

Computing and analyzing recoverable supports for sparse reconstruction

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Abstract Designing computational experiments involving ℓ_1 minimization with linear constraints in a finite-dimensional, real-valued space for receiving a sparse solution with a precise number k of nonzero entries is, in general, difficult. Several conditions were introduced which guarantee that, for example for small k or for certain matrices, simply placing entries with desired characteristics on a randomly chosen support will produce vectors which can be recovered by ℓ_1 minimization. In this work, we consider the case of large k and introduce a method which constructs vectors which support has the cardinality k and which can be recovered via ℓ_1 minimization. Especially, such vectors with largest possible support can be constructed. Further, we propose a methodology to quickly check whether a given vector is recoverable. This method can be cast as a linear program and we compare it with solving ℓ_1 minimization directly. Moreover, we gain new insights in the recoverability in a non-asymptotic regime. Our proposal for quickly checking vectors bases on optimality conditions for exact solutions of the ℓ_1 minimization. These conditions can be used to establish equivalence classes of recoverable vectors which have a support of the same cardinality. Further, by these conditions we deduce a geometrical interpretation which identifies an equivalence class with a face of a hypercube which is cut by a certain affine subspace. Due to the new geometrical interpretation we derive new results on the number of equivalence classes which are illustrated by computational experiments.

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1 Introduction

The difficulty of finding suitable test instances is a serious problem in the field of *Sparse Reconstruction*. A common and promising method to reconstruct a vector $x^* \in \mathbb{R}^n$ with only a few nonzero entries from a linear transformation, which is realized by a matrix $A \in \mathbb{R}^{m \times n}$ with $m < n$, is performing the ℓ_1 minimization, i.e.

$$x^* = \arg \min_y \|y\|_1 \text{ s.t. } Ay = Ax^*. \quad (1)$$

This optimization problem was introduced in [26] and is called *Basis Pursuit*. Under certain conditions (e.g. see [11, 21, 31]) the vector x^* is also a solution with the smallest number of nonzero entries; a vector x^* with exactly k nonzero entries is called k -sparse.

A popular method for finding a k -sparse vector x^* satisfying (1) for a given matrix $A \in \mathbb{R}^{m \times n}$ is to choose an index set $I \subset \{1, \dots, n\}$ with cardinality k and entries $x_i^*, i \in I$, randomly. For small k this procedure is promising especially if the conditions mentioned above are satisfied, but these conditions require small k . For large k it is more difficult to get suitable k -sparse vectors x^* . Besides the question how to compute x^* satisfying (1) for a given matrix, we state the question how many different pairs of index sets and signums do exist for a given sparsity k . We aim at partial answers to these questions in a non-asymptotic regime.

We denote by $I = \text{supp}(x^*)$ the support of a vector x^* and its complement by $I^c = \{1, \dots, n\} \setminus I$. By A_I we denote the submatrix of a matrix A , whose columns are indexed by I , by A_I^T its transpose, and set $s = \text{sign}(x^*)_I$. Further, the null space of A is denoted by $\mathcal{N}(A)$ and the range of A is denoted by $\text{rg}(A) = \{Ax : x \in \mathbb{R}^n\}$. For (1) to hold, it is necessary and sufficient (cf. [23, Theorem 2]) that

$$\exists w \in \mathbb{R}^m : A_I^T w = s, \|A_{I^c}^T w\|_\infty < 1 \text{ and } A_I \text{ has full rank.} \quad (2)$$

A vector w fulfilling (2) will be called *dual certificate for the support I and sign s* . Condition (2) shows that the recoverability of the solution x^* only depends on its support and its signum.

Definition 1 Let $A \in \mathbb{R}^{m \times n}$ and $k \leq m \leq n$. For $I \subset \{1, \dots, n\}$ and $s \in \{-1, 1\}^I$, a pair (I, s) satisfying (2) is called *Recoverable Support* of A . If I has the cardinality k , a Recoverable Support (I, s) has the *size k* .

Thus, finding $x^* \in \mathbb{R}^n$ which satisfies (1) for a given matrix $A \in \mathbb{R}^{m \times n}$ is equivalent to finding a corresponding Recoverable Support (I, s) such that $I = \text{supp}(x^*)$ and $s = \text{sign}(x^*)_I$. For the rest of this paper we will denote the cardinality of a

set I with $|I|$ and the i -th column of a matrix A with a_i . Moreover we will require $m \leq n$ for all $m \times n$ -matrices.

With a geometrical interpretation of (2), new insights to Basis Pursuit, including what kind of matrices can be used and how many Recoverable Supports do exist for a certain size k , can be developed. To that end, consider that $A^T w$ is a relative interior point of an $(n - |I|)$ -dimensional face of the n -dimensional hypercube $C^n := [-1, +1]^n$ and assume that the range of A^T is an m -dimensional subspace. Hence, condition (2) leads to the geometrical interpretation that the m -dimensional subspace $\text{rg}(A^T)$ cuts the relative interior of an $(n - |I|)$ -dimensional face of C^n . In [23], the resulting polytope emerging from the intersection of the m -dimensional subspace and C^n is considered. Counting all index sets $I \subset \{1, \dots, k\}$ with $|I| = k$, which satisfy this geometrical interpretation, one can give exact values for the numbers of recoverable vectors for a matrix A and a sparsity k . These values have been estimated in several papers (e.g. [5–7]) through Monte Carlo samplings. Further this interpretation brings *Sparse Reconstruction* together with the topic (*cross-sections of a hypercube* in Combinatorial Geometry).

A different geometrical interpretation has been given by Donoho in [3] through associating projected n -dimensional cross-polytopes with the Basis Pursuit problem, see also the accessible description in [8, Section 4.5]. The connection between *Sparse Reconstruction* and the theory of *convex polytopes* gave new insights in both fields. Our geometrical interpretation of Recoverable Supports, which is inspired by (2), is dual to this approach. Nonetheless our interpretation delivers additional insights to the questions posed above.

This paper is organized as follows. In Section 2 we develop conditions for the existence of Recoverable Supports. The geometrical aspect around the stated geometrical interpretation will be regarded more carefully in Section 3: a proof for the geometrical interpretation of Recoverable Supports will be given, and exact numbers of Recoverable Supports for certain types of matrices as well as a non-trivial upper bound for these numbers will be stated. Further we will introduce an algorithm to compute a Recoverable Support of a given matrix and a given size in Section 4. The theoretical results from these sections will be illustrated by Monte Carlo experiments in Section 5. Through numerical experiments we additionally provide evidence that checking (2) is considerably faster than solving Basis Pursuit as a linear program. In addition, our method stands out from recently done experiments since we can also ensure that a vector is the *unique* solution of Basis Pursuit, without restricting the test problems to a certain class of matrices (e.g. random matrices).

2 Existence of and conditions for recoverable supports

2.1 Establishing a partial order

The condition (2) for Recoverable Supports rests on two things: The injectivity of the submatrix A_I with I being the support of x^* and the existence of the dual certificate $w \in \mathbb{R}^m$. The following theorem shows that it is possible to shrink Recoverable

Supports and states conditions when it is possible to obtain a larger Recoverable Support from a given one.

Theorem 2 *Let $A \in \mathbb{R}^{m \times n}$ and let $S_1 = (I, s)$ be a Recoverable Support of A .*

1. *If for w satisfying (2) there is $z \in \ker A_I^T$ satisfying $\|A_I^T(w + z)\|_\infty = 1$ and A restricted to $J := \{i : |a_i^T(w + z)| = 1\}$ has full rank, then with $t = A_J^T(w + z)$ the pair $S_2 = (J, t)$ is a Recoverable Support of A and it holds $I \subset J$.*
2. *Let $|I| > 1$. For any $j_0 \in I$ there exists $\tilde{s} \in \{-1, 1\}^{I \setminus \{j_0\}}$ with $s_i = \tilde{s}_i$ for all $i \in I \setminus \{j_0\}$, such that the pair $S_3 = (I \setminus \{j_0\}, \tilde{s})$ is a Recoverable Support of A .*

Proof The existence of $z \in A_I^T$ for the first statement is obvious and the conclusion that (J, t) is a Recoverable Support follows directly by checking (2). For the second statement notice that $A_{I \setminus \{j_0\}}$ has full rank too and that $\ker(A_I^T) \subsetneq \ker(A_{I \setminus \{j_0\}}^T)$ holds. Hence, for $w \in \mathbb{R}^m$ satisfying (2) there exists $z \in \ker(A_{I \setminus \{j_0\}}) \setminus \{0\}$ with $a_{j_0}^T z \neq 0$. Choose $\lambda \neq 0$ such that $|\lambda| < (1 - |a_i^T w|) / |a_i^T z|$ for all $i \in I^c$ with $a_i^T z \neq 0$ and

$$\lambda a_{j_0}^T z \in \begin{cases} (-2, 0) & , \text{ if } a_{j_0}^T w = 1 \\ (0, 2) & , \text{ else} \end{cases}$$

holds. Considering all elements of $A^T(w + \lambda z)$ separately, it follows

$$\begin{aligned} &|a_{j_0}^T w + \lambda a_{j_0}^T z| < 1, \\ &a_i^T w + \lambda a_i^T z = a_i^T w = s_i \text{ for } i \in I \setminus \{j_0\}, \\ &|a_j^T w + \lambda a_j^T z| \leq |a_j^T w| + |\lambda| |a_j^T z| < 1 \text{ for } j \in I^c \end{aligned}$$

by construction. Hence with $\tilde{s} = A_{I \setminus \{j_0\}}^T(w + \lambda z)$ the pair $S_2 = (I \setminus \{j_0\}, \tilde{s})$ is a Recoverable Support of A . □

The following corollary can be obtained by applying the second statement in Theorem 2 recursively.

