_1 e^{h\lambda} + \gamma_2 + \gamma_3 e^{-h\lambda})(1-e^{-h\lambda})^\alpha. \end{aligned}$$

Furthermore, we get

$$\sum_{j=-\infty}^{+\infty} q_{m,j} = -(\gamma_1 e^{h\lambda} + \gamma_2 + \gamma_3 e^{-h\lambda})(1-e^{-h\lambda})^\alpha + \sum_{m=0}^{\infty} g_m^{(\alpha)} = 0.$$

By a straightforward calculation, and using Eq. 2.42, we get

$$\gamma_1 e^{h\lambda} + \gamma_2 + \gamma_3 e^{-h\lambda} = 2\gamma_1 \left(\cosh(h\lambda) - 1\right) + 1 + \frac{\alpha}{2}(1-e^{-h\lambda}) > 0 \text{ with } 1 < \alpha < 2 \text{ and } \gamma_1 > 0,$$

where $\cosh(h\lambda)$ denotes the hyperbolic cosine function $\cosh(h\lambda) = \frac{e^{h\lambda}+e^{-h\lambda}}{2}$. Noting that γ_1, γ_2 and γ_3 are chosen in set $\mathcal{S}_1^\alpha(\gamma_1,\gamma_2,\gamma_3)$ or $\mathcal{S}_2^\alpha(\gamma_1,\gamma_2,\gamma_3)$ or $\mathcal{S}_3^\alpha(\gamma_1,\gamma_2,\gamma_3)$, under the assumptions given in Lemma 4, we obtain $q_{m,m} < 0$, $m = 1, 2, \ldots, N_x - 1$. Hence,

$$-q_{m,m} > \sum_{j=0, j\neq m}^{m+1} q_{m,j},$$

which implies that the matrix Q is diagonally dominant. Using the Gershgorin theorem [32], we deduce that the eigenvalues of matrix Q are negative. □

Theorem 3 *Let* γ_1, γ_2 *and* γ_3 *be chosen in set* $\mathcal{S}_1^\alpha(\gamma_1,\gamma_2,\gamma_3)$ *with* $\max\left\{\frac{2(\alpha^2+3\alpha-4)}{\alpha^2+3\alpha+2}, \frac{\alpha^2+3\alpha}{\alpha^2+3\alpha+4}\right\} \le \gamma_1 \le \frac{3(\alpha^2+3\alpha-2)}{2(\alpha^2+3\alpha+2)}$, *or set* $\mathcal{S}_2^\alpha(\gamma_1,\gamma_2,\gamma_3)$ *with*

Table 7 Numerical errors and orders of accuracy for Example 2 computed by the Crank-Nicolson-tempered-WSGD schemes (3.5) at $t = 1$ with different weights and the fixed stepsizes $\tau = h$, $\lambda = 2.0$, $\alpha = 1.6$ and the parameters $(\gamma_1, \gamma_2, \gamma_3)$ are selected in set $\mathcal{S}_1^\alpha(\gamma_1, \gamma_2, \gamma_3)$

	$\gamma_1 = 0.7$		$\gamma_1 = 0.75$		$\gamma_1 = 0.8$	
h	$\Vert e\Vert_h$-error	order	$\Vert e\Vert_h$-error	order	$\Vert e\Vert_h$-error	order
1/10	4.64e−04		4.79e−04		4.98e−04	
1/20	1.30e−04	1.84	1.27e−04	1.92	1.25e−04	2.00
1/40	3.46e−05	1.91	3.26e−05	1.96	3.08e−05	2.02
1/80	8.92e−06	1.95	8.27e−06	1.98	7.63e−06	2.01

