Adv Comput Math (2015) 41:727–780
DOI 10.1007/s10444-014-9399-3

Regularization of divergent integrals: A comparison of the classical and generalized-functions approaches

V. V. Zozulya

Received: 13 March 2012 / Accepted: 29 August 2013 /
Published online: 19 June 2015

Abstract This article considers methods of weakly singular and hypersingular integral regularization based on the theory of distributions. For regularization of divergent integrals, the Gauss–Ostrogradskii theorem and the second Green's theorem in the sense of the theory of distribution have been used. Equations that allow easy calculation of weakly singular, singular, and hypersingular integrals in one- and two-dimensional cases for any sufficiently smooth function have been obtained. These equations are compared with classical methods of regularization. The results of numerical calculation using classical approaches and those based of the theory of generalized functions, along with a comparison for different functions, are presented in tables and graphs of the values of divergent integrals versus the position of the colocation point.

Keywords Divergent · Singular · Hypersingular · Regularization · Generalized function · BIE · BEM

Mathematics Subject Classification (2010) 35J08 · 35j25 · 45E05 · 65R20

1 Introduction

As mentioned in [7], divergent integrals were first considered by Cauchy, in 1826. He called such integrals "extraordinary." Cauchy also remarked that differentiation and integration with respect to a parameter are permissible with these "extraordinary"

Communicated by: Leslie Greengard

V. V. Zozulya (✉)
Centro de Investigacion Cientifica de Yucatan A.C.,
Calle 43, No 130, Colonia: Chuburná de Hidalgo, C.P. 97200, Mérida, Yucatán, México
e-mail: zozulya@cicy.mx

integrals. The next significant step in the definition and application of divergent integrals was taken by Hadamard, in about 1900. He extended the definition of divergent integrals to the multidimensional case and applied it to the solution of the Cauchy problem for differential equations of hyperbolic type [44]. But it is only recently, in connection with the development of boundary integral equations (BIE) and boundary element methods (BEM), that research in this area has been intensively pursued and applied to the solution of scientific and engineering problems.

Over the past few decades, thousands of publications on this topic in both theoretical and applied branches of mathematics have appeared. We cannot, therefore, give here anything approaching a comprehensive literature review, and so we instead consider issues related to the regularization and computation of divergent integrals. In this more reduced area of research, we consider in detail an approach associated with the use of generalized functions and compare it with the classical approach. We shall give references only to the most important works of the past, those works directly related to the topic of our research, and works of a fundamental nature, mostly monographs.

Divergent integrals arise in various fields of science and engineering. As mentioned above, divergent integrals were first introduced for the solution of differential equations [5, 7, 44, 99]. Today, they find application in many other fields, such as the fractional calculus [97] and relativistic and quantum field theory [8, 51]. Many textbooks and monographs have been written on theoretical aspects of divergent integrals and their applications, e.g., [54, 63, 64, 73, 83, 97, 105]. Divergent integrals are also often used in solving BIE and BEM problems in engineering mechanics; for references, see [2, 12, 35, 41, 48, 78–80, 82]. Comprehensive reviews and references are given in [13, 17, 21, 70, 101, 103]. Additionally, one should note the application of divergent integrals in solid [28, 47, 54, 64, 70, 81, 93] and fracture [2, 11, 16, 42, 43, 50, 92, 106] mechanics, gradient elasticity [89, 105], and piezoelasticity [14, 15, 90, 98], for example. Further references can be found in these cited publications.

Roughly speaking, the classical approach to the regularization of divergent integrals consists in extracting the singularity and considering the limit of such a modified integral together with additional compensation terms. This method was first used by Hadamard, who considered the finite part of hypersingular integrals [44]. Since then, many researchers have used such an approach, with the notion of the finite part being used for both theoretical study and practical calculation of divergent integrals, not only singular and hypersingular integrals, but also those with higher-order singularities [29, 30, 56, 76]. As for the practical computation of divergent integrals by classical methods, there are several possibilities. Relatively simple divergent integrals can be calculated analytically, employing the concept of the Hadamard finite part and calculating the corresponding limits. In the one-dimensional case, such an approach has been used by many authors; see, for example, the reviews [13, 103]. We might mention also [24, 55, 59, 68, 75], in which divergent integrals of importance in applications have been calculated and formulas for the differentiation of divergent integrals with respect to a parameter obtained. In the two-dimensional case, the situation becomes more complicated, and some divergent integrals over a circular region can be calculated relatively easily. For polygonal regions, the situation is more complicated, but simple integrals can be calculated analytically; see, for example, [19, 27, 93–96, 116–120, 122]. If a divergent integral cannot be calculated analytically, it must be transformed into a form suitable for numerical calculation. In such cases, the divergent integral is usually split into two parts: a regular part, which can be calculated using standard quadrature formulas, and a singular part, which is simpler and can be calculated analytically. We believe that such a method was first used by Kantorovich [55] for calculating various one-dimensional divergent integrals. He reported that the idea for such a separation came from the theory of divergent series [45]. Later, this method was applied to the regularization of multidimensional singular integrals by Michlin in [72] and explained in detail in [73]. He used series expansion of the singular function in polar coordinated and represented divergent integral as regular part and simple singular part which can be calculated analytically. Later, such a method was used by many authors, and it became widely used in BIE and BEM analysis after the publications [37, 38] and [2, 36].

Another method of regularization of divergent integrals consists in modifying the regular quadrature formulas to make them suitable for calculating such integrals. Many methods and formulas for calculating divergent integrals have been proposed by a variety of authors; for references, see, for example, [2, 13, 23, 25, 26, 52, 60, 74, 84, 103]. We note here as well the articles [4, 9, 61], in which quadrature formulas for one-dimensional divergent integrals are obtained based on orthogonal polynomials. In particular, the formulas from [61] are suitable for calculation of weakly singular, singular, and hypersingular integrals. They are based on Legendre polynomials and can be modified for functions with higher singularities. In the two-dimensional case, the situation become more complicated even for regular integrals. Most of the one-dimensional quadrature formulas can be easily extended over a rectangle to a two-dimensional numerical integration, but in the case of a triangle or other polygonal region, special techniques are needed [22, 49, 61, 65, 77]. Most of the quadrature formulas can be applied directly for calculating integrals over curvilinear one-dimensional or two-dimensional regions. For specific issues around integration over a curvilinear region, see [3, 57, 59, 91]. The monographs [20, 62, 65] can be used for reference and as introductions to the subject.

As in the theory of discontinuous functions, there are two approaches to the regularization of divergent integrals: the classical one and one based on the theory of distributions (generalized functions). In order to explain the generalized-functions approach, let us consider an analogy with calculating derivatives of discontinuous functions. The derivatives of discontinuous functions can be calculated using the classical approach or using the theory of generalized functions. The classical approach consists in separating out the singularities, differentiation of the smooth part, and then using a correct limit transition to obtain the derivative of the discontinuous part. A similar result can be easily obtained in the theory of generalized functions using generalized derivatives. A similar situation arises in the regularization of divergent integrals. Such integrals can be regularized by classical methods: separating out the singularities and then using a correct limit transition to obtain a regularized representation of the divergent integrals. There exists as well an approach based on the theory of generalized functions. That approach requires a consistent application of the rules established in the theory of distributions for calculating integrals of singular functions. Scientists and engineers have long made use of generalized functions for modeling physical quantities such as mass, force, and charge concentrated in the vicinity of a point. But the use of generalized functions established on a sound mathematical basis began with the introduction of generalized derivatives by Sobolev around 1930; see [102]. The first systematic presentation of the theory of distributions was given by Schwartz in [100]. We can also recommend the books [18, 32, 67, 109], which can be used as references and as introductions to the subject.

In our previous publications [41–43, 110–124], we used the theory of generalized functions for regularizing divergent integrals with different singularities that appear when boundary value problems are solved using BIE and BEM. We used theoretical concepts presented in [18] that allowed us to interpret definite integrals with singularities as distributions, and we applied those concepts to the regularization of divergent integrals. We applied these concepts in the solution of one-dimensional fracture dynamics problems for the first time in [110, 111]. Our techniques were further developed in [123] and [124] for the regularization of two-dimensional hypersingular integrals that appear respectively in static and dynamic problems of fracture mechanics. Regularized formulas for different types of divergent integrals were reported in [113, 117, 120, 122]. More applications of the regularization method can be found in the review articles [42, 43] and the book [41]. Further theoretical development and extension of the generalized-functions approach and application of the Gauss–Ostrogradskii and second Green's theorems in the sense of distributions was carried out in [112, 118, 120, 121]. Piecewise linear approximation for rectangular and triangular regions in the two-dimensional case was considered in [114, 116, 119]. Detailed descriptions of the methods of regularization in one- and two-dimensional elastostatics are given in [115] and [119] respectively. The formulas obtained in [116, 119, 122] transform divergent weakly singular, singular, and hypersingular integrals over an arbitrary polygonal area into regular contour integrals. That approach can be applied to the regularization not only of one- and two-dimensional weakly singular, singular, and hypersingular integrals, but also to divergent integrals with higher-order singularities, polynomial approximations, and curvilinear regions. We mention here that in relation to regularization based on the theory of generalized functions, we often refer to [85], where in fact, the classical approach has been used.

Closely related to the subject of this paper are the problem of calculating divergent integrals in the Galerkin formulation of BIE and BEM and the problem of calculating nearly singular integrals. We will not consider those issues here but content ourselves with giving references for the Galerkin formulation [6, 33, 34] and the evaluation of nearly singular integrals [10, 53, 86].

In this article, we review and further develop methods of regularization of divergent integrals based on the theory of generalized functions and compare them with the classical approach. Most of the equations related to the generalized-functions approach and all numerical results and graphs presented here are new. Even in the case of the classical approach, most of equations have been modified and adapted for our purposes. Most of the analytical and numerical computations and the plots of all the graphs were done using the computer algebra system *Mathematica*. The results of numerical calculations and comparison of the methods for different functions are presented in the tables. The graphs represent the values of divergent integrals versus the position of the colocation point.

2 Statement of the general elliptic boundary value problem and integral equations

Many stationary problems in science and engineering can be formulated in the form of a boundary value (BV) problem for a system of second-order elliptic partial differential equations in general form. The numerical solution of such problems by means of boundary integral equations (BIE) and approximation of their solution via boundary element methods (BEM) is well established in the academic community as well as in industry. The method of regularization of divergent integrals developed here is applicable to BIE developed from such general BV problems. Therefore, we consider here briefly the application of BIE and BEM in solving general systems of second-order elliptic partial differential equations and in particular forms in solving Poisson's and linear elasticity equations.

Let consider a homogeneous region, which in three-dimensional Euclidean space R^3 occupies volume V with smooth boundary ∂V. The region$V$is an open bounded subset of Euclidean space with a $C^{0,1}$ Lipschitz regular boundary ∂V. In the region V, we consider vector functions **u(x)** and **b(x)** that are subject to a system of second-order elliptic partial differential equations in general form

$$\mathbf{L} \cdot \mathbf{u} = \mathbf{b}, \tag{2.1}$$

where **u** and **b** are vector functions and **L** is a matrix differential operator of the form

$$\mathbf{L} = \begin{vmatrix} L_{11} & \cdots & L_{1n} \\ \vdots & \ldots & \vdots \\ L_{n1} & \cdots & L_{nn} \end{vmatrix}, \quad \mathbf{u} = \begin{vmatrix} u_1 \\ \vdots \\ u_n \end{vmatrix}, \quad \mathbf{f} = \begin{vmatrix} f_1 \\ \vdots \\ f_n \end{vmatrix} . \tag{2.2}$$

The coefficients of the matrix differential operator have the form

$$L_{lk} = \frac{\partial}{\partial x_j} c_{lkji} \frac{\partial}{\partial x_i} + b_{lki} \frac{\partial}{\partial x_i} + a_{lk} . \tag{2.3}$$

The coefficients c_{lkji}, b_{lki}, and a_{lk}can be constants or depend on their coordinates.