Corollary 3 *Let $A \in \mathbb{R}^{m \times n}$ and (I, s) be a Recoverable Support of A . Then for any $J \subset I, J \neq \emptyset$, there exists $\tilde{s} \in \mathbb{R}^n$ with $\tilde{s}_J = s_J$, such that the pair (J, \tilde{s}) is a Recoverable Support of A .*

By using the stated inclusion of Recoverable Supports, a partial order can be obtained through Theorem 2: for Recoverable Supports $S_1 = (I, s), S_2 = (J, \tilde{s})$ with $s_J = \tilde{s}_J$, it is $S_2 \leq S_1$ if and only if $J \subset I$. For example, the supports S_1, S_2 and S_3 from Theorem 2 fulfill $S_3 \leq S_1 \leq S_2$. Moreover, any Recoverable Support can be shrunk and enlarged under the assumption that the respective submatrix is injective. In other words, the set of all Recoverable Supports form a partially ordered set and may be visualized as a Hasse Diagram. Further, there exist Recoverable Supports which can not be enlarged, and we call them *Maximal Recoverable Supports*. Due to Corollary 3, the Maximal Recoverable Supports determine the full set of all Recoverable Supports.

The proof of Theorem 2 also provides a way to obtain a Recoverable Support if a pair (I, s) satisfies all requirements but having A_I as a full rank matrix.

Corollary 4 *Let $A \in \mathbb{R}^{m \times n}$, $I \subset \{1, \dots, n\}$ and $s \in \{-1, 1\}^I$. Further let there exist $w \in \mathbb{R}^m$ satisfying $A_I^T w = s$, $\|A_{I^c}^T w\|_\infty < 1$. If there exists $J \subset I$ such that the submatrix A_J has full rank, then there exists $\tilde{s} \in \{-1, 1\}^J$ with $\tilde{s}_J = s_J$ such that (J, \tilde{s}) is a Recoverable Support of A .*

2.2 Sufficient and necessary condition

Similar to Section 2.1, we will consider dual certificates to establish a sufficient and necessary condition for a pair (I, s) being a Recoverable Support of a given matrix. For this purpose, we introduce the pseudo-inverse $(A_I^T)^\dagger$ of A_I^T . The following theorem and its corollary are an extension of Fuchs' condition in [9].

Theorem 5 *Let $A \in \mathbb{R}^{m \times n}$, $I \subset \{1, \dots, n\}$ and $s \in \{-1, 1\}^I$. Then (I, s) is a Recoverable Support of A if and only if A_I has full rank and there exists $y \in \ker(A_I^T)$ such that*

$$\|A_{I^c}^T (A_I^T)^\dagger s + A_{I^c}^T y\|_\infty < 1.$$

Proof If (I, s) is a Recoverable Support, then A_I has full rank and there exists $w \in \mathbb{R}^m$ such that $A_I^T w = s$. With $\tilde{y} \in \ker(A_I^T)$ the vector w has the general representation $w = (A_I^T)^\dagger s + \tilde{y}$. Since there exists at least one w satisfying $\|A_{I^c}^T w\|_\infty < 1$, there exists $y \in \ker(A_I^T)$ proving the stated inequality.

Further, for $y \in \ker(A_I^T)$ consider $w = (A_I^T)^\dagger s + y$. Since A_I has full rank, A_I^T has linearly independent rows, so $A_I^T w = s$ holds as well as $\|A_{I^c}^T w\|_\infty < 1$. \square

Note that a conclusion of Theorem 5 is that

$$A_I \text{ has full rank and } \|A_{I^c}^T (A_I^T)^\dagger s\|_\infty < 1$$

is a sufficient condition for (I, s) being a Recoverable Support of A by choosing $y = 0$. For full rank matrices with $|I| = \text{rank}(A)$ this is also a necessary condition using the inverse A_I^{-T} of A_I^T .

Corollary 6 *Let $A \in \mathbb{R}^{m \times n}$ have a full rank, $I \subset \{1, \dots, n\}$ with $|I| = m$ and $s \in \{-1, 1\}^I$. Then A_I is invertible and $\|A_{I^c}^T A_I^{-T} s\|_\infty < 1$ holds if and only if (I, s) is a Maximal Recoverable Support of A .*

We close this section with the characterization on Recoverable Supports with size one when the corresponding column has the largest norm over all columns of the considered matrix. The following theorem will be considered in Algorithm 1.

Theorem 7 *Let $A \in \mathbb{R}^{m \times n}$ and $k \in \{1, \dots, n\}$ such that for all $j \neq k$ holds $\|a_j\| \leq \|a_k\|$. Then for $s \in \{-1, +1\}$ the pair $(\{k\}, s)$ is a Recoverable Support of A if and only if for any $j \neq k$ with $\|a_j\| = \|a_k\|$ it holds that $a_j \neq a_k$.*

Proof Let $(\{k\}, s)$ be a Recoverable Support of A and without loss of generality let $s = +1$. Assuming for $j \neq k$ it holds that $a_k = a_j$, then for all $y \perp a_k$ it follows

$$\left| \|a_k\|^{-2} a_j^T a_k + a_j^T y \right| = 1$$

which is a contradiction to Theorem 5.

For the converse implication let $a_j \neq a_k$ with $\|a_j\| = \|a_k\|$. With $w = \|a_k\|^{-2} a_k$ it holds that

$$|a_k^T w| = 1 \text{ and } |a_j^T w| = \frac{|a_j^T a_k|}{\|a_k\|^2} < \frac{\|a_j\|}{\|a_k\|} = 1$$

by applying Cauchy-Schwartz inequality. Further for any a_i satisfying $\|a_i\| < \|a_k\|$ the inequality $|a_i^T w| < 1$ holds. Trivially, the submatrix $A_{\{k\}}$ has full rank and with $s = a_k^T w$ it holds that the pair $(\{k\}, s)$ is a Recoverable Support of A . □

Hence, every matrix whose columns have the largest norm possesses a Recoverable Support if and only if one of these columns do not appear multiple times. That does not exclude matrices whose columns with largest norm have multiple appearance from having Recoverable Supports; see for example the matrix $A \in \mathbb{R}^{2 \times 4}$ whose first two columns are standard basis elements of \mathbb{R}^2 and $a_3 = a_4 = a_1 + a_2$ has still four Recoverable Supports with size one. Moreover, Theorem 7 will be useful as a starting point for the algorithm in Section 4.

3 Geometrical interpretation and number of recoverable supports

In this section, we deal with the geometrical interpretation of Recoverable Supports presented in Section 1 and its implications on their number. In the end of this section, we further derive a non-trivial, but heuristic upper bound on this number. As far as we know, this is a new bound.

Definition 8 For $A \in \mathbb{R}^{m \times n}$ the number $\Lambda(A, k)$ is defined as the number of all Recoverable Supports of A with size k , i.e.

$$\Lambda(A, k) := |\{(I, s) : (I, s) \text{ is a Recoverable Support of } A \text{ with size } k\}|.$$

Further let $\Xi(m, n, k)$ be defined as the maximum of Λ over all matrices of size $m \times n$, i.e.

$$\Xi(m, n, k) := \max\{\Lambda(A, k) : A \in \mathbb{R}^{m \times n}\}.$$

For some triples (m, n, k) , the values for Λ and Ξ will be derived in Sections 3.3 and 3.4. Prior, we briefly sketch some basics on *convex polytopes* in the next section.

3.1 Preliminaries

Let $x_1, \dots, x_m \in \mathbb{R}^n$, then its convex hull $P = \text{conv}(x_1, \dots, x_m)$ is called a *polytope*. The *dimension* of a polytope is the dimension of its affine hull; a polytope with dimension d is called *d-polytope*. We call P *centrally-symmetric* if for all $x \in P$ it

holds $-x \in P$. For $\lambda \in \mathbb{R}^n$ and $c \in \mathbb{R}$ we define the hyperplane $H_{\lambda,c} = \{x : \lambda^T x = c\}$. Further the intersection $F = H_{\lambda,c} \cap P$ is called a *face* of P if $\lambda^T x < c$ holds for all $x \notin F$. A face of P is also a polytope; more general, any intersection of a polytope with an affine subspace is a polytope. The set of all k -dimensional faces of P is denoted as $\mathcal{F}_k(P)$. A centrally symmetric polytope is called k -neighborly if any set of $k + 1$ vertices of P , not including an antipodal pair, spans a face of P .