$\frac{(\alpha-4)(\alpha^2+3\alpha+2)+24}{2(\alpha^2+3\alpha+2)} \le \gamma_2 \le \min\left\{\frac{(\alpha-2)(\alpha^2+3\alpha+4)+16}{2(\alpha^2+3\alpha+4)}, \frac{(\alpha-6)(\alpha^2+3\alpha+2)+48}{2(\alpha^2+3\alpha+2)}\right\}$, *or set* $\mathcal{S}_3^\alpha(\gamma_1, \gamma_2, \gamma_3)$ *with* $\max\left\{\frac{(2-\alpha)(\alpha^2+\alpha-8)}{\alpha^2+3\alpha+2}, \frac{(1-\alpha)(\alpha^2+2\alpha)}{2(\alpha^2+3\alpha+4)}\right\} \le \gamma_3 \le \frac{(2-\alpha)(\alpha^2+2\alpha-3)}{2(\alpha^2+3\alpha+2)}$. *Then the Crank-Nicolson-tempered-WSGD scheme* (3.5) *with* $\lambda \ge 0$ *and* $1 < \alpha < 2$ *is stable.*

Proof Denote $M = \frac{\tau}{2h^\alpha}(l\,A + r\,A^{\mathrm{T}}) + \frac{\tau\alpha\lambda^{\alpha-1}(r-l)}{4h}B$. Then the matrix form (3.6) of the scheme (3.5) can be rewritten as

$$(I - M)U^{n+1} = (I + M)U^n + \tau F^{n+1/2}. \tag{3.9}$$

If denote $\mu(M)$ as an eigenvalue of matrix M, then $\frac{1+\mu(M)}{1-\mu(M)}$ is the eigenvalue of matrix $(I - M)^{-1}(I + M)$. Combining Lemma 5, Lemma 6 and Theorem 2 shows that the eigenvalues of matrix $\frac{M+M^{\mathrm{T}}}{2} = \frac{\tau(l+r)}{4h^\alpha}(A + A^{\mathrm{T}}) = \frac{\tau(l+r)}{4h^\alpha}Q$ are negative and $\mathrm{Re}(\mu(\mathrm{M})) < 0$, which implies that $\left|\frac{1+\mu(M)}{1-\mu(M)}\right| < 1$. Therefore, the spectral radius of matrix $(I - M)^{-1}(I + M)$ is less than one; then the numerical scheme (3.5) is unconditionally stable. □

Table 8 Numerical errors and orders of accuracy for Example 2 computed by the Crank-Nicolson-tempered-WSGD schemes (3.5) at $t = 1$ with different weights and the fixed stepsizes $\tau = h$, $\lambda = 2.0$, $\alpha = 1.6$ and the parameters $(\gamma_1, \gamma_2, \gamma_3)$ are selected in set $\mathcal{S}_2^\alpha(\gamma_1, \gamma_2, \gamma_3)$

	$\gamma_2 = 0.2$		$\gamma_2 = 0.3$		$\gamma_2 = 0.4$	
h	$\Vert e\Vert_h$-error	order	$\Vert e\Vert_h$-error	order	$\Vert e\Vert_h$-error	order
1/10	4.98e−04		4.79e−04		4.64e−04	
1/20	1.25e−04	2.00	1.27e−04	1.92	1.30e−04	1.84
1/40	3.07e−05	2.02	3.26e−05	1.96	3.46e−05	1.91
1/80	7.62e−06	2.01	8.27e−06	1.98	8.92e−06	1.96

Define $V_h = \{v : v = \{v_j\}$ is a grid function defined on $\{x_j = a + jh\}_{j=1}^{N_x-1}$ and $v_0 = v_{N_x} = 0\}$. And we define the corresponding discrete L^2-norm $\|v\|_h = \left(h\sum_{j=1}^{N_x-1} v_j^2\right)^{1/2}$ for all $v = \{v_j\} \in V_h$.

Lemma 7 *For the iterative matrix M given in Eq. 3.9, there exists*

$$||(I-M)^{-1}||_2 \leq 1, \quad ||(I-M)^{-1}(I+M)||_2 \leq 1,$$

where $\|\cdot\|_2$ denotes the 2-norm (spectral norm).