If the region $V$is finite, it is necessary to establish boundary conditions. We consider mixed boundary conditions in the form

$$\mathbf{u}(\mathbf{x}) = \varphi(\mathbf{x}),\ \forall \mathbf{x} \in \partial V_u,\ \mathbf{p}(\mathbf{x}) = \mathbf{P} \cdot \mathbf{u}(\mathbf{x}) = \psi(\mathbf{x}),\ \forall \mathbf{x} \in \partial V_p . \tag{2.4}$$

The boundary contain two parts, ∂V_u and ∂V_p, such that $\partial V_u \cap \partial V_p = \emptyset$ and $\partial V_u \cup \partial V_p = \partial V$. On the part ∂V_u is prescribed unknown function **u(x)**, and on the part ∂V_p is prescribed its generalized normal derivative **p(x)**. The generalized normal derivative is defined by the matrix differential operator with coefficients

$$P_{lk} = n_j c_{lkji} \frac{\partial}{\partial x_i} . \tag{2.5}$$

Here the n_i are the components of the outward normal vector to the surface ∂V_p.

If the region V is infinite, then instead of boundary conditions, the solution of Eq. 2.1 must satisfy conditions at infinity of the form

$$\|\mathbf{u}(\mathbf{x})\| = O\left(r^{-1}\right), \quad \|\mathbf{P}\cdot\mathbf{u}(\mathbf{x})\| = O\left(r^{-2}\right) \quad \text{for} \quad r \to \infty\,, \tag{2.6}$$

where r is the Euclidian distance.

For a rigorous mathematical formulation of the BV problem Eqs. 2.1, 2.4, special functional spaces have to be introduced. For most applications, one may consider $\mathbf{u} \in \mathbf{H}^1(V)$ and $\mathbf{b} \in \mathbf{H}^{-1}(V)$, where $\mathbf{H}^1(V)$ is the Sobolev space

$$\mathbf{H}^1 = H_1^2(V) \oplus H_2^2(V) \oplus \cdots \oplus H_n^2(V) \tag{2.7}$$

with norm

$$\|u_j\|_{H_j^2(V)} = \left(\int\limits_V |u_j|^2 + |Du_j|^2\right)^{1/2}, \tag{2.8}$$

and $\mathbf{H}^{-1}(V)$ its dual space.

For functions defined on the boundary, one has, according to Sobolev's embedding theorem, $\mathbf{u} \in \mathbf{H}^{1/2}(\partial V)$ and $\mathbf{b} \in \mathbf{H}^{-1/2}(\partial V)$. For further details, refer to [48, 66, 102].

According to the generalized second Green's theorem,

$$\int\limits_V \left(\mathbf{u}^* \cdot \mathbf{L}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{L}^*\cdot\mathbf{u}^*\right) dV = \int\limits_{\partial V} \left(\mathbf{u}^*\cdot\mathbf{P}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{P}^*\cdot\mathbf{u}^*\right) dS\,, \tag{2.9}$$

we can obtain the following integral identity:

$$\int\limits_V \mathbf{u}\cdot\mathbf{L}^*\cdot\mathbf{u}^* dV = \int\limits_{\partial V} \left(\mathbf{u}\cdot\mathbf{P}^*\cdot\mathbf{u}^* - \mathbf{u}^*\cdot\mathbf{P}\cdot\mathbf{u}\right) dS - \int\limits_V \mathbf{u}^*\cdot\mathbf{b}dV\,, \tag{2.10}$$

where $\mathbf{L}^*$ is the operator adjoint to $\mathbf{L}$.

In the case of the scalar Poisson equation [39] and a system of Lamé's linear equations of elasticity [40], we have

$$L = L^* = \Delta,\ P = P^* = n_i\frac{\partial}{\partial x_i} \text{ and } L_{lk} = L_{lk}^* = c_{ljki}\partial_j\partial_i,\ P_{ij} = P_{lk}^* = n_k c_{ikjl}\partial_l \tag{2.11}$$

respectively. Here c_{ikjl} are elastic moduli.

Equation 2.10 is usually used to obtain integral representations for the solution of the boundary value problem Eqs. 2.1–2.4. In order to find such integral representations, we have to find a fundamental solution of the adjoint operator $\mathbf{L}^*$:

$$\mathbf{L}^*\cdot\mathbf{U} = \delta, \tag{2.12}$$

where $\mathbf{U}$ is the matrix of a fundamental solution and δ is the matrix delta function.

Hörmander [46] has developed an algorithm to construct fundamental solutions for systems of partial differential equations with constant coefficients. If the coefficients are not constant, then only in special circumstances can fundamental solutions be derived. Analytical expressions for different types of differential operators and systems of operators with applications in the sciences and engineering have appeared in numerous publications. A detailed list of fundamental solutions can be found in

[87, 88]. In the case of the two- and three-dimensional scalar Poisson equations and systems of linear equations of elasticity, the fundamental solutions have the form

$$U(\mathbf{x}-\mathbf{y}) = \frac{1}{2\pi}\ln\frac{1}{r}, \quad U_{ij}(\mathbf{x}-\mathbf{y}) = \frac{1}{8\pi\mu(1-\upsilon)}\left((3-4\upsilon)\delta_{ij}\ln\frac{1}{r}+\partial_i r\partial_j r\right),$$
$$U(\mathbf{x}-\mathbf{y}) = \frac{1}{4\pi r}, \quad U_{ij}(\mathbf{x}-\mathbf{y}) = \frac{1}{16\pi\mu(1-\upsilon)r}\left((3-4\upsilon)\delta_{ij}+\partial_i r\partial_j r\right). \quad (2.13)$$

Here $r=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$ and $r=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}$ denote the distance between points **x** and **y** in two- and three- dimensional Euclidean space.

Substituting a fundamental solution into Eq. 2.10, we obtain an integral representation for the solution of the general system of second-order elliptic partial differential Eq. 2.1 and its generalized normal derivative in the form

$$\mathbf{u}(\mathbf{y}) = \mathbf{U}(\mathbf{p},\mathbf{x},\partial V)-\mathbf{W}(\mathbf{u},\mathbf{x},\partial V)+\mathbf{U}(\mathbf{f},\mathbf{x},V),$$
$$\mathbf{p}(\mathbf{y}) = \mathbf{K}(\mathbf{p},\mathbf{x},\partial V)-\mathbf{F}(\mathbf{u},\mathbf{x},\partial V)+\mathbf{K}(\mathbf{f},\mathbf{x},V). \quad (2.14)$$

For the sake of concision, we introduce here the following notation for integral operators:

$$\mathbf{U}(\mathbf{f},\mathbf{x},V) = \int_V \mathbf{f}(\mathbf{x})\cdot\mathbf{U}(\mathbf{x}-\mathbf{y})dV, \quad \mathbf{K}(\mathbf{f},\mathbf{x},V) = \int_V \mathbf{f}(\mathbf{x})\cdot\mathbf{K}(\mathbf{x}-\mathbf{y})dV,$$
$$\mathbf{U}(\mathbf{p},\mathbf{x},\partial V) = \int_{\partial V} \mathbf{p}(\mathbf{x})\cdot\mathbf{U}(\mathbf{x}-\mathbf{y})dS, \quad \mathbf{W}(\mathbf{u},\mathbf{x},\partial V) = \int_{\partial V} \mathbf{u}(\mathbf{x})\mathbf{W}(\mathbf{x},\mathbf{y})dS,$$
$$\mathbf{K}(\mathbf{p},\mathbf{x},\partial V) = \int_{\partial V} \mathbf{p}(\mathbf{x})\cdot\mathbf{K}(\mathbf{x},\mathbf{y})dS, \quad \mathbf{F}(\mathbf{u},\mathbf{x},\partial V) = \int_{\partial V} \mathbf{u}(\mathbf{x})\mathbf{F}(\mathbf{x},\mathbf{y})dS. \quad (2.15)$$

It is well known that boundary integral operators are maps between the following functional spaces:

$$\mathbf{U}: \quad \mathbf{H}^{-1/2}(\partial V)\to\mathbf{H}^{1/2}(\partial V), \quad \mathbf{W}: \quad \mathbf{H}^{1/2}(\partial V)\to\mathbf{H}^{1/2}(\partial V),$$
$$\mathbf{K}: \quad \mathbf{H}^{-1/2}(\partial V)\to\mathbf{H}^{-1/2}(\partial V), \quad \mathbf{F}: \quad \mathbf{H}^{1/2}(\partial V)\to\mathbf{H}^{-1/2}(\partial V). \quad (2.16)$$

Together with boundary conditions (2.4), the integral representations (2.14) are used for composing the BIE for the general boundary value problem Eqs. 2.1, 2.4. Using Eq. 2.14 and boundary properties of the integral operators (2.15) we can construct various BIE for the BV problem Eqs. 2.1, 2.4. For example, if the first and second boundary integral representations are used on the respective parts ∂V_u and ∂V_p of the boundary, the BIE will assume the form

$$\frac{1}{2}\mathbf{u}(\mathbf{x})-\mathbf{U}(\mathbf{p},\mathbf{x},\partial V_u)+\mathbf{W}(\mathbf{u},\mathbf{x},\partial V_p) = \Phi(\mathbf{x}), \quad \forall\mathbf{x}\in\partial V_u,$$
$$\frac{1}{2}\mathbf{p}(\mathbf{x})-\mathbf{K}(\mathbf{p},\mathbf{x},\partial V_u)+\mathbf{F}(\mathbf{u},\mathbf{x},\partial V_p)) = \Psi(\mathbf{x}), \quad \forall\mathbf{x}\in\partial V_p, \quad (2.17)$$

where

$$\Phi(\mathbf{x}) = \mathbf{U}(\mathbf{f}, \mathbf{x}, V) + \mathbf{U}(\psi, \mathbf{x}, \partial V_p) - \mathbf{W}(\varphi, \mathbf{x}, \partial V_u),$$
$$\Psi(\mathbf{x}) = \mathbf{K}(\mathbf{f}, \mathbf{x}, V) + \mathbf{K}(\psi, \mathbf{x}, \partial V_p) - \mathbf{F}(\varphi, \mathbf{x}, \partial V_u). \quad (2.18)$$

More possibilities for creating different types of BIE have been considered in numerous books devoted to the BIE and BEM and their applications; see, for example, [2, 12, 41, 48, 64, 82].

The main idea of the BEM consists in approximation of the BIE and further solution of the approximated finite-dimensional boundary elements (BE) system of linear algebraic equations. The mathematical essence of this approach is called the projection method. Let us outline some results from the mathematical theory of the projection method related to the approximation of BIE. For more information, one can refer to [41, 48, 66, 102].