A face F of the hypercube $C^n := [-1, +1]^n$ is uniquely determined by a pair (I, s) consisting of an index set $I \subset \{1, \dots, n\}$ and $s \in \{-1, +1\}^I$: with (I, s) choose $\lambda \in \mathbb{R}^n$ through $\lambda_I = s, \lambda_j = 0$ if $j \notin I$. We see that for any $y \in \mathbb{R}^n$ with $\lambda^T y > n - |I|$ it holds $y \notin C^n$. Hence, it holds that $F = H_{\lambda,(n-|I|)} \cap C^n$ is an $|I|$ -dimensional face of C^n . For $F \subset C^n$ we note the following equivalence:

$$\begin{aligned}
 F \in \mathcal{F}_k(C^n) & \\
 \Leftrightarrow & \\
 \exists! I \subset \{1, \dots, n\}, |I| = n - k \ \forall v, w \in F : v_I \in \{-1, 1\}^I, v_I = w_I. & \tag{3}
 \end{aligned}$$

With $I(F)$ we denote the unique subset of $\{1, \dots, n\}$ determined by $F \in \mathcal{F}_k(C^n)$. Since the equivalence also holds for subsets $V \subset F$, we also use $I(V)$ to denote the unique subset. We collect these observations in the next lemma.

Lemma 9 *For $F \in \mathcal{F}_k(C^n)$ there exists $\lambda \in \mathbb{R}^n$ such that*

$$F = \{x \in C^n : \lambda^T x = n - k\} \text{ and } C^n \setminus F = \{y \in C^n : \lambda^T y < n - k\}.$$

On the basis of Lemma 9, we identify the relative interior of a face F with $\text{relint}(F) = \{x \in F : |x_i| < 1, i \notin I(F)\}$.

For an extensive overview in the field of convex polytopes, we refer to the books by Grünbaum [10] or Ziegler [32].

Finally, we have all tools for proving the geometrical interpretation of Recoverable Supports suggested in Section 1.

3.2 Geometrical interpretation of recoverable supports

With the introduced notation we will prove the following theorem. Note that the results are similar to the interpretation in [23].

Theorem 10 *Let $A \in \mathbb{R}^{m \times n}$ have rank l and let $k \leq l$. Then the following statements are equivalent:*

1. *There exists a Recoverable Support of A with size k .*
2. *There exists $F \in \mathcal{F}_{n-k}(C^n)$ such that $\text{relint}(F) \cap \text{rg}(A^T) \neq \emptyset$ and $A_{I(F)}^T$ has full rank.*
3. *There exists $V \in \mathcal{F}_{l-k}(C^n \cap \text{rg}(A^T))$ and $v \in V$ with $\|v_{I^c}\|_\infty < 1$ and A_I has full rank for $I := I(V)$.*

Proof First we state for any subset $I \subset \{1, \dots, n\}$ with $|I| \leq l$ that A_I has full rank if and only if A_I^T has full rank.

(1) \Rightarrow (2) : Let (I, s) be a Recoverable Support of A with size k . Choose $\lambda \in \mathbb{R}^n$ with $\lambda_I = s$ and $\lambda_j = 0$ for $j \in I^c$ and consider $F := \{x \in C^n : \lambda^T x = k\}$. It holds $F \in \mathcal{F}_{n-k}(C^n)$. By assumption there is $w \in \mathbb{R}^m$ such that $A^T w \in \text{relint}(F)$, hence $\text{relint}(F) \cap \text{rg}(A^T) \neq \emptyset$.

(2) \Rightarrow (3) : Denote $I = I(F)$ and choose $\lambda \in F$ with $\lambda_j = 0$ for $j \notin I$. Then $V = \{y \in P : \lambda^T y = k\}$ is a face of $P = C^n \cap \text{rg}(A^T)$ and further $V \subset F$. Hence $I = I(V)$ and there is $v \in V$ with $\|v_{I^c}\|_\infty < 1$. Since P is an l -polytope, it follows $V \in \mathcal{F}_{l-k}(P)$.

(3) \Rightarrow (1) : Let $I = I(V)$. There is $w \in \mathbb{R}^m$ such that $A^T w = v$ and further $\|A_{I^c}^T w\|_\infty < 1$. Hence, the pair (I, v_I) is a Recoverable Support of A with size k . □

Theorem 10 partitions solutions of (1) into equivalence classes separated into faces of C^n with different dimensions. For the rest of this section, we will use the notation of each polytope used in Theorem 10 and $P := C^n \cap \text{rg}(A^T)$. A first consequence of the latter theorem gives an equivalent expression of Definition 8: For $A \in \mathbb{R}^{m \times n}$ with rank l and $k \leq l$ it is $\Lambda(A, k) = |\mathcal{F}_{l-k}(P)|$.

Further the second statement from Theorem 2 delivers the following corollary.

Corollary 11 *Let $A \in \mathbb{R}^{m \times n}$ have rank l . Then the polytope $P = C^n \cap \text{rg}(A^T)$ is l -dimensional, centrally-symmetric, and simple, i.e. any vertex of P is adjacenced by l edges.*

With Corollary 11 we can link *Sparse Reconstruction* to simple, centrally-symmetric polytopes. Further with the two representations of the geometrical interpretation given by Theorem 10 we can involve the results from the field (*cross-*)sections of a hypercube from Combinatorial Geometry. This will be done in Sections 3.4 and 3.5.

3.3 Geometrical interpretation of basis pursuit by Donoho

In this subsection we briefly present the geometrical interpretation of Basis Pursuit by Donoho [3, 4].

With the cross-polytope $\mathcal{C} = \{x \in \mathbb{R}^n : \|x\|_1 \leq 1\}$ and the projection operator $A \in \mathbb{R}^{m \times n}$, we consider the projected cross-polytope $AC = \{Ax : x \in \mathcal{C}\}$ and further the following theorem.

Theorem 12 [3, Theorem 1] *Let $A \in \mathbb{R}^{m \times n}$. These two statements are equivalent:*

- *The polytope AC has $2n$ vertices and is k -neighborly.*
- *Any k -sparse vector solves Basis Pursuit uniquely.*

Theorem 12 connects Sparse Reconstruction with projected cross-polytopes. Thus, one can apply results from convex polytopes like the following necessary condition taken from [3] which is based on [28]. In the following, the floor function is denoted by $\lfloor \cdot \rfloor$.

Corollary 13 *Let $A \in \mathbb{R}^{m \times n}$ with $2 < m \leq n - 2$. If any k -sparse vector x^* solves (1) then $k \leq \lfloor (m + 1)/3 \rfloor$.*

Further in [4] tools from [27] are used to count the faces of randomly-projected cross-polytopes. Considering that any preimage of a face of AC is a face of C (e.g. see [32, Theorem 7.10]), the following lemma connects the property of k -neighborliness of AC and C . We need the term k -simplex describing a polytope with $k + 1$ vertices.

Lemma 14 [4, Lemma 2.1] *Let A be a projection and $P = AC$ such that for $k \in \mathbb{N}$ it holds $|\mathcal{F}_i(P)| = |\mathcal{F}_i(C)|$ for $i = 1, \dots, k - 1$. Then any $F \in \mathcal{F}_l(P)$ is an l -simplex for $l = 0, \dots, k - 1$ and P is k -neighborly.*

Hence, with Lemma 14 one can say rakishly that if we are losing faces through the projection, Basis Pursuit loses the power of reconstructing sparse vectors. Moreover, there exist explicit functions $\rho_N, \rho_F : (0, 1] \rightarrow [0, 1]$ (cf. [4, Section 3]) such that the following theorems hold.

Theorem 15 [4, Theorem 1] *Let $\rho < \rho_N(\delta)$ and $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ a uniformly-distributed random projection with $m \geq \delta n$. Then*

$$\text{Prob}\{|\mathcal{F}_l(C)| = |\mathcal{F}_l(AC)|, l = 0, \dots, \lfloor \rho m \rfloor\} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Theorem 16 [4, Theorem 2] *Let $m \sim \delta n$ and $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a uniform random projection. Then for k with $k/m \sim \rho, \rho < \rho_F(\delta)$ it holds*

$$|\mathcal{F}_k(AC)| = |\mathcal{F}_k(C)|(1 + o(1)).$$

The functions ρ_N, ρ_F are displayed in Fig. 1 and are known in the context of *Phase Transitions* [30]. Theorem 15 implies that for large m and n tending to infinity, with high probability any $\lfloor \rho m \rfloor$ -sparse vector x^* is recoverable. Donoho states [4, Section 1.5] that the result in Theorem 16 can be seen “as a weak kind of neighborliness [...] in which the overwhelming majority of (rather than all) k -tuples span $(k - 1)$ -faces”. Further he remarks that this result is “sharp in the sense that for sequences with $[k/m \sim \rho > \rho_F(\delta)]$, we do not have the approximate equality”. An additional result [4, Theorem 4] is the limit value consideration

$$\lim_{\delta \rightarrow 1} \rho_F(\delta) = 1.$$

This value combined with Theorem 16 implies that for $\delta \rightarrow 1$ and $n \rightarrow \infty$ almost all vectors x^* can be recovered through (1) since the number $|\mathcal{F}_k(AC)|$ tends to concentrate near its upper bound value $2^{k+1} \binom{n}{k+1}$.

