Adv Comput Math (2015) 41:1101–1118
DOI 10.1007/s10444-015-9402-7

Pairs of dual Gabor frames generated by functions of Hilbert-Schmidt type

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Received: 1 November 2013 / Accepted: 2 January 2015 /
Published online: 14 January 2015

Abstract We show that *any* two functions which are real-valued, bounded, compactly supported and whose integer translates each form a *partition of unity* lead to a pair of windows generating dual Gabor frames for $L^2(\mathbb{R})$. In particular we show that any such functions have families of dual windows where each member may be written as a linear combination of integer translates of any *B-spline*. We introduce functions of *Hilbert-Schmidt type* along with a new method which allows us to associate to certain such functions finite families of recursively defined dual windows of arbitrary smoothness. As a special case we show that *any* exponential *B-spline* has finite families of dual windows, where each member may be conveniently written as a linear combination of another exponential *B-spline*. Unlike results known from the literature we avoid the usual need for the *partition of unity constraint* in this case.

Keywords Gabor frames · Dual frame pairs · Dual windows · Exponential B-splines

Mathematics Subject Classification (2010) 42C15

1 Introduction and preliminaries

It is well-known that any real-valued, bounded and compactly supported function that satisfies the *partition of unity constraint*

Communicated by: L. L. Schumaker

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$$\sum_{k\in\mathbb{Z}} g(x+k) = 1, \ \ a.e.\ x \in [0,1], \tag{1.1}$$

generates a Gabor frame, to which one can associate a dual frame generated by a window which may be conveniently written as a simple linear combination of the integer translates of the function g itself. See [1, 3] and [4]. We generalize this fact by showing that *any* two real-valued, bounded and compactly supported functions which satisfy Eq. 1.1 lead to a pair of windows generating dual Gabor frames for $L^2(\mathbb{R})$. As a consequence we show that any such functions have dual windows which may be written as a linear combination of any classical *B-spline*. In particular this holds for any *B-spline* itself. In contrast to the constructions known from the literature we can therefore associate to any *B-spline* a dual window of arbitrary finite smoothness. Only relatively few concrete examples of functions satisfying Eq. 1.1 are known. The classical B-splines are one such example. However it was recently shown by O.Christensen and P.Massopust that certain exponential B-splines also satisfy the *partition of unity* constraint [4]. They used this to construct pairs of dual Gabor frames for $L^2(\mathbb{R})$. We introduce the notion of functions of *Hilbert-Schmidt type* and show that *any* exponential, as well as classical *B-spline* can be considered as such a function allowing us to treat both cases as one. As opposed to [1, 3] and [4] this allows us to associate to *any* (exponential) *B-spline* a family of dual windows of *Hilbert-Schimidt type* which are compactly supported and belong to $C^k(\mathbb{R})$, $k \in \mathbb{N}$. In particular we show that *any* (exponential) *B-spline* has a dual window of arbitrary finite smoothness which may be conveniently written as a linear combination of another exponential *B-spline*. In contrast to the constructions in [4] we therefore circumvent the need for the *partition of unity constraint*. Other and quite different approaches which do not rely on Eq. 1.1 have been proposed by R.S Laugesen [13] and I. Kim [12]. Our approach differs by offering a unified approach to constructing dual pairs of Gabor frames for any exponential, as well as classical *B-spline* by means of functions of *Hilbert-Schmidt type*.

The paper is organized as follows. In Section 2 we prove that *any* two real-valued, bounded and compactly supported functions which satisfy Eq. 1.1 lead to a pair of windows generating dual Gabor frames for $L^2(\mathbb{R})$. We introduce functions of *Hilbert-Schmidt type* and establish key properties which will be needed throughout the paper. We state sufficient conditions on a function of *Hilbert-Schmidt type* to generate a Gabor frame for appropriate parameters $a, b > 0$. In Section 3 we present the new method for constructing pairs of dual Gabor frames generated by functions of *Hilbert-Schmidt type*. Finally in Section 4 we apply this method to the (exponential) B-splines constructing dual pairs of Gabor frames.

In the remaining part of this section we give a short introduction to Gabor frames. For $a, b \in \mathbb{R}$ we consider the translation operator $(T_a f)(x) = f(x-a)$ and modulation operator $(E_b f)(x) = e^{2\pi i b x} f(x)$, both acting on $L^2(\mathbb{R})$. The collection of functions $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is referred to as the *Gabor system* generated by the function g and the parameters a, b. In particular we will consider frames having Gabor structure:

Definition 1.1 Given $g \in L^2(\mathbb{R})$, $a, b \in \mathbb{R}$, we say that the collection of functions $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$ if there exist constants $A, B > 0$, such that

$$A\|f\|^2 \leq \sum_{n,m\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 \leq B\|f\|^2, \ \forall\ f \in L^2(\mathbb{R}).$$

If at least the upper frame condition is satisfied $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is called a Bessel sequence.

Gabor frames allow convenient representations of functions $f \in L^2(\mathbb{R})$ similar to those of orthonormal bases for $L^2(\mathbb{R})$:

Theorem 1.2 *Assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame. Then there exists a function $h \in L^2(\mathbb{R})$, such that*

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g, \ \forall\ f \in L^2(\mathbb{R}). \tag{1.2}$$

It can be shown that any two Bessel sequences $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ satisfying Eq. 1.2 are in fact frames. Such frames are said to be *dual frames*. The function g is referred to as the *generator* or *window* function, whereas the function h is called the *dual generator* or *dual window*. Gabor frames and their dual generators have been characterized by Ron and Shen [18], resp. Janssen [11]:

Theorem 1.3 *Two Bessel sequences $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}and\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$form dual frames for $L^2(\mathbb{R})$ if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g(x - n/b - ka)}h(x - ka) = b\delta_{n,0}, \ a.e.\ x \in [0, a].$$

For more on Gabor frames and frames in general we refer to the monographs [8, 9] and [10]. In the following we focus on Gabor frames, however we note that the exponential B-splines have also been used in the construction of wavelets and wavelet frames (see resp. [14] and [5]). For more on exponential B-splines and their applications to the theory of approximation and interpolation we refer to the following extensive, however incomplete, list of references [6, 7, 15–17, 19–21].

2 Functions of Hilbert-Schmidt type

In the following we consider functions $g \in L^2(\mathbb{R})$ which are real-valued, bounded and compactly supported. For any $N_1, N_2 \in \mathbb{Z}$ we denote the set of all such functions with supp $g \subseteq [N_1, N_2]$ by V_{N_1,N_2}. By this notation we have chosen to put emphasis on the support of the functions as it will play a key role in the construction of the associated dual windows. In this context we will throughout the paper tacitly assume $N_1 < N_2$. Functions $g \in V_{N_1,N_2}$ which satisfy Eq. 1.1 will be of special interest and we define

$$W_{N_1,N_2} := \left\{ g \in V_{N_1,N_2} \mid \sum_{k\in\mathbb{Z}} g(x+k) = 1,\ a.e.\ x \in [0,1] \right\}.$$

It is well-known that any $g \in W_{N_1,N_2}$ generates a Gabor frame and has a dual frame generated by a window, given as a linear combination of the integer translates of the function itself. See [1, 3]. The following theorem generalizes this fact stating that *any* two functions $g \in W_{N_1,N_2}$ and $L \in W_{Q_1,Q_2}$ give rise to a pair of windows generating dual Gabor frames for $L^2(\mathbb{R})$.