We consider two Banach spaces $\mathbf{X} = \mathbf{H}^{1/2}(\partial V_p) \oplus \mathbf{H}^{-1/2}(\partial V_u)$and $\mathbf{Y} = \mathbf{H}^{-1/2}(\partial V_p) \oplus \mathbf{H}^{1/2}(\partial V_u)$ and the functional equation in those spaces:

$$\mathbf{A} \cdot \mathbf{u} = \mathbf{f}, \quad \mathbf{u} \in D(\mathbf{A}) \subset \mathbf{X}, \quad \mathbf{f} \in R(\mathbf{A}) \subset \mathbf{Y}. \quad (2.19)$$

Here $\mathbf{A} : \mathbf{X} \to \mathbf{Y}$ is the linear operator mapping from the Banach space $\mathbf{X}$ to the Banach space $\mathbf{Y}$, $D(\mathbf{A})$ is a domain, and $R(\mathbf{A})$ is the range of the operator $\mathbf{A}$. Equation 2.19 called an exact equation, and its solution is called an exact solution. We denote by $L(\mathbf{X}, \mathbf{Y})$ the Banach space of the linear operators mapping from $\mathbf{X}$ to $\mathbf{Y}$.

There is an action on the functional spaces $\mathbf{X}$ and $\mathbf{Y}$ by sequences of the projection operators $\mathbf{P}_h$ and $\mathbf{P}'_h$ such that

$$\mathbf{P}_h^2 = \mathbf{P}_h, \quad \mathbf{P}_h\mathbf{X} = \mathbf{X}_h, \quad \mathbf{X}_h \subset \mathbf{X},$$
$$\left(\mathbf{P}'_h\right)^2 = \mathbf{P}'_h, \quad \mathbf{P}'_h\mathbf{Y} = \mathbf{Y}_h, \quad \mathbf{Y}_h \subset \mathbf{Y}, \quad (2.20)$$

where $\mathbf{X}_h$ and $\mathbf{Y}_h$ are finite-dimensional subspaces of the Banach spaces $\mathbf{X}$ and $\mathbf{Y}$, and $h \in R^1$ is a parameter of discretization.

Now we consider the operator $\mathbf{A}_h \in L(\mathbf{X}_h, \mathbf{Y}_h)$ mapping the finite-dimensional subspaces $\mathbf{X}_h$ and $\mathbf{Y}_h$, and we approximate Eq. 2.19 by a finite-dimensional equation of the form

$$\mathbf{A}_h \cdot \mathbf{u}_h = \mathbf{f}_h, \quad \mathbf{A}_h = \mathbf{P}'_h \cdot \mathbf{A} \cdot \mathbf{P}_h, \quad \mathbf{u}_h = \mathbf{P}_h \cdot \mathbf{u}, \quad \mathbf{f}_h = \mathbf{P}'_h \cdot \mathbf{f}. \quad (2.21)$$

The solution $\mathbf{u}_h$ of Eq. 2.21 is the approximate solution of Eq. 2.19. The general scheme of constructing approximate equations is shown in the following diagram:

$$\begin{array}{ccc} \mathbf{X} \supset D(\mathbf{A}) & \xrightarrow{\mathbf{A}} & R(\mathbf{A}) \subset \mathbf{Y} \\ P_h \downarrow & & P'_h \downarrow \\ \mathbf{X}_h \supset D(\mathbf{A}_h) & \xrightarrow{\mathbf{A}_h} & R(\mathbf{A}_h) \subset \mathbf{Y}_h \end{array} \quad (2.22)$$

Now let us consider in detail scheme (2.22) for constructing approximate equations in the case of BIE and BEM. Obviously, the operator $\mathbf{A}$ corresponds to the boundary integral operators defined in Eq. 2.17. The operator $\mathbf{A}_h \in L(\mathbf{X}_h, \mathbf{Y}_h)$ maps the finite-dimensional subspaces $\mathbf{X}_h$ and $\mathbf{Y}_h$. Clearly, it corresponds to an approximate finite-dimensional equation of the BEM.

To transform the integral Eq. 2.17 into finite-dimensional equations of the BEM, we have to construct finite-dimensional functional spaces that correspond to infinite-dimensional functional spaces $\mathbf{X}(\partial V)$ and $\mathbf{Y}(\partial V)$ and construct the corresponding projection operators.

We first split the boundary ∂V into finite boundary elements of the form

$$\partial V = \bigcup_{n=1}^{N} \partial V_n, \partial V_n \cap \partial V_k = \varnothing, \text{ if } n \neq k, \tag{2.23}$$

and represent the infinite-dimensional functional spaces $\mathbf{X}(\partial V)$ and $\mathbf{Y}(\partial V)$ in the form

$$\mathbf{X}(\partial V) = \mathbf{X}\left(\bigcup_{n=1}^{N} \partial V_n\right), \quad \mathbf{Y}(\partial V) = \mathbf{Y}\left(\bigcup_{n=1}^{N} \partial V_n\right). \tag{2.24}$$

On each boundary element, choose Q interpolation nodes. Local projection operators act from the infinite-dimensional functional spaces $\mathbf{X}(\partial V_n)$ and $\mathbf{Y}(\partial V_n)$ to the finite-dimensional functional spaces $\mathbf{X}_q(\partial V_n)$ and $\mathbf{Y}_q \partial V_n)$as follows:

$$\begin{aligned} \mathbf{P}_q^u &: \mathbf{X}(\partial V_n) \to \mathbf{X}_q(\partial V_n) \forall \mathbf{x} \in \partial V_n, \\ \mathbf{P}_q^p &: \mathbf{Y}(\partial V_n) \to \mathbf{Y}_q(\partial V_n) \forall \mathbf{x} \in \partial V_n. \end{aligned} \tag{2.25}$$

Global projection operators are defined as the sum of the local projection operators $\mathbf{P}_{nq}^u = \sum_{n=1}^{N} \mathbf{P}_q^u, \quad \mathbf{P}_{nq}^p = \sum_{n=1}^{N} \mathbf{P}_q^p$. They map from the infinite-dimensional functional spaces $\mathbf{X}(\partial V)$ and $\mathbf{Y}(\partial V)$ to the finite-dimensional functional spaces in the following way:

$$\begin{aligned} \mathbf{P}_{nq}^u &: \mathbf{X}(\partial V) \to \mathbf{X}_q\left(\bigcup_{n=1}^{N} \partial V_n\right) \forall \mathbf{x} \in \partial V, \\ \mathbf{P}_{nq}^p &: \mathbf{Y}(\partial V) \to \mathbf{Y}_q\left(\bigcup_{n=1}^{N} \partial V_n\right) \forall \mathbf{x} \in \partial V. \end{aligned} \tag{2.26}$$

The local projection operators $\mathbf{P}_n^u$ and $\mathbf{P}_n^p$ establish a correspondence between vectors $\mathbf{u}$ and $\mathbf{p}$ and their values on the nodes of interpolation of the boundary in the form

$$\begin{aligned} \mathbf{P}_{nq}^u \cdot \mathbf{u}_i(\mathbf{x}) &= \left\{\mathbf{u}_i^n\left(\mathbf{x}_q\right), q = 1, \ldots, Q; n = 1, \ldots, N\right\} \quad \forall \mathbf{x} \in \partial V, \\ \mathbf{P}_{nq}^p \cdot \mathbf{p}_i(\mathbf{x}) &= \left\{\mathbf{p}_i^n\left(\mathbf{x}_q\right), q = 1, \ldots, Q; n = 1, \ldots, N\right\} \forall \mathbf{x} \in \partial V. \end{aligned} \tag{2.27}$$

The inverse operators $\left(\mathbf{P}_q^u\right)^{-1}$ and $\left(\mathbf{P}_q^p\right)^{-1}$ are called interpolation operators. They map from the finite-dimensional functional spaces $\mathbf{X}_q(\partial V_n)$ and $\mathbf{Y}_q(\partial V_n)$ to the infinite-dimensional functional spaces $\mathbf{X}(\partial V_n)$ and $\mathbf{Y}(\partial V_n)$. In order to determinate the interpolation operators, we introduce on each boundary element the interpolation polynomials or shape functions $\phi_{nq}(\mathbf{x})$, which map from the finite-dimensional functional spaces $\mathbf{X}_q(\partial V_n)$ and $\mathbf{Y}_q(\partial V_n)$ to the infinite-dimensional functional spaces $\mathbf{X}(\partial V_n)$ and $\mathbf{Y}(\partial V_n)$. Then the vectors of displacement and traction will be represented approximately in the form

$$u_i(\mathbf{x}) \approx \sum_{n=1}^{N}\sum_{q=1}^{Q} u_i^n\left(\mathbf{x}_q\right)\varphi_{nq}(\mathbf{x}),\ \mathbf{x} \in \bigcup_{n=1}^{N} \partial V_n,$$
$$p_i(\mathbf{x}) \approx \sum_{n=1}^{N}\sum_{q=1}^{Q} p_i^n\left(\mathbf{x}_q\right)\varphi_{nq}(\mathbf{x}),\ \mathbf{x} \in \bigcup_{n=1}^{N} \partial V_n. \tag{2.28}$$

Finite-dimensional analogies for the integral operators are operators that map between the finite-dimensional functional spaces $\mathbf{X}_q(\bigcup_{n=1}^{N} \partial V_n)$ and $\mathbf{Y}_q(\bigcup_{n=1}^{N} \partial V_n)$ in the following way:

$$\mathbf{U}_{ij}^{nq} = \mathbf{P}_{nq}^{p} \cdot \mathbf{U}_{ij} \cdot \mathbf{P}_{nq}^{u} : \mathbf{Y}_q(\partial V_n) \to \mathbf{X}_q(\partial V_n),$$
$$\mathbf{W}_{ij}^{nq} = \mathbf{P}_{nq}^{p} \cdot \mathbf{W}_{ij} \cdot \mathbf{P}_{nq}^{u} : \mathbf{X}_q(\partial V_n) \to \mathbf{Y}_q(\partial V_n)$$
$$\mathbf{K}_{ij}^{nq} = \mathbf{P}_{nq}^{p} \cdot \mathbf{K}_{ij} \cdot \mathbf{P}_{nq}^{u} : \mathbf{Y}_q(\partial V_n) \to \mathbf{X}_q(\partial V_n),$$
$$\mathbf{F}_{ij}^{nq} = \mathbf{P}_{nq}^{p} \cdot \mathbf{F}_{ij} \cdot \mathbf{P}_{nq}^{u} : \mathbf{X}_q(\partial V_n) \to \mathbf{Y}_q(\partial V_n). \tag{2.29}$$

Substituting expressions Eq. 2.28 into the BIE of the form Eq. 2.17 gives us the finite-dimensional equations of the BEM and representation of the displacement and traction vectors on the boundary in the form

$$\frac{1}{2}\mathbf{u}(\mathbf{y}_r) = \sum_{n=1}^{N}\sum_{q=1}^{Q}\left[\mathbf{U}^n(\mathbf{y}_r, \mathbf{x}_q) \cdot \mathbf{p}(\mathbf{x}_q) - \mathbf{W}^n(\mathbf{y}_r, \mathbf{x}_q) \cdot \mathbf{u}(\mathbf{x}_q)\right] + \Phi\left(\mathbf{f}, \mathbf{y}_r, V_n\right),$$
$$\frac{1}{2}\mathbf{p}(\mathbf{y}_r) = \sum_{n=1}^{N}\sum_{q=1}^{Q}\left[\mathbf{K}^n(\mathbf{y}_r, \mathbf{x}_q) \cdot \mathbf{p}(\mathbf{x}_q) - \mathbf{F}^n(\mathbf{y}_r, \mathbf{x}_q) \cdot \mathbf{u}(\mathbf{x}_q)\right] + \Psi\left(\mathbf{f}, \mathbf{y}_r, V_n\right), \tag{2.30}$$

where

$$\mathbf{U}^n(\mathbf{y}_r, \mathbf{x}_q) = \int_{\partial V_n} \mathbf{U}(\mathbf{y}_r, \mathbf{x})\varphi_{nq}(\mathbf{x})\, dS, \quad \mathbf{W}^n(\mathbf{y}_r, \mathbf{x}_q) = \int_{\partial V_n} \mathbf{W}(\mathbf{y}_r, \mathbf{x})\varphi_{nq}(\mathbf{x})\, dS,$$
$$\mathbf{K}^n(\mathbf{y}_r, \mathbf{x}_q) = \int_{\partial V_n} \mathbf{K}(\mathbf{y}_r, \mathbf{x})\varphi_{nq}(\mathbf{x})\, dS, \quad \mathbf{F}^n(\mathbf{y}_r, \mathbf{x}_q) = \int_{\partial V_n} \mathbf{F}(\mathbf{y}_r, \mathbf{x})\varphi_{nq}(\mathbf{x})\, dS. \tag{2.31}$$

The volume potentials $\Phi\left(\mathbf{f}, \mathbf{y}_r, V_n\right)$ and $\Psi\left(\mathbf{f}, \mathbf{y}_r, V_n\right)$ depend on discretization of the V domain. They will not be considered here. More detailed information about the relationship of BIE to the BEM, their creation and applications in science and engineering, can be found in [2, 12, 41, 42, 48, 64, 82].