Taking up our geometrical perspective, we introduce the *polar set* K^* of $K \subset \mathbb{R}^m$ as

$$K^* := \{w \in \mathbb{R}^m : x^T w \leq 1 \text{ for all } x \in K\}$$

and see with

$$A^T(AC)^* = \{A^T w \in \mathbb{R}^m : |a_i^T w| \leq 1, i = 1, \dots, n\} = \text{rg}(A^T) \cap C^n$$

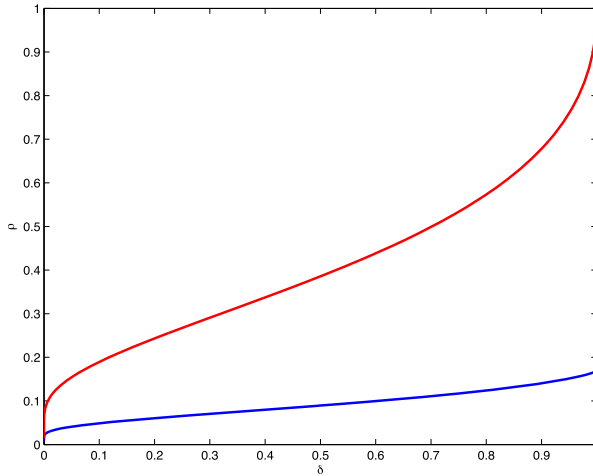


Fig. 1 Functions ρ_N (blue) and ρ_F (red) in Theorems 15 and 16

that the projected cross-polytope AC and $C^n \cap \text{rg}(A^T)$ are dual to each other, see also [23], which means that both polytopes have isomorphic face lattices. Hence, our approach simply differs that we additionally consider unique solutions of Basis Pursuit.

For further considerations, we denote the cross-section of an m -dimensional subspace K of \mathbb{R}^n and C^n as *regular* if K has no point in common with any $(n - m - 1)$ -dimensional face of C^n . The second statement in Theorem 10 connects regular cross-sections of the hypercube to Recoverable Supports. In general, we can not assume that the sections occurring through regarding the range of A^T are regular but we still can use some basic result from literature and connect them to *Sparse Reconstruction*. This is done in Section 3.4 and 3.5.

3.4 Values for Λ

In this subsection, we give some values of Λ for specified matrices and sizes of their Recoverable Supports. In general, the polytope $P = \text{rg}(A^T) \cap C^n$ is not a regular cross-section. Thus, the already difficult problem of counting k -faces of a (simple) polytope becomes even more difficult counting only all k -faces of P intersecting with $(n - m + k)$ -faces of C^n in case of full rank matrices. Different from Ξ (cf. Section 3.5), using past results for a lower bound of Λ over all $m \times n$ -matrices is, as far as we know, only possible under certain assumptions, as the following corollary states.

Corollary 17 *Let $A \in \mathbb{R}^{m \times n}$ with rank l and assume $\text{rg}(A^T) \cap C^n$ is a regular cross-section. Then*

$$\Lambda(A, l) \geq 2^l.$$

Proof The result follows from Statement 3 of Theorem 10 and [17, Corollary 2]. \square

With the same assumptions, Euler’s relation [24, 25] and Steinitz’ characterization for 3-polytopes [29] can be applied, but the practicability is limited since for every matrix the regularity of its corresponding cross-section has to be checked. Considering the cross-section as a simple polytope delivers a different lower bound, which is only dependent on the value $\Lambda(A, 1)$.

Corollary 18 *Let $A \in \mathbb{R}^{m \times n}$ with rank l . Then*

$$\Lambda(A, l) \geq (l - 1)\Lambda(A, 1) - (l + 1)(l - 2).$$

Proof Combining [1, Theorem 1] and Corollary 11 proves the result. □

Note that Corollary 18 provides a lower bound on the number of Recoverable Supports of a matrix if the number of Recoverable Supports of size one is known. However, there are no more than $2n$ possibilities and these can be checked easily for any matrix.

For the rest of this section we consider two types of matrices: *Equiangular tight frames* and *Gaussian matrices*. The term *equiangular tight frame* will be dwelled on later; a Gaussian matrix means that its entries are independant and standard normally distributed random variables, i.e. having mean zero and variance one.

First we consider Gaussian matrices and regard the work of Lonke in [15]. With erf we denote the *Gauss Error function* and $\mathcal{E}(Z)$ describes the expected value of Z .

Corollary 19 *Let $A \in \mathbb{R}^{m \times n}$ be a randomly drawn Gaussian matrix. Then*

$$\mathcal{E}(\Lambda(A, m)) = 2^m \binom{n}{m} \sqrt{\frac{2m}{\pi}} \int_0^\infty e^{-mt^2/2} \left[\operatorname{erf} \left(\frac{t}{\sqrt{2}} \right) \right]^{n-m} dt.$$

Further it holds that

$$\mathcal{E}(\Lambda(A, m)) \geq \binom{n}{n - m} 2^n \left(\frac{1}{\pi} \arctan \frac{1}{\sqrt{m}} \right)^{n-m}, \tag{4}$$

where equality holds for $m = n - 1$.

Proof The result follows from [15, Proposition 2.2, Proposition 2.5] and the second statement of Theorem 10. □

In Section 5 we will match (4) with Monte-Carlo samplings. Additionally, Lonke delivers an asymptotic behavior for sizes $k \neq m$.

Corollary 20 *Let $A \in \mathbb{R}^{m \times n}$ be a randomly drawn Gaussian matrix. Then for $k \neq m$ it holds that*

$$\lim_{n \rightarrow \infty} \mathcal{E}(\Lambda(A, k))(2n)^{-k} k! = 1.$$

Proof Combining [15, Corollary 3.4] and the second statement of Theorem 10 proves the assertion. □

As Lonke says [15, Section 3], the value $\Lambda(A, k)$ “tends to concentrate near the value $[2^k \binom{n}{k}]$, which bounds it from above” (cf. the statement of Donoho [4] as an implication of Theorem 16).

For the rest of this section we regard *equiangular tight frames* $\{a_i\}_{1 \leq i \leq n}$ in \mathbb{R}^m , where the vector a_i forms the i -th column of the $m \times n$ -matrix. Among other things, these frames have the property that any pair of columns has the same inner product. In case of minimally redundant matrices, i.e. $m = n - 1$, the only equiangular tight frame is (up to rotation) the so-called *Mercedes-Benz frame*, see [19, Section 3.2] and [18]. Particularly, Mercedes-Benz frames have an additionally property: Each row of such a matrix has the mean value equal to zero, in other words, the kernel is spanned by the vector of all ones. This property can be used to give the exact number of Maximal Recoverable Supports. Let n be odd. Since any $v \in \text{rg}(A^T)$ has the mean value zero, any vertex of P has the same property. We construct these vertices combinatorically by choosing an index set $J \subset \{1, \dots, n\}$ with $|J| = (n - 1)/2$; there are $\binom{n}{(n-1)/2}$ different possibilities choosing J . Further there are $(n + 1)/2$ different possibilities choosing one $l \in \{1, \dots, n\} \setminus J$. For, say, the Mercedes-Benz frame $A \in \mathbb{R}^{n-1 \times n}$ it holds that $v \in \mathbb{R}^n$, with $v_i = 1$ for $i \in J$ and $v_l = 0$ as well as the remaining entries having the value -1 , is a vertex of P . Hence

$$\Lambda(A, n - 1) = \binom{n + 1}{2} \binom{n}{\frac{n-1}{2}}. \tag{5}$$

Using the same argument for n even, we get $\Lambda(A, n - 1) = 0$ but $\Lambda(A, n - 2) = (n/2) \binom{n}{n/2}$. Keeping in mind that the combinatorical amount increases with a decreasing number of ± 1 , we can construct any Recoverable Support of A with any size, e.g. for n even it holds

$$\Lambda(A, n - 2) = \binom{n - 1}{2} \binom{n + 1}{2} \binom{n}{\frac{n-1}{2}}.$$

The theoretical results so far are illustrated in Fig. 2 by Monte Carlo experiments with Mercedes-Benz frames and randomly drawn Gaussian matrices. One may observe that the empirical results agree with the theoretical statements.

For n even we can also construct a matrix $A \in \mathbb{R}^{n-1 \times n}$ similar to the formula (5), this will be revisited in Section 3.5.

Lemma 21 *Let $A \in \mathbb{R}^{m \times n}$ then there exists a matrix $B \in \mathbb{R}^{m+1 \times n+1}$ such that $\Lambda(B, m + 1) = 2\Lambda(A, m)$.*

Proof Consider the set $W = \{w \in \mathbb{R}^m : w \text{ satisfies (2) for some } (I, s)\}$ and for $\alpha \neq 0$ the matrix

$$B = \begin{bmatrix} A & 0 \\ 0 & \alpha \end{bmatrix}.$$

Then for any $w \in W$ the elements $w^{(1)} = (w, \alpha^{-1})^T$, $w^{(2)} = (-w, \alpha^{-1})^T$ satisfy (2) for B . Hence there are $2\Lambda(A, m)$ Recoverable Supports of B with size $m + 1$. \square