Theorem 2.1 *Let $N_1, N_2, Q_1, Q_2 \in \mathbb{Z}$. Let $g \in W_{N_1,N_2}$ and $L \in W_{Q_1,Q_2}$. Let $b \in]0, \frac{1}{N_2-N_1+Q_2-Q_1-1}]$ and define*

$$h(x) := b \sum_{j=N_1-(Q_2-1)}^{N_2-(Q_1+1)} L(x-j),\ x \in \mathbb{R}.$$

Then h is compactly supported and the functions g, h generate dual frames $\{E_{\ell b}T_j g\}_{\ell,j\in\mathbb{Z}}$ and $\{E_{\ell b}T_j h\}_{\ell,j\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof By assumption g, L are real-valued, bounded and compactly supported functions. The same is therefore also true for the function h. It follows that g, h generate Bessel sequences $\{E_{\ell b}T_j g\}_{\ell,j\in\mathbb{Z}}$ and $\{E_{\ell b}T_j h\}_{\ell,j\in\mathbb{Z}}$. By Theorem 1.3 it will therefore suffice to prove that

$$\sum_{k\in\mathbb{Z}} g(x - j/b - k)h(x-k) = b\delta_{j,0},\ a.e.\ x \in [0,1]. \tag{2.1}$$

By assumption g is compactly supported with supp $g \subseteq [N_1, N_2]$. We therefore see that Eq. 2.1 is satisfied for all $j \neq 0$ whenever $\frac{1}{b} \geq N_2 - N_1 + Q_2 - Q_1 - 1$.
Let $j = 0$. By assumption L satisfies Eq. 1.1. Furthermore since
supp $L \subset [Q_1, Q_2]$ it follows that for $a.e.\ y \in [N_1, N_2]$ we have

$$\sum_{j\in\mathbb{Z}} L(y-j) = \sum_{j=N_1-(Q_2-1)}^{N_2-(Q_1+1)} L(y-j) = 1.$$

Let $x \in [0,1]$. By the compact support of g we only obtain non-zero contributions to the sum Eq. 2.1 whenever $x + k \in [N_1, N_2]$. We therefore see that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} g(x-k)h(x-k) &= \sum_{k\in\mathbb{Z}} g(x+k) \left(b \sum_{j=N_1-(Q_2-1)}^{N_2-(Q_1+1)} L(x+k-j) \right) \\
&= b \sum_{k\in\mathbb{Z}} g(x+k) \sum_{j=N_1-(Q_2-1)}^{N_2-(Q_1+1)} L([x+k]-j) \\
&= b \sum_{k\in\mathbb{Z}} g(x+k) = b.
\end{aligned}$$

□

As an immediate consequence of Theorem 2.1 simple linear combinations of a fixed function $L \in W_{Q_1,Q_2}$ act as dual generators for all other functions $g \in W_{N_1,N_2}$, $N_1, N_2 \in \mathbb{Z}$. In particular we can associate to any such function g a *family* of dual windows where each member is given as a linear combination of the integer translates of a classical *B-spline*

$$B_{m+1} = B_m * B_1,\ \ B_1 = \chi_{[0,1]},\ \ m \in \mathbb{N}.$$

Corollary 2.2 *Let $N_1, N_2 \in \mathbb{Z}$ and $g \in W_{N_1,N_2}$. Let $m \in \mathbb{N}$ and let B_m denote the classical B-spline of order m. Let $b_m \in]0, \frac{1}{N_2-N_1+m-1}]$ and define*

$$h_m(x) = b_m \sum_{j=N_1-(m-1)}^{N_2-1} B_m(x-j),\ \ m \in \mathbb{N}.$$

For each $m \in \mathbb{N}$ the functions g, h_m generate dual frames $\{E_{\ell b_m} T_j g\}_{\ell, j \in \mathbb{Z}}$ and $\{E_{\ell b_m} T_j h_m\}_{\ell, j \in \mathbb{Z}}$ for $L^2(\mathbb{R})$.

In particular the result of Corollary 2.2 holds for any *B-spline* itself. In contrast to the constructions known from the literature, we may for any *B-spline* B_m, $m \in \mathbb{N}$, construct a dual window of arbitrary finite smoothness given as a linear combination of (possibly) another *B-spline*.

Example 2.3 We consider the classical *B-spline*

$$B_2(x) = \begin{cases} x, & x \in [0,1[\\ 2-x, & x \in [1,2] \\ 0, & x \notin [0,2]. \end{cases}$$

Let $b_m \in]0, \frac{1}{m+1}]$. By Corollary 2.2 we can associate to B_2 the dual windows

$$h_m(x) = b_m \sum_{j=N_1-(m-1)}^{N_2-1} B_m(x-j),\ \ m \in \mathbb{N}. \tag{2.2}$$

In other words B_2 is a common dual generator for all functions (2.2). Since $B_m \in C^{m-2}(\mathbb{R})$ for all $m \in \mathbb{N}$ it follows that we can make these dual windows arbitrarily smooth by considering sufficiently large $m \in \mathbb{N}$.

The result of Corollary 2.2 suffers one major insufficiency. As the smoothness of the dual window increases so does its support and the domain of the modulation parameter b becomes smaller. The following lemma resolves this issue.

Lemma 2.4 *Let $N_1, N_2 \in \mathbb{Z}$ and $K \in W_{N_1,N_2}$. Define*

$$g(x) := \int_0^1 K(x-y) f(y) dy,\ \ x \in \mathbb{R}, \tag{2.3}$$

for some $f \in V_{0,1}$ *such that* $\int_0^1 f(y)dy = C$. *Then the following hold*

(i) *The function* g *belongs to* V_{N_1,N_2+1}. *If* $C = 1$ *then* g *belongs to* W_{N_1,N_2+1}.
(ii) *Let* $k \in \mathbb{N} \cup \{\infty\}$. *If* $f \in V_{0,1} \cap C^k(\mathbb{R})$ *then* g *belongs to* $C^k(\mathbb{R})$.

