Electromagnetic inverse shape problem
for coated obstacles

N. Chaulet · H. Haddar

Abstract We address the inverse problem of retrieving the shape of an obstacle with
impedance in the form of a surface wave operator using the knowledge of electromag-
netic scattering amplitude at a fixed frequency. We prove unique reconstructions from
infinitely many measures. We then provide a characterization of the scattering ampli-
tude derivative with respect to the obstacle shape. This derivative includes the case
of shape dependent impedance parameters. We then employ a gradient-descent algo-
rithm with H^1 boundary regularisation of the descent direction to numerically solve
the inverse problem. The procedure is validated for three dimensional geometries
using synthetic data.

Keywords Inverse scattering problem · Maxwell's equations · Generalized
impedance boundary conditions · Shape derivative · Steepest descent method

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1 Introduction

We investigate the inverse problem of retrieving the shape of coated obstacles from electromagnetic measurements of the scattering amplitude at a fixed frequency. The specificity of our work compared to the vast literature on inverse scattering problems [14], is to address coating models for which the impedance is written as a surface wave operator. They can also be referred to as generalized impedance boundary conditions [4, 16]. They provide accurate models for thin dielectric coatings, imperfectly conducting obstacles or corrugated surfaces [17] and they can also be used as models for plasmonic waves [30]. Considering inverse problems with generalised impedance boundary conditions has been proposed in [7] and further developed in [5, 9, 10, 12].

The focus of our work is on the use of shape optimisation techniques to solve the above mentioned inverse problem. Using an adaptation of the mixed-reciprocity technique [23, 29], we first investigate the identifiability issue and prove in particular uniqueness of the shape reconstruction (independently from the operator coefficients) from the knowledge of infinitely many scattering amplitudes associated with different directions of incident waves. The case of finitely many measures is known to be challenging and left open.

We then address the main topic of this work: characterise the shape derivative of the electromagnetic scattering amplitude. This problem is technically hard, first due to the inherent complexity of Maxwell's equations and second due to the presence of a surface wave operator in the boundary condition. We employ a methodology similar to the one in [18, 25] where the expression of the derivative is determined using the integral representation of the solution in terms of the Green's function of the unperturbed domain. Although based on integral representations of the solutions, the method leads to an explicit expression of the derivative in terms of surface differential operators. Moreover, one is able to deal with cases where the impedance operator coefficients are also unknown and to define a derivative that depends on the geometry and the impedance operator. This type of derivative has been proposed in [5] for the scalar case and shown to be useful in simultaneous reconstructions of the geometry and the impedance coefficients.

We then exploit the derivative to solve the inverse scattering problem using a gradient descent technique associated with a least squares misfit functional. We employ an adjoint-state technique in order to compute the cost functional's derivative. The slightly uncommon feature of our algorithm is the incorporation of a surface H^1 regularisation of the descent direction by solving a surface Laplace Beltrami problem. This smoothing is stronger than the one usually used in shape optimisation problems [1, 2] and turned out to be essential in order to stabilise the inversion. This methodology has the advantage of freeing the number of parameters (mesh points) used to parametrise the geometry during iterations. Indeed the expression of the shape derivative can also be used in parametrisation-dependent techniques, such as Newton type methods [19, 21, 22, 26], but this issue will not be discussed in the present work.

We conclude our paper with validating 3D numerical examples for generalised impedance boundary conditions. The problem is solved using FreeFem++ [20] and Nédélec's edge elements. The radiation condition is ensured using a first order radiation condition on a surface that encloses the computational domain. Inverse crime

is avoided in our simulations since the 3D mesh is different at each iteration step and the final mesh (and reconstructed domain) are quite different from the one used to compute the synthetic data.

The outline of our article is as follows. We formulate the direct and inverse scattering problems in Section 2. We investigate in Section 3 the uniqueness issue for infinitely many measures after proving mixed reciprocity relations. Section 4 contains the main result of this paper related to the scattering amplitude derivative with respect to the obstacle shape. Section 5 is dedicated to the presentation of the inversion scheme based on gradient descent with regularised descent direction and an adjoint technique method to compute the cost functional derivative. We end this section with some three dimensional validating numerical results. The proofs of some technical results for surface differential operators is given in an Appendix.

2 The forward and inverse problems

2.1 The generalized impedance boundary condition scattering problem

Let Ω be a bounded open set of $\mathbb{R}^3$ with C^2 boundary Γ. We denote by ν the outward unit normal to Γ and by $\Omega_{\text{ext}} := \mathbb{R}^3 \setminus \overline{\Omega}$ that we assume to be a connected set. The scattering of an incident electromagnetic wave $(\mathrm{E}^i, \mathrm{H}^i)$ by an obstacle characterised by a Generalized Impedance Boundary Condition (GIBC) gives rise to a scattered electromagnetic wave $(\mathrm{E}^s, \mathrm{H}^s)$ that solves

$$\begin{cases} \mathbf{curl}\, \mathrm{H}^s + ik\mathrm{E}^s = 0 & \text{in } \Omega_{\text{ext}}, \\ \mathbf{curl}\, \mathrm{E}^s - ik\mathrm{H}^s = 0 & \text{in } \Omega_{\text{ext}}, \\ \nu \times \mathrm{E}^s + \mathbf{curl}_\Gamma(\eta \mathrm{curl}_\Gamma \mathrm{H}^s_T) + \lambda \mathrm{H}^s_T = f & \text{on } \Gamma, \\ \lim_{R \to \infty} \int_{\partial B_R} |\mathrm{H}^s \times \hat{x} - (\hat{x} \times \mathrm{E}^s) \times \hat{x}|^2 ds = 0 \end{cases} \tag{1}$$

with

$$f := -\left(\nu \times \mathrm{E}^i + \mathbf{curl}_\Gamma(\eta \mathrm{curl}_\Gamma \mathrm{H}^i_T) + \lambda \mathrm{H}^i_T\right) \quad \text{on } \Gamma. \tag{2}$$

In these equations, B_R is the ball of radius R, $\hat{x} = x/|x|$, k is the wavenumber of the incident wave and for any vector field $V \in (L^2(\Gamma))^3$ we denote by $V_T := (\nu \times V) \times \nu$ its tangential component that belongs to $L^2_t(\Gamma) := \{V \in (L^2(\Gamma))^3 \mid V \cdot \nu = 0\}$. The differential operators curl_Γ and $\mathbf{curl}_\Gamma$ are respectively the scalar and vector surface curl operators which are adjoint to each other. We refer to [28, Section 2.5.6] for a precise definition of these operators (see also Eq. 3 below). Finally, the parameters λ and η in the boundary condition are two complex valued functions of $L^\infty(\Gamma)$. Problem (1) is well defined for any $f \in H_{\mathrm{curl}_\Gamma}(\Gamma)^*$ the dual space of $H_{\mathrm{curl}_\Gamma}(\Gamma) := \{V \in L^2_t(\Gamma) \mid \mathrm{curl}_\Gamma V \in L^2(\Gamma)\}$ which is endowed with the graph norm. Let us introduce the spaces

$$H^{\text{ext}}_{\mathbf{curl}}(\Omega_{\text{ext}}) := \{V \in (\mathcal{D}'(\Omega_{\text{ext}}))^3 \mid \varphi V \in H_{\mathbf{curl}}(\Omega_{\text{ext}}), \ \forall \varphi \in C_0^\infty(\mathbb{R}^3\}$$

$$V_{\mathrm{H}}(\Omega_{\text{ext}}) := \{V \in H^{\text{ext}}_{\mathbf{curl}}(\Omega_{\text{ext}}) \mid V_T \in H_{\mathrm{curl}_\Gamma}(\Gamma)\}$$

where for any open set D, $H_{\mathbf{curl}}(D) := \{V \in (L^2(D))^3 \mid \mathbf{curl}\, V \in (L^2(D))^3\}$. We also need the Hilbert space

$$V_{H,R} := \{V \in H_{\mathbf{curl}}(\Omega \cap B_R) \mid V_T \in H_{\mathrm{curl}_\Gamma}(\Gamma)\}.$$

endowed with the graph norm denoted by $\|\cdot\|_{V_{H,R}}$. Let us recall the following theorem from [11].

Theorem 2.1 *Let* $(\lambda, \eta) \in (L^\infty(\Gamma))^2$ *be such that*

$$\Re(\lambda) \geq 0, \quad \Re(\eta) \geq 0, \quad |\lambda| \geq c \text{ and } |\eta| \geq c \quad \text{a.e. on } \Gamma$$

for some constant $c > 0$, *and assume the imaginary parts of* λ *and* η *do not change sign on* Γ. *Then for all* $f \in H_{\mathrm{curl}_\Gamma}(\Gamma)^*$ *problem* (1) *has a unique solution* $(E^s, H^s) \in H_{\mathbf{curl}}^{ext}(\Omega_{\mathrm{ext}}) \times V_{\mathrm{H}}(\Omega_{\mathrm{ext}})$. *Moreover, for all ball* B_R *that contains* $\overline{\Omega}$ *there exists* $C_R > 0$ *independent of* f *such that*

$$\|E^s\|_{H_{\mathbf{curl}}(\Omega_R)} + \|H^s\|_{V_{H,R}} \leq C_R \|f\|_{H_{\mathrm{curl}_\Gamma}(\Gamma)^*}$$

where $\Omega_R := \Omega \cap B_R$.

From now on, we assume that the impedance functions λ and η satisfy the assumptions of Theorem 2.1. In the following we will need additional regularity for the solution to problem (1). In this view, let us introduce the surface divergence operator div_Γ which is the negative adjoint of the surface gradient ∇_Γ. We recall the following algebraic relations that link the different differential operators introduced here above

$$\nu \cdot (\mathbf{curl}\, V)|_\Gamma = \mathrm{curl}_\Gamma V_T = -\mathrm{div}_\Gamma(\nu \times V) \quad \text{and} \quad \mathbf{curl}_\Gamma \nu = -\nu \times \nabla_\Gamma \nu \qquad (3)$$

where V is vector field defined in a neighbourhood of Γ and v is a function defined on Γ. For regular boundaries (C^∞) we also define fractional Sobolev spaces $H_{\mathrm{div}_\Gamma}^s(\Gamma) := \{V \in H_t^s(\Gamma) \mid \mathrm{div}_\Gamma V \in H^s(\Gamma)\}$ and $H_{\mathrm{curl}_\Gamma}^s(\Gamma) := \{V \in H_t^s(\Gamma) \mid \mathrm{curl}_\Gamma V \in H^s(\Gamma)\}$ where $H_t^s(\Gamma)$ is the closure of $\{V \in (C^\infty(\Gamma))^3 \mid V \cdot \nu = 0\}$ in $(H^s(\Gamma))^3$ and $s \in \mathbb{R}$.

Under additional regularity assumptions on λ, η and Γ we have the following regularity property for the electromagnetic field.

Proposition 2.2 *Let* Γ *be of class* C^{s+2}, *and let us assume that* λ *and* η *are in* $C^{s+1}(\Gamma)$. *For any* $f \in H_{\mathrm{curl}_\Gamma}(\Gamma)^* \cap H_{\mathrm{div}_\Gamma}^{s-1/2}(\Gamma)$ *for* $s \geq 0$, *if* $(E, H) \in H_{\mathbf{curl}}^{ext}(\Omega_{\mathrm{ext}}) \times V_{\mathrm{H}}(\Omega_{\mathrm{ext}})$ *satisfies problem* (1) *then* $(E, H) \in (H^{s+1}(\Omega_R))^3 \times (H^{s+1}(\Omega_R))^3$ *for all ball* B_R *of radius* R *such that* $\overline{\Omega} \subset B_R$.

Proof Using $\mathrm{div}_\Gamma \mathbf{curl}_\Gamma = 0$, the boundary condition satisfied by (E, H) implies $\mathrm{div}_\Gamma(\lambda H_T) = -\mathrm{div}_\Gamma(\nu \times E) + \mathrm{div}_\Gamma f \in H^{-1/2}(\Gamma)$ since $E \in H_{\mathbf{curl}}^{ext}(\Omega_{\mathrm{ext}})$ and f is in $H_{\mathrm{div}_\Gamma}^{-1/2}(\Gamma)$. Combined with $\mathrm{curl}_\Gamma H_T \in L^2(\Gamma)$ because $H_T \in H_{\mathrm{curl}_\Gamma}(\Gamma)$, this implies $H_T \in (H^{1/2}(\Gamma))^3$. Hence, applying classical regularity results for Maxwell's equations [3, Corollary 2.15] to the magnetic field H, we obtain $H \in (H^1(\Omega_R))^3$. In

addition, $\mathbf{curl}_\Gamma(\eta \mathrm{curl}_\Gamma H_T) = -\nu \times E - \lambda H_T + f \in H^{-1/2}(\Gamma)$ whence $\mathrm{curl}_\Gamma H_T \in H^{1/2}(\Gamma)$ and from Maxwell's equations we have

$$\nu \cdot E = -\frac{1}{ik}\nu \cdot \mathbf{curl}\, H = -\frac{1}{ik}\mathrm{curl}_\Gamma H_T \in H^{1/2}(\Gamma). \tag{4}$$

Therefore, using again regularity results [3, Corollary 2.15], we deduce that $E \in (H^1(\Omega_R))^3$. To obtain further regularity let us remark that similarly to Eq. 4 we have

$$\mathrm{div}_\Gamma(\nu \times E) = -\nu \cdot \mathbf{curl}\, E = -ik\nu \cdot H \in H^{1/2}(\Gamma).$$

We then obtain the desired result by induction on s. $\qquad\square$

2.2 Statement of the inverse problem

We recall that any solution (E, H) to Maxwell's equations

$$\mathbf{curl}\, H + ikE = 0, \quad \mathbf{curl}\, E - ikH = 0 \tag{5}$$

outside some bounded Lipschitz domain D that also satisfies the Silver-Müller radiation condition (the last equation in system (1)) admits the following asymptotic behaviour

$$E(x) = \frac{e^{ik|x|}}{|x|}\left(E^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right)\right), \quad H(x) = \frac{e^{ik|x|}}{|x|}\left(H^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right)\right) \quad |x| \to +\infty$$

uniformly for all direction $\hat{x} \in S^2$ where S^2 denotes the unit sphere of $\mathbb{R}^3$. This asymptotic behaviour uniquely defines the far field patterns E^∞ and H^∞ as functions of $L_t^2(S^2)$ and we have the following representation formula

$$E^\infty(\hat{x}) = \frac{ik}{4\pi}\hat{x} \times \int_{\partial D}\{\nu(y) \times E(y) + [\nu(y) \times H(y)] \times \hat{x}\}e^{-ik\hat{x}\cdot y}ds(y),$$

and $H^\infty(\hat{x}) = \hat{x} \times E^\infty(\hat{x})$. We refer to [14] or [24] for general results about electromagnetic scattering theory.