Proof Following the idea given in [31], we prove this lemma. From Theorem 2, we know that $M + M^{\mathrm{T}} = \frac{\tau(l+r)}{2h^\alpha}(A + A^{\mathrm{T}})$ is negative semi-definite and symmetric. So for any $v = (v_1, v_2, \ldots, v_{N_x-1})^T \in \mathbb{R}^{N_x-1}$, we have

$$v^T v \leq v^T(I - M^T)(I - M)v.$$

Substituting v^T and v by $v^T(I - M^T)^{-1}$ and $(I - M)^{-1}v$ in above inequality, respectively, we get

$$v^T(I-M^T)^{-1}(I-M)^{-1}v \leq v^T v.$$

Therefore, there exists

$$||(I-M)^{-1}||_2 = \sup_{v\neq 0}\sqrt{\frac{v^T(I-M^T)^{-1}(I-M)^{-1}v}{v^T v}} \leq 1.$$

Similarly, we have

$$v^T(I+M^T)(I+M)v \leq v^T(I-M^T)(I-M)v.$$

Replacing v by $(I - M)^{-1}v$ in above inequality, we then have that

$$v^T(I-M^T)^{-1}(I+M^T)(I+M)(I-M)^{-1}v \leq v^T v.$$

Furthermore, we obtain

$$||(I-M)^{-1}(I+M)||_2 = \sup_{v\neq 0}\sqrt{\frac{v^T(I-M^T)^{-1}(I+M^T)(I+M)(I-M)^{-1}v}{v^T v}} \leq 1.$$

□

Theorem 4 *Denote u_j^n as the exact solution of problem (3.1), U_j^n the solution of the numerical scheme (3.5). Let γ_1, γ_2 and γ_3 be chosen in set $\mathcal{S}_1^\alpha(\gamma_1, \gamma_2, \gamma_3)$ with $\max\left\{\frac{2(\alpha^2+3\alpha-4)}{\alpha^2+3\alpha+2}, \frac{\alpha^2+3\alpha}{\alpha^2+3\alpha+4}\right\} \leq \gamma_1 \leq \frac{3(\alpha^2+3\alpha-2)}{2(\alpha^2+3\alpha+2)}$, or set $\mathcal{S}_2^\alpha(\gamma_1, \gamma_2, \gamma_3)$ with $\frac{(\alpha-4)(\alpha^2+3\alpha+2)+24}{2(\alpha^2+3\alpha+2)} \leq \gamma_2 \leq \min\left\{\frac{(\alpha-2)(\alpha^2+3\alpha+4)+16}{2(\alpha^2+3\alpha+4)}, \frac{(\alpha-6)(\alpha^2+3\alpha+2)+48}{2(\alpha^2+3\alpha+2)}\right\}$, or set $\mathcal{S}_3^\alpha(\gamma_1, \gamma_2, \gamma_3)$ with $\max\left\{\frac{(2-\alpha)(\alpha^2+\alpha-8)}{\alpha^2+3\alpha+2}, \frac{(1-\alpha)(\alpha^2+2\alpha)}{2(\alpha^2+3\alpha+4)}\right\} \leq \gamma_3 \leq \frac{(2-\alpha)(\alpha^2+2\alpha-3)}{2(\alpha^2+3\alpha+2)}$. Then we get*

$$\|u^n - U^n\|_h \leq c(\tau^2 + h^2),\ 1 \leq n \leq N_t, \tag{3.10}$$

Table 9 Numerical errors and orders of accuracy for Example 2 computed by the Crank-Nicolson-tempered-WSGD schemes (3.5) at $t = 1$ with different weights and the fixed stepsizes $\tau = h,\ \lambda = 2,\ \alpha = 1.6$ and the parameters $(\gamma_1, \gamma_2, \gamma_3)$ are selected in set $\mathcal{S}_3^{\alpha}(\gamma_1, \gamma_2, \gamma_3)$

	$\gamma_3 = -0.04$		$\gamma_3 = 0$		$\gamma_3 = 0.04$	
h	$\\|e\\|_h$-error	order	$\\|e\\|_h$-error	order	$\\|e\\|_h$-error	order
1/10	4.82e−04		4.98e−04		5.16e−04	
1/20	1.26e−04	1.93	1.25e−04	2.00	1.23e−04	2.07
1/40	3.22e−05	1.97	3.08e−05	2.02	2.94e−05	2.07
1/80	8.14e−06	1.99	7.63e−06	2.01	7.13e−06	2.04

where c denotes a positive constant and $\| \cdot \|_h$ the discrete L^2-norm; u^n stands for $(u_1^n, u_2^n, \cdots, u_{N_x-1}^n)^T$.