Proof We begin by proving (i). By definition the functions K and f are both real-valued. Thus g is real-valued. Furthermore by the compact support of K we see that $\operatorname{supp} K(\cdot - y) \subseteq [N_1, N_2 + 1], \ \forall\, y \in [0, 1]$. Thus whenever $x > N_2 + 1$ or $x < N_1$ we have $K(x - y) = 0, \ \forall\, y \in [0, 1]$ and we therefore see that

$$g(x) = \int_0^1 K(x-y)f(y)dy = 0, \ \forall\, x \in]-\infty, N_1[\cup]N_2+1, \infty[.$$

Using the support of g we have

$$\sup_{x\in\mathbb{R}} |g(x)| = \sup_{x\in[N_1,N_2+1]} \left| \int_0^1 K(x-y)f(y)dy \right| \le \sup_{(x,y)\in I\times[0,1]} |K(x-y)| \left(\int_0^1 |f(y)|dy \right),$$

and we conclude that g is bounded. Assuming that $C = 1$ we now show that g satisfies Eq. 1.1. The function K is assumed to satisfy Eq. 1.1. Hence

$$\sum_{k\in\mathbb{Z}} g(x+k) = \int_0^1 \sum_{k\in\mathbb{Z}} K(x-y+k)f(y)dy = \int_0^1 f(y)dy = 1.$$

This proves (i). We now prove (ii). Let $k \in \mathbb{N}\cup\{\infty\}$ and assume that $f \in C^k(\mathbb{R})$. Let $x_0 \in \mathbb{R}$ and let $\{x_n\}$ be any sequence converging to x_0. By the mean-value theorem

$$\left| \frac{f(x_n - y) - f(x_0 - y)}{x_n - x_0} \right| |K(y)| \le \sup_{x\in\mathbb{R}} \left| \frac{df}{dx}(x-y) \right| |K(y)] = C|K(y)| < \infty, \ \forall\, y \in \mathbb{R}.$$

Since K is compactly supported it follows that $CK(\cdot) \in L^1(\mathbb{R})$. By the dominated convergence theorem we therefore obtain

$$\begin{aligned} &\lim_{n\to\infty} \int \left[\frac{f(x_n - y) - f(x_0 - y)}{x_n - x_0} \right] K(y)dy \\ &= \int \lim_{n\to\infty} \left[\frac{f(x_n - y) - f(x_0 - y)}{x_n - x_0} \right] K(y)dy = \int f'(x_0 - y)K(y)dy. \end{aligned}$$

Noting that $g(x) = (K * f)(x) = (f * K)(x)$, we conclude that $g'(x_0) = \int f'(x_0 - y)K(y)dy$. The general result follows by induction. □

By Lemma 2.4 we see that functions of the type (2.3) provide a convenient way of constructing numerous new examples of compactly supported functions satisfying Eq. 1.1. Furthermore by an appropriate choice of the function $f \in V_{0,1}$ we may control the smoothness properties of the function g *independently* of the associated kernel K, avoiding the issue of increased support.

Example 2.5 Let $k \in \mathbb{N}$ and consider the function

$$f_k(y) = y^k(1-y)^k \chi_{[0,1]}(y), \ \ y \in \mathbb{R}.$$

Then f_k belongs to $C^{k-1}(\mathbb{R})$. Let $C_k := \int_0^1 f_k(y)dy$ and $K := \chi_{[0,1]}$. Now define

$$g_k(x) = \frac{1}{C_k}\int_0^1 K(x-y)f_k(y)dy,\ \ x \in \mathbb{R}.$$

By Lemma 2.4 it follows that $g_k \in C^{k-1}(\mathbb{R})$. Furthermore g_k satisfies Eq. 1.1 and is compactly supported in $I = [0, 2]$. In the concrete case $k = 3$ we have

$$g_3(x) = \begin{cases} -20x^7 + 70x^6 - 84x^5 + 35x^4, & x \in [0, 1[\\ 20x^7 - 210x^6 + 924x^5 - 2205x^4 + 3080x^3 - 2520x^2 + 1120x - 20x, & x \in [1, 2] \\ 0, & x \notin [1, 2]. \end{cases}$$

Let B_2 denote the classical *B-spline* of order 2. Let $\ell \in \mathbb{N}$ and consider

$$\tilde{f}_\ell(x) := \sin(\pi x)^\ell \chi_{[0,1]}(x).$$

We see that $\tilde{f}_\ell \in C^{\ell-1}(\mathbb{R}),\ \ell \in \mathbb{N}$. Let $\tilde{C}_\ell := \int_0^1 \sin(\pi y)^\ell dy$ and define

$$L_\ell(x) = \frac{1}{\tilde{C}_\ell}(B_2 * \tilde{f}_\ell)(x) = \int_0^1 B_2(x-y)\sin(\pi y)^\ell dy.$$

In the concrete case of $\ell = 2$ we have

$$L_2(x) = \begin{cases} \frac{1}{2}\frac{\cos(\pi x)^2 - 1 + (\pi x)^2}{\pi^2}, & x \in [0, 1[\\ -\frac{1}{2}\frac{\pi^2(2x^2-6x+3)+2\cos(\pi x)^2-2}{\pi^2}, & x \in [1, 2[\\ \frac{1}{2}\frac{\pi^2(x^2-6x-1)+\cos(\pi x)^2+9\pi^2}{\pi^2}, & x \in [2, 3] \\ 0, & x \notin [0, 3]. \end{cases}$$

By Lemma 2.4 we have for each $\ell \in \mathbb{N}$ that $L \in C^{\ell-1}(\mathbb{R})$, supp $L \subseteq [0, 3]$ and L satisfies Eq. 1.1. By Theorem 2.2 it follows that for any $b \in]0, \frac{1}{4}]$ the functions g_k and h_2 defined by $h_2(x) := b\sum_{j=-2}^{1} L_2(x-j),\ x \in \mathbb{R}$ generate frame pairs $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Motivated by Lemma 2.4 we will consider functions of the type (2.3) in more detail. We refer to these as functions of *Hilbert-Schmidt type* or simply functions of type $\mathcal{HS}$.

Definition 2.6 Let $N_1, N_2 \in \mathbb{Z}$ and $K \in V_{N_1,N_2}$. A function g is of Hilbert-Schmidt type if it has a representation of the form

$$g(x) = \int_0^1 K(x-y)f(y)dy, \tag{2.4}$$

for some real-valued function $f \in V_{0,1}$.

We note that the (exponential) B-splines

$$\mathcal{E}_{m,a} = \mathcal{E}_{m-1,a} * e^{a_m(\cdot)}\chi_{[0,1]},\ \ \mathcal{E}_{1,a} = e^{a_1(\cdot)}\chi_{[0,1]},$$

where $\mathbf{a} = (a_1, a_2, ..., a_m)$ is some m-tuple of real numbers, are functions of type $\mathcal{HS}$. Indeed by repeated use of Lemma 2.4 each member of the recursively defined family

$$g_n(x) = \int_0^1 g_{n-1}(x-y) f_n(y) dy, \ n \in \mathbb{N}, \ g_0(x) := K(x), \tag{2.5}$$