Following the notations in [24] let us introduce the incident electromagnetic plane waves with incidence direction $\hat{\theta} \in S^2$ as being described by the matrices $\mathcal{E}^i(x, \hat{\theta})$ and $\mathcal{H}^i(x, \hat{\theta})$ which are defined for a polarisation $\mathbf{p} \in S^2$ by

$$\begin{aligned}\mathcal{E}^i(x, \hat{\theta})\mathbf{p} &:= -\frac{1}{ik}\mathbf{curl}_x\mathbf{curl}_x(\mathbf{p}e^{ikx\cdot\hat{\theta}}) = ik((\hat{\theta} \times \mathbf{p}) \times \hat{\theta})e^{ikx\cdot\hat{\theta}},\\ \mathcal{H}^i(x, \hat{\theta})\mathbf{p} &:= \mathbf{curl}_x(\mathbf{p}e^{ikx\cdot\hat{\theta}}) = ik(\hat{\theta} \times \mathbf{p})e^{ikx\cdot\hat{\theta}}.\end{aligned} \tag{6}$$

Each pair of corresponding columns of $\mathcal{E}^i(\cdot, \hat{\theta})$ and $\mathcal{H}^i(\cdot, \hat{\theta})$ satisfy Maxwell's Eq. 5 in $\mathbb{R}^3$. Since problem (1) is linear with respect to the right-hand side f, we can define the scattered field matrices $\mathcal{E}^s(\cdot, \hat{\theta})$ and $\mathcal{H}^s(\cdot, \hat{\theta})$ where for any polarisation $\mathbf{p} \in S^2$, the electromagnetic field $(\mathcal{E}^s(x, \hat{\theta})\mathbf{p}, \mathcal{H}^s(x, \hat{\theta})\mathbf{p})$ solves problem (1) with f being given by Eq. 2 for $E^i(x) = \mathcal{E}^i(x, \hat{\theta})\mathbf{p}$ and $H^i(x) = \mathcal{H}^i(x, \hat{\theta})\mathbf{p}$. We also denote by $\mathcal{E}^\infty(\hat{x}, \hat{\theta})$ the matrix representation of the far field pattern associated with $\mathcal{E}^s(x, \hat{\theta})$ and by

$$\mathcal{E}(x, \hat{\theta}) = \mathcal{E}^i(x, \hat{\theta}) + \mathcal{E}^s(x, \hat{\theta}) \quad \text{and} \quad \mathcal{H}(x, \hat{\theta}) = \mathcal{H}^i(x, \hat{\theta}) + \mathcal{H}^s(x, \hat{\theta})$$

the total fields. In what follows, we will frequently make use this matrix field notation. We define the **curl** of a matrix $M = (m_1, m_2, m_3)$ where m_1, m_2 and m_3 are column vectors as being given by the matrix $\mathbf{curl}M := (\mathbf{curl}\, m_1, \mathbf{curl}\, m_2, \mathbf{curl}\, m_3)$. Similarly, for $M = (m_1, m_2, m_3)^{\mathrm{T}}$ where m_1, m_2 and m_3 are row vectors we define the divergence of M as the column vector $\mathrm{div}M := (\mathrm{div}(m_1), \mathrm{div}(m_2), \mathrm{div}(m_3))^{\mathrm{T}}$ where $^{\mathrm{T}}$ stands for the transpose of a vector or a matrix. Moreover, the surface differential operator curl_Γ is defined for M by the row vector field $\mathrm{curl}_\Gamma M := (\mathrm{curl}_\Gamma m_1, \mathrm{curl}_\Gamma m_2, \mathrm{curl}_\Gamma m_3)$ and the vector $\mathbf{curl}_\Gamma$ operator is defined for a row vector field $V = (v_1, v_2, v_3)$ by the matrix $\mathbf{curl}_\Gamma V := (\mathbf{curl}_\Gamma v_1, \mathbf{curl}_\Gamma v_2, \mathbf{curl}_\Gamma v_3)$. Finally, for a column vector V and a matrix M we set $V \times M = -M \times V := (V \times m_1, V \times m_2, V \times m_3)$ and by extension $M_T := (\nu \times M) \times \nu$ where the normal vector ν is considered as a column vector.

For fixed λ and η we address the question of reconstructing the shape Γ from the knowledge of the far field pattern $\mathrm{E}^\infty(\hat{x}, \hat{\theta})$ for any $(\hat{x}, \hat{\theta})$. We establish in the next section a uniqueness result for this inverse problem and in Section 4 we compute the derivatives of the far field patterns with respect to Γ. This derivative can then be used to solve the inverse problem with a non linear optimisation technique as the one presented in Section 5. We shall numerically demonstrate that in practice a small number of incident waves would be sufficient to obtain accurate shape reconstructions.

3 Uniqueness for infinitely many incident waves

We prove in this section that the knowledge of $\mathscr{E}^\infty(\hat{x}, \hat{\theta})$ for all $(\hat{x}, \hat{\theta}) \in (S^2)^3$ uniquely determines Γ. The main result of this section is stated in the theorem below and can be seen as an extension of the case $\eta = 0$ treated in [8, Theorem 3.1] for example.

Theorem 3.1 *Let Γ_1 and Γ_2 be two boundaries of class C^2 and λ and η be two complex valued functions defined on Γ_1 and Γ_2 such that $\lambda \in C^1(\Gamma_1) \cap C^1(\Gamma_2)$ and $\eta \in C^1(\Gamma_1) \cap C^1(\Gamma_2)$ satisfy the assumptions of Theorem 2.1. Let us denote by $\mathscr{E}_1^\infty$ (respectively $\mathscr{E}_2^\infty$) the far field pattern associated with Γ_1, λ and η (respectively Γ_2, λ and η). If $\mathscr{E}_1^\infty(\hat{x}, \hat{\theta}) = \mathscr{E}_2^\infty(\hat{x}, \hat{\theta})$ for all $\hat{x}, \hat{\theta} \in S^2$ then $\Gamma_1 = \Gamma_2$.*

To prove this result we need first to establish a mixed reciprocity relation (see Lemma 3.2) that has been first obtained by Potthast in [29]. This result is the corner stone of the proof of uniqueness for Γ which does not depend on the values of λ and η. Let us first introduce the electromagnetic dipole located at point $z \in \Omega_{\mathrm{ext}}$ that we represent with the matrices $\mathcal{E}^i(\cdot, z)$ and $\mathcal{H}^i(\cdot, z)$ defined for $\mathbf{p} \in S^2$ by

$$\mathcal{E}^i(x, z)\mathbf{p} := -\frac{1}{ik}\mathbf{curl}_x\mathbf{curl}_x(\mathbf{p}\Phi(x, z)),$$

$$\mathcal{H}^i(x, z)\mathbf{p} := \mathbf{curl}_x(\mathbf{p}\Phi(x, z)) = \nabla_x(\Phi(x, z)) \times \mathbf{p}$$

where

$$\Phi(x, z) = \frac{1}{4\pi}\frac{e^{ik|x-z|}}{|x - z|} \quad \text{for } x \neq z$$

is the outgoing Green's function for the Helmholtz' equation in $\mathbb{R}^3$. By linearity, we define the scattered field matrices $\mathcal{E}^s(\cdot, z)$ and $\mathcal{H}^s(\cdot, z)$ such that $(\mathcal{E}^s(\cdot, z)\mathbf{p}, \mathcal{H}^s(\cdot, z)\mathbf{p})$ solves problem (1) with f being given by Eq. 2 for $\mathrm{E}^i = \mathcal{E}^i(\cdot, z)\mathbf{p}$ and $\mathrm{H}^i = \mathcal{H}^i(\cdot, z)\mathbf{p}$. We denote by $\mathcal{E}^\infty((\cdot, z)$ and $\mathcal{H}^\infty(\cdot, z)$ the associated matrix far field patterns and by $\mathcal{E} := \mathcal{E}^i + \mathcal{E}^s$ and $\mathcal{H} = \mathcal{H}^i + \mathcal{H}^s$ the matrix total fields. With this notation, the following mixed reciprocity relation holds.

Lemma 3.2 *For all $x \in \Omega_{\text{ext}}$ and $\hat{\theta} \in S^2$ we have*

$$[\mathscr{E}^s(x, -\hat{\theta})]^{\mathrm{T}} = 4\pi \mathcal{E}^\infty(\hat{\theta}, x).$$

Proof For (E, H) solution to Maxwell's equations outside Ω that satisfy the Silver-Müller radiation condition, the Stratton-Chu formula (see e.g. [24, Corollary 4.1.2.3]) writes as

$$\mathrm{E}(x) = \int_\Gamma \{[\mathcal{E}^i(z, x)]^{\mathrm{T}}[v \times \mathrm{H}(z)] + [\mathcal{H}^i(z, x)]^{\mathrm{T}}[v \times \mathrm{E}(z)]\}\, ds(z) \quad \text{for } x \in \Omega_{\text{ext}}. \quad (7)$$

The associated far field pattern is given by (see e.g. [24, Corollary 4.1.3.1])

$$4\pi \mathrm{E}^\infty(\hat{\theta}) = \int_\Gamma \{[\mathscr{H}^i(z, -\hat{\theta})]^{\mathrm{T}}[v \times \mathrm{E}(z)] + [\mathscr{E}^i(z, -\hat{\theta})]^{\mathrm{T}}[v \times \mathrm{H}(z)]\}\, ds(z) \text{ for } \hat{\theta} \in S^2. \quad (8)$$

Moreover, for $(\mathrm{E}_1^i, \mathrm{H}_1^i)$ and $(\mathrm{E}_2^i, \mathrm{H}_2^i)$ solutions to Maxwell's equations inside Ω,

$$\int_\Gamma \{\mathrm{E}_1^i \cdot [v \times \mathrm{H}_2^i] + \mathrm{H}_1^i \cdot [v \times \mathrm{E}_2^i]\}\, ds = 0, \quad (9)$$

and the same applies for solutions $(\mathrm{E}_1^s, \mathrm{H}_1^s)$ and $(\mathrm{E}_2^s, \mathrm{H}_2^s)$ to Maxwell's equations outside Ω that satisfy the Silver-Müller radiation condition:

$$\int_\Gamma \{\mathrm{E}_1^s \cdot [v \times \mathrm{H}_2^s] + \mathrm{H}_1^s \cdot [v \times \mathrm{E}_2^s]\}\, ds = 0. \quad (10)$$

Formulas (7) and (8) applied to the expressions of $\mathcal{E}^s(z, -\hat{\theta})$ and $\mathcal{E}^\infty(\hat{\theta}, z)$ imply that

$$[\mathscr{E}^s(x, -\hat{\theta})]^{\mathrm{T}} - 4\pi \mathcal{E}^\infty(\hat{\theta}, x) = \int_\Gamma \{[\mathscr{H}^s(z, -\hat{\theta})]^{\mathrm{T}}[\mathcal{E}^i(z, x) \times v]$$
$$+ [\mathscr{E}^s(z, -\hat{\theta})]^{\mathrm{T}}[\mathcal{H}^i(z, x) \times v]$$
$$+ [\mathscr{H}^i(z, -\hat{\theta})]^{\mathrm{T}}[\mathcal{E}^s(z, x) \times v]$$
$$+ [\mathscr{E}^i(z, -\hat{\theta})]^{\mathrm{T}}[\mathcal{H}^s(z, x) \times v]\}\, ds(z)$$

for all $x \in \Omega_{\text{ext}}$ and $\hat{\theta} \in S^2$. By using identities Eqs. 9 and 10 we then obtain

$$[\mathscr{E}^s(x, -\hat{\theta})]^{\mathrm{T}} - 4\pi \mathcal{E}^\infty(\hat{\theta}, x) = \int_\Gamma \{[\mathscr{H}(z, -\hat{\theta})]^{\mathrm{T}}[\mathcal{E}(z, x) \times v]$$
$$+ [\mathscr{E}(z, -\hat{\theta})]^{\mathrm{T}}[\mathcal{H}(z, x) \times v]\}\, ds(z).$$

Since the columns of $(\mathscr{E}(\cdot, \hat{\theta}), \mathscr{H}(\cdot, \hat{\theta}))$ and $(\mathcal{E}(\cdot, x), \mathcal{H}(\cdot, x))$ satisfy

$$v \times \mathrm{E} = -\{\mathbf{curl}_\Gamma(\eta \mathrm{curl}_\Gamma \mathrm{H}_T) + \lambda \mathrm{H}_T\} \quad \text{on } \Gamma,$$

the previous identity and the symmetry of the operator $\mathbf{curl}_\Gamma(\eta\mathrm{curl}_\Gamma\cdot)+\lambda\cdot$ imply that

$$[\mathscr{E}^s(x,-\hat\theta)]^{\mathrm{T}} - 4\pi\mathscr{E}^\infty(\hat\theta, x) = 0 \quad \text{for all } (x,\hat\theta) \in \Omega_{\mathrm{ext}} \times S^2.$$

$\square$

We have now the necessary tools to prove Theorem 3.1.