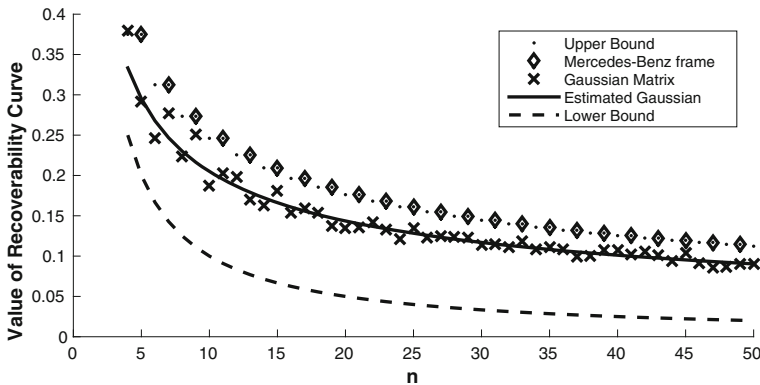


Fig. 2 Monte Carlo Sampling versus Theoretical Results in Section 3.4 and 3.5. Results from Monte Carlo experiments for Mercedes-Benz frame and Gaussian matrix (displayed as crosses) of the size $(n - 1) \times n$. For any $n \geq 4$, one thousand pairs (I, s) with $I \subset \{1, \dots, n\}$, $|I| = n - 1$, $s \in \{-1, +1\}^I$ were taken randomly and tested whether (I, s) is a Recoverable Support. The y-axis displays the proportion of Recoverable Supports versus all tested pairs. For Mercedes Benz-frames only results for n odd are displayed. The formula (4) is plotted as a straight line, and formula (5) is displayed as points. The lower bound from Corollary 17 is displayed in the dashed line

Since for n even it holds that

$$\left(n + 2 - \frac{n}{2}\right) \binom{n + 2}{\frac{n}{2}} = 2 \left(n + 1 - \frac{n}{2}\right) \binom{n + 1}{\frac{n}{2}},$$

and, by denoting $\lfloor \cdot \rfloor$ as the *Floor function*, we can state matrices $A \in \mathbb{R}^{n-1 \times n}$ satisfying

$$\Lambda(A, n - 1) = \left(n - \left\lfloor \frac{n}{2} \right\rfloor\right) \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor}.$$

This formula will be important in Corollary 26.

Up to here, the partial order in the set of all Recoverable Supports of a certain matrix has not been used. The following lemma enters this subject. It will be helpful for bounding Λ and Ξ and further gives some characteristics about the actual recoverability which is the number of Recoverable Supports in proportion to the total number of $(n - k)$ -faces of C^n (where k is the size of the appropriate Recoverable Support).

Lemma 22 *Let $A \in \mathbb{R}^{m \times n}$, then for any $k \leq \text{rank}(A)$ with $\Lambda(A, k) \neq 0$, there exists a positive number $\lambda \leq 2(n - k + 1)$ satisfying*

$$\lambda \Lambda(A, k - 1) = k \Lambda(A, k).$$

Proof Regarding the lattice of all Recoverable Supports of A , Theorem 2 states that any Recoverable Support with size k is adjacent to k Recoverable Supports with size $k - 1$, i.e. the number $k \Lambda(A, k)$ states the number of all adjacences between Recoverable Supports with size k and $k - 1$. Hence, there is a positive number λ satisfying the desired equation. Any Recoverable Support (I, s) with size $k - 1$ is

adjacent to no more than $2(n - k + 1)$ Recoverable Supports with size k , since $|I^c| = n - k + 1$ and each new $s_j, j \in I^c$, in a Recoverable Support with size k can adopt both signs: a positive or a negative sign. Hence, it follows $\lambda \leq 2(n - k + 1)$. \square

The number λ from Lemma 22 states the averaged number of outgoing adjacences from a Recoverable Support with size $k - 1$ to Recoverable Supports with size k . The upper bound for λ implies a statement for the probability that an appropriate pair (I, s) is a Recoverable Support.

Proposition 23 *Let $A \in \mathbb{R}^{m \times n}$, then the mapping*

$$k \mapsto \left[2^k \binom{n}{k} \right]^{-1} \Lambda(A, k) \tag{6}$$

is monotonically nonincreasing.

Proof Assume there is $k \leq \text{rank}(A)$ satisfying

$$\left[2^{k-1} \binom{n}{k-1} \right]^{-1} \Lambda(A, k-1) > \left[2^k \binom{n}{k} \right]^{-1} \Lambda(A, k).$$

Since there is $\lambda \in \mathbb{R}$ such that $\lambda \Lambda(A, k-1) = k \Lambda(A, k)$, it follows that $\lambda > 2(n - k + 1)$, which is a contradiction to Lemma 22. \square

The mapping (6) states the ratio between the actual number of Recoverable Supports of A with size k and the total number of all pairs (I, s) with $I \subset \{1, \dots, n\}, s \in \{-1, +1\}^I$, previously introduced as the recoverability. The second proposition aims at an actual number of Λ for sparsity $\text{rank}(A) - 1$ if the number of Maximal Recoverable Supports is known.

Proposition 24 *Let $A \in \mathbb{R}^{m \times n}$ with rank l and assume $\Lambda(A, l) \neq 0$. Then $\Lambda(A, l - 1) = \frac{l}{2} \Lambda(A, l)$.*

Proof Regarding any Recoverable Support (I, s) with size $l - 1$, it holds that the null space of A_I^T is one-dimensional. Since there exist at least one Recoverable Support with size l , we can enlarge it, due to Theorem 2, in two different directions. \square

Proposition 24 states another interesting fact about the number of Maximal Recoverable Support: Noticing that all values of Λ are even due to the symmetry of the underlying polytope, we observe for an odd rank l of a matrix A that $\Lambda(A, l)$ is divisible by four or even a higher even number.

3.5 Bounds and values for Ξ

In this subsection, we give bounds and values for the largest possible number of Recoverable Supports of all matrices with a certain size, i.e. Ξ (cf. Definition 8).

It is obvious that we can slice the three dimensional cube C^3 with a hyperplane in maximal six edges, see Fig. 3. As Fig. 3 prompts it is not possible to slice less

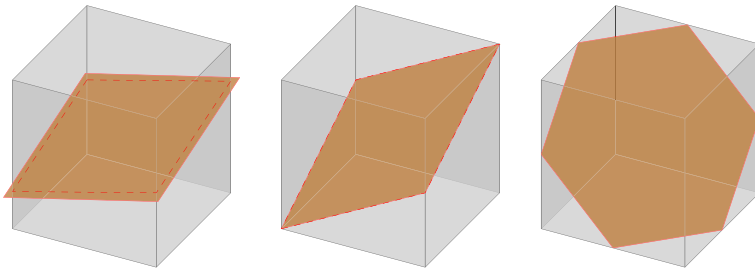


Fig. 3 Examples for Sections $\text{rg}(A^T) \cap C^3$. *Left:* Regular Section; *Center:* Not a Regular Section; *Right:* Regular Section with Mercedes Benz Frame

than four edges without failing the origin, the graphics in the middle shows that it is possible to touch also vertices of the hypercube. Despite Theorem 10 implies that the results from the field *cross-sections of a hypercube* can be used for our issues, these results often require a regular cross-section while, in general, the section $\text{rg}(A^T) \cap C^n$ is not regular. In contrast to lower bounds (cf. Section 3.4), results for an upper bound can be used, as regarded in the following of this subsection. Note that McMullens Upper Bound Theorem [20] can not be used as a typical choice, since it exceeds the trivial bound.

Firstly we give an upper bound for Ξ if k is large. This result is already known [3, Corollary 1.3] (cf. Corollary 13) in the field of *Sparse Reconstruction*.

Corollary 25 *Let $0 < m < n - 1$. If $k > \frac{m+1}{3}$ then $\Xi(m, n, k) < 2^k \binom{n}{k}$.*

Proof This result follows from [14, 28] and the second statement in Theorem 10. □

Considering minimally redundant matrices, remind $m = n - 1$, we get the following value for Maximal Recoverable Supports.

Corollary 26 *It holds that*

$$\Xi(n - 1, n, n - 1) = \left(n - \left\lfloor \frac{n}{2} \right\rfloor \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Proof Combining [22] and Statement 2 of Theorem 10 proves the result. □

In Section 3.4 we have seen that the Mercedes-Benz frame with an odd number of columns and the construction in Lemma 21 reaches this value. Additionally, with the *mutual coherence* slightly more than half of the values $\Xi(n - 1, n, k)$ for variable k are known from the following result.

Corollary 27 *It holds that*

$$\Xi(m, n, k) = 2^k \binom{n}{k} \text{ if } k < \frac{1}{2} \left(1 + \sqrt{\frac{m(n-1)}{n-m}} \right).$$

Proof This follows from [11, 12]. □

The bound in Corollary 27 can be reached by equiangular tight frames, see [12]. As a further consequence of the bound in Lemma 22, the following proposition delivers an upper bound for Ξ .

Proposition 28 *For $k \leq m$ it holds that*

$$\Xi(m, n, k) \leq \frac{2(n - k + 1)}{k} \Xi(m, n, k - 1).$$

Proof Assume there is $k \leq m$ for $A \in \mathbb{R}^{m \times n}$ with $\Xi(m, n, k - 1) = \Lambda(A, k - 1)$ and $\tilde{A} \in \mathbb{R}^{m \times n}$ with $\Xi(m, n, k) = \Lambda(\tilde{A}, k)$ satisfying

$$\Lambda(\tilde{A}, k) > \frac{2(n - k + 1)}{k} \Lambda(A, k - 1),$$

then it holds that

$$\frac{2(n - k + 1)}{k} \Lambda(A, k - 1) < \Lambda(\tilde{A}, k) \leq \frac{2(n - k + 1)}{k} \Lambda(\tilde{A}, k - 1)$$

with Lemma 22, which is a contradiction to $\Xi(m, n, k - 1) = \Lambda(A, k)$. □

Similarly to the value Λ , the latter result implies further statements about Ξ , which are similar to Propositions 23 and 24.