is a function of type $\mathcal{HS}$. Also, by appropriate choices of the functions f_n and the kernel K in Eq. 2.5, the result of Lemma 2.4 ensures that each member g_n satisfies Eq. 1.1 and that g_n can be made arbitrarily smooth. Further the proof of Lemma 2.4 implies that any function of type $\mathcal{HS}$ is bounded and compactly supported. Any such function g will generate a Bessel sequence $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, for any choices of $a, b > 0$. We now state a sufficient condition on the kernel K such that g for appropriate parameters $a, b > 0$ generates a frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proposition 2.7 *Let $N_1, N_2 \in \mathbb{Z}$ and $K \in V_{N_1,N_2}$.*
Let $(a,b) \in]0, N_2 - N_1 + 1[\times]0, \frac{1}{N_2-N_1+1}]$ and assume that K is continuous and strictly positive on the open interval $I =]N_1, N_2[$. Choose $f \in V_{0,1}$ such that $f(y) > 0, \ \forall \ y \in]0, 1[$. Now define g by Eq. 2.4. Then g is a function of Hilbert-Schimidt type and generates a frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof Since $g \in L^2(\mathbb{R})$ is the convolution of $K, f \in L^1(\mathbb{R})$ it follows that g is continuous on $\mathbb{R}$. By Lemma 2.4 supp $g \subseteq [N_1, N_2 + 1]$. Thus by Theorem 9.1.5 in [2] it suffices to prove that $g(x) > 0, \ \forall \ x \in]N_1, N_2 + 1[$. Consider $x \in]N_1, N_2 + 1[$. Then there exists at least one $y_0 \in]0, 1[$ such that $x - y_0 \in]N_1, N_2[$. By assumption this implies that $\epsilon := K(x - y_0) > 0$. By continuity of K on the open interval $I = [N_1, N_2][$, it follows that there exists a $\delta_1 > 0$ such that $|K(x-y) - K(x-y_0)| < \epsilon$, whenever$|y - y_0| < \delta_1$. Since $y_0 \in]0, 1[$ there exists a $\delta_2 > 0$ such that $|y_0 + h \in]0, 1[, \ \forall \ |h| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. When $|y - y_0| < \delta$ it now follows that $|K(x - y) - K(x - y_0)| < \epsilon$. We therefore conclude that $K(x - y) > 0$, $\forall y \in]y_0 - \delta, y_0 + \delta[\subseteq [0, 1]$. Since K is non-negative and f is strictly positive on the interval $I =]0, 1[$ it follows that $g(x) = \int_0^1 K(x-y) f(y) dy > 0 \forall x \in]N_1, N_2 + 1[$. □

By Proposition 2.7 we may construct families $\{g_n\}_{n\in\mathbb{N}}$ of functions of type $\mathcal{HS}$ by Eq. 2.5, such that each member generates a Gabor frame for $L^2(\mathbb{R})$. In particular any (exponential) *B-spline* $\mathcal{E}_{N,a}$ of order $N \in \mathbb{N}$ is covered by the proposition, which follows since any such spline is given recursively by Eq. 2.5, where $f_n = e^{a_n(\cdot)}\chi_{[0,1]}$, for some N-tuple $\mathbf{a} = (a_1, a_2, ..., a_N)$.

3 Pairs of dual Gabor frames generated by functions of Hilbert-Schmidt type

By Lemma 2.4 any (exponential) *B-spline* of the form

$$\mathcal{E}_{m,\mathbf{a}} := \chi_{[0,1]} * e^{a_1(\cdot)}\chi_{[0,1]} * e^{a_2(\cdot)}\chi_{[0,1]} * ... * e^{a_m(\cdot)}\chi_{[0,1]}, \quad m \in \mathbb{N}, \ a_i \in \mathbb{R} \ \forall \ i \in \{1, ..., m\},$$

satisfies the *partition of unity constraint* (up to a multiplicative constant). By virtue of this property one knows how to construct dual generators. See [1, 3] and [4]. The result of Lemma 2.4 explains why such B-splines satisfy Eq. 1.1 directly in terms of the kernel $K(x) := \chi_{[0,1]}(x),\ x \in \mathbb{R}$. It therefore seems reasonable to presume that the dual properties of functions of type $\mathcal{HS}$ are related to the properties of their kernels and not directly to Eq. 1.1 in itself.

The following is dedicated to formalize this idea in the general setting of functions of type $\mathcal{HS}$. Preliminary and technical results are presented in Lemmas 3.1-3.3 and Proposition 3.4, composing the building blocks of the main result (Theorem 3.5). The theorem allows us to construct dual windows of arbitrary finite smoothness associated with certain functions of type $\mathcal{HS}$, including *any* exponential, as well as classical B-spline. In the case of exponential B-splines we therefore avoid the usual need of Eq. 1.1.

Lemma 3.1 *Let* $N_1, N_2, Q_1, Q_2 \in \mathbb{Z}$. *Let* $\tilde{K} \in W_{N_1,N_2}$ *and* $L \in W_{Q_1,Q_2}$. *Let* $M \in \mathbb{N}$, $a \in \mathbb{R}$ *and* $F(x) := e^{-ax}, G(x) := e^{ax},\ x \in \mathbb{R}$.
Let $\{\tilde{f}_n\}_{n=1}^M,\ \tilde{f}_n \in V_{0,1}$ *be such that* $\int_0^1 F(y)\tilde{f}_n(y)dy = 1,\ \forall\, n \in \{1, ..., M\}$ *and define*

$$\tilde{F}_n(y_n) = \int_{[0,1]^{n-1}} F(y_1 + ... + y_n) \prod_{j=1}^{n-1} \tilde{f}_j(y_j)dy_j.$$

$$K(x, y) := G(x - y)\tilde{K}(x - y), \qquad x, y \in \mathbb{R}.$$

$$L(x, z) := F(x - z) \sum_{j=N_1-M-(Q_2-1)}^{N_2+M-(Q_1+1)} L(x - z - j), \qquad x, z \in \mathbb{R}.$$

Then the following hold

(i) $\sum_{k\in\mathbb{Z}} K(x + k, y)L(x + k, z) = F(y)G(z),\ a.e.\ x \in [0, 1],\ y, z \in [0, M]$.

(ii) $\int_0^1 \tilde{F}_n(y_n)\tilde{f}(y_n)dy_n = 1,\ \forall\, n \in \{1, ..., M\}$.

Proof We begin by proving (i). By assumption we have supp $L \subset [Q_1, Q_2]$. Combined with the *partition of unity property* it follows that for $a.e.\ y \in [N_1 - M, N_2 + M]$ we have

$$\sum_{j\in\mathbb{Z}} L(y - j) = \sum_{j=N_1-M-(Q_2-1)}^{N_2+M-(Q_1+1)} L(y - j) = 1. \tag{3.1}$$

By assumption we have supp $K \subseteq [N_1, N_2]$.
Since $|(x - z + k) - (x - y + k)| = |y - z| \leq M$,
it follows that if $x + k - y \in [N_1, N_2]$, then $x + k - z \in [N_1 - M, N_2 + M]$.