Proof of Theorem 3.1 Let us assume that $\Omega_1 \neq \Omega_2$ and let us define $\widetilde\Omega = \mathbb{R}^3 \setminus \overline{\Omega_1 \cup \Omega_2}$. From Rellich's Lemma and the unique continuation principle we have

$$\mathscr{E}_1^s(y,\hat\theta) = \mathscr{E}_2^s(y,\hat\theta) \quad \text{for all } (y,\hat\theta) \in \widetilde\Omega \times S^2.$$

The mixed reciprocity principle (Lemma 3.2) then implies

$$\mathcal{E}_1^\infty(-\hat\theta, y) = \mathcal{E}_2^\infty(-\hat\theta, y) \quad \text{for all } (y,\hat\theta) \in \widetilde\Omega \times S^2.$$

Applying once more Rellich's Lemma and the unique continuation principle we deduce from the last identity that

$$\mathcal{E}_1^s(x, y) = \mathcal{E}_2^s(x, y) \quad \text{for all } (x, y) \in (\widetilde\Omega)^2. \tag{11}$$

We now make use of the singular behaviour of the dipole functions $\mathcal{E}^i$ and $\mathcal{H}^i$ to complete the proof. Let us assume that $\Omega_1 \not\subset \Omega_2$. Then there exists $x_* \in (\Gamma_1 \cap \partial\widetilde\Omega) \setminus \overline{\Omega_2}$ and $r_* > 0$ such that $B(x_*, r_*) \subset \mathbb{R}^3 \setminus \overline{\Omega_2}$. Let us define the sequence of points $z_n := x_* + \nu/n$ which approaches x_* from outside $\widetilde\Omega$ (at least for large n). For n sufficiently large we deduce from Eq. 11 and the boundary condition satisfied by $(\mathcal{E}_1^s(\cdot, z_n), \mathcal{H}_1^s(\cdot, z_n))$ on Γ_1 that

$$\begin{aligned}
&\nu \times \mathcal{E}_2^s(\cdot, z_n) + \mathbf{curl}_\Gamma(\eta\mathrm{curl}_\Gamma\mathcal{H}_{2,T}^s)(\cdot, z_n) + \lambda\mathcal{H}_{2,T}^s(\cdot, z_n) \\
&= \nu \times \mathcal{E}_1^s(\cdot, z_n) + \mathbf{curl}_\Gamma(\eta\mathrm{curl}_\Gamma\mathcal{H}_{1,T}^s)(\cdot, z_n) + \lambda\mathcal{H}_{1,T}^s(\cdot, z_n) \\
&= -\{\nu \times \mathcal{E}^i(\cdot, z_n) + \mathbf{curl}_\Gamma(\eta\mathrm{curl}_\Gamma\mathcal{H}_T^i)(\cdot, z_n) + \lambda\mathcal{H}_T^i(\cdot, z_n)\}
\end{aligned} \tag{12}$$

on $\Gamma_1 \cap B(X_*, r_*)$. Let us denote $P[E, H] := \nu \times E + \mathbf{curl}_\Gamma(\eta\mathrm{curl}_\Gamma H_T) + \lambda H_T$. Since the scattering problem associated with Γ_2 is well posed and since $\eta \in C^1(\Gamma_1)$ and $\lambda \in C^1(\Gamma_1)$, the sequence $P[\mathcal{E}_2^s(\cdot, z_n), \mathcal{H}_2^s(\cdot, z_n)]$ converges in $(L^2(\Gamma_1 \cap B(x_*, r_*)))^{3\times 3}$ towards $P[\mathcal{E}_2^s(\cdot, x_*), \mathcal{H}_2^s(\cdot, x_*)]$ as $n \to \infty$. Moreover, for $x_1 \in \Gamma_1 \cap B(X_*, r_*) \setminus x_*$ we have

$$\lim_{n\to\infty} P[\mathcal{E}^i(x_1, z_n), \mathcal{H}^i(x_1, z_n)] = P[\mathcal{E}^i(x_1, x_*), \mathcal{H}^i(x_1, x_*)].$$

Therefore, by Eq. 12 we deduce that

$$P[\mathcal{E}^i(\cdot, x_*), \mathcal{H}^i(\cdot, x_*)] \in (L^2(\Gamma_1 \cap B(x_*, r_*)))^{3\times 3}.$$

Similarly we obtain that $\mathrm{div}_\Gamma[P[\mathcal{E}^i(\cdot, x_*), \mathcal{H}^i(\cdot, x_*)]] \in (L^2(\Gamma_1 \cap B(x_*, r_*)))^3$ and since Γ_1 is of class C^2, by using Proposition 2.2 we deduce that $\mathcal{E}^i(\cdot, x_*) \in (H^1(\mathbb{R}^3 \setminus \overline{\Omega_1} \cap B(x_*, r_*)))^3$. This is in contradiction with the fact that $\mathcal{E}^i(\cdot, x_*)$ is singular at point x_*. Therefore $\Omega_1 \subset \Omega_2$. The reverse inclusion also holds by symmetry and hence $\Gamma_1 = \Gamma_2$.

$\square$

Remark 3.3 The uniqueness result stated in Theorem 3.1 only rely on the symmetry of the impedance operator **curl**$_\Gamma \eta$curl$_\Gamma + \lambda$ and therefore can be extended to boundary conditions of the form

$$\nu \times \mathrm{E} + \mathcal{Z}\mathrm{H}_T = 0 \quad \text{on } \Gamma$$

where $\mathcal{Z} : V(\Gamma) \to V(\Gamma)^*$ is a linear, bounded and symmetric operator from an Hilbert space $V(\Gamma) \subset H^{-1/2}_{\mathrm{curl}_\Gamma}(\Gamma)$ into its dual space $V(\Gamma)^*$. The operator $\mathcal{Z}$ has also to satisfy sign conditions which ensure that the scattering problem is well-posed (see [11]).

Remark 3.4 For a known shape Γ of class C^2 one can establish uniqueness for the impedance functions λ and η. To be more precise, let us denote Λ a subset of $C^0(\Gamma) \cap C^2(\Gamma)$ that is such that any $(\lambda, \eta) \in \Lambda$ satisfy assumptions of Theorem 2.1. Then the map

$$\Lambda \quad \longrightarrow \quad \left(L^2_t \left(S^2 \times S^2 \right) \right)^3; \qquad (\lambda, \eta) \quad \longmapsto \quad \mathscr{E}^\infty(\hat{x}, \hat{\theta})$$

is injective. The proof relies on a density result for the total magnetic field stipulating that that if $f \in H_{\mathrm{curl}_\Gamma}(\Gamma)^*$ satisfies

$$\int_\Gamma \mathscr{H}(x, \hat{\theta})^\mathrm{T} f(x) \, ds = 0 \quad \text{for all } \hat{\theta} \in S^2$$

then $f = 0$ (see [12, Section 6.2] for further details). The question of identifiability for a finite number of incident waves is still open. Even if one assumes that the boundary is known numerical simulations suggest that a single incident wave is not sufficient to uniquely determine λ and η. See [7] for a discussion of the scalar case.

4 Shape derivative of the far field pattern

For a given direction of incidence $\hat{\theta} \in S^2$ and a given polarisation $\mathbf{p} \in S^2$ let us define the non linear functional

$$T : (\lambda, \eta, \Gamma) \longrightarrow \mathscr{E}^\infty(\cdot, \hat{\theta})\mathbf{p}$$

where $\mathscr{E}^\infty(\cdot, \hat{\theta})\mathbf{p} \in L^2_t(S^2)$ is the far field pattern of the solution to problem (1) with f given by Eq. 2 and $(\mathrm{E}^i, \mathrm{H}^i)$ given by Eq. 6. In the following we assume that $\hat{\theta}$ and $\mathbf{p}$ are fixed and we do not mention explicitly the dependence of T on $\hat{\theta}$ and $\mathbf{p}$. We begin this section by giving an explicit characterization of the Fréchet derivative of T with respect to both, the shape in Theorem 4.1, and the impedance coefficient in Theorem 4.3. We then pursue in Section 4.2 with the proof of Theorem 4.1 and we conclude in Section 4.3 with a numerical validation of the result obtained in Theorem 4.1.

4.1 Characterization of the partial derivatives of the far field pattern

Before giving the derivative of T with respect to Γ, let us introduce some notations and give a precise definition of the shape derivative of the far-field pattern. Let B^∞_r be

the open ball of $(C^{1,\infty}(\mathbb{R}^3))^3$ of radius r where $C^{1,\infty}(\mathbb{R}^3) := C^1(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$ and $(C^{1,\infty}(\mathbb{R}^3))^3$ is endowed with the norm

$$\|V\| := \|V\|_{(L^\infty(\mathbb{R}^3))^3} + \|\nabla V\|_{(L^\infty(\mathbb{R}^3))^{3 \times 3}}.$$

For any $\varepsilon \in B_1^\infty$ we denote $f_\varepsilon := \mathrm{Id} + \varepsilon$ and $\lambda_\varepsilon := \lambda \circ f_\varepsilon^{-1}$, $\eta_\varepsilon := \eta \circ f_\varepsilon^{-1}$. We note that since $\|\varepsilon\| < 1$, the map f_ε is a C^1 diffeomorphism from $\mathbb{R}^3$ into $\mathbb{R}^3$. This map is used to define a perturbed geometry $\Omega_\varepsilon := f_\varepsilon(\Omega)$ with boundary $\Gamma_\varepsilon := f_\varepsilon(\Gamma)$.

For $r < 1$ small enough we define the map

$$T_{\lambda,\eta} \ : \ B_r^\infty \longrightarrow L_t^2(S^2); \quad \varepsilon \longmapsto T(\lambda_\varepsilon, \eta_\varepsilon, \Gamma_\varepsilon).$$

Then, for fixed λ and η, the shape derivative of T at Γ is defined as the Fréchet derivative of $T_{\lambda,\eta}$ at 0 that we denote by $T'_{\lambda,\eta}$. Moreover, we have the following characterization of $T'_{\lambda,\eta}$.

Theorem 4.1 *For an analytic boundary Γ and two analytic functions λ and η, the map $T_{\lambda,\eta}$ is Fréchet differentiable at 0 and its Fréchet derivative is given by*

$$T'_{\lambda,\eta}(0) \cdot \varepsilon = U_\varepsilon^\infty$$

where U_ε^∞ is the far field pattern of the electric field that solves problem (1) with f given by

$$\begin{aligned}
f := & -ik(\nu \cdot \varepsilon)\mathrm{H}_T + \mathbf{curl}_\Gamma[(\nu \cdot \varepsilon)(\nu \cdot \mathrm{E})] + \lambda(\nu \cdot \varepsilon)(2R - 2H_\Gamma Id)\mathrm{H}_T \\
& - \lambda \nabla_\Gamma[(\nu \cdot \varepsilon)(\nu \cdot \mathrm{H})] + 2\mathbf{curl}_\Gamma[H_\Gamma(\nu \cdot \varepsilon)\eta\mathrm{curl}_\Gamma(\mathrm{H}_T)] + ik\mathcal{Z}[(\nu \cdot \varepsilon)\mathcal{Z}\mathrm{H}_T] \\
& + (\nabla_\Gamma \lambda \cdot \varepsilon)\mathrm{H}_T + \mathbf{curl}_\Gamma[(\nabla_\Gamma \eta \cdot \varepsilon)\mathrm{curl}_\Gamma\mathrm{H}_T],
\end{aligned}$$

where $\mathcal{Z}\cdot = \mathbf{curl}_\Gamma(\eta\mathrm{curl}_\Gamma\cdot) + \lambda\cdot$, $R = \nabla_\Gamma\nu$, $2H_\Gamma = \mathrm{div}_\Gamma\nu$, $\mathrm{E} := \mathscr{E}(\cdot, \hat{\theta})\boldsymbol{p}$ and $\mathrm{H} := \mathscr{H}(\cdot, \hat{\theta})\boldsymbol{p}$.

Let us mention that this result coincide with the result obtained in [18, Theorem 3.4] for $\eta = 0$ and λ being constant (($\nabla_\Gamma\lambda \cdot \varepsilon)\mathrm{H}_T = 0$).

Remark 4.2 We would like to emphasise that due to our definition of the shape derivative, its expression contains two non standard terms, namely $(\nabla_\Gamma\lambda \cdot \varepsilon)\mathrm{H}_T$ and $\mathbf{curl}_\Gamma[(\nabla_\Gamma\eta \cdot \varepsilon)\mathrm{curl}_\Gamma\mathrm{H}_T]$ that involve the tangential component of the perturbation ε. It has been numerically demonstrated in [6], for the scalar case, that these additional terms improve the speed of convergence of iterative schemes for simultaneous reconstruction of the impedance coefficients and obstacle shape.

Another possible choice for defining the shape derivative would be to first extend the impedance parameters λ and η in a neighbourhood of Γ and then study the variation of $T(\lambda, \eta, \Gamma)$. In this case the expression of the shape derivative coincides with the one in Theorem 4.1 but without the above mentioned additional terms.

For the sake of completeness we also give the derivative of T with respect to λ and η, with Γ being fixed. We introduce the functional

$$T_\Gamma \ : \ \Lambda \longrightarrow L_t^2(S^2); \quad (\lambda, \eta) \longmapsto T(\lambda, \eta, \Gamma),$$

where $\Lambda \subset (L^\infty(\Gamma))^2$ is the subset of (λ, η) satisfying the hypothesis of Theorem 2.1. The Fréchet derivative of T_Γ is given below.

Theorem 4.3 *The function T_Γ is Fréchet differentiable on Λ and for all $(\lambda, \eta) \in \Lambda$ its Fréchet derivative is the map $T_\Gamma'(\lambda, \eta) : L^\infty(\Gamma) \times L^\infty(\Gamma) \to (L_t^2(S^2))^3$ defined by*

$$T_\Gamma'(\lambda, \eta) \cdot (h, l) = U_{h,l}^\infty \quad \text{for all } (h, l) \in (L^\infty(\Gamma))^2,$$

where $U_{h,l}^\infty$ is the far field pattern of the solution to problem (1) *with*

$$f = -(\mathbf{curl}_\Gamma(l\mathrm{curl}_\Gamma \mathrm{H}_T) + h\mathrm{H}_T)$$

and $\mathrm{H} := \mathscr{H}^s(\cdot, \hat{\theta})\boldsymbol{p} + \mathscr{H}^i(\cdot, \hat{\theta})\boldsymbol{p}$.