Proof Let $e_j^n = u_j^n - U_j^n$. Combining (3.4) and (3.5) leads to

$$(I - M)E^{n+1} = (I + M)E^n + \rho^n, \tag{3.11}$$

where

$$E^n = \left(u_1^n - U_1^n, u_2^n - U_2^n, \cdots, u_{N_x-1}^n - U_{N_x-1}^n\right)^{\mathrm{T}},\ \rho^n = \left(\rho_1^n, \rho_2^n, \cdots, \rho_{N_x-1}^n\right)^{\mathrm{T}},$$

and $\rho_j^n = O(\tau^3 + \tau h^2)$ is the local truncation error. Equation 3.11 can be rewritten as

$$E^{n+1} = (I - M)^{-1}(I + M)E^n + (I - M)^{-1}\rho^n.$$

Taking the Euclidean norm $||\cdot||$ on both sides of the above equation, we have

$$\begin{aligned} ||E^n|| &\le ||(I - M)^{-1}(I + M)E^{n-1}|| + ||(I - M)^{-1}\rho^n|| \\ &\le ||(I - M)^{-1}(I + M)||_2\|E^{n-1}|| + ||(I - M)^{-1}||_2\|\rho^n||, \end{aligned}$$

where we used the fact that the matrix norm $\| \cdot \|_2$ is consistent with the vector norm $\| \cdot \|$ [11]. Noting that $|| \cdot ||_h = h^{1/2}|| \cdot ||$ and thanks to Lemma 7, we further find that

$$||E^n||_h \le ||(I-M)^{-1}(I+M)\|_2\|E^{n-1}||_h+||(I-M)^{-1}\|_2\|\rho^n||_h \le ||E^{n-1}||_h+||\rho^n||_h.$$

Since the truncation error gives $|\rho_j^n| \le C\tau(\tau^2 + h^2)$, replacing n by k and iterating for all $0 \le k \le n - 1$, we conclude that

$$||E^n||_h \le ||E^{n-1}||_h + ||\rho^n||_h \le \sum_{k=0}^{n-1} ||\rho^k||_h \le C(\tau^2 + h^2).$$

□

Table 10 Numerical errors and orders of accuracy for Example 3 computed by the Crank-Nicolson-tempered-WSGD schemes (3.5) at $t = 1$ with different weights and the fixed stepsizes $\tau = h$, $\lambda = 1.0$, $\alpha = 1.2$ and the parameters $(\gamma_1, \gamma_2, \gamma_3)$ are selected in set $\mathcal{S}_1^{\alpha}(\gamma_1, \gamma_2, \gamma_3)$

	$\gamma_1 = 0.7$		$\gamma_1 = 0.75$		$\gamma_1 = 0.8$	
h	$\|e\|_h$-error	order	$\|e\|_h$-error	order	$\|e\|_h$-error	order
1/10	3.94e−03		4.18e−03		4.43e−03	
1/20	9.22e−04	2.09	9.53e−04	2.14	9.85e−04	2.17
1/40	2.18e−04	2.07	2.20e−04	2.12	2.22e−04	2.16
1/80	5.30e−05	2.04	5.25e−05	2.06	5.21e−05	2.10

4 Numerical results

In this section, we perform the numerical experiments to verify the approximation orders of the tempered-WSGD operators to the tempered fractional calculus in Example 1; in Examples 2 and 3, to show the powerfulness of the presented Crank-Nicolson-tempered-WSGD schemes for the tempered fractional diffusion equations with the left tempered fractional derivative and the right tempered fractional derivative, respectively; in particular, the desired convergence orders of the Crank-Nicolson-tempered-WSGD schemes are carefully confirmed.