3 Boundary elements and approximation

The BEM is an approximate method for the solution of BIE, which includes approximation of the functions that belong to some functional space by a discrete finite model. It is important to mention that local approximation of the considered function on each BE can be done independently from other BE. Hence, it is possible to create a catalogue of various BE interpolation functions with arbitrary node values. Then from this catalogue can be chosen the BE that are needed for approximating the function in the domain of its definition. The same BE can be used for discrete models of various functions or physical fields by determining the required locations of the nodes in the model and further defining the node values of the function or physical field. Thus, finite-dimensional models of a domain and its boundary need not depend on the functions and physical fields for which they can be a domain of definition.

Let us consider the question of how to construct a BE model of the boundary $\partial V \subset R^{n-1}$ of the domain $V \subset R^n (n = 2, 3)$. We first fix on the boundary of the domain ∂Va finite number of points $\mathbf{x}^q$ $(q = 1, \dots, Q)$. These points are referred to as global nodal points $\partial V(q) = \{\mathbf{x}^q \in V : q = 1, \dots, Q\}$. Then we divide the boundary ∂Vinto a finite number of subdomains $\partial V_n (n = 1, \dots, N)$, which are called BE. They have to satisfy the following conditions:

$$\partial V_n \cup \partial V_m = \emptyset, m \neq n, m,\ n = 1, 2, \dots, N, \partial V = \bigcup_{n=1}^{N} \partial V_n. \tag{2.32}$$

On each BE we introduce a local coordinate system ξ. The local coordinates ξ_i are functions of the global coordinates(ξ_i (x_1, x_2, x_3)), and conversely, the global coordinates are functions of the local coordinates (x_i (ξ_1, ξ_2, ξ_3)). In order for these functions to be bijective, it is necessary and sufficient that the Jacobians of the transformations be nonzero:

$$J = \det \left| \frac{\partial x_i}{\partial \xi_j} \right| \neq 0,\ J^{-1} = \det \left| \frac{\partial \xi_i}{\partial x_j} \right| \neq 0. \tag{2.33}$$

The borders of the BE and the position of the nodal points should be such that after they are joined, the separate elements form a discrete model of the boundary∂V.

After creation of the finite-dimensional model of the boundary∂V, we consider approximation of a function $f(\mathbf{x})$ that belongs to some functional space. We denote by $f^n(\mathbf{x})$ the restriction of the function $f(\mathbf{x})$ to the BE $\partial V_n f^n(\mathbf{x})$. Then

$$f(\mathbf{x}) = \sum_{n=1}^{N} f^n(\mathbf{x}). \tag{2.34}$$

On each BE, the local functions $f^n(\mathbf{x})$ may be represented in the form

$$f^n(\mathbf{x}) \approx \sum_{q=1}^{Q} f^n(\mathbf{x}^q)\phi_{nq}(\xi). \tag{2.35}$$

At the nodal point with coordinates $\mathbf{x}^q$, the interpolation polynomials $\phi_{nq}(\xi)$are equal to 1, and at other nodal points are equal to zero. Taking into account Eqs. 2.34 and 2.35, the global approximation of the function $f(\mathbf{x})$ looks like

$$f(\mathbf{x}) \approx \sum_{n=1}^{N} \sum_{q=1}^{Q} f^n \left(\mathbf{x}_q\right) \phi_{nq}(\xi). \tag{2.36}$$

If the nodal point q belongs to several BE, it is nonetheless considered in these sums Eq. 2.36 only once.

The BE may have different forms, shapes, and sizes, and their surfaces can be curvilinear. The curvilinear BE are very important in the BEM, because the domain boundary is usually curvilinear. But in some cases, it is more convenient to use a flat BE whose surface coincides with the planes of the local coordinate system.

3.1 One-dimensional boundary elements and approximation

In order to calculate the integrals in Eq. 2.31 in the one-dimensional case, let us introduce the local coordinate $\xi \in [-1, 1]$. Then the global coordinates can be expressed in the form

$$x_i(\xi) = \sum_{q=0}^{Q} x_i^q \varphi_q(\xi), \;\; y_i(\xi) = \sum_{q=0}^{Q} y_i^q \varphi_q(\xi), \tag{2.37}$$

where x_i^q and y_i^q are global coordinates of the nodal points.

Because we will use the same shape functions on each BE, the index n will be omitted, and instead of $\phi_{nq}(\xi)$, we will write$\phi_q(\xi)$. In the general case, on a BE with $Q+1$ nodes of interpolation, an approximation polynomial (shape function) has degree Q. To obtain explicit expressions for the shape functions, we represent them in the form

$$\varphi_q(\xi) = \sum_{k=0}^{Q} \alpha_k^q \xi^k. \tag{2.38}$$

The shape functions $\phi_q(\xi)$ must satisfy the following conditions:

$$\varphi_q(\xi_r) = \delta_{qr}, \tag{2.39}$$

where $\xi_0, \xi_1, \xi_2, \ldots, \xi_Q$ are the nodes of interpolation on the BE in the local system of coordinates, and δ_{qr} is the Kronecker delta.

Using the conditions Eq. 2.39, the unknown coefficients can be determined, and the interpolation polynomial of degree Q can be presented in the form

$$\phi_q(\xi) = \frac{(\xi - \xi_0)(\xi - \xi_1)\ldots\left(\xi - \xi_{q-1}\right)\left(\xi - \xi_{q+1}\right)\ldots\left(\xi - \xi_Q\right)}{\left(\xi_q - \xi_0\right)\left(\xi_q - \xi_1\right)\ldots\left(\xi_q - \xi_{q-1}\right)\left(\xi_q - \xi_{q+1}\right)\ldots\left(\xi_q - \xi_Q\right)}. \tag{2.40}$$

For calculation of the integrals in Eq. 2.31, we need to calculate the functions that they contain, such as the distance between points $\mathbf{x}$ and $\mathbf{y}$, $r(\mathbf{x}, \mathbf{y})$, elements of length dS, and the components $n_1(\mathbf{x})$ and $n_2(\mathbf{x})$ of the normal vector in local coordinates.

The distance between points$\mathbf{x}$ and $\mathbf{y}$ in local coordinates is

$$r(\mathbf{x}(\xi), \mathbf{y}(\zeta)) = \sqrt{\left((x_1^q \varphi_q(\xi) - y_1^r \varphi_r(\zeta)\right)^2 + \left(x_2^q \varphi_q(\xi) - y_2^r \varphi_r(\zeta)\right)^2}, \tag{2.41}$$

where $x_i(\xi) = \sum_{q=0}^{Q} x_i^q \phi_q(\xi) = x_i^q \phi_q(\xi) y_i(\zeta) = \sum_{q=0}^{Q} y_i^q \phi_q(\zeta) = y_i^q \phi_q(\zeta)$.

It is important to mention that if the collocation and integration points belong to the same BE, then Eq. 2.41 can be represented in the form

$$r(\xi,\zeta) = |\xi - \zeta| \, \Phi\left(\xi, \zeta, \mathbf{x}^q\right), \tag{2.42}$$

where $\Phi\left(\xi, \zeta, \mathbf{x}^q\right)$ is a regular function.

The curved length element presented in Eq. 2.31 has in local coordinates the form

$$dS = J(\xi)d\xi, \;\; J(\xi) = \sqrt{(dx_1(\xi)/d\xi)^2 + (dx_2(\xi)/d\xi)^2}. \tag{2.43}$$

Here the Jacobian may also be represented in the form

$$J(\xi) = \left[\left(\sum_{q=0}^{Q} x_1^q \frac{d\varphi_q(\xi)}{d\xi}\right)^2 + \left(\sum_{q=0}^{Q} x_2^q \frac{d\varphi_q(\xi)}{d\xi}\right)^2\right]^{1/2}. \tag{2.44}$$

In order to calculate the normal vectors $n_1(\mathbf{x})$ and $n_2(\mathbf{x})$ in local coordinates, we will use the representations

$$\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2, \;\; \boldsymbol{\tau} = \frac{dx_1}{d\xi}\mathbf{e}_1 + \frac{dx_2}{d\xi}\mathbf{e}_2. \tag{2.45}$$

Taking into account orthogonality of normal and the tangential vector and their unit length, we obtain the following equations:

$$(\mathbf{n}\cdot\boldsymbol{\tau}) = n_1\frac{dx_1}{d\xi} + n_2\frac{dx_2}{d\xi} = 0, \;\; |\mathbf{n}| = \sqrt{(n_1)^2 + (n_2)^2} = 1. \tag{2.46}$$

Solution of the system of Eqs. 2.45 and 2.46 gives us

$$n_1 = \frac{-1}{J(\xi)}\frac{dx_2}{d\xi}, \;\; n_2 = \frac{1}{J(\xi)}\frac{dx_1}{d\xi}. \tag{2.47}$$

Taking into account that

$$\frac{dx_i(\xi)}{d\xi} = \frac{d}{d\xi}\sum_{q=0}^{Q} x_i^q \varphi_q(\xi) = \sum_{q=0}^{Q} x_i^q \frac{d\varphi_q(\xi)}{d\xi}, \tag{2.48}$$

the normal and tangential vectors can be represented in the form

$$\begin{aligned} \mathbf{n}(\xi) &= \tfrac{-1}{J(\xi)} \sum_{q=0}^{Q} x_2^q \tfrac{d\varphi_q(\xi)}{d\xi}\,\mathbf{e}_1 + \tfrac{1}{J(\xi)} \sum_{q=0}^{Q} x_2^q \tfrac{d\varphi_q(\xi)}{d\xi}\,\mathbf{e}_2, \\ \boldsymbol{\tau}(\xi) &= \sum_{q=0}^{Q} x_1^q \tfrac{d\varphi_q(\xi)}{d\xi}\,\mathbf{e}_1 + \sum_{q=0}^{Q} x_2^q \tfrac{d\varphi_q(\xi)}{d\xi}\,\mathbf{e}_2. \end{aligned} \tag{2.49}$$

The obtained formulas can be used for calculating regular and divergent integrals over the standard curvilinear and flat BE used in the BEM. We should mention that the BE approximation has to be linearly independent and compact in the corresponding functional space.