Proof The proof of this result is very similar to the proof of Proposition 6 in [7] in the scalar case and is not detailed here. See also [12]. □

4.2 Proof of Theorem 4.1

In this section we assume that λ, η and Γ are analytic and characterise the shape derivative of the far field pattern. We chose to follow the constructive procedure first proposed in [25] and then extended in [18] to treat the case of a classical impedance boundary condition ($\eta = 0$ and λ being constant).

We recall that $\mathscr{E}^s(\cdot, \hat{\theta})\boldsymbol{p}$ is the solution to problem (1) with f being given by Eq. 2 for a given incident field $(\mathscr{E}^i(\cdot, \hat{\theta})\boldsymbol{p}, \mathscr{H}^i(\cdot, \hat{\theta})\boldsymbol{p})$. Moreover, we denote $(\mathscr{E}_\varepsilon^s(\cdot, \hat{\theta})\boldsymbol{p}, \mathscr{H}_\varepsilon^s(\cdot, \hat{\theta})\boldsymbol{p})$ the scattered field induced by the scattering of plane waves on Γ_ε that solves

$$\begin{cases} \mathbf{curl}\, \mathrm{H}_\varepsilon + ik\mathrm{E}_\varepsilon = 0 & \text{in } \Omega_{\text{ext}}^\varepsilon, \\ \mathbf{curl}\, \mathrm{E}_\varepsilon - ik\mathrm{H}_\varepsilon = 0 & \text{in } \Omega_{\text{ext}}^\varepsilon, \\ \nu_\varepsilon \times \mathrm{E}_\varepsilon + \mathbf{curl}_{\Gamma_\varepsilon}(\eta_\varepsilon \mathrm{curl}_{\Gamma_\varepsilon} \mathrm{H}_{\varepsilon, T_\varepsilon}) + \lambda_\varepsilon \mathrm{H}_{\varepsilon, T_\varepsilon} = f & \text{on } \Gamma_\varepsilon, \\ \lim_{R \to \infty} \int_{\partial B_R} |\mathrm{H}_\varepsilon \times \hat{x} - (\hat{x} \times \mathrm{E}_\varepsilon) \times \hat{x}|^2 ds = 0, \end{cases}$$

where ν_ε is the outward normal to Γ_ε, $\Omega_{\text{ext}}^\varepsilon := \mathbb{R}^3 \setminus \overline{\Omega}_\varepsilon$ is the exterior domain to Ω_ε and f is given by

$$f := -\left(\nu_\varepsilon \times \mathscr{E}^i(\cdot, \hat{\theta})\boldsymbol{p} + \mathbf{curl}_\Gamma(\eta_\varepsilon \mathrm{curl}_\Gamma \mathscr{H}_{T_\varepsilon}^i(\cdot, \hat{\theta})\boldsymbol{p}) + \lambda_\varepsilon \mathscr{H}_{T_\varepsilon}^i(\cdot, \hat{\theta})\boldsymbol{p}\right).$$

For all column vectors or matrices V defined on Γ_ε, we denote $V_{T_\varepsilon} := (\nu_\varepsilon \times V) \times \nu_\varepsilon$.

The scheme of the proof mainly consists in writing an appropriate integral representation formula for the difference $\mathscr{E}_\varepsilon^s(\cdot, \hat{\theta})\boldsymbol{p} - \mathscr{E}^s(\cdot, \hat{\theta})\boldsymbol{p}$ on Γ_ε and then performing an asymptotic expansion for small ε of this integral representation. In the remaining of the section we do not mention the dependence of the electromagnetic fields on the position x, the direction of incidence $\hat{\theta}$ and the polarisation $\boldsymbol{p}$. The notations $(\mathrm{E}^s, \mathrm{H}^s)$ and $(\mathrm{E}_\varepsilon^s, \mathrm{H}_\varepsilon^s)$ always refer to the scattered field associated with a plane wave of direction of incidence $\hat{\theta}$ and polarisation $\boldsymbol{p}$ given by Eq. 6. The notations $\mathcal{E}$ and $\mathcal{H}$ refer to the matrix representations of the total fields associated with the scattering of

an electromagnetic dipole. To begin with, we assume that $\overline{\Omega} \subset \Omega_\varepsilon$ and give a first representation formula for $\mathrm{E}_\varepsilon^s - \mathrm{E}^s$.

Lemma 4.4 *For any $z \in \mathbb{R}^3 \setminus \overline{\Omega_\varepsilon}$,*

$$\mathrm{E}_\varepsilon^s(z) - \mathrm{E}^s(z) = -\int_{\Gamma_\varepsilon} [\nu_\varepsilon \times \mathcal{E}(y,z) + \mathbf{curl}_{\Gamma_\varepsilon}(\eta_\varepsilon \mathrm{curl}_{\Gamma_\varepsilon} \mathcal{H}_{T_\varepsilon}(y,z))$$
$$+ \lambda_\varepsilon \mathcal{H}_{T_\varepsilon}(y,z)]^{\mathrm{T}} \mathrm{H}_\varepsilon(y)\, ds(y). \tag{13}$$

Proof Let $z \in \mathbb{R}^3 \setminus \overline{\Omega_\varepsilon}$. From the Stratton Chu integral representation (7) we first have

$$\mathrm{E}^s(z) = \int_\Gamma \{[\mathcal{E}^i(y,z)]^{\mathrm{T}}[\nu(y) \times \mathrm{H}^s(y)] + [\mathcal{H}^i(y,z)]^{\mathrm{T}}[\nu(y) \times \mathrm{E}^s(y)]\}\, ds(y),$$

which can be equivalently written as (using properties Eqs. 9 and 10)

$$\mathrm{E}^s(z) = -\int_\Gamma \{[\mathcal{E}^s(y,z)]^{\mathrm{T}}[\nu(y) \times \mathrm{H}^i(y)] + [\mathcal{H}^s(y,z)]^{\mathrm{T}}[\nu(y) \times \mathrm{E}^i(y)]\}\, ds(y)$$
$$+ \int_\Gamma \{[\mathcal{E}(y,z)]^{\mathrm{T}}[\nu(y) \times \mathrm{H}(y)] + [\mathcal{H}(y,z)]^{\mathrm{T}}[\nu(y) \times \mathrm{E}(y)]\}\, ds(y).$$

In this expression the second line is indeed equal to zero from the boundary conditions satisfied by $(\mathcal{E}, \mathcal{H})$ and (E, H) on Γ. Gauss' divergence theorem applied in $\Omega_\varepsilon \setminus \overline{\Omega}$ then implies

$$\mathrm{E}^s(z) = -\int_{\Gamma_\varepsilon} \{[\mathcal{E}^s(y,z)]^{\mathrm{T}}[\nu_\varepsilon(y) \times \mathrm{H}^i(y)] + [\mathcal{H}^s(y,z)]^{\mathrm{T}}[\nu_\varepsilon(y) \times \mathrm{E}^i(y)]\}\, ds(y).$$

Using Eq. 10 we finally obtain

$$\mathrm{E}^s(z) = -\int_{\Gamma_\varepsilon} \{[\mathcal{E}^s(y,z)]^{\mathrm{T}}[\nu_\varepsilon(y) \times \mathrm{H}_\varepsilon(y)] + [\mathcal{H}^s(y,z)]^{\mathrm{T}}[\nu_\varepsilon(y) \times \mathrm{E}_\varepsilon(y)]\}\, ds(y).$$

On the other hand, by Stratton-Chu formula (7) and by using Eq. 10 we have for E_ε^s

$$\mathrm{E}_\varepsilon^s(z) = \int_{\Gamma_\varepsilon} \{[\mathcal{E}^i(y,z)]^{\mathrm{T}}[\nu_\varepsilon(y) \times \mathrm{H}_\varepsilon(y)] + [\mathcal{H}^i(y,z)]^{\mathrm{T}}[\nu_\varepsilon(y) \times \mathrm{E}_\varepsilon(y)]\}\, ds(y).$$

By adding up the two last equations we obtain

$$\mathrm{E}_\varepsilon^s(z) - \mathrm{E}^s(z) = \int_{\Gamma_\varepsilon} \{[\mathcal{E}(y,z)]^{\mathrm{T}}[\nu_\varepsilon(y) \times \mathrm{H}_\varepsilon(y)] + [\mathcal{H}(y,z)]^{\mathrm{T}}[\nu_\varepsilon(y) \times \mathrm{E}_\varepsilon(y)]\}\, ds(y).$$

We conclude the proof by using the boundary conditions satisfied by $(\mathrm{E}_\varepsilon, \mathrm{H}_\varepsilon)$ on Γ_ε together with two integrations by part on Γ_ε. $\square$

We can actually substitute H_ε with H in formula (13) with an error of order $\|\varepsilon\|^2$. This relies of on the following continuity result.

Lemma 4.5 *Let B_R be a ball of radius R sufficiently large so that $\overline{\Omega} \subset B_{R/2}$. It exists $C_R > 0$ such that for $\varepsilon \in B_1^\infty$ small enough we have*

$$\|\mathrm{E}_\varepsilon \circ f_\varepsilon - \mathrm{E}\|_{(H_{\mathbf{curl}}(B_R \setminus \overline{\Omega}))^3} \le C_R \|\varepsilon\|$$

and

$$\|H_\varepsilon \circ f_\varepsilon - H\|_{(H_{\mathrm{curl}_\Gamma}(\Gamma))^3} + \|H_\varepsilon \circ f_\varepsilon - H\|_{(H_{\mathbf{curl}}(B_R\setminus\overline{\Omega}))^3} \leq C_R\|\varepsilon\|.$$

Proof The proof of this result is rather straightforward. We only give here the two main ingredients. We first use that the forward problem has a unique solution that depends continuously on the boundary right hand side. We second use that the boundary operator $\mathbf{curl}_{\Gamma_\varepsilon}\eta_\varepsilon\mathrm{curl}_{\Gamma_\varepsilon} + \lambda_\varepsilon$ has a Lipschitz continuous dependence with respect to the perturbation ε. A detailed proof of a similar result in the case of Helmholtz' equation can be found in [5]. See also [12]. □

In the sequel we denote by $\mathcal{O}(\cdot)$ a $C^\infty([0, +\infty[)$ function for which there exists $C > 0$ such that for all $x \in [0, +\infty[$

$$|\mathcal{O}(x)| \leq C|x|.$$

Then by using the two previous lemmas, we obtain the following result.

Lemma 4.6 *The following representation formula holds,*

$$E_\varepsilon^s(z) - E^s(z) = -\int_{\Gamma_\varepsilon} [\nu_\varepsilon \times \mathcal{E}(y, z) + \mathbf{curl}_{\Gamma_\varepsilon}(\eta_\varepsilon\mathrm{curl}_{\Gamma_\varepsilon}\mathcal{H}_{T_\varepsilon}(y, z))$$

$$+\lambda_\varepsilon\mathcal{H}_{T_\varepsilon}(y, z)]^{\mathrm{T}}H(y)\, ds(y) + O(\|\varepsilon\|^2),$$

uniformly for z in a compact set $K \subset \mathbb{R}^3 \setminus \overline{\Omega_\varepsilon}$.

Proof The proof relies on the continuity result of Lemma 4.5. We refer to Lemma 4.3 in [6] in the scalar case and to [12]. □

Before we proceed any further with the proof of Theorem 4.1 we need to define an appropriate extension of λ_ε, η_ε, ν_ε and $\mathrm{curl}_{\Gamma_\varepsilon}$ in the domain $\Omega_\varepsilon \setminus \overline{\Omega}$. For any $x_0 \in \Gamma$, there exists a local parametrisation of Γ, i.e. two open sets $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^3$ which are neighbourhoods of 0 and x_0 respectively as well as a function $\varphi \in C^1(U; V)$ such that $\varphi(0) = x_0$ and

$$\Gamma \cap V = \{\varphi(\xi) : \xi \in U\}.$$

For $t \in [0, 1]$ let us define

$$f_t := \mathrm{Id} + t\varepsilon, \quad \varphi_t := f_t \circ \varphi.$$

For readability, the dependence of f_t and φ_t on ε is not written explicitly. The function φ_t defines a parametrisation of $\Gamma_t := (\mathrm{Id} + t\varepsilon)(\Gamma)$ around $x_0^t := f_t(x_0)$, and the vectors

$$e_j^t := \frac{\partial\varphi_t}{\partial\xi_j} = (\mathrm{Id} + t\nabla\varepsilon)\frac{\partial\varphi}{\partial\xi_j} = (\mathrm{Id} + t\nabla\varepsilon)e_j, \quad \text{for } j = 1, 2 \qquad (14)$$

define a basis of tangent plane to Γ_t at x_0^t where $(\nabla\varepsilon)_{i,j} = \partial\varepsilon_i/\partial x_j$ is the Jacobian matrix of ε. We then define the associated covariant basis (e_t^i) on Γ_t at point x_0^t by

$$e_t^i \cdot e_j^t = \delta^i_j, \quad \text{pour} \quad i, j = 1, 2. \qquad (15)$$

The outward unit normal to Γ_t at point x_0^t is then given by

$$\nu_t = \frac{e_1^t \times e_2^t}{|e_1^t \times e_2^t|}.$$

We extend the surface scalar curl by using the following formula

$$\text{curl}_{\Gamma_t} := \nu_t \cdot \mathbf{curl}$$

and for a row vector field or a matrix V defined on Γ_t we define

$$V_{T_t} := (\nu_t \times V) \times \nu_t$$

the tangent component of V on Γ_t. Finally, the impedance functions λ_ε and η_ε are extended in $\Omega_\varepsilon \setminus \overline{\Omega}$ in the following consistent way:

$$\lambda_t := \lambda \circ f_t^{-1}, \quad \eta_t := \eta \circ f_t^{-1}.$$

Let us recall the technical Lemma 4.5 from [6].