Example 1 We numerically test the approximation accuracy of the tempered-WSGD operators to the left and right Riemann-Liouville tempered fractional derivatives; and also the approximation accuracy of the corresponding tempered-WSGD operators to the left and right Riemann-Liouville tempered fractional integrals. Using

$${}_aD_x^{\alpha}\left[(x-a)^{\mu}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(x-a)^{\mu-\alpha}, \quad {}_xD_b^{\alpha}\left[(b-x)^{\mu}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(b-x)^{\mu-\alpha},$$

$${}_aD_x^{-\sigma}\left[(x-a)^{\mu}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\sigma+1)}(x-a)^{\mu+\sigma}, \quad {}_xD_b^{-\sigma}\left[(b-x)^{\mu}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\sigma+1)}(b-x)^{\mu+\sigma},$$

Table 11 Numerical errors and orders of accuracy for Example 3 computed by the Crank-Nicolson-tempered-WSGD schemes (3.5) at $t = 1$ with different weights and the fixed stepsizes $\tau = h$, $\lambda = 1.0$, $\alpha = 1.2$ and the parameters $(\gamma_1, \gamma_2, \gamma_3)$ are selected in set $\mathcal{S}_2^{\alpha}(\gamma_1, \gamma_2, \gamma_3)$

	$\gamma_2 = 0.2$		$\gamma_2 = 0.3$		$\gamma_2 = 0.4$	
h	$\|e\|_h$-error	order	$\|e\|_h$-error	order	$\|e\|_h$-error	order
1/10	3.94e−03		3.69e−03		3.46e−03	
1/20	9.22e−04	2.09	8.95e−04	2.05	8.70e−04	1.99
1/40	2.18e−04	2.07	2.17e−04	2.04	2.17e−04	2.01
1/80	5.29e−05	2.04	5.35e−05	2.02	5.40e−05	2.00

Table 12 Numerical errors and orders of accuracy for Example 3 computed by the Crank-Nicolson-tempered-WSGD schemes (3.5) at $t = 1$ with different weights and the fixed stepsizes $\tau = h$, $\lambda = 1.0$, $\alpha = 1.2$ and the parameters $(\gamma_1, \gamma_2, \gamma_3)$ are selected in set $\mathcal{S}_3^{\alpha}(\gamma_1, \gamma_2, \gamma_3)$

	$\gamma_3 = -0.04$		$\gamma_3 = 0$		$\gamma_3 = 0.04$	
h	$\|e\|_h$-error	order	$\|e\|_h$-error	order	$\|e\|_h$-error	order
1/10	3.29e−03		3.46e−03		3.65e−03	
1/20	8.53e−04	1.95	8.70e−04	1.99	8.89e−04	2.04
1/40	2.16e−04	1.98	2.17e−04	2.00	2.17e−04	2.03
1/80	5.45e−05	1.99	5.40e−05	2.00	5.36e−05	2.02

we obtain the analytical/exact results

$$ {}_aD_x^{\alpha,\lambda}\left[e^{-\lambda x}(x-a)^{\mu}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}e^{-\lambda x}(x-a)^{\mu-\alpha}, \quad {}_xD_b^{\alpha,\lambda}\left[e^{\lambda x}(b-x)^{\mu}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}e^{\lambda x}(b-x)^{\mu-\alpha}, $$

$$ {}_aD_x^{-\sigma,\lambda}\left[e^{-\lambda x}(x-a)^{\mu}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\sigma+1)}e^{-\lambda x}(x-a)^{\mu+\sigma}, \quad {}_xD_b^{-\sigma,\lambda}\left[e^{\lambda x}(b-x)^{\mu}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\sigma+1)}e^{\lambda x}(b-x)^{\mu+\sigma}, $$

where $\mu > -1$.

The numerical values are computed in the finite interval [0, 1]; the numerical errors and orders of accuracy are shown in Tables 1, 2, 3, 4, 5 and 6, which confirm the desired second order accuracy.