3.2 Two-dimensional boundary elements and approximation

In order to calculate the integrals in Eq. 2.31 in the two-dimensional case, let us introduce local coordinates $\xi(\xi_1, \xi_2, \xi_3)$ associated with the BE that depend on their shape. The coordinates (ξ_1, ξ_2) parameterize points that belong to the BE, and the coordinate ξ_3 is perpendicular to the BE surface. Then the global coordinates of the points that belong to the BE can be expressed in the form

$$x_i(\xi) = \sum_{q=0}^{Q} x_i^q \varphi_q(\xi), \;\; y_i(\xi) = \sum_{q=0}^{Q} y_i^q \varphi_q(\xi), \tag{3.1}$$

where x_i^q and y_i^q are the global coordinates of the nodal points.

At the interpolation nodes of a BE in general form containing Q, the shape function is an interpolation polynomial of degree Q. In the general case, it has the form

$$\varphi_{qi}(\xi_i) = \frac{\left(\xi_i - \xi_i^1\right)\cdots\left(\xi_i - \xi_i^{q-1}\right)\left(\xi_i - \xi_i^{q+1}\right)\cdots\left(\xi_i - \xi_i^Q\right)}{\left(\xi_i^q - \xi_i^1\right)\cdots\left(\xi_i^q - \xi_i^{q-1}\right)\left(\xi_i^q - \xi_i^{q+1}\right)\cdots\left(\xi_i^q - \xi_i^Q\right)}. \tag{3.2}$$

We can write the shape function $\phi_q(\xi_1, \xi_2)$, which depends on two variables, as a product of two interpolation polynomials $\phi_{q1}(\xi_1)$ and $\phi_{q2}(\xi_2)$ of one variable in the form

$$\varphi_q(\xi_1, \xi_2) = \varphi_{q1}(\xi_1)\varphi_{q2}(\xi_2). \tag{3.3}$$

For calculating integrals in Eq. 2.31, we must also calculate some functions, such as the distance $r(\mathbf{x}, \mathbf{y})$ between points $\mathbf{x}$ and $\mathbf{y}$, $r(\mathbf{x}, \mathbf{y})$ area and volume elements $dVdV$, components $n_1(\mathbf{x})$ and $n_2(\mathbf{x})$ of the normal vector, and surface elements dS, in local coordinates.

The distance between points$\mathbf{x}$ and $\mathbf{y}$ in local coordinates is

$$r(\mathbf{x}(\xi), \mathbf{y}(\zeta)) = \sqrt{\left(\left(x_1^q\varphi_q(\xi) - y_1^r\varphi_r(\zeta)\right)^2 + \left(x_2^q\varphi_q(\xi) - y_2^r\varphi_r(\zeta)\right)^2 + \left(x_3^q\varphi_q(\xi) - y_3^r\varphi_r(\zeta)\right)^2}. \tag{3.4}$$

It is important to mention that if collocation and integration points belong to the same BE, then Eq. 3.4 can be represented in the form

$$r(\xi, \zeta) = \sqrt{(\xi_1 - \zeta_1)^2 + (\xi_2 - \zeta_2)^2}\Phi(\xi, \zeta, \mathbf{x}^q), \tag{3.5}$$

where $\Phi(\xi, \zeta^q)$ is a regular function.

The differential elements along the global and local coordinate axes are related by

$$d\mathbf{x} = \mathbf{e}_i\left(\partial x_i/\partial \xi_j\right)d\xi_j, d\xi = \mathbf{i}_i\left(\partial \xi_i/\partial x_j\right)dx_j, \tag{3.6}$$

where the elements of the coordinate basis in global and local coordinates are related by the equations

$$\mathbf{i}_i = \mathbf{e}_j\frac{\partial \xi_j}{\partial x_i}, \quad \mathbf{e}_i = \mathbf{i}_j\frac{\partial x_j}{\partial \xi_i}. \tag{3.7}$$

The volume element in the three-dimensional case is transformed under the formula

$$dV = dx_1 dx_2 dx_3 = J(\xi) d\xi_1 d\xi_2 d\xi_3, \tag{3.8}$$

and the area of a two-dimensional flat BE is transformed under the formula

$$dA = dx_1 dx_2 = \det \left| \frac{\partial x_\alpha}{\partial \xi_\beta} \right| d\xi_1 d\xi_2, \alpha, \beta = 1, 2. \tag{3.9}$$

The differential of a surface located in three-dimensional space is defined by the expression

$$dS = |\mathbf{e}_1 \times \mathbf{e}_2| \, d\xi_1 d\xi_2 = \left(n_1^2 + n_2^2 + n_3^2 \right)^{1/2} d\xi_1 d\xi_2, \tag{3.10}$$

where

$$\begin{aligned} n_1 &= \tfrac{\partial x_1}{\partial \xi_1} \tfrac{\partial x_3}{\partial \xi_2} - \tfrac{\partial x_2}{\partial \xi_2} \tfrac{\partial x_3}{\partial \xi_1}, \\ n_2 &= \tfrac{\partial x_3}{\partial \xi_1} \tfrac{\partial x_1}{\partial \xi_2} - \tfrac{\partial x_1}{\partial \xi_1} \tfrac{\partial x_3}{\partial \xi_2}, \\ n_3 &= \tfrac{\partial x_1}{\partial \xi_1} \tfrac{\partial x_2}{\partial \xi_2} - \tfrac{\partial x_2}{\partial \xi_1} \tfrac{\partial x_1}{\partial \xi_2}. \end{aligned} \tag{3.11}$$

Obviously, the vectors $\mathbf{e}_1$ and$\mathbf{e}_2$ are tangential to the surface of the BE, and therefore, the vector $\mathbf{e}_1 \times \mathbf{e}_2$is perpendicular to the surface of the BE. It coincides with the vector normal to the surface of the BE, and its components can be calculated using Eq. 3.11.

The element of length of the BE's contour in three dimensions along the coordinate ξ_α is defined by expression

$$dl = \sqrt{(dx_1/d\xi_\alpha)^2 + (dx_2/d\xi_\alpha)^2 + (dx_2/d\xi_\alpha)^2} d\xi_\alpha. \tag{3.12}$$

Substituting global coordinates from Eq. 3.1 into Eqs. 3.8, 3.11, and 3.12, taking into account the expression for the derivatives

$$\frac{\partial x_i(\xi)}{\partial \xi_j} = \sum_{q=1}^{Q} x_i^q \frac{\partial \phi_q(\xi)}{\partial \xi_j}, \tag{3.13}$$

we will get the coordinates of the unit normal vectors

$$\begin{aligned} n_1 &= \sum_{q=1}^{Q} x_2^q \frac{\partial \phi_q(\xi)}{\partial \xi_1} \sum_{q=1}^{Q} x_3^q \frac{\partial \phi_q(\xi)}{\partial \xi_2} - \sum_{q=1}^{Q} x_2^q \frac{\partial \phi_q(\xi)}{\partial \xi_2} \sum_{q=1}^{Q} x_3^q \frac{\partial \phi_q(\xi)}{\partial \xi_1}, \\ n_2 &= \sum_{q=1}^{Q} x_3^q \frac{\partial \phi_q(\xi)}{\partial \xi_1} \sum_{q=1}^{Q} x_1^q \frac{\partial \phi_q(\xi)}{\partial \xi_2} - \sum_{q=1}^{Q} x_1^q \frac{\partial \phi_q(\xi)}{\partial \xi_1} \sum_{q=1}^{Q} x_3^q \frac{\partial \phi_q(\xi)}{\partial \xi_2}, \\ n_3 &= \sum_{q=1}^{Q} x_1^q \frac{\partial \phi_q(\xi)}{\partial \xi_1} \sum_{q=1}^{Q} x_2^q \frac{\partial \phi_q(\xi)}{\partial \xi_2} - \sum_{q=1}^{Q} x_2^q \frac{\partial \phi_q(\xi)}{\partial \xi_1} \sum_{q=1}^{Q} x_1^q \frac{\partial \phi_q(\xi)}{\partial \xi_2}, \end{aligned} \tag{3.14}$$

the Jacobian

$$J(\xi) = Det \left| \sum_{q=1}^{Q} x_i^q \frac{\partial \phi_q(\xi)}{\partial \xi_j} \right|, \tag{3.15}$$

and length element of the BE contour

$$dl = \sqrt{\left(\sum_{q=1}^{Q} x_1^q \frac{\partial \phi_q(\xi)}{\partial \xi_j}\right)^2 + \left(\sum_{q=1}^{Q} x_2^q \frac{\partial \phi_q(\xi)}{\partial \xi_j}\right)^2 + \left(\sum_{q=1}^{Q} x_2^q \frac{\partial \phi_q(\xi)}{\partial \xi_j}\right)^2} d\xi_\alpha \tag{3.16}$$

expressed in local coordinates.

The formulas thus obtained can be used for calculating regular and divergent integrals over the standard curvilinear and flat BE used in BEM. We should mention that the BE approximation must be linearly independent and compact in the corresponding functional space.

4 Calculation of one-dimensional integrals

In this and the following section, we consider some issues related to the calculation of divergent integrals. The main emphasis will be on the computational aspects of the problem. Therefore, we assume that all functions considered here are well defined and possess all the requisite properties of continuity and differentiability in the classical or generalized sense to ensure that all operations that we perform are justified.

We begin our study of divergent integrals with a general question. Let a function $f(x)$ of one variable with singularities be defined in the region $x \in \Omega = [-a, a]$. How should we define the *integral*

$$I_0 = \int_a^b f(x)dx \tag{4.1}$$

of the singular function, and how should it be calculated? For a regular function $f(x)$, we may consider at least two answers to this question. The first is that if there exists a function for which we have a closed-form representation such that

$$f(x) = \frac{dg(x)}{dx}, \tag{4.2}$$

we can use Newton–Leibniz formula and define the integral (4.1) analytically as

$$I_0 = \int_a^b f(x)dx = g(x)\Big|_a^b = g(b) - g(a). \tag{4.3}$$

Using this formula, one can easily calculate the regular integral (4.1).

The second answer is this: If function (4.2) $g(x)$ does not exist, then formula Eq. 4.3 is of no use. In that case, we can use the definition of the integral as the limit of finite sums and use a finite number of terms in that sum to calculate the integral of the regular function $f(x)$ numerically. In the simplest form it looks like this:

$$I_0 = \int_a^b f(x)dx = \lim_{\Delta x_i \to 0} \sum f(x_i)\Delta x_i \approx \sum_{i=1}^{N} f(x_i)\Delta x_i. \tag{4.4}$$

For a singular function $f(x)$, neither approach can be applied directly. The first fails because the derivative of a singular function does not exist in the classical sense, and so formulas Eq. 4.2 and Eq. 4.3 cannot be applied. The second approach does not work for a singular function because the sum in Eq. 4.4 contain infinite members and is infinite.