Lemma 4.7 *Let λ be a $C^1(\Gamma)$ function and define $\lambda_t := \lambda \circ f_t^{-1}$. Then the following identity holds on Γ,*

$$(\varepsilon \cdot \nu)(\nabla \lambda_t \cdot \nu_t)|_{t=0} = -(\nabla_\Gamma \lambda \cdot \varepsilon).$$

Lemma 4.8 *The representation formula of Lemma 4.6 yields*

$$E_\varepsilon^s(z) - E^s(z) = - \int_\Gamma (\varepsilon \cdot \nu) \text{div}\{[\mathcal{E} \times H]^{\mathrm{T}} + \eta_t [\nu_t \text{curl}_{\Gamma_t} \mathcal{H}_{T_t}]^{\mathrm{T}} \text{curl}_{\Gamma_t} H_{T_t}$$
$$+ \lambda_t [\mathcal{H} \times (\nu_t \times H)]^{\mathrm{T}}\}|_{t=0} ds + \mathcal{O}(\|\varepsilon\|^2),$$

uniformly for z in a compact set $K \subset \mathbb{R}^3 \setminus \overline{\Omega_\varepsilon}$.

Proof From Lemma 4.6 we have

$$E_\varepsilon^s(z) - E^s(z) = - \int_{\Gamma_\varepsilon} [\nu_\varepsilon \times \mathcal{E} + \mathbf{curl}_{\Gamma_\varepsilon}(\eta_\varepsilon \text{curl}_{\Gamma_\varepsilon} \mathcal{H}_{T_\varepsilon}) + \lambda_\varepsilon \mathcal{H}_{T_\varepsilon}]^{\mathrm{T}} H(y) \, ds(y)$$
$$= - \int_{\Gamma_\varepsilon} \{[\mathcal{E} \times H]^{\mathrm{T}} \nu_\varepsilon + \eta_\varepsilon [\text{curl}_{\Gamma_\varepsilon} \mathcal{H}_{T_\varepsilon}]^{\mathrm{T}} \text{curl}_{\Gamma_\varepsilon} H_{T_\varepsilon}$$
$$+ \lambda_\varepsilon [\mathcal{H} \times (\nu_\varepsilon \times H)]^{\mathrm{T}} \nu_\varepsilon\} \, ds + \mathcal{O}(\|\varepsilon\|^2).$$

The Gauss divergence theorem together with the boundary conditions satisfied by $(\mathcal{E}, \mathcal{H})$ on Γ imply

$$E_\varepsilon^s(z) - E^s(z) = - \int_{\Omega_\varepsilon \setminus \overline{\Omega}} \text{div}\{[\mathcal{E} \times H]^{\mathrm{T}} + \eta_t [\nu_t \text{curl}_{\Gamma_t} \mathcal{H}_{T_t}]^{\mathrm{T}} \text{curl}_{\Gamma_t} H_{T_t}$$
$$+ \lambda_t [\mathcal{H} \times (\nu_t \times H)]^{\mathrm{T}}\} dx + \mathcal{O}(\|\varepsilon\|^2).$$

By using the change of variable $(x_\Gamma, t) \mapsto x_\Gamma + t\varepsilon(x_\Gamma)$ between $\Gamma \times (0, 1)$ and $\Omega_\varepsilon \setminus \overline{\Omega}$, and a Taylor expansion, this integral becomes after regrouping the $\mathcal{O}(\|\varepsilon\|^2)$ terms

$$E_\varepsilon^s(z) - E^s(z) = - \int_\Gamma (\varepsilon \cdot \nu) \mathrm{div}\{[\mathcal{E} \times H]^\mathrm{T} + \eta_t[\nu_t \mathrm{curl}_{\Gamma_t} \mathcal{H}_{T_t}]^\mathrm{T} \mathrm{curl}_{\Gamma_t} H_{T_t}$$
$$+ \lambda_t[\mathcal{H} \times (\nu_t \times H)]^\mathrm{T}\}|_{t=0} ds + \mathcal{O}(\|\varepsilon\|^2).$$

$\square$

In order to express the divergence term in Lemma 4.8 we need the following technical lemma which is proven in the

Lemma 4.9 *For $V \in (C^2(\overline{\Omega_{\mathrm{ext}}}))^3$,*

$$(\varepsilon \cdot \nu)\mathbf{curl}(\nu_t \times V)|_{t=0} = (\varepsilon \cdot \nu)(\mathrm{div}_\Gamma V_T)\nu + (\varepsilon \cdot \nu)\left(R - 2H_\Gamma Id - \frac{\partial}{\partial \nu}\right)V_T + [\nabla_\Gamma(\nu \cdot \varepsilon) \times V] \times \nu$$

and

$$(\varepsilon \cdot \nu)\frac{\partial}{\partial \nu}(\mathrm{curl}_{\Gamma_t} V_{T_t})|_{t=0} = -(\varepsilon \cdot \nu)\mathrm{div}_\Gamma(\mathbf{curl}\, V)_T - 2H_\Gamma(\varepsilon \cdot \nu)\mathrm{curl}_\Gamma V_T - \nabla_\Gamma(\nu \cdot \varepsilon) \cdot (\mathbf{curl}\, V),$$

where $R = \nabla_\Gamma \nu$ and $2H_\Gamma = \mathrm{div}_\Gamma \nu$.

We shall now use this lemma to express d the integral representation of $E_\varepsilon^s - E^s$ obtained in Lemma 4.8 in terms of boundary values of E and H. We split the process into two parts: Lemma 4.10 and Lemma 4.11.

Lemma 4.10 *The following first identity holds for $(\mathcal{E}, \mathcal{H})$ and (E, H)*

$$\int_\Gamma (\nu \cdot \varepsilon)\mathrm{div}\{[\mathcal{E} \times H + \lambda_t \mathcal{H} \times (\nu_t \times H)]^\mathrm{T}\}|_{t=0} ds = \int_\Gamma \mathcal{H}^\mathrm{T}\{ik(\nu \cdot \varepsilon)H_T$$
$$- ik\mathcal{Z}[(\nu \cdot \varepsilon)(\nu \times E)] - \mathbf{curl}_\Gamma[(\nu \cdot \varepsilon)(\nu \cdot E)] - \lambda(\nu \cdot \varepsilon)(2R - 2H_\Gamma)H_T$$
$$- ik\lambda(\nu \cdot \varepsilon)\mathcal{Z}H_T + \lambda\nabla_\Gamma[(\nu \cdot \varepsilon)(\nu \cdot H)] - ik\mathcal{Z}[\lambda(\nu \cdot \varepsilon)H_T]\}\, ds.$$

Proof Let us first recall that for two vector fields V and W and for a function φ,

$$\mathrm{div}(V \times W) = \mathbf{curl}\, V \cdot W - V \cdot \mathbf{curl}\, W, \quad \mathbf{curl}(\varphi V) = \nabla\varphi \times V + \varphi\mathbf{curl}\, V.$$

Therefore, from Maxwell's equations

$$\mathrm{div}([\mathcal{E} \times H]^\mathrm{T})|_{t=0} = ik(\mathcal{H}^\mathrm{T}H + \mathcal{E}^\mathrm{T}E)|_\Gamma.$$

From the boundary conditions satisfied by (E, H) and the formula $\mathrm{curl}_\Gamma V = \nu \cdot \mathbf{curl}\, V$ we have

$$\mathcal{E}^\mathrm{T}E = [\nu \times \mathcal{E}]^\mathrm{T}(\nu \times E) + (\nu \cdot E)[\mathcal{E}^\mathrm{T}\nu] = -[\mathcal{Z}\mathcal{H}_T]^\mathrm{T}(\nu \times E) - \frac{1}{ik}(\nu \cdot E)[\mathrm{curl}_\Gamma \mathcal{H}_T]^\mathrm{T}$$

where $\mathcal{Z} = \mathbf{curl}_\Gamma \eta\mathrm{curl}_\Gamma + \lambda$. Similarly

$$\mathrm{div}\{\lambda_t[\mathcal{H} \times (\nu_t \times H)]^T\}|_{t=0} = [\mathbf{curl}\, \mathcal{H}|_\Gamma]^\mathrm{T}(\lambda_t\nu \times H)|_\Gamma - \mathcal{H}^\mathrm{T}|_\Gamma\mathbf{curl}[\lambda_t(\nu_t \times H)]|_{t=0}$$
$$= -ik\lambda[\mathcal{E}^\mathrm{T}(\nu \times H)]|_\Gamma - \mathcal{H}^\mathrm{T}|_\Gamma\mathbf{curl}[\nu_t \times (\lambda_t H)]|_{t=0}.$$

By using the first formula of Lemma 4.9 we obtain

$$\text{div}\{\lambda_t[\mathcal{H} \times (\nu_t \times H)]^T\}|_{t=0} = -ik\lambda\mathcal{E}^T(\nu \times H)|_\Gamma$$
$$-\mathcal{H}^T\left\{\text{div}_\Gamma(\lambda H_T)\nu + \left(R - 2H_\Gamma - \frac{\partial}{\partial\nu}\right)(\lambda_t H_T)|_{t=0} + \frac{\lambda}{\nu \cdot \varepsilon}[\nabla_\Gamma(\nu \cdot \varepsilon) \times H] \times \nu\right\}$$

and by using the boundary conditions

$$\text{div}\{\lambda_t[\mathcal{H} \times (\nu_t \times H)]^T\}|_{t=0} = -ik\lambda[(\mathcal{Z}H_T)^T H_T]|_\Gamma$$
$$-\mathcal{H}^T\left\{\text{div}_\Gamma(\lambda H_T)\nu + \left(R - 2H_\Gamma - \frac{\partial}{\partial\nu}\right)(\lambda_t H_T)|_{t=0} + \frac{\lambda}{\nu \cdot \varepsilon}[\nabla_\Gamma(\nu \cdot \varepsilon) \times H] \times \nu\right\}.$$

We expand $\frac{\partial(\lambda_t H_T)}{\partial\nu}|_{t=0}$ by using Lemma 4.7

$$\text{div}\{\lambda_t[\mathcal{H} \times (\nu_t \times H)]^T\}|_{t=0} = -ik\lambda(\mathcal{Z}H_T)^T H_T) - \frac{\nabla_\Gamma\lambda \cdot \varepsilon}{\nu \cdot \varepsilon}\mathcal{H}^T H_T$$
$$-\mathcal{H}^T\left\{\text{div}_\Gamma(\lambda H_T)\nu + \lambda\left(R - 2H_\Gamma - \frac{\partial}{\partial\nu}\right)H_T + \frac{\lambda}{\nu \cdot \varepsilon}[\nabla_\Gamma(\nu \cdot \varepsilon) \times H] \times \nu\right\}.$$

From [28, formula 2.5.225]

$$-\frac{\partial}{\partial\nu}H_T = \nu \times \mathbf{curl}\,H - \nabla_\Gamma(\nu \cdot H) + RH_T,$$

we get, using the boundary conditions,

$$-\frac{\partial}{\partial\nu}H_T = ik\mathcal{Z}H_T - \nabla_\Gamma(\nu \cdot H) + RH_T.$$

The identity $U \times (V \times W) = (U \cdot W)V - (U \cdot V)W$ yields

$$-\lambda[\nabla_\Gamma(\nu \cdot \varepsilon) \times H] \times \nu = \lambda(\nabla_\Gamma(\nu \cdot \varepsilon))(\nu \cdot H),$$

whence

$$-\lambda[\nabla_\Gamma(\nu \cdot \varepsilon) \times H] \times \nu + \lambda(\nu \cdot \varepsilon)\nabla_\Gamma(\nu \cdot H) = \lambda\nabla_\Gamma[(\nu \cdot \varepsilon)(\nu \cdot H)].$$

We finally get, using the symmetry of $\mathcal{Z}$,

$$\int_\Gamma (\nu \cdot \varepsilon)\text{div}\{[\mathcal{E} \times H + \lambda_t\mathcal{H} \times (\nu_t \times H)]^T\}|_{t=0}ds = \int_\Gamma \mathcal{H}^T\{ik(\nu \cdot \varepsilon)H - (\nu \cdot \varepsilon)\text{div}_\Gamma(\lambda H_T)\nu$$
$$-ik\mathcal{Z}[(\nu \cdot \varepsilon)(\nu \times E)] - \mathbf{curl}_\Gamma[(\nu \cdot \varepsilon)(\nu \cdot E)] - \lambda(\nu \cdot \varepsilon)(2R - 2H_\Gamma)H_T$$
$$-ik\lambda(\nu \cdot \varepsilon)\mathcal{Z}H_T + \lambda\nabla_\Gamma[(\nu \cdot \varepsilon)(\nu \cdot H)] - ik\mathcal{Z}[\lambda(\nu \cdot \varepsilon)H_T] - (\nabla_\Gamma\lambda \cdot \varepsilon)H_T\}\,ds.$$

We conclude the proof by noticing that, using Maxwell's equations,

$$ik(\nu \cdot H) - \text{div}_\Gamma(\lambda H_T) = (-\text{div}_\Gamma(\nu \times E) - \text{div}_\Gamma(\lambda H_T)) = \text{div}_\Gamma(\mathbf{curl}_\Gamma(\eta\text{curl}_\Gamma H_T)) = 0$$

since $\text{div}_\Gamma\mathbf{curl}_\Gamma = 0$.   □

Lemma 4.11 *The following second identity holds for* $(\mathcal{E}, \mathcal{H})$ *and* (E, H)

$$\int_\Gamma (\nu \cdot \varepsilon)\mathrm{div}\{\eta_t[\nu_t\mathrm{curl}_{\Gamma_t}\mathcal{H}_{T_t}]^\mathrm{T}\mathrm{curl}_{\Gamma_t}\mathcal{H}_{T_t}\}|_{t=0}ds$$

$$= \int_\Gamma \mathcal{H}^\mathrm{T}\{-2\mathbf{curl}_\Gamma[H_\Gamma(\nu \cdot \varepsilon)\eta\mathrm{curl}_\Gamma(H_T)]$$

$$- ik\mathcal{Z}[(\nu \cdot \varepsilon)\mathbf{curl}_\Gamma(\eta\mathrm{curl}_\Gamma H_T)] - ik\mathbf{curl}_\Gamma[\eta\mathrm{curl}_\Gamma[(\nu \cdot \varepsilon)\mathcal{Z}H_T]]$$

$$- \mathbf{curl}_\Gamma[(\nabla_\Gamma\eta \cdot \varepsilon)\mathrm{curl}_\Gamma H_T]\}ds\}ds.$$