Corollary 29 *It holds that $\Xi(m, n, m - 1) = \frac{m}{2} \Xi(m, n, m)$.*

Additionally, we get a similar statement to Proposition 23 about an upper bound of the recoverability.

Corollary 30 *The mapping*

$$k \mapsto \left[2^k \binom{n}{k} \right]^{-1} \Xi(m, n, k)$$

is monotonically nonincreasing.

To the end of this section, we develop a heuristic upper bound of Ξ . Considering λ in Lemma 22, we can establish an upper bound of Ξ by assuming that λ can be bounded from below, i.e. $\lambda \geq 2(l - k + 1)$ for matrices with rank l . Conveniently, we derive this heuristic bound for full rank, minimally redundant matrices A , i.e. $l = n - 1$, but the construction can be adapted straightforward to other instances. Assume $\lambda \geq 2(n - k)$, then for a positive integer $v < n$ it follows

$$\Lambda(A, n - 1) \geq 2^{v-1} \frac{(v - 1)!(n - v)!}{(n - 1)!} \Lambda(A, n - v)$$

by applying the lower bound recursively. Through substituting $k = n - v$ and bounding $\Lambda(A, k - 1)$ by Corollary 26, we obtain

$$\Lambda(A, k) \leq 2^{k+1-n} \binom{n-1}{k} \left(n - \lfloor \frac{n}{2} \rfloor\right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Since the right-hand side of the latter inequality exceeds the trivial bound $2^k \binom{n}{k}$ for small k , we postulate the following heuristic upper bound:

$$\Xi(n - 1, n, k) \leq \min \left\{ 2^k \binom{n}{k}, 2^{k+1-n} \binom{n-1}{k} \left(n - \lfloor \frac{n}{2} \rfloor\right) \binom{n}{\lfloor \frac{n}{2} \rfloor} \right\}. \tag{7}$$

In general, the inequality $\lambda \geq 2(l - k + 1)$ is not true, but we motivate this bound by the observation that the transition from all pairs (I, s) are Recoverable Supports to none of the pairs (I, s) are Recoverable Supports is rapid, e.g. [5, 6], and, furthermore, this bound is true and strict in case that $k = l$, cf. Corollary 29. As far as we know, there is no matrix exceeding this heuristic; it will be considered in the computational experiments in Section 5. Moreover, this bound is also strict due to Corollary 26, 27 and 29 for some values of k .

In the context of the Hasse Diagram of all Recoverable Supports, the maximum Ξ also states a geometrical question: what is the maximal λ such that the ratio λ/k between the outgoing edges of all Recoverable Supports with size $k - 1$ and the incoming edges of all Recoverable Supports with size k ? Results for this question would give further insights about Ξ and improve a non-trivial upper bound. Combining Corollary 30 with Corollary 27 delivers an interesting insight for $A \in \mathbb{R}^{m \times n}$: if $\Lambda(A, \tilde{k}) = 2^k \binom{n}{\tilde{k}}$ holds for some $\tilde{k} \leq m$, then equality holds in Lemma 22 for all $k \leq \tilde{k}$, and also for Ξ .

4 Computing a recoverable support

In general, generating test instances for computational experiments is an expensive problem in Basis Pursuit. Even for, say, Gaussian matrices, where one only has to find an instance satisfying the optimality condition for ℓ_1 minimization derived by its subdifferential, it is not straightforward to find a suitable x^* satisfying (1) if the desired x^* shall not be very sparse.

One naïve way to generate a test instance is to choose an arbitrary k -sparse vector x^* , solve (1) with some solver and then check whether the solution is equal to x^* . This may work well for small k but usually becomes computationally expensive for larger k . Moreover, this construction suffers from a “trusted method bias”, i.e. the method used to solve (1) may work better on instances which inherit a particular structure (something which may not be under control of the experimenter). Another approach has been proposed in [16]: Choose a pair (I, s) randomly and construct a dual certificate, i.e. find w as in (2). This problem could be seen as a convex feasibility problem [2] and can be solved, e.g., by alternating projections as outlined in [16]. This approach often leads to dual certificates w such that the value $\|A_{I^c}^T w\|_\infty$ is close to one and hence, the result may not be trustworthy due to numerical errors. A more

favorable way to check the reconstructability using (2) would be to check if for some (I, s) the optimal value of

$$\min_w \|A_{I^c}^T w\|_\infty \text{ subject to } A_I^T w = s_I \tag{8}$$

is less or equal one. Similar to the ℓ_1 minimization problem (1), this may be cast as a linear program. However, there are important differences to the naïve approach: First, the number of variables is m which may be much smaller than n . Moreover, one does not rely on the entries of x^* but only on its sign and the support.

However, in all the above methods one generates some trial support (I, s) and then checks whether it is recoverable. Derived from Corollary 19, the probability for an appropriate pair (I, s) , $|I| = n - 1$, being a Maximal Recoverable Support of a randomly drawn Gaussian matrix of the size $(n - 1) \times n$ tends to zero for huge n . Hence, one may never find any $(n - 1)$ -sparse vector by any trial-and-error method and a similar conclusion is true for k -sparse vectors for $m \times n$ matrices if k is sufficiently large. But in view of Theorem 2, there is a systematic way to generate Recoverable Supports (I, s) with maximal size by selecting a 1-sparse recoverable vector, computing a corresponding dual certificate and incrementally increasing the support while maintaining a valid dual certificate (according to Theorem 2, 1.). The method is outlined in Algorithm 1. Note that there is considerable freedom in lines 7 and 8 of the algorithm on how to continue.

Algorithm 1 Computing a Recoverable Support

```

Input :  $A \in \mathbb{R}^{m \times n}, k \leq \text{rank}(A)$ 
Output: Recoverable Support  $(I, s)$  of  $A$  with size  $k$ 
1  $a_k = \arg \max_{a_i} \|a_i\|_2^2$  // The  $i$ -th column of  $A$  is denoted by  $a_i$ 
2  $w \leftarrow \|a_k\|_2^{-2} a_k$ 
3  $s \leftarrow A^T w$ 
4  $I \leftarrow \{k\}$ 
5  $I^c \leftarrow \{1, \dots, n\} \setminus \{k\}$ 
6 while  $|I| < k$  do
7   Choose a vector  $y \in \ker A_I^T$ 
8   Choose  $\lambda \in \mathbb{R}$  such that  $\|A_{I^c}^T(w + \lambda y)\|_\infty = 1$ 
9    $J \leftarrow \{i : |a_i^T(w + \lambda z)| = 1\}$ 
10  if  $|J| \leq k$  and  $A_J$  has full rank then
11     $I \leftarrow J$ 
12     $I^c \leftarrow \{1, \dots, n\} \setminus I$ 
13     $w \leftarrow w + \lambda y$ 
14     $s \leftarrow A^T w$ 
15  else Return to line 7
16 end
    
```

Algorithm 1 is designed for arbitrary matrices of arbitrary sizes. However, it is possible that the algorithm does not deliver a desired Recoverable Support if it gets stuck in line 10. To protect against these cases the method could be extended by including the second statement of Theorem 2; this extension would deliver more freedom to jump between different index sets I but requires elaborate bookkeeping of previously visited index sets. We experienced that this extension is not necessary in most cases.

The first issue about the algorithm might be the question, for what kind of matrices does the method compute a Recoverable Support. Theorem 7 gives an answer: matrices whose columns with maximal Euclidean norm are pairwise linearly independent. The construction in the proof of Theorem 7 for a Recoverable Support with size one is used in the first three lines. Hence, for these matrices the variable s in line 3 has only one entry equal to one in absolute value, the rest of the absolute entries are less than one; this occasions the clauses in line 4 and 5.

Theorem 10 gives a geometrical interpretation of Algorithm 1. In line 3 we start on one facet of the hypercube and by line 14 we walk along the range of the transposed matrix to the next lower-dimensional face of the hypercube. Consequently, the method requires at least $k - 1$ iterations for computing a Recoverable Support with size k . Experiences show that mostly only $k - 1$ iterations are required. The if-clause in line 10 saves for being *stuck* in an unsuitable face.

In any iteration step of the while loop, an element of the corresponding null space is chosen. To choose such a vector it is advantageous to maintain an orthonormal basis for the kernel of A_I^T during the iteration in the form of some decomposition. In our setting, we are calling up a rank one update to a QR decomposition. In the worst-case scenario it may happen that one needs to check several vectors y in line 7, however, using an orthonormal basis of the kernel one can just try all of the basis vectors one after another. This worst case would lead to an iteration number $\mathcal{O}(l^2)$ for computing a Recoverable Support with size l . Actually, we were not able to construct such an instance and usually the iteration number is $\mathcal{O}(l)$. Our setting of this method, implemented as a MATLAB program, can be found online at <http://wwwopt.mathematik.tu-darmstadt.de/spear/>. Further, experiments with Gaussian matrices and Mercedes-Benz frames are evaluated in one of the author's PhD thesis [13].

5 Computational experiments

In this section, we present computational experiments for the topics of the previous sections. The optimization problem (8) delivers an alternative method to perform numerical experiments in Basis Pursuit. A comparison of solving (8) and solving the ℓ_1 minimization in (1) will be done in the following subsection. In Section 5.2 we will highlight the theoretical results from Section 3 with Monte Carlo experiments and will show the behaviour of the heuristic upper bound from (7).