There are at least two approaches that can help us overcome these difficulties. One is based on the theory of generalized functions. It consists in considering the generalized derivative instead of the classical derivative in Eq. 4.2 and extending the Leibniz formula to the case of singular functions. We can consider this approach as a type of analytical regularization. Another approach is based on the definition of the integral as the limit of finite sums and consists in changing definition (4.4) in such way that the limit becomes finite.

We will consider here both approaches to the issue of regularization of divergent integrals, compare the formulas that arise, and calculate divergent integrals with different types of singularities using the regularized formulas obtained. In order to verify and compare the regularization formulas, we will consider an approach based on the direct calculation of divergent integrals using the special Gaussian quadrature interpolation formulas developed in [61].

Because the regularized formulas obtained in this section contain regular integrals that have to be calculated, we present here some brief information related to the numerical calculation of one-dimensional regular integrals.

4.1 Numerical calculation of one-dimensional regular integrals

We consider here briefly Gaussian quadrature interpolation formulas based on Legendre polynomials. Using this approach, an arbitrary interval $[a, b]$ is transformed to the interval $[-1, 1]$, and the integral (4.1) is represented approximately in the form

$$I_0 = \int_a^b f(x)dx \approx \sum_{i=1}^{N} w_i f(\xi_i), \tag{4.5}$$

where the points $\xi_i \in [-1, 1]$ and the coefficients $w_i \in R^1$ are referred to respectively as the *nodes* and *weights* of the quadrature.

The Legendre polynomials have an explicit representation given by the equation

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n. \tag{4.6}$$

The nodes $\xi_1, \xi_1, ..., \xi_N$ are the roots of the Nth Legendre polynomial $P_N(\xi)$, and the weights $w_1, w_1, ..., w_N$ are all positive. Their values for different N can be found in many publications, for example in [2, 62, 65]. The roots of the Nth $N - th$ Legendre polynomial also can be easily calculated in *Mathematica* or other mathematically

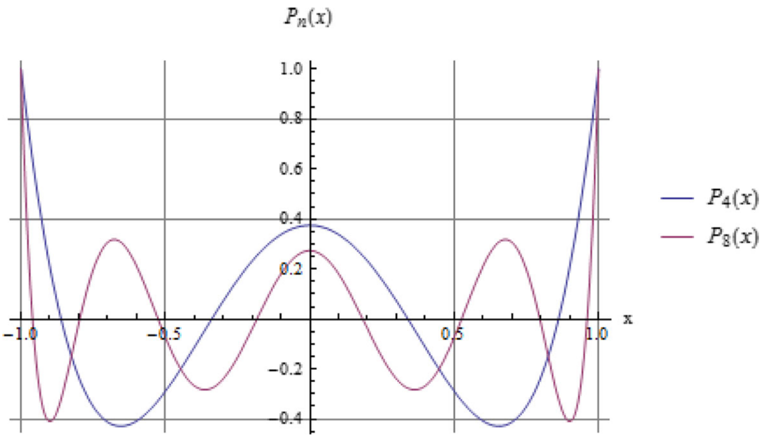

Fig. 1 Legendre polynomials

oriented software used for finding roots of polynomials. The weights can be calculated using the following formula:

$$w_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}}^{N} \left(\frac{\xi - \xi_j}{\xi_i - \xi_j} \right)^2 d\xi, \; i = 1, 2, ..., N. \tag{4.7}$$

In Fig. 1, we present plots of the Legendre polynomials for $N = 4$ and$N = 8$ respectively, generated by *Mathematica*'s built-in function **Plot**[ ].

From that plot, one can see the location of the roots of the Legendre polynomials. The quadrature rule (4.5) is called *N-point Gaussian quadrature*. It is constructed to yield an exact result for polynomials of degree $2N - 1$or less, while for other functions, it gives an approximate value. For further details, including error estimation, one can refer to [20, 62, 65].

Quadrature formula (4.5) will be used here for calculating regularized divergent integrals. Sometimes, for comparing results and error estimation, *Mathematica*'s built-in function **NIntegrate**[ ] will also be used.

4.2 The classical approach to the regularization of one-dimensional divergent integrals

There are many different definitions of divergent integrals; see the above-mentioned references. We will use the idea of finite-part integrals, introduced by Hadamard [44]. In order to illustrate Hadamard's ideas, let us consider the integral

$$I(y) = \int_{y}^{b} \frac{dx}{(x-y)^{1/2}} = 2\sqrt{b-y}\,, \quad y < b. \tag{4.8}$$

Differentiation of this expression leads to

$$\frac{dI(y)}{dy} = \frac{1}{2}\int_{y}^{b} \frac{dx}{(x-y)^{3/2}} - \left.\frac{1}{(x-y)^{1/2}}\right|_{x=y} = -\frac{1}{\sqrt{b-y}}, \quad y < b. \tag{4.9}$$

From this equation, it follows that the derivative of $I(y)$ is different from the integral and the term outside the integral. The integral is divergent, and the term outside the integral is infinite. Nevertheless, the limit of their difference exists and is finite. Hadamard called this limit the finite part (FP) of the divergent integral. It is equal to the derivative of the integral $I(y)$:

$$F.P.\int_{y}^{b} \frac{dx}{(x-y)^{3/2}} = \lim_{\varepsilon\to 0}\left[\int_{y+\varepsilon}^{b} \frac{dx}{(x-y)^{3/2}} - \frac{2}{\sqrt{\varepsilon}}\right] = -\frac{2}{\sqrt{b-y}}, \quad y < b. \tag{4.10}$$

For the divergent integral with fixed limits of integration, the finite part is equal to

$$\begin{aligned} F.P.\int_{a}^{b} \frac{dy}{(x-y)^{3/2}} &= F.P.\int_{a}^{y} \frac{dx}{(x-y)^{3/2}} + F.P.\int_{y}^{b} \frac{dy}{(x-y)^{3/2}} \\ &= -\frac{2}{\sqrt{b-y}} - \frac{2}{\sqrt{y-a}}, \quad a < y < b. \end{aligned} \tag{4.11}$$

We will use here Hadamard's idea of the finite part to obtain a valid definition of divergent integrals and to come up with regularized quadrature rules for their computation.

4.2.1 *Regularization of weakly singular integrals*

To define weakly singular (WS) integrals, we consider the limit

$$W.S.\int_{y}^{b} \ln\frac{1}{x-y}dx = \lim_{\varepsilon\to 0}\left(\int_{y+\varepsilon}^{b} \ln\frac{1}{x-y}dx\right) = (b-y)\left(1+\ln\frac{1}{b-y}\right). \tag{4.12}$$

Because of this limit is finite, Eq. 4.12 can be considered the classical definition of the weakly singular integral. Thus we can calculate weakly singular integrals with fixed limits:

$$W.S.\int_{a}^{b} \ln\frac{1}{y-x}dx = (b-a) + (b-y)\left(1+\ln\frac{1}{y-b}\right) - (a-y)\left(1+\ln\frac{1}{y-a}\right). \tag{4.13}$$

For the case $a = -1$ and$b = 1$, we have

$$I_0(y) = W.S. \int_{-1}^{1} \ln \frac{1}{y-x} dx = 2 + (y+1) \ln \left| \frac{1}{1+y} \right| - (y-1) \ln \left| \frac{1}{y-1} \right|. \quad (4.14)$$

Many formulas have been proposed for calculating weakly singular integrals that contain a sufficiently smooth function $f(x)$. We use here the following one:

$$W.S. \int_{-1}^{1} f(x) \ln \frac{1}{y-x} dx = \int_{-1}^{1} (f(x) - f(y)) \ln \frac{1}{y-x} dx + f(y)$$
$$W.S. \int_{-1}^{1} \ln \frac{1}{y-x} dx. \quad (4.15)$$

In order to illustrate some of the problems that occur in the numerical calculation of weakly singular integrals, let us calculate integral (4.14) using the Gaussian quadrature interpolation formula (4.5), *Mathematica*'s built-in function **NIntegrate**[ ], and the regularized formula (4.15). The results of these calculations for $N = 8$ are shown in Fig. 2. Here, the black lines correspond to the exact expression (4.14), calculation with *Mathematica*'s built-in function **NIntegrate**[ ], and the regularized formula (4.15). The red line corresponds to calculations by the Gaussian interpolation quadrature formula (4.5).

From these results, it follows that even in the case of weakly singular integrals, direct numerical calculations using the Gaussian quadrature formula give incorrect results, and regularization is therefore needed. *Mathematica*'s built-in function **NIntegrate**[ ] can calculate weakly singular integrals.

In the Section 4.5, we will present further results on the calculation of weakly singular integrals using the regularized formula (4.15) and compare it with other approaches.

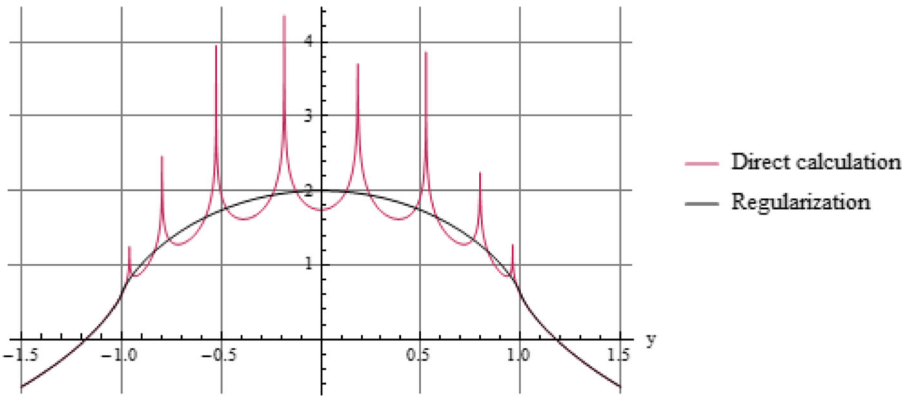

Fig. 2 Value of the weakly singular integral versus the parameter y

4.2.2 Regularization of singular integrals

For our definition of the singular integral, we consider the limits

$$P.V.\int_{y}^{b}\frac{1}{x-y}dx=\lim_{\varepsilon\to 0}\left(\int_{y+\varepsilon}^{b}\frac{1}{x-y}dx-\ln\varepsilon\right)=\ln\frac{1}{b-y} \tag{4.16}$$

and

$$P.V.\int_{a}^{b}\frac{1}{x-y}dx=\lim_{\varepsilon\to 0}\left(\int_{a}^{y-\varepsilon}\frac{1}{x-y}dx+\int_{y+\varepsilon}^{b}\frac{1}{x-y}dx\right)=\ln\frac{y-a}{b-y}. \tag{4.17}$$

Equations 4.16 and 4.17 can be considered the classical definition of the singular integral. One can see that definition (4.17) coincides with the definition of the Cauchy principal value (PV).

For the case $a=-1$ and $b=1$, we have

$$I_1(y)=P.V.\int_{-1}^{1}\frac{1}{y-x}dx=\ln\frac{1+y}{1-y}. \tag{4.18}$$

For calculation of singular integrals that contain a sufficiently smooth function $f(x)$, we use here a formula similar to Eq. 4.15 in the form

$$P.V.\int_{-1}^{1}f(x)\frac{1}{x-y}dx=\int_{-1}^{1}(f(x)-f(y))\frac{1}{x-y}dx+f(y)P.V.\int_{-1}^{1}\frac{1}{x-y}dx. \tag{4.19}$$

In Fig. 3, we present the results of calculating a singular integral of type (4.18) directly using an analytical expression and using the regularized formula (4.19).