Proof First of all

$$\mathrm{div}\{\eta_t[\nu_t\mathrm{curl}_{\Gamma_t}\mathcal{H}_{T_t}]^\mathrm{T}\mathrm{curl}_{\Gamma_t}\mathcal{H}_{T_t}\}|_{t=0} = 2H_\Gamma\eta\mathrm{curl}_\Gamma H_T[\mathrm{curl}_\Gamma\mathcal{H}_T]^\mathrm{T}$$

$$+ \frac{\partial\eta_t}{\partial\nu}|_{t=0}\mathrm{curl}_\Gamma H_T[\mathrm{curl}_\Gamma\mathcal{H}_T]^\mathrm{T} + \eta\frac{\partial}{\partial\nu}(\mathrm{curl}_{\Gamma_t}\mathcal{H}_{T_t}[\mathrm{curl}_{\Gamma_t}\mathcal{H}_{T_t}]^\mathrm{T})|_{t=0},$$

where by using Lemma 4.7 once more we have

$$\frac{\partial\eta_t}{\partial\nu}|_{t=0} = \frac{\nabla_\Gamma\eta \cdot \varepsilon}{\nu \cdot \varepsilon}.$$

From the second equality in Lemma 4.9 we have by using Maxwell's equations

$$\frac{\partial}{\partial\nu}(\mathrm{curl}_{\Gamma_t}H_{T_t}\mathrm{curl}_{\Gamma_t}\mathcal{H}_{T_t})|_{t=0}$$

$$= \mathrm{curl}_\Gamma H_T\frac{\partial}{\partial\nu}(\mathrm{curl}_{\Gamma_t}\mathcal{H}_{T_t})|_{t=0} + \mathrm{curl}_\Gamma\mathcal{H}_T\frac{\partial}{\partial\nu}(\mathrm{curl}_{\Gamma_t}H_{T_t})|_{t=0}$$

$$= \mathrm{curl}_\Gamma H_T\left\{ik\mathrm{div}_\Gamma\mathcal{E}_T - 2H_\Gamma\mathrm{curl}_\Gamma\mathcal{H}_T + \frac{ik}{\nu \cdot \varepsilon}\mathcal{E}^\mathrm{T}\nabla_\Gamma(\nu \cdot \varepsilon)\right\}$$

$$+ \mathrm{curl}_\Gamma\mathcal{H}_T\left\{ik\mathrm{div}_\Gamma E_T - 2H_\Gamma\mathrm{curl}_\Gamma H_T + \frac{ik}{\nu \cdot \varepsilon}\nabla_\Gamma(\nu \cdot \varepsilon) \cdot E\right\}.$$

The boundary conditions satisfied by (E, H) and $(\mathcal{E}, \mathcal{H})$ implies

$$\frac{\partial}{\partial\nu}(\mathrm{curl}_{\Gamma_t}H_{T_t}\mathrm{curl}_{\Gamma_t}\mathcal{H}_{T_t})|_{t=0}$$

$$= \mathrm{curl}_\Gamma H_T\left\{-ik\mathrm{curl}_\Gamma(\mathcal{Z}\mathcal{H}_T) - 2H_\Gamma\mathrm{curl}_\Gamma\mathcal{H}_T + \frac{ik}{\nu \cdot \varepsilon}[\mathcal{Z}\mathcal{H}_T]^\mathrm{T}\mathbf{curl}_\Gamma(\nu \cdot \varepsilon)\right\}$$

$$+ \mathrm{curl}_\Gamma\mathcal{H}_T\left\{-ik\mathrm{curl}_\Gamma(\mathcal{Z}H_T) - 2H_\Gamma\mathrm{curl}_\Gamma H_T + \frac{ik}{\nu \cdot \varepsilon}(\mathcal{Z}H_T) \cdot \mathbf{curl}_\Gamma(\nu \cdot \varepsilon)\right\}.$$

We recall that for a function φ and a vector field V we have $\mathrm{curl}_\Gamma(\varphi V) = -V \cdot \mathbf{curl}_\Gamma(\varphi) + \varphi\mathrm{curl}_\Gamma(V)$, therefore the previous relation gives

$$\frac{\partial}{\partial\nu}(\mathrm{curl}_{\Gamma_t}H_{T_t}\mathrm{curl}_{\Gamma_t}\mathcal{H}_{T_t})|_{t=0} = -4H_\Gamma\mathrm{curl}_\Gamma\mathcal{H}_T\mathrm{curl}_\Gamma H_T$$

$$- \frac{ik}{\nu \cdot \varepsilon}\mathrm{curl}_\Gamma[(\nu \cdot \varepsilon)\mathcal{Z}\mathcal{H}_T]\mathrm{curl}_\Gamma H_T - \frac{ik}{\nu \cdot \varepsilon}\mathrm{curl}_\Gamma\mathcal{H}_T\mathrm{curl}_\Gamma[(\nu \cdot \varepsilon)\mathcal{Z}H_T].$$

Therefore

$$\int_\Gamma (\nu \cdot \varepsilon)\mathrm{div}\{\eta_t[\nu_t\mathrm{curl}_{\Gamma_t}\mathcal{H}_{T_t}]^{\mathrm{T}}\mathrm{curl}_{\Gamma_t}H_{T_t}\}|_{t=0}ds$$

$$= \int_\Gamma \{-2H_\Gamma(\nu\cdot\varepsilon)\eta[\mathrm{curl}_\Gamma\mathcal{H}_T]^{\mathrm{T}}\mathrm{curl}_\Gamma H_T - ik\eta[\mathrm{curl}_\Gamma[(\nu\cdot\varepsilon)\mathcal{Z}\mathcal{H}_T]]^{\mathrm{T}}\mathrm{curl}_\Gamma H_T$$

$$- ik\eta[\mathrm{curl}_\Gamma\mathcal{H}_T]^{\mathrm{T}}\mathrm{curl}_\Gamma[(\nu\cdot\varepsilon)\mathcal{Z}H_T] - (\nabla_\Gamma\eta\cdot\varepsilon)[\mathrm{curl}_\Gamma\mathcal{H}_T]^{\mathrm{T}}\mathrm{curl}_\Gamma H_T\}ds$$

and integration by part on Γ gives the final expression. $\qquad\square$

Gathering the results of Lemmas 4.10 and 4.11 we obtain the following proposition.

Proposition 4.12 *We have the following formula for the discrepancy between* E_ε^s *and* E^s:

$$\mathrm{E}_\varepsilon^s(z) - \mathrm{E}^s(z) = \int_\Gamma \mathcal{H}(\cdot, z)^{\mathrm{T}} B_\varepsilon(\mathrm{E}, \mathrm{H})ds + \mathcal{O}(\|\varepsilon\|^2)$$

uniformly for z in a compact set $K \subset \mathbb{R}^3 \setminus \overline{\Omega_\varepsilon}$ *where*

$$B_\varepsilon(\mathrm{E}, \mathrm{H}) := -ik(\nu\cdot\varepsilon)\mathrm{H}_T + \mathbf{curl}_\Gamma[(\nu\cdot\varepsilon)(\nu\cdot\mathrm{E})] + \lambda(\nu\cdot\varepsilon)(2R - 2H_\Gamma)\mathrm{H}_T$$

$$- \lambda\nabla_\Gamma[(\nu\cdot\varepsilon)(\nu\cdot\mathrm{H})] + 2\mathbf{curl}_\Gamma[H_\Gamma(\nu\cdot\varepsilon)\eta\mathrm{curl}_\Gamma\mathrm{H}_T] + ik\mathcal{Z}[(\nu\cdot\varepsilon)\mathcal{Z}\mathrm{H}_T]$$

$$+ (\nabla_\Gamma\lambda\cdot\varepsilon)\mathrm{H}_T + \mathbf{curl}_\Gamma[(\nabla_\Gamma\eta\cdot\varepsilon)\mathrm{curl}_\Gamma\mathrm{H}_T].$$

We recall that $\mathcal{Z}\cdot = \mathbf{curl}_\Gamma(\eta\mathrm{curl}_\Gamma\cdot) + \lambda\cdot$, $R = \nabla_\Gamma\nu$ *and* $2H_\Gamma = \mathrm{div}_\Gamma\nu$.

Proof First of all, by Lemmas 4.10 and 4.11:

$$\int_\Gamma (\varepsilon\cdot\nu)\mathrm{div}\{\mathcal{E}\times\mathrm{H} + \lambda\mathcal{H}\times(\nu_t\times\mathrm{H}) + \eta\nu_t(\mathrm{curl}_{\Gamma_t}\mathrm{H}_{T_t}\mathrm{curl}_{\Gamma_t}\mathcal{H}_{T_t})\}|_{t=0}ds$$

$$= \int_\Gamma \mathcal{H}^{\mathrm{T}}\{ik(\nu\cdot\varepsilon)\mathrm{H}_T - ik\mathcal{Z}[(\nu\cdot\varepsilon)\nu\times\mathrm{E})] - \mathbf{curl}_\Gamma[(\nu\cdot\varepsilon)(\nu\cdot\mathrm{E})]$$

$$- \lambda(\nu\cdot\varepsilon)(2R - 2H_\Gamma)\mathrm{H}_T - ik\lambda(\nu\cdot\varepsilon)\mathcal{Z}\mathrm{H}_T + \lambda\nabla_\Gamma[(\nu\cdot\varepsilon)(\nu\cdot\mathrm{H})]$$

$$- ik\mathcal{Z}[\lambda(\nu\cdot\varepsilon)\mathrm{H}_T] - 2\mathbf{curl}_\Gamma[H_\Gamma(\nu\cdot\varepsilon)\eta\mathrm{curl}_\Gamma\mathrm{H}_T]$$

$$- ik\mathcal{Z}[(\nu\cdot\varepsilon)\mathbf{curl}_\Gamma(\eta\mathrm{curl}_\Gamma\mathrm{H}_T)] - ik\mathbf{curl}_\Gamma[\eta\mathrm{curl}_\Gamma[(\nu\cdot\varepsilon)\mathcal{Z}\mathrm{H}_T]]\}ds.$$

The boundary conditions satisfied by (E, H) implies that

$$-ik\mathcal{Z}[(\nu\cdot\varepsilon)(\nu\times\mathrm{E})] - ik\mathcal{Z}[\lambda(\nu\cdot\varepsilon)\mathrm{H}_T] - ik\mathcal{Z}[(\nu\cdot\varepsilon)\mathbf{curl}_\Gamma(\eta\mathrm{curl}_\Gamma\mathrm{H}_T)] = 0.$$

We also have

$$-ik\mathbf{curl}_\Gamma[\eta\mathrm{curl}_\Gamma[(\nu\cdot\varepsilon)\mathcal{Z}(\mathrm{H}_T)]] - ik\lambda(\nu\cdot\varepsilon)\mathcal{Z}\mathrm{H}_T = -ik\mathcal{Z}[(\nu\cdot\varepsilon)\mathcal{Z}\mathrm{H}_T].$$

We then obtain the result by using Lemma 4.8. $\qquad\square$

We now have all the necessary results to conclude the proof of Theorem 4.1.

Proof of Theorem 4.1 First of all, since λ, η and Γ are analytical, we can extend Proposition 4.12 to the case where Ω is not included in Ω_ε (see [18] for the details).

Moreover, by using the Stratton-Chu representation formula (7) we prove that for $f \in H_{\text{curl}_\Gamma}(\Gamma)^*$, the electric field E_f solution to problem (1) is such that

$$\mathrm{E}_f(z) = \int_\Gamma \mathcal{H}(y, z)^{\mathrm{T}} f(y) ds(y) \quad \text{for all } z \in \Omega_{\text{ext}}. \tag{16}$$

Therefore, Proposition 4.12 together with formula (16) concludes the proof. □

4.3 Numerical validation of the shape derivative

In this section we describe the numerical procedure we use to compute the shape derivative of the scattered field. We also present numerical experiments that we use to validate the expression of the shape derivative given in Theorem 4.1. To be more precise we compute the shape derivative of the following L^2 cost functional

$$F(\Gamma) := \frac{1}{2} \left\| T(\lambda, \eta, \Gamma) - \mathrm{E}_{\text{obs}}^\infty \right\|_{L_t^2(S^2)}^2.$$

where $\mathrm{E}_{\text{obs}}^\infty := T(\lambda, \eta, \Gamma_{\text{obs}})$ for given shape Γ_{obs}, impedance coefficients λ and η, direction of incidence $\hat{\theta}$ and polarisation $\mathbf{p}$. In the next section we show how to solve a shape identification problem by minimizing such cost functional.