All experiments were done with Matlab R2012b employed on a desktop computer with 4 CPUs, each Intel® Core™ i5-750 with 2.67GHz, and 5.8 GB RAM; the ℓ_1 and ℓ_∞ minimization problems were solved as linear programs with Mosek 6.

In the Monte Carlo experiments it will be tested whether a pair (I, s) , with $I \subset \{1, \dots, n\}$, $s \in \{-1, 1\}^I$, is a Recoverable Support of a given matrix. The experiments were done as follows: For a given matrix $A \in \mathbb{R}^{m \times n}$ and $k \leq m$, we generate $I \subset \{1, \dots, n\}$ with $|I| = k$ randomly by choosing I uniformly at random over $\{1, \dots, n\}$ and assure whether the submatrix A_I has full rank through the Matlab function `rank`. If A_I has no full rank, then (I, s) is not a Recoverable Support of A ; otherwise we also choose $s \in \{-1, 1\}^I$ randomly and solve the ℓ_∞ minimization problem (8) with

$s = \text{sign}(x_I^*)$. If the optimization problem is feasible, solved with status 'optimal' and its optimization value is strictly less than one, the pair (I, s) will be recorded as a Recoverable Support of A . For each size k , we perform M repetitions and average the results; the number M varies from experiment to experiment and may be obtained from the descriptions to each experiment. For reproducibility the code for all tests is at <http://wwwopt.mathematik.tu-darmstadt.de/spear/>.

5.1 Comparing ℓ_1 and ℓ_∞ solver in Mosek

To check whether a pair (I, s) is a Recoverable Support, there are different methods, e.g. outlined in Section 4. In this subsection, we compare the naïve approach, i.e. solving (2) for some x^* with the desired signum s , with solving Eq. 8. For comparison, we decided to perform a similar setup as in typical studies of the *Phase Transition*, see e.g. [30]. We chose, as in [30], Gaussian matrices $A \in \mathbb{R}^{m \times n}$ for fixed $n = 1600$ and varying m such that $\delta = m/n \in (0, 1]$ is chosen in forty equidistant steps. The tests were realized as Monte Carlo experiments with varying $|I| = k$ such that for any m the value $\rho = k/m \in (0, 1]$ is chosen in forty equidistant steps. For any triple (m, n, k) , we did the following testing. We chose $A \in \mathbb{R}^{m \times n}$ as a randomly drawn Gaussian matrix, and performed the Monte Carlo sampling as described above by firstly check whether (I, s) is a Recoverable Support of A , then choose x^* with $\text{supp}(x^*) = I$, $\text{sign}(x^*)_I = s$, and solve Basis Pursuit with the right-hand side Ax^* . This procedure is done with $M = 10$ repetitions. Remarkably, both approaches can be cast as solutions of linear programs and hence, we used the same solver for linear programs. More precisely, testing whether (I, s) is a Recoverable Support by solving (8) was implemented as a linear program and solved with the Mosek routine `mosekopt` with all tolerances set to default. We decide that the pair (I, s) is a Recoverable Support if A_I has full rank, the optimization problem is feasible, it is solved with a status 'Optimal', and its objective value is strictly less than $1 - 10^{-12}$. On the other hand, we checked whether x^* satisfies (1) by solving the constrained ℓ_1 minimization as a linear program with the Mosek routine `mosekopt`; again all tolerances were set to default. We judge a calculated solution \tilde{x} to be exact if $\|\tilde{x} - x^*\| < 10^{-5}$.

First we observe that all calculated solutions were solved with the status "Optimal". Figure 4 displays the averaged results of the decision whether a calculated solution of ℓ_1 minimization is the desired solution (left) and a tested pair is a Recoverable Support (right). The miss-fit between the figures comes from the fact that the solutions of (1) are not accurate enough to fulfill the desired tolerance of 10^{-5} . Relaxing the bound from 10^{-5} to 10^{-3} would lead to almost identical figures in this case but may lead to more errors in other circumstances. Alternatively, instead of measuring the Euclidean distance between the calculated solution \tilde{x} and the actual solution x^* , one may compare whether the support of x^* and the support of \tilde{x} coincide; however, to determine the support, another tolerance would be needed to identify the nonzero entries. In perspective to previous experiments, e.g. [30], the results as in Fig. 4 are as expected. Further, we see agreement to previous testings as the phase

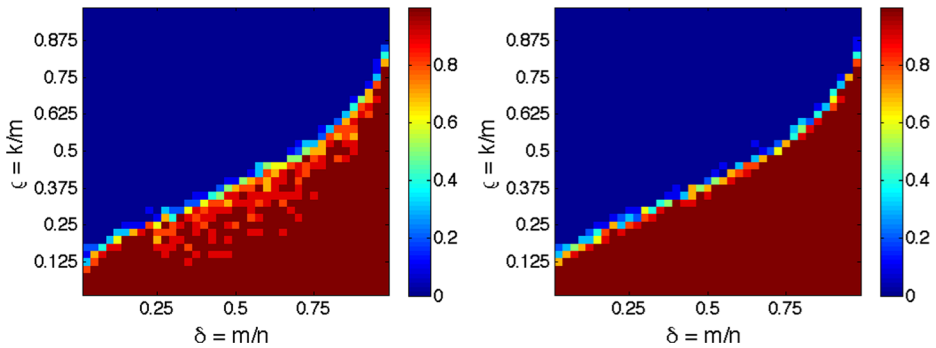


Fig. 4 Averaged results from Monte Carlo experiments whether test instance is solved by ℓ_1 minimization (left) and (8) (right). The values reach from zero (none of the instances where solutions) to one (all instances were solutions)

transition between one to zero is displayed by the curve ρ_F from Theorem 16 (cf. Fig. 1).

For measuring the performance of both procedures, we measure the time it took to solve each linear program. We excluded all operations to formulate the constraints of the linear programs from the time measurement. Additionally before solving (8), we checked whether A_I has full rank and measure its duration. If A_I is not a full rank matrix, the problem (8) would not be solved. Since we are only considering Gaussian random matrices, which are full spark matrices with probability 1, we could have skipped the testing of the rank (and we would have saved about 0.7 percent of the entire run time of the test) but we decided to present the test without any restrictions to specific test problems.

In dependence of δ and ρ , Fig. 5 shows the averaged duration of solving (8) and calculating the rank of the submatrix divided by the averaged duration of the ℓ_1 minimization. One may observe that all quotients are less than one which means that in all cases solving (8) and checking the injectivity of the submatrix is faster than solving Basis Pursuit as a linear program. Figure 6 illustrates that the duration of both methods do increase with an increasing δ , but while solving (8) seems to depend only on δ , the ℓ_1 minimization depends on δ and also on ρ . Moreover, the contours of ρ_F from Theorem 16 can be seen in the duration of time at the ℓ_1 minimization as well as in the Fig. 4: one may say that, on average, solving Basis Pursuit at $\rho = \rho_F(\delta)$ takes more time than solving it at any different ρ in the neighborhood of $\rho_F(\delta)$. Additionally, for small δ only small differences up to a quotient of 4/5 appear in the comparison of the time duration. In total, the use of checking (8) instead of doing ℓ_1 minimization reduces the computational time by a factor of 0.29 (which amounts to a total save of 16 hours of computational time in our experiments).

Furthermore, one may observe that the quotients decrease between $\delta = 0.225$ and $\delta = 0.25$. This phenomenon stems from the duration of the ℓ_1 minimization program, cf. Fig. 6. We believe that an internal change in Mosek, where it is decided whether the primal or the dual problem should be solved, causes this behaviour.

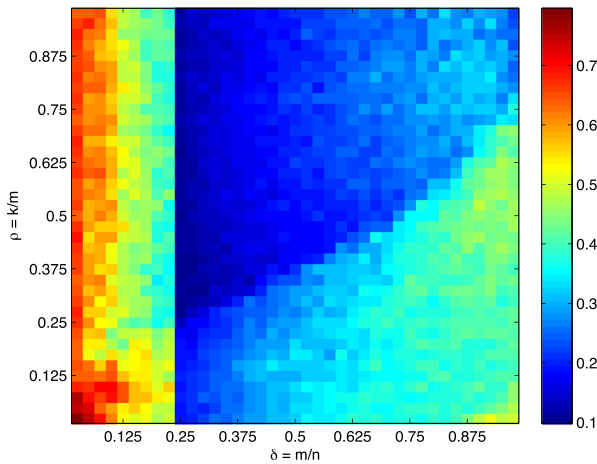


Fig. 5 Comparison of duration performances between ℓ_1 minimization and (8) and checking injectivity

5.2 Number of recoverable supports for certain types of matrices

In this subsection, we compare computational experiments on the number of Recoverable Supports of several types of matrices with results from Section 3 whereas we restrict our experiments to minimally redundant matrices. The computational experiments were done by Monte Carlo experiments described above. Since in the previous sections only Gaussian matrices as well as Mercedes-Benz frames were considered, we will use these types as test problems. Note that in any repetition of the Monte Carlo procedure, a new Gaussian matrix is drawn. Further note that the calculated value approximates the expected number of Recoverable Supports divided by the total number of different pairs (I, s) , $I \subset \{1, \dots, n\}$, $|I| = k, s \in \{-1, +1\}^I$, as in (6).