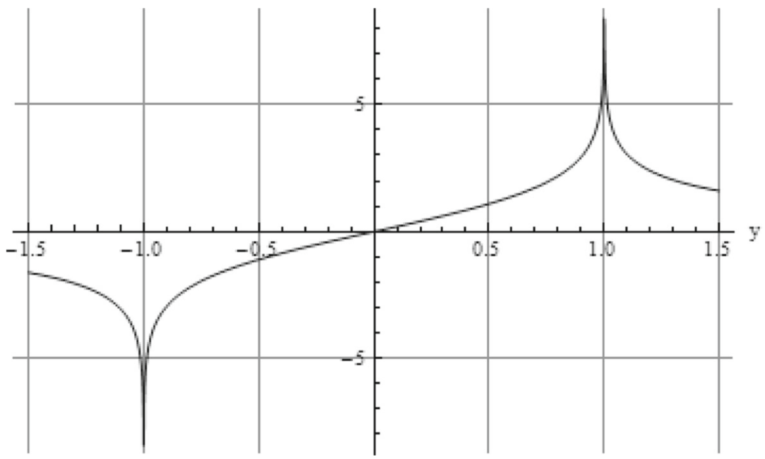

Fig. 3 The Cauchy principal value versus the parameter y

It should be noted that numerical calculation of the singular integrals here using the Gaussian quadrature interpolation formula (4.5) and *Mathematica*'s built-in function **NIntegrate**[ ] gives incorrect results.

In Section 4.5, we will present further results on the calculation of singular integrals using the regularized formula (4.19) and compare that approach with other approaches.

4.2.3 Regularization of hypersingular integrals

For the definition of the hypersingular integral, we consider the limits

$$\int_{y}^{b} \frac{1}{(x-y)^2}dx = \lim_{\varepsilon\to 0}\left(\int_{y+\varepsilon}^{b} \frac{1}{(x-y)^2}dx - \frac{1}{\varepsilon}\right) = -\frac{1}{b-y} \tag{4.20}$$

and

$$F.P.\int_{a}^{b} \frac{1}{(x-y)^2}dx = \lim_{\varepsilon\to 0}\left(\int_{a}^{y-\varepsilon} \frac{1}{(y-x)^2}dx + \int_{y+\varepsilon}^{b} \frac{1}{(y-x)^2}dx\right) = -\frac{1}{b-y} - \frac{1}{y-a}. \tag{4.21}$$

Equations 4.20 and 4.21 can be considered the classical definition of the hypersingular integral.

For the case $a = -1$ and$b = 1$, we have

$$I_2(y) = F.P.\int_{-1}^{1} \frac{1}{(x-y)^2}dx = \frac{2}{y^2-1}. \tag{4.22}$$

For calculating hypersingular integrals that contain a sufficiently smooth function $f(x)$, we use here a formula similar to Eqs. 4.15 and 4.19. Then for a hypersingular integral, the regularization formula takes the form

$$F.P.\int_{-1}^{1} f(x)\frac{dx}{(x-y)^2} = \int_{-1}^{1} \frac{f(x)-f(y)-\frac{df(y)}{dx}(y-x)}{(x-y)^2}dx + f(y)$$
$$F.P.\int_{-1}^{1} \frac{dx}{(x-y)^2} + \frac{df(y)}{dx}P.V.\int_{-1}^{1} \frac{dx}{y-x}. \tag{4.23}$$

In Fig. 4, we present results of calculating a singular integral of type (4.22) directly using an analytical expression and using the regularized formula (4.23). It should be noted that numerical calculation of the hypersingular integral here using the Gaussian quadrature interpolation formula (4.5) and *Mathematica*'s built-in function **NIntegrate**[ ] gives us incorrect results.

In Section 4.5, we will present further results on the calculation of hypersingular integrals using the regularized formula (4.19) and compare that approach with other approaches.

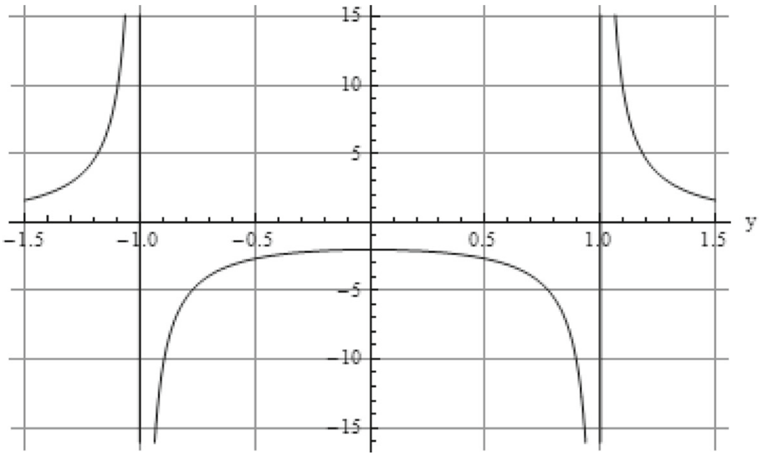

Fig. 4 Value of the Hadamard finite part versus the parameter y

4.2.4 Generalization to divergent integrals with higher-order singularities

Divergent integrals with higher-order singularities can be defined using the formula

$$\frac{d}{dy} F.P. \int_{-1}^{1} \frac{1}{(x-y)^k} dx = F.P. \int_{-1}^{1} \frac{k}{(x-y)^{k+1}} dx. \tag{4.24}$$

One must take into account that the differential operator here has to be applied not directly, but in the sense of the Hadamard finite part, as explained above.

For calculation of divergent integrals with higher-order singularities that involve a sufficiently smooth function $f(x)$, one can use a generalization of formulas (4.15), (4.19), and (4.23). The formula ultimately assumes the following form:

$$F.P. \int_{-1}^{1} f(x) \frac{1}{(x-y)^{n+1}} dx = \int_{-1}^{1} \frac{f(x) - \sum_{k=0}^{r} f^k(y)(y-x)^k/k!}{(x-y)^{n+1}} dx + \sum_{k=0}^{r} \frac{f^k(y)}{k!} P.F. \int_{-1}^{1} \frac{1}{(x-y)^{n+1-r}} dx. \tag{4.25}$$

Using this formula, one can calculate divergent integrals containing singularities of arbitrary order.

4.3 Generalized-functions approach to the regularization of one-dimensional divergent integrals

Let us consider again the integral (4.1) and consider how the theory of generalized functions (distributions) allows us to define the definite integral of a singular function $f(x)$. But first, we introduce some basic concepts related to the theory of distributions. For more detailed information, one can refer to a number of books devoted to the theory of generalized functions, for example [18, 32, 67, 100, 102, 109].

Generalized functions, or distributions, were introduced in order to extend and justify the operation of differentiation in the case of nonsmooth and even discontinuous functions. They are used to calculate derivatives of functions with singularities and to manipulate generalized functions with concentrated loads. It is important to mention that one cannot talk about the value of a generalized function at a given point; they are defined as functionals on the entire space. For instance, one cannot say that a generalized function is equal to zero at the point x_0. However, it is possible to give meaning to the statement that the generalized function $f(x)$ is equal to zero in the vicinity of the point x_0. For every $\varphi(x) \in C^{\infty}(R^1)$, we can consider the functional

$$(f, \varphi) = \int\limits_{R^1} f(x)\varphi(x)dx, \tag{4.26}$$

which vanishes in the vicinity of the point x_0, and say that it equals zero in the vicinity of x_0.

The derivative of the generalized function $f(x)$ is defined by the equation

$$D_x f(x) = \left(\frac{df(x)}{dx}, \varphi(x)\right) = \left(f(x), \frac{d\varphi}{dx}(x)\right), \forall\varphi(x) \in C^{\infty}(R^1). \tag{4.27}$$

The derivative of order $k$of the generalized function $f(x)$ is defined by

$$D_x^k f(x) = \left(\frac{d^k f(x)}{dx^k}, \varphi(x)\right) = (-1)^k \left(f(x), \frac{d^k\varphi(x)}{dx^k}\right), \forall\varphi(x) \in C^{\infty}(R^1). \tag{4.28}$$

For a regular function, its ordinary derivative is equal to its generalized derivative. For the regularization of divergent integrals, we do not need to consider generalized functions in the most general case. The situation become much simpler if we restrict our attention to generalized functions that can be represented in the form

$$f(x) = \frac{d^r g(x)}{dx^r}, \tag{4.29}$$

where $g(x)$ is any continuous or even piecewise continuous function, usually called a *generating function*.

Now formula (4.28) for the derivative of order $k$of the generalized function $f(x)$ can be represented in the form

$$D_x^k f(x) = \frac{d^{r+k} g(x)}{dx^{r+k}}. \tag{4.30}$$

For our purposes it is important to consider generalized functions of the type (4.30 that are ordinary functions everywhere except on a subdomain Ω^{ε} of a larger domain $\Omega = [a, b]$ such that outside of $\Omega^{\varepsilon}_{,}$ the generating function $g(x)$ possesses continuous derivatives. In many cases of importance for applications, the subdomain Ω^{ε} consists of isolated points.

The antiderivative, or indefinite integral, of the generalized function $f(x)$ represented in the form Eq. 4.29 can be defined as

$$\int f(x)dx = D_x^{-1} f(x) = \frac{d^{r-1}g(x)}{dx^{r-1}}. \tag{4.31}$$

In the case $r = 1$, we have

$$\int f(x)dx = D_x^{-1} f(x) = g(x). \tag{4.32}$$

Equations 4.31 and 4.32 can be considered the definition of the indefinite integral for the case of functions with singularities. Wide classes of divergent integrals can be considered as generalized functions, more specifically as linear functionals defined on appropriate functional spaces.

Following [18, 32, 41, 112, 121], we consider the definition of the definite integral of a generalized function, which is of great importance in deriving the regular representation of a divergent integral. Let $f(x)$ be a function of one variable defined in the region $x \in \Omega = [-a, a]$. All singularities of the function $f(x)$ are concentrated in the subregion $\Omega^\varepsilon = [-\varepsilon, \varepsilon] \subset \Omega$. In the region $\Omega \backslash \Omega^\varepsilon$ including the boundary, the function $f(x)$ is regular and possesses all necessary derivatives. The function $f(x)$ is clearly a generalized function.

Let us consider the definite integral of the function $f(x)$ over the finite interval Ω:

$$I_0 = \int_{-a}^{a} f(x)dx. \tag{4.33}$$

What does the symbol I_0 mean for such singular function? In the general case, the classical approach cannot answer this question. Only for special types of singularity does an answer exist, and each type of singularity must be considered separately, since no general theory based on the classical approach exists.