For analytic λ, η and Γ, by using the result of Theorem 4.1 and since the forward problem is linear we prove that the shape derivative of F in the direction ε is given by

$$F'(\Gamma) \cdot \varepsilon = \frac{1}{4\pi} \Re \left(\int_\Gamma \mathcal{G}(y) \cdot B_\varepsilon(\mathrm{E}, \mathrm{H})(y) ds(y) \right), \tag{17}$$

where B_ε is given in Proposition 4.12, $\mathrm{E}(y) = \mathscr{E}^s(y, \hat{\theta})\mathbf{p} + \mathscr{E}^i(y, \hat{\theta})\mathbf{p}$, $\mathrm{H}(y) = \mathscr{H}^s(y, \hat{\theta})\mathbf{p} + \mathscr{H}^i(y, \hat{\theta})\mathbf{p}$ and the adjoint state $\mathcal{G}$ is given by

$$\mathcal{G}(y) := \int_{S^2} \mathscr{H}(y, -\hat{x})\mathbf{g}(\hat{x}) ds(\hat{x}),$$

where

$$\mathbf{g}(\hat{x}) := \overline{(T(\lambda, \eta, \Gamma) - \mathrm{E}_{\text{obs}}^\infty)}(\hat{x}).$$

From Eq. 17, evaluating the shape derivative of F requires the knowledge of two total fields: (E, H) and $\mathcal{G}$ that we numerically compute by solving two scattering problems: a first one associated with the incident plane wave $(\mathscr{E}^i(y, \hat{\theta})\mathbf{p}, \mathscr{H}^i(y, \hat{\theta})\mathbf{p})$ and a second one associated with an electromagnetic Herglotz wave $(\mathcal{G}_{\mathrm{E}}^i, \mathcal{G}_{\mathrm{H}}^i)$ given by

$$\mathcal{G}_{\mathrm{E}}^i(y) := \int_{S^2} \mathrm{E}^i(y, -\hat{x})\mathbf{g}(\hat{x}) ds(\hat{x}), \quad \mathcal{G}_{\mathrm{H}}^i(y) := \int_{S^2} \mathrm{H}^i(y, -\hat{x})\mathbf{g}(\hat{x}) ds(\hat{x}).$$

Then $\mathcal{G} = \mathcal{G}_{\mathrm{H}}^i + \mathcal{G}_{\mathrm{H}}^s$ is the total magnetic field associated with the solution $(\mathcal{G}_{\mathrm{E}}^s, \mathcal{G}_{\mathrm{H}}^s)$ to Eq. 1 for f being given by Eq. 2 with $\mathrm{E}^i = \mathcal{G}_{\mathrm{E}}^i$ and $\mathrm{H}^i = \mathcal{G}_{\mathrm{H}}^i$.

We choose here to use edge finite elements to solve the second order system associated with the Maxwell system (1) which can be written as

$$\begin{cases} \mathbf{curl\, curl\,} \mathrm{H}^s - k^2 \mathrm{H}^s = 0 & \text{in } \Omega_{\text{ext}}, \\ \frac{i}{k} \nu \times \mathbf{curl\,} \mathrm{H}^s + \mathbf{curl}_\Gamma(\eta \mathrm{curl}_\Gamma \mathrm{H}_T^s) + \lambda \mathrm{H}_T^s = f & \text{on, } \Gamma \end{cases} \tag{18}$$

where f is given by Eq. 2 in term of the incident field (E^i, H^i). We complement this system with the approximate first order radiation condition

$$\hat{x} \times \mathbf{curl}\, H^s + ik(\hat{x} \times H^s) \times \hat{x} = 0 \quad \text{on } \Sigma, \tag{19}$$

where Σ is is closed regular surface that is far from Γ. Typically, Σ is a sphere or an ellipsoid such that the distance between Γ and Σ is of the order of one wavelength. We discretise the domain contained between Σ and Γ with tetrahedron with approximately twelve tetrahedron per wavelength ($l := 2\pi/k$) and we approximate the field H with the Nédélec edge elements of the first kind (see [27]). We use the finite elements software Freefem++ (see [20]) to set up and solve the discrete system. Finally, the far-field pattern is computed by using the integral representation formula (8). The solver has been validated against Mie's serie solutions in the case of the scattering by a sphere and we obtained relative errors smaller than 5 % on the L^2 norm of the far field pattern in several test cases. Then we evaluate (17) by using numerical integration. The only problematic point would be the numerical evaluation of the operator B_ε since it contains fourth order derivatives. First we transform (17) by using integration by part on Γ:

$$F'(\Gamma) \cdot \varepsilon = \frac{1}{4\pi} \Re \int_\Gamma (\nu \cdot \varepsilon) \{ -ik H_T \cdot \mathcal{G} + (\nu \cdot E) \text{curl}_\Gamma \mathcal{G}_T + \lambda (2R - 2H_\Gamma) H_T \cdot \mathcal{G}$$
$$+ \lambda (\nu \cdot H) \text{div}_\Gamma \mathcal{G}_T + 2H_\Gamma \eta \text{curl}_\Gamma H_T \text{curl}_\Gamma \mathcal{G}_T + ik \mathcal{Z} H_T \cdot \mathcal{Z} \mathcal{G}_T \}.$$

We evaluate $\mathcal{Z} H_T$ by first computing H_η, the L^2 projection of $\eta \text{curl}_\Gamma H_T$ on piecewise linear functions that solves

$$\int_\Gamma H_\eta v \, ds = \int_\Gamma (\eta \text{curl}_\Gamma H_T) v \, ds,$$

for all v in the space of piecewise linear functions defined on the triangular discretisation of Γ. We then compute $\mathcal{Z} H_T$ by evaluating $\mathbf{curl}_\Gamma H_\eta$ as a piecewise constant function.

We numerically validate both, the characterization of the shape derivative and its numerical evaluation, by comparing expression (17) with a numerical evaluation of

$$\partial_\varepsilon F(\Gamma) := \frac{F(\Gamma) - F(\Gamma_t)}{t}, \tag{20}$$

for $\Gamma_t := \Gamma + t\varepsilon(\Gamma)$ where t is a small positive number and ε is a vector field defined on Γ. We also used finite elements to compute $\partial_\varepsilon F(\Gamma)$.

Numerical Results

We will consider data E^∞_{obs} that correspond with the scattering of an incident plane wave of frequency $k = 4$, of direction of direction of incidence $\hat{\theta} = (0, 0, 1)$ and of polarisation $\mathbf{p} = (1, 0, 0)$. Moreover we set $\eta = -\lambda = i/k$ and the obstacle Ω_{obs} is a sphere of radius $0.3l$ where $l := 2\pi/k$ is the wavelength of the incident wave. The obstacle Ω is taken as being a sphere of radius $0.5l$ and the deformation is $\varepsilon = \hat{x} = x/|x|$. In this case, Eq. 20 can also be computed via Mie's series for $\varepsilon = \hat{x}$ (we denote by $\widetilde{\partial F}_\varepsilon$ this value) and we have the following results.

$$\partial_\varepsilon F(\Gamma) = 11.7, \quad \widetilde{\partial F}_\varepsilon(\Gamma) = 11.3, \quad F'(\Gamma) = 10.9.$$

These three values are fairly close given that we have a precision of approximately 5 % on the forward solver. Considering an artificial boundary Σ as close as one wavelength from the scatterer is certainly not enough to achieve such precision for more complex objects such as the cube use below. Nevertheless, the accuracy we obtained is certainly good enough for illustration purposes.

5 Application to shape identification of obstacles with GIBC

We propose in this section to use the characterization of the shape derivative obtained previously to reconstruct the shape Γ of a scatterer with boundary conditions of the form

$$\nu \times E + \mathbf{curl}_\Gamma (\eta \mathrm{curl}_\Gamma H_T) + \lambda H_T = 0 \quad \text{sur } \Gamma,$$

with constant λ and η, from the knowledge of the far field produced by the scattering of J incident plane waves. We first present the proposed regularised steepest descent minimization procedure and then show some numerical validating results.

5.1 Reconstruction procedure

The technique we used to solve the inverse problem consists in minimizing the following cost functional

$$F_J(\Gamma) := \frac{1}{2} \sum_{j=1}^J \left\| T(\Gamma, \hat{\theta}_j, \mathbf{p}_j) - \mathrm{E}_{j,\delta}^\infty \right\|_{L_t^2(S^2)}^2 ,$$

where the far field patterns $\mathrm{E}_{j,\delta}^\infty$ represent the data and we do not mention the dependence of T on λ and η for simplicity reasons. For more clarity we made explicit the dependence of T on the direction of incidence $\hat{\theta}_j$ and the polarisation $\mathbf{p}_j$ of the incident plane wave. Therefore, we obtain the shape derivative of F_J in the direction ε as being given by

$$F_J'(\Gamma) \cdot \varepsilon := \frac{1}{4\pi} \sum_{j=1}^J \Re \left(\int_\Gamma \mathcal{G}_j(y) \cdot B_\varepsilon(\mathrm{E}_j, \mathrm{H}_j)(y) ds(y) \right), \qquad (21)$$

where B_ε is given in Theorem 4.1, $\mathrm{E}_j(y) = \mathscr{E}^s(y, \hat{\theta}_j)\mathbf{p}_j + \mathscr{E}^i(y, \hat{\theta}_j)\mathbf{p}_j$, $\mathrm{H}_j(y) = \mathrm{H}^s(y, \hat{\theta}_j)\mathbf{p}_j + \mathrm{H}^i(y, \hat{\theta}_j)\mathbf{p}_j$ and the adjoint state $\mathcal{G}_j$ is given by

$$\mathcal{G}_j(y) := \int_{S^2} \mathscr{H}_j(y, -\hat{x})\mathbf{g}_j(\hat{x}) ds(\hat{x}),$$

where $\mathbf{g}_j(\hat{x}) := \overline{(T(\Gamma, \hat{\theta}_j, \mathbf{p}_j) - \mathrm{E}_{j,\delta}^\infty)}(\hat{x})$. We assume moreover that the impedance coefficients λ and η are constant, fixed and known and that for $j = 1, \cdots, J$, the far field patterns $\mathrm{E}_{j,\delta}^\infty$ are

$$\mathrm{E}_{j,\delta}^\infty = T(\Gamma_0, \hat{\theta}_j, \mathbf{p}_j) + \delta_j,$$

for some surface Γ_0 and "small" functions $\delta_j \in L_t^2(\Gamma)$ (the noise). The case of known but non constant coefficients requires a careful treatment since the coefficients

would be known and defined on a shape that is unknown. One could for example assume that the unknown object can be described by a function of two variable X and Y and define the coefficients as functions of X and Y. Another solution would consist in defining the coefficients in $\mathbb{R}^3$ but their physical significance would not be completely clear anymore. For more details on non constant impedances we refer to [6] that considers a similar problem for the scaler Helmholtz equation.

In the experiments hereafter we numerically compute an exact data set $T(\Gamma_0, \hat{\theta}_j, \mathbf{p}_j)$ by using finite elements and the δ_i are modeled by a realisation of a Gaussian random variable with mean 0 and variance 1. The δ_j are then normalised such that

$$\frac{\|\delta_j\|_{L^2_t(S^2)}}{\|T(\Gamma_0, \hat{\theta}_j, \mathbf{p}_j)\|_{L^2_t(S^2)}} = \delta_0,$$

for a given noise level $\delta_0 > 0$.

We propose to use a steepest descent method to minimize F_J. At each step n we update the current shape Γ_n by moving it with a regular representation of the shape derivative of F. More precisely, we begin with a given initial mesh M_0 obtained from the discretisation of the domain contained between an initial guess for the geometry Γ and the auxiliary boundary Σ. Then we move this mesh iteratively until convergence. For a given step n of the algorithm we obtain the mesh at step $n+1$, which is denoted M_{n+1}, by applying the following deformation to the current mesh M_n:

$$M_{n+1} := (\mathrm{Id} + \alpha_n V) M_n,$$

where $V \in X_h$ solves

$$(V, W)_{(H^1(M_n))^3} + \sigma_\Gamma (\nabla_\Gamma V, \nabla_\Gamma W)_{(L^2_t(\Gamma_n))^{3 \times 3}} + (V, W)_{(L^2_t(\Gamma_n))^3} = F'_J(\Gamma_n) \cdot W, \quad (22)$$

for all $W \in X_h$ where X_h is the set of piecewise linear Lagrange finite elements defined on the mesh M_n that vanish on the exterior boundary Σ (see [2] for a similar approach). The parameter $\alpha_n > 0$ is the step size, $\sigma_\Gamma > 0$ is a regularisation coefficient and $F'_J(\Gamma_n) \cdot v$ is given by formula (21). One advantage of this approach is that it is a parametrisation free procedure (see Fig. 1 for an example of successive meshes obtained during the iterative process). The inverse crime is automatically avoided since the mesh is different at each step.