By the *partition of unity property* and Eq. 3.1 we see that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} K(x+k,y)L(x+k,z)\\
&= \sum_{k\in\mathbb{Z}} e^{a(x+k-y)}K(x+k-y)e^{-a(x+k-z)} \sum_{j=N_1-M-(Q_2-1)}^{N_2+M-(Q_1+1)} L([x+k-z]-j)\\
&= e^{-ay}e^{az}\sum_{k\in\mathbb{Z}} K(x+k-y) \sum_{j=N_1-M-(Q_2-1)}^{N_2+M-(Q_1+1)} L([x+k-z]-j)\\
&= e^{-ay}e^{az}\sum_{k\in\mathbb{Z}} K([x-y]+k) = e^{-ay}e^{az},\ \ a.e.\ x\in[0,1],\ \ y,z\in[0,M].
\end{aligned}$$

This proves (i). We now prove (ii). By assumption $\int_0^1 e^{-ay}\tilde{f}_n(y)dy = 1,\ \forall\, n\in\{1,...,M\}$. We therefore see that

$$\begin{aligned}
\int_0^1 \tilde{F}_n(y_n)\tilde{f}_n(y_n)dy &= \int_0^1 \left(\int_{[0,1]^{n-1}} F(y_1+...+y_n)\prod_{j=1}^{n-1}\tilde{f}_j(y_j)dy_j\right)\tilde{f}_n(y_n)dy_n\\
&= \int_{[0,1]^n} F(y_1+...+y_n)\prod_{j=1}^{n}\tilde{f}_j(y_j)dy_j\\
&= \int_{[0,1]^n} e^{-a(y_1+...+y_n)}\prod_{j=1}^{n}\tilde{f}_j(y_j)dy_j\\
&= \int_{[0,1]^n} e^{-ay_1}\dots e^{-ay_n}\prod_{j=1}^{n}\tilde{f}_j(y_j)dy_j\\
&= \prod_{j=1}^{n}\left(\int_0^1 e^{-ay_j}\tilde{f}_j(y_j)dy_j\right) = 1.
\end{aligned}$$

□

For any fixed $M\in\mathbb{N}$ we now consider families $\{g_n\}_{n=1}^M$, $\{h_n\}_{n=1}^M$, of the type (2.5). Our goal is to show that dual properties of members g_n, h_m are related to their initial kernels. More precisely we set out to find sufficient conditions on the initial kernels to ensure that any member of the family $\{h_n\}_{n=1}^M$ will act as a common dual generator for each function $g_n,\ n\in\{1,...,M\}$ and vice versa.

Lemma 3.2 *Let $M\in\mathbb{N}$ and $N_1,N_2\in\mathbb{Z}$. Let g,h be functions of Hilbert-Schmidt type. Let $K\in V_{N_1,N_2}$ and $L\in V_{M_1,M_2}$ denote their respective kernels and $\tilde{f},\tilde{g}$ the associated $V_{0,1}$ functions, where*

$$M_1 = N_1 - M - Q + 1,\ \ M_2 = N_2 + M + Q - 1,$$

for some $Q\in\mathbb{N}$. Let $b\in]0,\frac{1}{N_2-N_1+M+Q}]$ and $F,G\in V_{0,1}$. Assume that *(i)* $\sum_{k\in\mathbb{Z}} K(x+k,y)L(x+k,z) = bF(y)G(z),\ a.e.\ x \in [0,1],\ \ y,z \in [0,M].$

(ii) $\int_0^1 F(y)\tilde{f}(y)dy = \int_0^1 G(z)\tilde{g}(z)dz = 1.$

Then the functions g, h *generate dual frames* $\{E_{\ell b}T_j g\}_{\ell,j\in\mathbb{Z}}$ *and* $\{E_{\ell b}T_j h\}_{\ell,j\in\mathbb{Z}}$ *for* $L^2(\mathbb{R})$.

Proof By construction g, h are real-valued, bounded and compactly supported functions. It follows that $\{E_{\ell b}T_j g\}_{\ell,j\in\mathbb{Z}}$ and $\{E_{\ell b}T_j h\}_{\ell,j\in\mathbb{Z}}$ are Bessel sequences. By Theorem 1.3 these sequences form pairs of dual frames if and only if

$$\sum_{k\in\mathbb{Z}} g(x-j/b-k)h(x-k) = b\delta_{j,0},\ a.e.\ x \in [0,1]. \tag{3.2}$$

Let $j \neq 0$. By construction we have supp $g \subseteq [N_1, N_2+1]$ and supp $h \subseteq [N_1 - M - Q + 1, N_2 + M + Q]$. We therefore see that Eq. 3.2 is satisfied whenever $\frac{1}{b} \geq N_2 - N_1 + M + Q$. Let $j = 0$. Since g, h are compactly supported we can interchange the order of summation and integration in Eq. 3.2. We thereby obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z}} g(x+k)h(x+k) &= \sum_{k\in\mathbb{Z}}\left(\int_0^1 K(x+k,y)\tilde{f}(y)dy\right)\left(\int_0^1 L(x+k,z)\tilde{g}(z)dz\right)\\ &= \sum_{k\in\mathbb{Z}}\int_0^1\int_0^1 K(x+k,y)L(x+k,z)\tilde{f}(y)\tilde{g}(z)dydz \qquad (3.3)\\ &= \int_0^1\int_0^1\left(\sum_{k\in\mathbb{Z}} K(x+k,y)L(x+k,z)\right)\tilde{f}(y)\tilde{g}(z)dydz.\end{aligned}$$

Combining Eq. 3.3 with assumptions (i) and (ii) it follows that for $a.e.\ x \in [0,1]$ we have

$$\sum_{k\in\mathbb{Z}} g(x+k)h(x+k) = b\left(\int_0^1 F(y)\tilde{f}(y)dy\right)\left(\int_0^1 G(z)\tilde{g}(z)dz\right) = b.$$

□

We now provide a convenient way of constructing families of functions of type $\mathcal{HS}$ satisfying Lemma 3.2(i):

Lemma 3.3 *Let* $M \in \mathbb{N}$ *and* $N_1, N_2 \in \mathbb{Z}$. *Let* $g_0 \in V_{N_1,N_2}$ *and* $h_0 \in V_{M_1,M_2}$ *satisfy the conditions of Lemma 3.2 with* $b > 0$. *For* $n \in \{1, ..., M\}$ *define*

$$g_n(x) = \int_0^1 g_{n-1}(x-y)\tilde{f}_n(y)dy,\ \ h_n(x) = \int_0^1 h_{n-1}(x-z)\tilde{g}_n(z)dz, \tag{3.4}$$

for some $\tilde{f}_n, \tilde{g}_n \in V_{0,1}$ *and let*

$$\tilde{F}_n(y_n) := \int_{[0,1]^{n-1}} F(y_1 + ... + y_n) \prod_{j=1}^{n-1} \tilde{f}_j(y_j) dy_j, \quad \tilde{G}_n(z_n) := \int_{[0,1]^{n-1}} G(z_1 + ... + z_n) \prod_{j=1}^{n-1} \tilde{g}_j(z_j) dz_j. \tag{3.5}$$

Let $n, m \in \{1, ...M\}$. *Then*

$$\sum_{k\in\mathbb{Z}} g_{n-1}(x+k-y) h_{m-1}(x+k-z) = b\tilde{F}_n(y)\tilde{G}_m(z), \ a.e. \ x \in [0, 1], \ \ y, z \in [0, M].$$

Proof We begin by establishing the formulas

$$g_n(x) = \int_{[0,1]^n} g_0(x - \sum_{j=1}^{n} y_j) \prod_{j=1}^{n} \tilde{f}_j(y_j) dy_j, \quad h_m(x) = \int_{[0,1]^m} h_0(x - \sum_{j=1}^{m} z_j) \prod_{j=1}^{m} \tilde{g}_j(z_j) dz_j.$$