Example 2 We consider the following tempered fractional diffusion equation with the left tempered fractional derivative

$$ \begin{aligned} \frac{\partial u(x,t)}{\partial t} &= {}_0\mathbf{D}_x^{\alpha,\lambda}u(x,t) + e^{-\lambda x - t}\Big((\lambda^{\alpha} - \alpha\lambda^{\alpha} - 1)x^{1+\alpha} - \Gamma(2+\alpha)x \\ &\quad + \alpha(\alpha+1)\lambda^{\alpha-1}x^{\alpha}\Big), \quad (x,t) \in (0,1) \times (0,1], \quad 1 < \alpha < 2, \end{aligned} \tag{4.1} $$

with the boundary conditions

$$ u(0,t) = 0, \quad u(1,t) = e^{-\lambda - t}, \quad t \in [0,1], $$

and the initial value

$$ u(x,0) = e^{-\lambda x}x^{1+\alpha}, \quad x \in [0,1]. $$

We can check that the exact solution of Eq. 4.1 is $u(x,t) = e^{-\lambda x - t}x^{1+\alpha}$.

Equation 4.1 is solved by the Crank-Nicolson-tempered-WSGD scheme (3.5); and the numerical results are collected in Tables 7, 8 and 9. It can be seen that the numerical results with second order accuracy are obtained.

Example 3 Finally, we consider the following tempered fractional diffusion equation with the right tempered fractional derivative

$$\frac{\partial u(x,t)}{\partial t} = {}_x\mathbf{D}_b^{\alpha,\lambda}u(x,t) + e^{\lambda x-t}\Big((\lambda^\alpha - \alpha\lambda^\alpha - 1)(1-x)^{1+\alpha} - \Gamma(2+\alpha)(1-x) + \alpha(\alpha+1)\lambda^{\alpha-1}(1-x)^\alpha\Big), \quad (x,t) \in (0,1)\times(0,1],, \quad 1<\alpha<2, \tag{4.2}$$

with the boundary conditions

$$u(0,t) = e^{\lambda x}(1-x)^{1+\alpha}, \;\; u(1,t) = 0, \quad t \in [0,1],$$

and the initial value

$$u(x,0) = e^{\lambda x}(1-x)^{1+\alpha}, \quad x \in [0,1].$$

With the help of the formulae given in Example 1, we get the exact solution of Eq. 4.2: $u(x,t) = e^{\lambda x-t}(1-x)^{1+\alpha}$.

Tables 10, 11, and 12 present the numerical errors and the convergence behaviors of the Crank-Nicolson-tempered-WSGD schemes (3.5). These confirm the results given in Theorem 4.

Remark 13 Note that the homogeneous left and right boundary conditions are, respectively, required for Examples 2 and 3 to make the corresponding scheme converge. For dealing with the nonhomogeneous boundary conditions, some techniques are needed; for the details, refer to [34].

5 Concluding remarks

Lévy flight models suppose that the particles have very large jumps; and they have infinite moments. But many realistically non-Brownian (at least converge to the Brownian ultraslowly and it is not possible to observe the Brownian behaviors in the finite observing time) physical processes just lie in the bounded physical domain. So some techniques to modify the Lévy flight models are introduced. The most popular one seems to be exponentially tempering the probability of large jumps of Lévy flight, which leads to the tempered fractional diffusion equation being used to describe the probability density function of the positions of the particles. With this model, the tempered fractional calculus are introduced; they are very similar to but still different from the fractional substantial calculus. The fractional substantial calculus are time-space coupled operators; and their discretizations are in the time direction. The tempered fractional derivative used in this paper is a space operator without coupling with time. On one hand, we need to derive its high order discretizations, which can greatly improve the accuracy but without introducing new computational cost comparing with the first order scheme; on the other hand, the numerical stability of the derived schemes is a key issue.

This paper derive a series of high order discetizations for the tempered fractional calculus, including the left Riemann-Liouville tempered fractional derivative and integral and the right Riemann-Liouville tempered fractional derivative and integral. In particular, the superconvergent point still exists for the first order discretization of left/right Riemann-Liouville tempered fractional derivative/integral. The stability domains of the schemes are analytically derived and clearly illustrated in figures. A family of second order schemes are used to numerically solve the tempered fractional diffusion equation. And the stability and convergence of the numerical schemes are theoretically proved and numerically verified.

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