Similar to Section 5.1, the experiments were done by checking (2) through checking whether the corresponding submatrix is injective and solving (8) afterwards. If

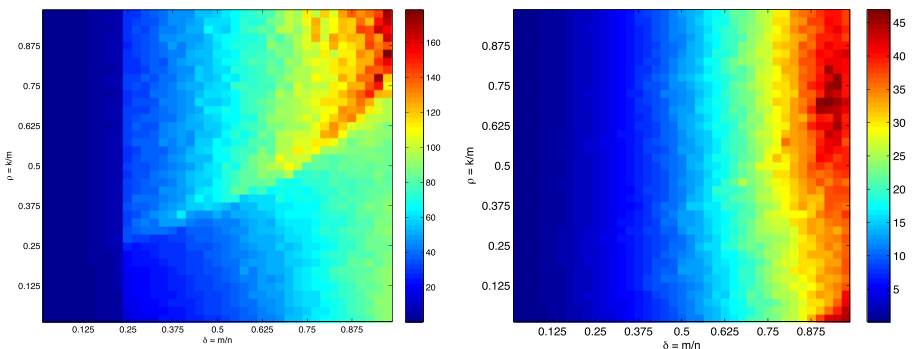


Fig. 6 Duration of ℓ_1 minimization (left) and solving (8) with injectivity check (right) in seconds

the optimal value is strictly less than $1 - 10^{-12}$, we record the chosen pair (I, s) as a Recoverable Support. We did the experiments with $n = 15, 34, 155$ and $n = 555$ and all $|I| = k \leq n - 1$. For each k we did $M = 1000$ repetitions. In Figs. 7-9 all results are shown averaged. The size k of the desired Recoverable Support is given on the x-axis, on the y-axis the probability of recoverability is shown in percent. These functions are empirical approximations of the mapping (6). For comparison, the heuristic upper bound from (7) in proportion to the total number $2^k \binom{n}{k}$ is also displayed. Additionally, a circle for each type of matrix denotes the size k when the recoverability at $k + 1$ is less than one hundred percent (*Empirical Bound*). The empirical bounds are upper bounds for the smallest value k where the actual recoverability (6) at $k + 1$ is less than one hundred percent, since there exists one pair (I, s) which is not a Recoverable Support and the recoverability curve (6) is monotonically nonincreasing by Proposition 23. Note that in almost all cases (e.g. $n = 155, k = 111$ in Fig. 8) the empirical recoverability curves are not monotonically nonincreasing due their empirical nature. The black cross denotes the last k for which the recoverability guarantee for small sizes in Corollary 27 holds (*Bound Mutual Coherence*). All figures only show results from the smallest of all displayed bounds to $n - 1$, since the tests deliver a recoverability of one hundred percent for the missing sizes. Besides the empirical results for the Mercedes-Benz frame A , Fig. 7 shows the actual ratio $\Lambda(A, k) [2^k \binom{n}{k}]^{-1}$ in black with respect to k for $n = 15$. For these results each of the $2^{|I|} \binom{15}{|I|}$ pairs (I, s) with $|I| \leq 14$ have been checked solving (8) if it was a Recoverable Support.

We emphasize that Mosek solved all problems with the status 'Optimal'. In Fig. 7 one can see for the Mercedes-Benz frame that the results of the Monte Carlo sampling (blue) coincide with the actual values (black) up to an error of 10^{-1} . We tolerate this margin of error since improving the precision on one-tenth, we need to increase the number of samplings M a hundredfold. All results are bounded by the Upper Bound (red) except for the Mercedes-Benz frame in this case, which obviously is

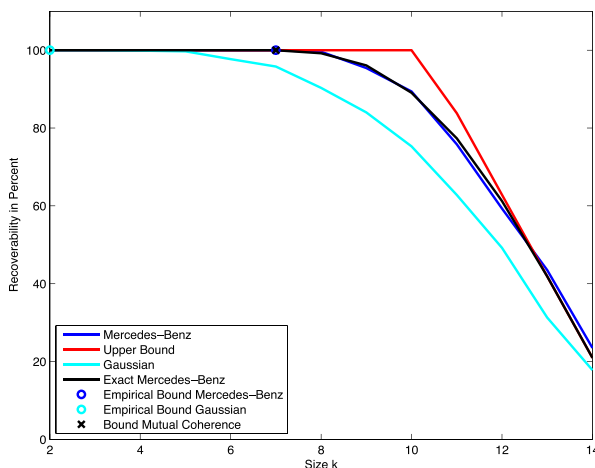


Fig. 7 Monte Carlo Sampling for $n = 15, m = 14$. The black curve represents the actual number of Recoverable Supports of the Mercedes-Benz frame proportional to $2^k \binom{15}{k}$

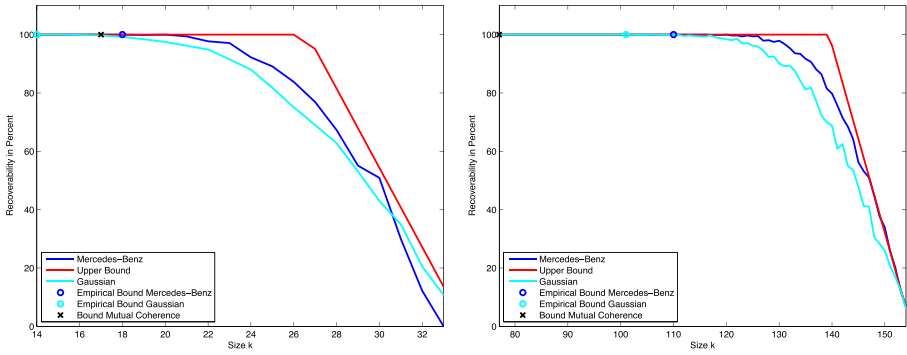


Fig. 8 Monte Carlo Sampling for $n = 34, m = 33$ (left) and $n = 155, m = 154$ (right)

owed by the lack of accuracy. Further the “Bound Mutual Coherence” coincides with the empirical bound for the Mercedes-Benz frame, which is not the case in the other cases. Only in the case $n = 155$ the “Bound Mutual Coherence” is the weakest bound, but as expected the distance to the “Empirical Bound Mercedes-Benz” increases with increasing n . In all cases, Mercedes-Benz has the largest empirical bound. At $n = 155$, this value is $k = 151$, while for $n = 555$ it is $k = 543$. However, the distance between the ‘Empirical Bound Mercedes-Benz’ and the Upper Bound reaching one hundred percent increases with increasing n . Additionally, Proposition 24 holds for all suitable cases except an error of at most 10^{-2} . Hence, the results underlay the expectation that (7) is a good bound for k close to $n - 1$.

Regarding Gaussian matrices, we observe that these matrices do not exceed the empirical recoverability curve of the Mercedes-Benz frame if n is odd. Contrary, it is expected that, at least with k close to $n - 1$, the recoverability curves of the Gaussian matrices exceed the curve of the Mercedes-Benz frame in case n even; this behaviour may be observed in Fig. 8.

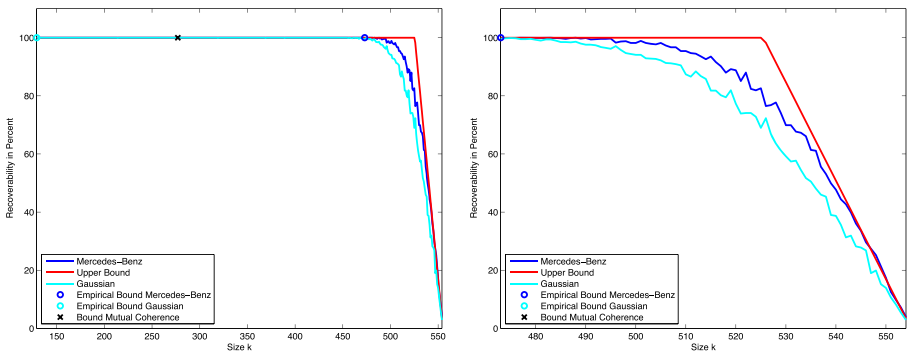


Fig. 9 Monte Carlo Sampling for $n = 555, m = 554$. Left: segment from the “Empirical Bound Gaussian” to $k = 554$. Right: segment from the “Empirical Bound Mercedes-Benz” to $k = 554$, this graphics is a segment of the left graphics

As also observed in the past similar experiments (e.g. [5, 6]), in all cases one can notice a rapid transition from one hundred to zero percent as k increases.

6 Conclusion

In this paper, we gave further insight in the apparently difficult question which vectors are recoverable by ℓ_1 minimization for a given matrix A . Through arranging recoverable vectors in equivalence classes (Recoverable Supports), dependent on A , it follows from Theorem 2 that the Recoverable Supports form a partial ordered set, which is completely known if its maximal elements, i.e. Maximal Recoverable Supports, are known. Although Algorithm 1 is able to compute such a Maximal Recoverable Support quite quickly, even for rather large matrices, we are still far away from any computational method which can result in an exhausting description of the set of Recoverable Supports (and such a method seems to be out of reach).

Moreover, we elaborated on a geometrical viewpoint on sparse recovery which is dual to the view through the projected cross polytope. Exact values and new bounds on the number of Recoverable Supports were derived by connecting ℓ_1 minimization to the dual approach via cross sections of the hypercube which has impact on probability whether a given vector can be reconstructed.

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