We do so by induction. When $n = 1$ we have $g_1(x) = \int g_0(x - y_1) \tilde{f}_1(y_1) dy_1$. Now assume that the formula holds for some $n \in \{1, ..., M - 1\}$. We then obtain

$$\begin{aligned} g_{n+1}(x) &= \int g_n(x - y_{n+1}) \tilde{f}_{n+1}(y_{n+1}) dy_{n+1} \\ &= \int_0^1 \left(\int_{[0,1]^n} g_0(x - y_{n+1} - \sum_{j=1}^{n} y_j) \prod_{j=1}^{n} \tilde{f}_j(y_j) dy_j \right) \tilde{f}_{n+1}(y_{n+1}) dy_{n+1} \\ &= \int_{[0,1]^{n+1}} g_0(x - \sum_{j=1}^{n+1} y_j) \prod_{j=1}^{n+1} \tilde{f}_j(y_j) dy_j. \end{aligned}$$

The formula for h_m holds by a similar argument. Combining the results we see that

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} g_{n-1}(x + k - y_n) h_{m-1}(x + k - z_m) \\ &= \sum_{k\in\mathbb{Z}} \left(\int_{[0,1]^{n-1}} g_0(x + k - \sum_{j=1}^{n} y_j) \prod_{j=1}^{n-1} \tilde{f}_j(y_j) dy_j \right) \left(\int_{[0,1]^{m-1}} h_0(x + k - \sum_{j=1}^{m} z_j) \prod_{j=1}^{m-1} \tilde{g}_j(z_j) dz_j \right) \\ &= \int_{[0,1]^{n+m-2}} \sum_{k\in\mathbb{Z}} g_0(x + k - \sum_{j=1}^{n} y_j) h_0(x + k - \sum_{j=1}^{m} z_j) \prod_{j=1}^{n-1} \tilde{f}_j(y_j) \prod_{j=1}^{m-1} \tilde{g}_j(z_j) dy_j dz_j. \end{aligned}$$

Let $y := \sum_{j=1}^{n} y_j, \quad z := \sum_{j=1}^{m} z_j$. Since $n, m \leq M$ it follows that $y, z \in [0, M]$.
By the assumptions (i) and (ii) we therefore obtain

$$\sum_{k\in\mathbb{Z}} g_{n-1}(x+k-y_n)h_{m-1}(x+k-z_m)$$
$$= b \int_{[0,1]^{n+m-2}} F(y_1+...+y_n)G(z_1+...+z_m) \prod_{j=1}^{n-1} \tilde{f}(y_j) \prod_{j=1}^{m-1} \tilde{g}(z_j) dy_j dz_j$$
$$= b \left(\int_{[0,1]^{n-1}} F(y_1+...+y_n) \prod_{j=1}^{n-1} \tilde{f}(y_j) dy_j \right) \left(\int_{[0,1]^{m-1}} G(z_1+...+z_n) \prod_{j=1}^{m-1} \tilde{g}(z_j) dz_j \right)$$
$$= b\tilde{F}_n(y_n)\tilde{G}_m(z_m).$$

□

Combining Lemmas 3.2 and 3.3 we get sufficient conditions on the kernels g_0, h_0 for *any* two members g_n, h_m of the families (2.5) to form pairs of dual generators:

Proposition 3.4 *Let $M \in \mathbb{N}$ and $N_1, N_2 \in \mathbb{Z}$. Let $g_0 \in V_{N_1,N_2}$ and $h_0 \in V_{M_1,M_2}$ satisfy the conditions of Lemma 3.2 with $b \in]0, \frac{1}{N_2-N_1+2M+Q-1}]$.*
Choose $\tilde{f}_n, \tilde{g}_n \in V_{0,1}, \ n \in \{1, ..., M\}$, such that

$$\int_0^1 \tilde{F}_n(y)\tilde{f}_n(y)dy = \int_0^1 \tilde{G}_n(z)\tilde{g}_n(z)dz = 1,$$

where $\tilde{F}_n$ and $\tilde{G}_n$ are given by Eq. 3.5. Define g_n, h_n recursively by Eq. 3.4. Let $n, m \in \{1, ...M\}$. Then the functions g_n, h_m generate dual frames $\{E_{\ell b}T_j g_n\}_{\ell,j\in\mathbb{Z}}$ and $\{E_{\ell b}T_j h_m\}_{\ell,j\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof Let $n, m \in \{1, ..., M\}$. By repeated use of Lemma 2.4 it follows that $\operatorname{supp} g_n \subseteq [N_1, N_2+n]$ and $\operatorname{supp} h_m \subseteq [N_1 - M - Q + 1, N_2 + M + m + Q - 1]$. Let $j \neq 0$. Since $n, m \leq M$ we see that

$$\sum_{k\in\mathbb{Z}} g_n(x - j/b - k)h_m(x-k) = b\delta_{j,0}, \ a.e. \ x \in [0,1]$$

is satisfied whenever $\frac{1}{b} \geq N_2 - N_1 + 2M + Q - 1$. Furthermore by Lemma 3.3 we have

$$\sum_{k\in\mathbb{Z}} g_{n-1}(x+k-y)h_{m-1}(x+k-z) = b\tilde{F}_n(y)\tilde{G}_m(z), \ a.e. \ x \in [0,1], \ \ y, z \in [0, M].$$

Since $\int_0^1 \tilde{F}_n(y)\tilde{f}_n(y)dy = \int_0^1 \tilde{G}_m(z)\tilde{g}_m(z)dz = 1$ the result follows by Lemma 3.2. □

Finally, combining Lemma 3.1 and Proposition 3.4 we obtain the main result describing how to construct finite families of dual windows associated with certain functions of type $\mathcal{HS}$:

Theorem 3.5 *Let $M \in \mathbb{N}$, $N_1, N_2, Q_1, Q_2 \in \mathbb{Z}$. Let $K \in W_{N_1,N_2}$ and $L \in W_{Q_1,Q_2}$. Let $a \in \mathbb{R}$, $b \in]0, \frac{1}{N_2-N_1+2M+Q_2-Q_1-1}]$ and define*

$$g_0(x) := e^{ax}K(x), \quad h_0(x) = be^{-ax} \sum_{j=N_1-M-(Q_2-1)}^{N_2+M-(Q_1+1)} L(x-j), \; x \in \mathbb{R}.$$

Now define g_n, h_n recursively by Eq. 3.4 for some $\tilde{f}_n, \tilde{g}_n \in V_{0,1}$ such that $\int_0^1 e^{-ay}\tilde{f}_n(y)dy = \int_0^1 e^{ay}\tilde{g}_n(y)dy = 1$. Let $n, m \in \{1, ..., M\}$ and $k \in \mathbb{N} \cup \{\infty\}$. Then the following hold

(i) The functions g_n, h_m are compactly supported and if $\tilde{f}_n, \tilde{g}_m$ belong to $C^k(\mathbb{R})$ then g_n, h_m belong to $C^k(\mathbb{R})$.
(ii) The functions g_n, h_m generate dual frames $\{E_{\ell b}T_j g_n\}_{\ell,j\in\mathbb{Z}}$ and $\{E_{\ell b}T_j h_m\}_{\ell,j\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof By assumption we have supp $L \subseteq [Q_1, Q_2]$. We therefore see that h_0 is compactly supported with

$$\text{supp } h_0 \subseteq [N_1 - M - (Q_2 - Q_1 - 1), N_2 + M + Q_2 - Q_1 - 1].$$

Combined with Lemma 2.4 this proves (i). By Lemma 3.1 we have

$$\text{(i)} \quad \sum_{k\in\mathbb{Z}} g_0(x+k-y)h_0(x+k-z) = be^{-ay}e^{az}, \; a.e. \; x \in [0,1], \; y, z \in [0, M],$$

$$\text{(ii)} \quad \int_0^1 \tilde{F}_n(y_n)\tilde{f}_n(y_n)dy_n = \int_0^1 \tilde{G}_n(z_n)\tilde{g}_n(z_n)dz_n = 1, \; \forall\, n \in \{1, ..., M\}.$$

By Proposition 3.4 this proves duality of g_n and h_m. □

In the next section we apply Theorem 3.5 to derive the results proclaimed at the beginning of this section, i.e. we associate to *any* (exponential) B-spline finite families of compactly supported dual windows of arbitrary finite smoothness.