The regularisation parameter is kept fixed during iterations while the step size evolves in the following way: we initialise it with a relatively small value and increase it by a multiplicative parameter $r > 1$ if the cost functional decreases between mesh
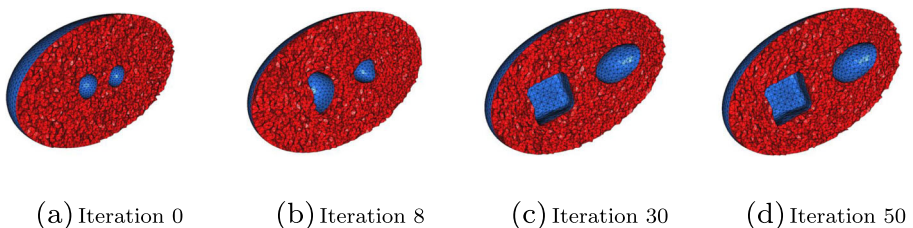

(a) Iteration 0   (b) Iteration 8   (c) Iteration 30   (d) Iteration 50

Fig. 1 Different meshes obtained during the minimization procedure corresponding to Fig. 4

M_n and M_{n+1}. On the contrary, we decrease it by the same multiplicative parameter r if the cost functional increases between meshes M_n and M_{n+1} and we move back to the mesh M_n. A direct consequence of such approach is that the algorithm does not diverge and we can choose to stop the iterations when the step size is too small. To keep reasonably short computations (a few hours on a twelve core workstation) we choose to stop the algorithm after a sufficiently large number of iterations. In the experiments hereafter we observed that 40 iterations is generally sufficient to obtain a shape which barely evolves between iterations. We observe for instance tiny differences between Figs. 1c and d. Finally, let us mention that moving meshes in dimension three is certainly a complicated subject and we used the free software mmg3d (see [15]) to perform this step. We also used the mesh adaptation capabilities of mmg3d at every step to further ensure that the mesh stays regular enough after each iteration. We can summarise the algorithm as follows:

Initialisation: pick an initial mesh M, an initial step size α, solve Eqs. 18–19 for all incident plane waves and evaluate the initial cost functional F_J

Iterations: until $\alpha < \alpha_{\min}$ or the maximum number of iterations is reached
    1. compute the I adjoint states $\mathcal{G}$ (one for each incident wave)
    2. compute a descent direction V by solving Eq. 22
    3. define a new mesh $\tilde{M} := (\mathrm{Id} + \alpha V)M$
    4. solve Eqs. 18–19 in $\tilde{M}$ for all incident plane waves and
       evaluate the cost functional $\tilde{F}$
    5. if $\tilde{F} < F$
$$\alpha := r\alpha, \; M := \tilde{M}, \; F := \tilde{F},$$
    else
$$\alpha := \alpha/r \text{ and go back to step 3.}$$

5.2 Numerical results

We now present three examples of reconstructions obtained using the procedure described here above. Our goal is to show how the algorithm behaves with respect to
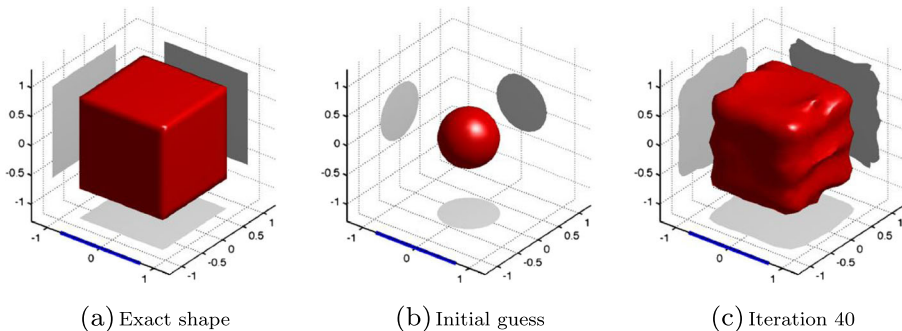

(a) Exact shape   (b) Initial guess   (c) Iteration 40

Fig. 2 Reconstruction with $\delta_0 = 2\,\%$ of noise and for two incident waves : $\hat{\theta}_1 = -\hat{\theta}_2 = (0, 0, 1)$ and $\mathbf{p}_1 = \mathbf{p}_2 = (0, 1, 0)$. The impedance parameters are $\lambda = 0.5$ and $\eta = 0$
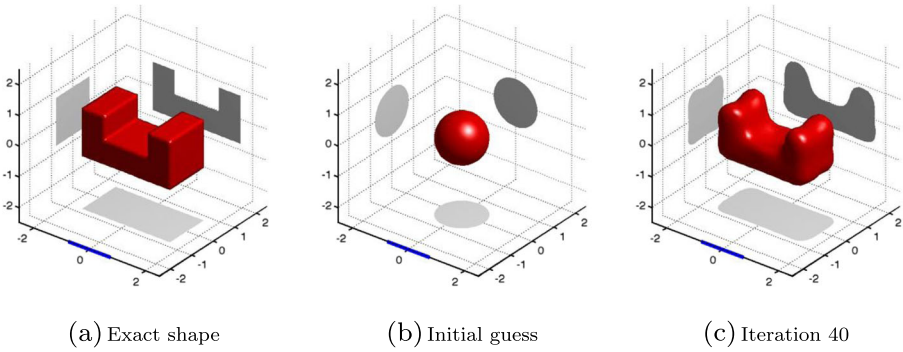
(a) Exact shape   (b) Initial guess   (c) Iteration 40

Fig. 3 Reconstruction with $\delta_0 = 2$ % of noise and for four incident waves : $\hat{\theta}_1 = -\hat{\theta}_2 = (0,0,1)$, $\hat{\theta}_3 = -\hat{\theta}_4 = (0,1,0)$, $\mathbf{p}_1 = \mathbf{p}_2 = (0,1,0)$ and $\mathbf{p}_3 = \mathbf{p}_4 = (1,0,0)$. The impedance parameters are $\lambda = -0.25i$ and $\eta = 0.25i$

the shape complexity and to demonstrate that our choice of minimization procedure (regularisation of the gradient and parametrisation free optimisation) gives valuable results. For each example we plot three shapes: the exact shape Γ_0 used to simulate the data, the initial guess and the shape obtained at the 40th iteration of the minimization algorithm. The blue line represents the wavelength of the incident wave, in all cases we considered an obstacle which was about the size of the wavelength and we used the same parameters: the regularisation coefficient for the computation of the shape gradient is $\sigma_\Gamma = 0.3$, the initial step size is $\alpha_0 = 0.1$, the actualisation rate for the step size is $r = 1.2$ and the level of noise is $\delta_0 = 2$ %. We recall that the incident plane waves that are used to generate the data are of the form

$$\mathcal{E}^i(x, \hat{\theta})\mathbf{p} := -\frac{1}{ik}\mathbf{curl}_x\mathbf{curl}_x\left(\mathbf{p}e^{ikx\cdot\hat{\theta}}\right) = ik((\hat{\theta} \times \mathbf{p}) \times \hat{\theta})e^{ikx\cdot\hat{\theta}},$$

$$\mathcal{H}^i(x, \hat{\theta})\mathbf{p} := \mathbf{curl}_x\left(\mathbf{p}e^{ikx\cdot\hat{\theta}}\right) = ik(\hat{\theta} \times \mathbf{p})e^{ikx\cdot\hat{\theta}},$$

and are characterised by their direction of incidence $\hat{\theta}$ and polarisation $\mathbf{p}$.

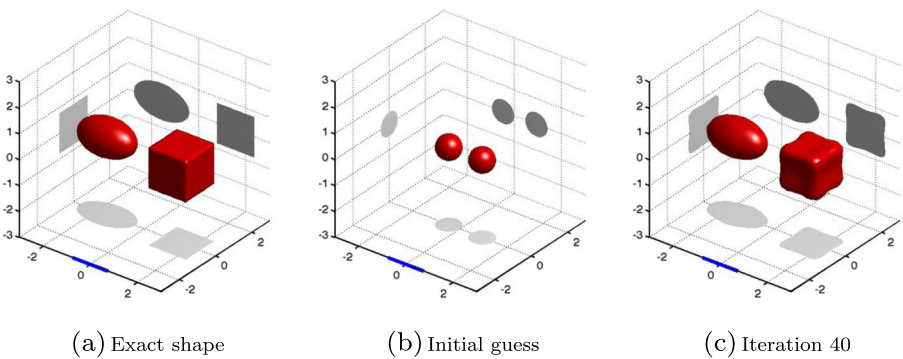(a) Exact shape   (b) Initial guess   (c) Iteration 40

Fig. 4 Reconstruction with $\delta_0 = 2$ % of noise and for four incident waves : $\hat{\theta}_1 = -\hat{\theta}_2 = (0,0,1)$, $\hat{\theta}_3 = -\hat{\theta}_4 = (0,1,0)$, $\mathbf{p}_1 = \mathbf{p}_2 = (0,1,0)$ and $\mathbf{p}_3 = \mathbf{p}_4 = (1,0,0)$. The impedance parameters are $\lambda = -0.25i$ and $\eta = 0.25i$.

In the first test (Fig. 2) we considered data from 2 incident waves while in the second and third one we considered four incident waves since the shape reconstruction is more challenging (Γ_0 is non convex in Fig. 3 or non connected in Fig. 4). In the three cases the reconstructions are accurate and comparable to the one obtained with the so called Linear Sampling Method (see [13]) that requires the use of many more incident waves (but indeed is much less expensive and does not need a priori initial guesses).

Appendix

Proof of Lemma 4.9 We use the notation introduced in Section 4.2. We complement the tangent basis (e_1, e_2) (associated with the local parametrisation φ (14)) with $e_3 = \varepsilon$. This defines a non orthogonal basis of $\mathbb{R}^3$ and we denote by (f^1, f^2, f^3) the associated covariant basis. Let us also introduce (e^1, e^2, e^3) the covariant basis associated with (e_1, e_2, ν). One easily verifies (see also [6]):

$$f^1 = e^1 - \frac{1}{\nu \cdot \varepsilon}(e^1 \cdot \varepsilon)\nu , \ \ f^2 = e^2 - \frac{1}{\nu \cdot \varepsilon}(e^2 \cdot \varepsilon)\nu \ \ \text{and} \ f^3 = \frac{1}{\nu \cdot \varepsilon}\nu.$$

We also denote by $\widetilde{\nu}_t$ the extension of the normal vector ν given by

$$\widetilde{\nu}_t(x_\Gamma + t\nu(x_\Gamma)) = \nu(x_\Gamma) \quad \text{for } x_\Gamma \in \Gamma.$$

Let us prove the first identity of Lemma 4.9. By using [28, Theorem 2.5.20] we have

$$\mathbf{curl}(\nu_t \times V)|_{t=0} = \text{div}_\Gamma(V_T)\nu + \left(R - 2H_\Gamma \text{Id} - \frac{\partial}{\partial \nu} \right)[(\nu_t \times V) \times \widetilde{\nu}_t]|_{t=0}$$

since

$$\mathbf{curl}_\Gamma([\nu_t \times V] \cdot \widetilde{\nu}_t)|_{t=0} = \mathbf{curl}_\Gamma([\nu \times V] \cdot \nu) = 0.$$

Moreover,

$$\frac{\partial}{\partial \nu}[(\nu_t \times V) \times \widetilde{\nu}_t]|_{t=0} = \frac{\partial}{\partial \nu}[((\nu_t - \widetilde{\nu}_t) \times V) \times \widetilde{\nu}_t]|_{t=0} + \frac{\partial}{\partial \nu}V_T.$$

For any vector field W,

$$\frac{\partial W}{\partial \nu}|_\Gamma = \sum_{i=1}^{2} \frac{\partial W}{\partial \xi_i}(f^i \cdot \nu) + \frac{\partial W}{\partial t}(f^3 \cdot \nu).$$

Hence we end up with

$$\frac{\partial}{\partial \nu}[((\nu_t - \widetilde{\nu}_t) \times V) \times \widetilde{\nu}_t]|_{t=0} = \left[\frac{\partial}{\partial t}(\nu_t - \widetilde{\nu}_t)|_{t=0} \times V \right] \times \nu = -[\nabla_\Gamma(\nu \cdot \varepsilon) \times V] \times \frac{\nu}{\nu \cdot \varepsilon}$$

where for the first equality we used that for $i = 1, 2$

$$\frac{\partial}{\partial \xi_i}[((\nu_t - \widetilde{\nu}_t) \times V) \times \widetilde{\nu}_t]|_{t=0} = 0$$

and for the second equality we used [18, Lemma 2.3]. This gives the first identity of Lemma 4.9.

For the second identity of Lemma 4.9 we first remark that

$$\frac{\partial}{\partial \nu}(\text{curl}_{\Gamma_t} V_{T_t})|_{t=0} = \frac{\partial}{\partial \nu}(\nu_t \cdot \mathbf{curl} V)|_{t=0} = \frac{\partial}{\partial \nu}(\widetilde{\nu}_t \cdot \mathbf{curl} V)|_{t=0} + \frac{\partial}{\partial \nu}[(\nu_t - \widetilde{\nu}_t) \cdot \mathbf{curl} V]|_{t=0}.$$
(23)

Similarly we prove that

$$\frac{\partial}{\partial \nu}[(\nu_t - \widetilde{\nu}_t) \cdot \mathbf{curl} V]|_{t=0} = -\frac{1}{\nu \cdot \varepsilon} \nabla_\Gamma (\nu \cdot \varepsilon) \cdot (\mathbf{curl} V)|_\Gamma.$$
(24)

We denote

$$\text{rot}_{\widetilde{\Gamma}_t}((\widetilde{\nu}_t \times V) \times \widetilde{\nu}_t) := \widetilde{\nu}_t \cdot \mathbf{curl}\ V$$

and

$$\text{div}_{\widetilde{\Gamma}_t}(V \times \widetilde{\nu}_t) := \text{rot}_{\widetilde{\Gamma}_t}((\widetilde{\nu}_t \times V) \times \widetilde{\nu}_t).$$

From [28, Lemma 2.5.10] we have

$$\frac{\partial}{\partial \nu}[\text{div}_{\widetilde{\Gamma}_t}(V \times \widetilde{\nu}_t)]|_{t=0} - \text{div}_{\widetilde{\Gamma}_t}\left[\frac{\partial}{\partial \nu}(V \times \widetilde{\nu}_t)\right]|_{t=0} = -\text{div}_\Gamma(RV) + 2[(V \times \nu) \cdot \nabla_\Gamma H_\Gamma].$$
(25)

We also observe that

$$\text{div}_{\widetilde{\Gamma}_t}\left[\frac{\partial}{\partial \nu}(V \times \widetilde{\nu}_t)\right]|_{t=0} = \text{div}_\Gamma\left[\frac{\partial}{\partial \nu}(V \times \nu)\right]$$
$$= \text{div}_\Gamma\left[-\mathbf{curl}\ V + (\text{curl}_\Gamma V_T)\nu + (R - 2H_\Gamma)(V \times \nu)\right]$$

and

$$-(\mathbf{curl}\ V) \cdot \nu + \text{curl}_\Gamma V_T = 0.$$

Whence

$$\text{div}_{\widetilde{\Gamma}_t}\left[\frac{\partial}{\partial \nu}(V \times \widetilde{\nu}_t)\right]|_{t=0} = \text{div}_\Gamma[-(\mathbf{curl}\ V)_T + (R - 2H_\Gamma)(V \times \nu)].$$

Combining this last relation together with Eq. 25 gives

$$\frac{\partial}{\partial \nu}(\widetilde{\nu}_t \cdot \mathbf{curl} V)|_{t=0} = \frac{\partial}{\partial \nu}[\text{div}_{\widetilde{\Gamma}_t}(V \times \widetilde{\nu}_t)]|_{t=0} = -\text{div}_\Gamma(\mathbf{curl}\ V)_T - 2H_\Gamma \text{div}_\Gamma(V \times \nu).$$

Finally, we can conclude by plugging this last identity and Eq. 24 into Eq. 23. $\quad\square$

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