4 Gabor frames generated by (exponential) B-splines

It has recently been shown in [4] that if $a_i \neq 0 \,\forall \in \{1, ..., N\}$ then the exponential B-spline $\mathcal{E}_{N,\mathbf{a}}$ cannot satisfy Eq. 1.1. Using Theorem 3.5 we now show that *any* exponential *B-spline* $\mathcal{E}_{N,\mathbf{a}}$ can be associated with a finite family of dual windows. We therefore avoid the usual need for the *partition of unity property*. Prior to formally stating the result we fix some notation. For any $N \in \mathbb{N}$ and any N-tuple $\mathbf{a} = (a_1, a_2, ..., a_N)$ of real numbers we let $Q_{m,\mathbf{a}} = \{j \in \{2, ..., m+1\} \,|a_j - a_1 \neq 0\}$, $m \in \{1, ..., N-1\}$. For each $m \in \{1, ..., N-1\}$ we define the corresponding constants

$$C_{m,\mathbf{a}} = \begin{cases} \prod_{j\in Q_m} \frac{a_j - a_1}{e^{a_j - a_1} - 1}, & Q_{m,\mathbf{a}} \neq \emptyset \\ 1, & Q_{m,\mathbf{a}} = \emptyset. \end{cases}$$

Furthermore for any $n \in \{2, ..., N\}$ we shall write $\mathcal{E}_{n,\mathbf{a}}$ to denote the exponential *B-spline* associated with the n-tuple $\mathbf{a}_n = (a_1, a_2, ..., a_n)$, i.e.

$$\mathcal{E}_{n,\mathbf{a}} := e^{a_1(\cdot)}\chi_{[0,1]} * e^{a_2(\cdot)}\chi_{[0,1]} * ... * e^{a_n(\cdot)}\chi_{[0,1]}.$$

Theorem 4.1 *Let $N \in \mathbb{N}$ and $\mathbf{a} = (a_1, a_2, ..., a_N)$. Let $\mathcal{E}_{n,\mathbf{a}}$ denote the exponential B-spline of order n, $n \in \{2, ..., N\}$. Let $Q_1, Q_2 \in \mathbb{Z}$ and $L \in W_{Q_1,Q_2}$. Let $b \in]0, \frac{1}{2(N-1)+Q_2-Q_1}]$ and define*

$$h_0(x) = C_{n,\mathbf{a}} b e^{-a_1 x} \sum_{j=-N+1-(Q_2-1)}^{N-(Q_1+1)} L(x-j). \tag{4.1}$$

Take $\{\tilde{g}_m\}_{m=1}^{N-1}$ such that $\int_0^1 e^{a_1 y}\tilde{g}_m(y)dy = 1$, $\forall\, m \in \{1, ..., N-1\}$ and define h_m, $m \in \{1, ..., N-1\}$ recursively by Eq. 3.4. Then the functions $\mathcal{E}_{n,\mathbf{a}}, h_m$ generate dual frames $\{E_{\ell b}T_j\mathcal{E}_{n,\mathbf{a}}\}_{\ell,j\in\mathbb{Z}}$ and $\{E_{\ell b}T_j h_m\}_{\ell,j\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof For any $m \in \{1, ..., N-1\}$ we let

$$\tilde{f}_m(y) = \begin{cases} \frac{a_{m+1}-a_1}{e^{a_{m+1}-a_1}-1} e^{a_{m+1}y}, & m \in Q_{m,\mathbf{a}} \\ 1, & m \notin Q_{m,\mathbf{a}}. \end{cases}$$

A straightforward calculation shows that $\int_0^1 e^{a_1 y}\tilde{f}_m(y)dy = 1$. By Theorem 3.5 it follows that the scaled exponential B-splines $\tilde{\mathcal{E}}_{m,\mathbf{a}} = C_{m,\mathbf{a}}\mathcal{E}_{m,\mathbf{a}}$ have dual windows

$$h_m(x) = \int_0^1 h_{m-1}(x-z)\tilde{g}_m(z)dz, \;\; m \in \{1, ..., N-1\}, \tag{4.2}$$

where h_0 is given by Eq. 4.1. By Theorem 1.3 we therefore have

$$\sum_{k\in\mathbb{Z}} \mathcal{E}_{m,\mathbf{a}}(x-n/b-k)C_{n,\mathbf{a}}h_m(x-k) = \sum_{k\in\mathbb{Z}} \tilde{\mathcal{E}}_{m,\mathbf{a}}(x-n/b-k)h_m(x-k) = b\delta_{n,0}.$$

Appealing to Theorem 1.3 once more concludes the proof. □

By Theorem 4.1 any exponential B-spline has dual windows which may be conveniently written as simple linear combinations of other exponential B-splines:

Theorem 4.2 *Let $N \in \mathbb{N}$ and $\mathbf{a} = (a_1, a_2, ..., a_N)$. Let $\mathcal{E}_{n,\mathbf{a}}$ denote the exponential B-spline of order n, $n \in \{2, ..., N\}$. Take any $\mathbf{b} = (b_1, b_2, ..., b_N)$, where $b_1 := -a_1$. Let $b \in]0, \frac{1}{2N-1}]$ and define*

$$h_m(x) := bC_{n,\mathbf{a}}C_{m,\mathbf{b}} \sum_{j=-N+1}^{N-1} e^{-a_1 j}\mathcal{E}_{m+1,\mathbf{b}}(x-j), \;\; m \in \{1, .., N-1\}.$$

Then $\mathcal{E}_{n,\mathbf{a}}$ and h_m generate dual frames $\{E_{\ell b}T_j\mathcal{E}_{n,\mathbf{a}}\}_{\ell,j\in\mathbb{Z}}$ and $\{E_{\ell b}T_j h_m\}_{\ell,j\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof For any $m \in \{1, ..., N-1\}$ we let

$$\tilde{g}_m(z) = \begin{cases} \frac{b_{m+1}-a_1}{e^{b_{m+1}-a_1}-1} e^{b_{m+1}z}, & m \in Q_m \\ 1, & m \notin Q_m. \end{cases}$$

By Theorem 4.1 the exponential B-splines $\mathcal{E}_{n,\mathbf{a}},\ n \in \{2, ..., N\}$ have dual windows (4.2) where h_0 is given by Eq. 4.1 with $L = \chi_{[0,1]}$. We now find an explicit expression for Eq. 4.2. We see that

$$\begin{aligned} h_1(x) &= bC_{n,\mathbf{a}} \int_0^1 h_0(x-z)\tilde{g}_1(z)dz \\ &= bC_{n,\mathbf{a}} \int_0^1 e^{-a_1(x-z)} \sum_{j=-N+1}^{N-1} \chi_{[0,1]}(x-z-j)\tilde{g}_1(z)dz \\ &= bC_{n,\mathbf{a}} \sum_{j=-N+1}^{N-1} \int_0^1 e^{-a_1(x-j-z)} e^{-a_1 j} \chi_{[0,1]}(x-z-j)\tilde{g}_1(z)dz, \\ &= bC_{n,\mathbf{a}} C_{1,\mathbf{b}} \sum_{j=-N+1}^{N-1} e^{-a_1 j} \int_0^1 e^{-a_1(x-j-z)} \chi_{[0,1]}(x-z-j) e^{b_2 z} dz \\ &= bC_{n,\mathbf{a}} C_{1,\mathbf{b}} \sum_{j=-N+1}^{N-1} e^{-a_1 j} \mathcal{E}_{2,\mathbf{b}}(x-j). \end{aligned}$$

By repetition of the above argument we obtain the desired result

$$h_m(x) = bC_{n,\mathbf{a}} C_{m,\mathbf{b}} \sum_{j=-N+1}^{N-1} e^{-a_1 j} \mathcal{E}_{m+1,\mathbf{b}}(x-j),\ \ m \in \{1, .., N-1\}.$$

□

We note that the results of Theorems 4.1 and 4.2 encompass the classical B-splines. Indeed let $\mathbf{a} = \mathbf{b} = \underbrace{(0, 0, ..., 0)}_{N \text{ zeros}}$. We then see that

$$\mathcal{E}_{n,\mathbf{a}} = e^{a_1(\cdot)}\chi_{[0,1]} * e^{a_2(\cdot)}\chi_{[0,1]} * ... * e^{a_n(\cdot)}\chi_{[0,1]} = \underbrace{\chi_{[0,1]} * \chi_{[0,1]} * ... * \chi_{[0,1]}}_{n \text{ terms}} = B_n.$$

By Theorem 4.1 and Lemma 2.4 any (exponential) B-spline has a finite family of dual windows of arbitrary finite smoothness. Also, by Theorem 4.2 any (exponential) B-spline has a dual window $h \in C^k(\mathbb{R}),\ k \in \mathbb{N}$ which may be written as a linear combination of another B-spline.

Example 4.3 We consider the exponential B-spline

$$\mathcal{E}_{2,\mathbf{a}}(x) = e^{a_1 x}\chi_{[0,1]}(x) * e^{a_2 x}\chi_{[0,1]}(x),\ \ x \in \mathbb{R},\ \ a_1 - a_2 \neq 0.$$

In Theorem 4.1 let $N = 2$ and take $L(x) := \chi_{[0,1]}(x),\ x \in \mathbb{R}$. We may then choose $b \in]0, \frac{1}{3}]$. We then get the associated kernel

$$h_0(x) = C_{2,\mathbf{a}} b e^{-a_1 x} \sum_{j=-1}^{1} \chi_{[0,1]}(x-j) = \frac{a_2 - a_1}{e^{a_2 - a_1} - 1} \sum_{j=-1}^{1} \chi_{[0,1]}(x-j).$$

Let $k \in \mathbb{N}$ and consider $\tilde{f}_k(x) := x^k(1-x)^k \chi_{[0,1]}(x),\ x \in \mathbb{R}$. Then f_k belongs to $C^{k-1}(\mathbb{R})$. Letting $C_k = \int_0^1 e^{a_1 y} y^k (1-y)^k dy$ we obtain for each $k \in \mathbb{N}$ the $(k-1)$-times continuously differentiable dual window

$$h_{1,k}(x) = \int_0^1 h_0(x-y) f_k(y) dy.$$

In the concrete case $a_1 := 3, a_2 := -3, b = 1/3$ and $k = 2$ we obtain

$$h_{1,2}(x) = \frac{1}{3} \begin{cases} \frac{-26}{81} e^{-3x} + \left(\frac{-2}{81} + \frac{1}{3}x^4 + \frac{2}{9}x^3 + \frac{1}{9}x^3 - \frac{2}{127}x \right) e^3 & x \in [-1, 0[\\ \frac{2}{81} e^{3-3x} - \frac{26}{81} e^{-3x} & x \in [0, 2[\\ \frac{1}{81} \left(-27x^4 + 2e^{9-3x} + 306x^3 - 1305x^2 + 2490x - 1802 \right) e^{-6} & x \in [2, 3] \\ 0 & x \notin [-1, 3]. \end{cases}$$

Example 4.4 We consider once again the exponential *B-spline*

$$\mathcal{E}_{2,\mathbf{a}}(x) = e^{a_1 x} \chi_{[0,1]}(x) * e^{a_2 x} \chi_{[0,1]}(x),\ x \in \mathbb{R},\ a_1 - a_2 \neq 0.$$

By Theorem 4.2 it follows that $\mathcal{E}_{2,\mathbf{a}}$ has finite families of dual windows $\{h_m\}_{m=1}^N$ where $N \in \mathbb{N},\ n \geq 2$ is kept fixed and $b \in]0, \frac{1}{2N-1}]$. Here each member h_m can be written as a linear combination of another exponential *B-spline*. In the concrete case of $N = 4,\ \mathbf{a} = (2, 3)$ we take $\mathbf{b} = (-2, 1, -1, 2)$ and let $b = \frac{1}{7}$. We then obtain the dual windows

$$h_m(x) = bC_{n,\mathbf{a}} C_{m,\mathbf{b}} \sum_{j=-3}^{3} e^{-2j} \mathcal{E}_{m+1,\mathbf{b}}(x-j),\ m \in \{1, 2, 3\}.$$

These can all be written out explicitly using the formula given in [4]. For simplicity we only write out h_1 which is given explicitly by

$$h_1(x) = b \frac{a_2 - a_1}{e^{a_2 - a_1} - 1} \frac{b_2 + a_1}{e^{b_2 + a_1} - 1} \sum_{j=-3}^{3} e^{-2j} \mathcal{E}_{2,\mathbf{b}}(x-j) = \frac{1}{7} \begin{cases} \frac{e^{x+9} - e^{-2x}}{(e^3-1)(e-1)} & x \in [-3, -2[\\ \frac{e^{-2x+3} - e^{-2x}}{(e^3-1)(e-1)} & x \in [-2, 4[\\ \frac{e^{-2x+3} - e^{-12+x}}{(e^3-1)(e-1)} & x \in [4, 5] \\ 0 & x \notin [-3, 5]. \end{cases}$$

Acknowledgments The author thanks the reviewers for suggestions leading to an improved manuscript. Further he thanks Ole Christensen for invaluable discussions and comments.

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