We introduce a finite test function $\phi(x) \in C^\infty(R)$ such that $\phi(x) = 1 \quad \forall x \in \Omega$ and extend it arbitrarily to the region Ω^0. Clearly, for such a function $\phi(x)$, its derivatives are equal to zero in the region Ω including the endpoints $\partial\Omega = \{-a, a\}$:

$$\frac{d^k\phi(x)}{dx^k} = 0, \quad x \in \bar{\Omega} = [-a, a]. \tag{4.34}$$

Now let us consider a scalar product that is a functional and define the function $f(x)$ in the sense of distributions:

$$(f, \phi) = \int_R f(x)\phi(x)dx = \int_R \phi(x)\frac{d^k g(x)}{dx^k}dx. \tag{4.35}$$

Integrating by parts, taking into account properties (4.34) and the finiteness of the test function, we obtain

$$\int_R \phi(x)\frac{d^k g(x)}{dx^k}dx = (-1)^k \int_{\Omega\cup\Omega^0} g(x)\frac{d^k\phi(x)}{dx^k}dx = (-1)^k \int_{\Omega^0} g(x)\frac{d^k\phi(x)}{dx^k}dx. \tag{4.36}$$

The last equality holds because of $\phi(x) = 1$ and the validity of Eq. 4.34 in the region Ω. Performing the last integral again along the path of integration in reverse order and taking into account that $\phi(-a) = \phi(a) = 1$, we obtain

$$\int_{\Omega^0} g(x) \frac{d^k \phi(x)}{dx^k} dx = \left. \frac{d^{i-1} g(x)}{dx^{i-1}} \right|_{x=-a}^{x=a} + (-1)^k \int_{\Omega^0} \phi(x) \frac{d^k g(x)}{dx^k} dx. \tag{4.37}$$

The integral (4.33) in the sense of distributions can be written as

$$I_0 = \int_R f(x)\phi(x)dx - \int_{\Omega^0} f(x)\phi(x)dx. \tag{4.38}$$

The first term here is

$$\int_R f(x)\phi(x)dx = \left. \frac{d^{i-1} g(x)}{dx^{i-1}} \right|_{x=-a}^{x=a} + \int_{\Omega^0} \phi(x) \frac{d^k g(x)}{dx^k} dx, \tag{4.39}$$

and the second term is

$$\int_{\Omega^0} f(x)\phi(x)dx = \int_{\Omega^0} \phi(x) \frac{d^k g(x)}{dx^k} dx. \tag{4.40}$$

As a result, from Eqs. 4.38, 4.39, and 4.40 we will obtain the finite part of the divergent integral in the form

$$I_0 = F.P. \int_{-a}^{a} f(x)dx = \left. \frac{d^{k-1} g(x)}{dx^{k-1}} \right|_{x=-a}^{x=a}. \tag{4.41}$$

This is the definition of the divergent integrals in the sense of distributions. We can use this equation for calculating in the same way that we calculated weakly singular, singular, and hypersingular integrals.

Obviously for $r = 1$ we have

$$F.P. \int_{-a}^{a} f(x)dx = g(x) \Big|_{x=-a}^{x=a} \tag{4.42}$$

For regular functions this is usual formula from integral calculus for calculation of the definite integrals, it coincide with formula (4.3). In this case , let's call it generalized Newton-Leibnitz formula.

We apply formulas (4.41) and (4.42) to determinate and regularization of the one-dimensional divergent integrals.

4.3.1 Weakly singular integrals regularization

In order to define weakly singular integrals of type (4.13), we consider in (4.42) functions $f(x) = \ln\left|\frac{1}{x-y}\right|$ and $g(x) = (x - y) + (x - y)\ln\frac{1}{|x-y|}$.

Then the corresponding *weakly singular* (WS) integral can be represented in the form

$$W.S.\int_a^b \ln\frac{1}{x-y}dx = (x-y) + (x-y)\ln\frac{1}{|x-y|}\Big|_a^b$$
$$= (b-a) + (b-y)\left(1+\ln\frac{1}{b-y}\right) - (a-y)\left(1+\ln\frac{1}{y-a}\right), \tag{4.43}$$

which obviously coincides with Eq. 4.13.

For calculating weakly singular integrals that contain a sufficiently smooth function $f(x)$, we use integration by parts. The regularized representation of the weakly singular integral then takes the form

$$W.S.\int_a^b f(x)\ln\frac{1}{x}dx = -f(x)\left((y-x)+(y-x)\ln\frac{1}{|x-y|}\right)\Big|_a^b$$
$$+\int_a^b \frac{df(x)}{dx}\left((y-x)+(y-x)\ln\frac{1}{|x-y|}\right)dx. \tag{4.44}$$

In Section 4.5, we will present more results on calculating weakly singular integrals using the regularized formula (4.44) and compare that approach with other approaches.

4.3.2 Regularization of singular integrals

To define the singular integral of type (4.17), we consider in Eq. 4.42 functions $f(x) = \frac{1}{x-y}$ and $g(x) = -\ln\frac{1}{x-y}$. Then the corresponding singular integral can be represented as follows:

$$P.V.\int_a^b \frac{1}{x-y}dx = -\ln\frac{1}{x-y}\Big|_a^b = \ln\frac{y-a}{b-y}, \tag{4.45}$$

which obviously coincides with (4.17).

For calculating singular integrals that contain a sufficiently smooth function $f(x)$, we use integration by parts twice. As a result, the regularized representation of the singular integral takes the form

$$P.V.\int_a^b \frac{f(x)}{x-y}dx = \left(\frac{df(x)}{dx}\left((y-x)+(y-x)\ln\frac{1}{|x-y|}\right) - \left(f(x)\ln\frac{1}{|x-y|}\right)\right)\Big|_a^b$$
$$+\int_a^b \frac{d^2f(x)}{dx^2}\left((y-x)+(y-x)\ln\frac{1}{|x-y|}\right)dx. \tag{4.46}$$

In Section 4.5, we will present more results on calculating weakly singular integrals using the regularized formula (4.46) and compare that approach with other approaches.

4.3.3 Regularization of hypersingular integrals

To definite the hypersingular integral of type (4.21), we consider in Eq. 4.42 functions $f(x) = \frac{1}{(x-y)^2}$ and $g(x) = -\frac{1}{x-y}$. Then the corresponding hypersingular integral can be represented in the form

$$F.P.\int_a^b \frac{1}{(x-y)^2}dx = -\frac{1}{x-y}\bigg|_a^b = -\frac{1}{b-y} - \frac{1}{y-a}, \tag{4.47}$$

which obviously coincides with Eq. 4.21.

For calculating singular integrals that contain a sufficiently smooth enough function $f(x)$, we use integration by parts twice. As result, the regularized representation of the hypersingular integral takes the form

$$F.P.\int_a^b \frac{f(x)}{(x-y)^2}dx = \left(\frac{d^2 f(x)}{dx^2}\left((y-x) + (y-x)\ln\frac{1}{|x-y|}\right) - \left(\frac{df(x)}{dx}\ln\frac{1}{|x-y|}\right) + \frac{f(x)}{x-y}\right)\bigg|_a^b + \int_a^b \frac{d^3 f(x)}{dx^3}\left((y-x) + (y-x)\ln\frac{1}{|x-y|}\right)dx. \tag{4.48}$$

In Section 4.5, we will present further results on calculating weakly singular integrals using the regularized formula (4.48) and compare that approach with other approaches.

4.3.4 Generalization to divergent integrals with higher-order singularities

For calculating divergent integrals with higher-order singularities that contain a sufficiently smooth function $f(x)$, one can use a generalization of formulas (4.44), (4.46), and (4.48). This case has been already analyzed using the generalized-functions approach in [112]. The corresponding regularization formula can be represented in the following form:

$$F.P.\int_{-a}^{a} \frac{f(x)}{r^m}dx = \sum_{i=0}^{k-1}(-1)^{i+1}\frac{d^i}{dx^i}\frac{P_i}{r^{m-k}}\frac{d^{k-1-i}f(x)}{dx^{k-1-i}}\bigg|_{x=-a}^{x=a} + (-1)^k\int_{-a}^{a}\frac{P_k}{r^{m-k}}\frac{d^k f(x)}{dx^k}, \tag{4.49}$$

where $P_k = (-1)^k \prod_{i=0}^{k-1} \frac{1}{(m+i)}$ for $k, m > 1$.

Using this formula, we can calculate divergent integrals containing singularities of any order.

4.4 A Legendre-polynomials-based approach to the regularization of one-dimensional divergent integrals

In [61], a Gaussian quadrature interpolation formula was developed, based on Legendre polynomials, for the numerical evaluation of divergent integrals. According to that formula, an arbitrary interval $[a, b]$ is transformed to the interval $[-1, 1]$, and a divergent integral can be approximately represented as

$$I_k = \int_{=1}^{1} f(\xi)w(\xi)d\xi \approx \sum_{i=1}^{N} w_{ik} f(\xi_i), \quad k = 0, 1, 2, \tag{4.50}$$

with weights w_{ik}defined by the formula

$$w_{ik} = w_i \sum_{j=1}^{N-2} \left(\frac{2j+1}{2} P_j(\xi_i) \int_{-1}^{1} w(\xi)P_j(\xi)d\xi \right), \tag{4.51}$$

where w_i and ξ_i are the weights and nodes defined in regular Gaussian quadrature interpolation formula (4.5), and the weighted function $w(\xi)$ has the form

$$w(\xi) = \ln \frac{1}{(\xi - y)}, \text{ for} k = 0, \text{ and } w(\xi) = \frac{1}{(\xi - y)^k}, \text{ for} k = 1, 2. \tag{4.52}$$

We should mention that slightly modified formulas adapted for BIE applications are presented in [9, 10]. Below, we present only the final formulas required for calculating divergent integrals. For details, one can refer to [61].

4.4.1 Formulas for the calculation of weakly singular integrals

For calculating weakly singular integrals, the weights w_{i0} have to be calculated by the formula

$$w_{i0} = w_i \Bigg((P_0(\xi_i) - P_1(\xi_i)R_0(y)) + \sum_{j=1}^{N-2} \big(P_{j-1}(\xi_i) - P_{j+1}(\xi_i)R_0(y)\big)$$
$$+P_{N-2}(\xi_i)R_{N-1}(y) - P_{N-1}(\xi_i)R_N(y) \Bigg), \tag{4.53}$$

where $R_j = Q_j(y) + \frac{1}{4}\ln(y-1)^2$ and $Q_j(y)$ is the Legendre function of the second kind [1, 31].

4.4.2 Formulas for the calculation of singular integrals

For singular integrals, the weights w_{i0} are calculated using the simple formula

$$w_{i1} = w_i \sum_{j=1}^{N-2} (2j+1)P_j(\xi_i)Q_j(y), \tag{4.54}$$

where the following formula [1, 58] has been used:

$$P.V.\int_{-1}^{1}\frac{P_n(x)}{x-y}dx = 2Q_n(y). \tag{4.55}$$

4.4.3 Formulas for calculating hypersingular integrals

For singular integrals, the weights w_{i0} are calculated using the simple formula

$$w_{i1} = w_i\sum_{j=1}^{N-2}(2j+1)P_j(\xi_i)\frac{2(j+1)}{1-y^2}\big(yQ_j(y)-Q_{j+1}(y)\big), \tag{4.56}$$

where the following formula [58] has been used:

$$F.P.\int_{-1}^{1}\frac{P_n(x)}{(x-y)^2}dx = \frac{2(n+1)}{1-y^2}\left(yQ_n(y)-Q_{n+1}(y)\right). \tag{4.57}$$

4.4.4 Generalization to divergent integrals with higher-order singularities

The approach proposed in [61] can be easily extended to the calculation of divergent integrals with higher-order singularities. The simplest way to proceed is to explore the formula

$$F.P.\int_{-1}^{1}\frac{P_n(x)}{(x-y)^{p+1}}dx = \frac{1}{p!}\frac{d^p}{dy^p}P.V.\int_{-1}^{1}\frac{P_n(x)}{(y-x)}dx = \frac{2}{p!}\frac{d^p}{dy^p}Q_n(y). \tag{4.58}$$

Indeed, one can easily use formula (4.57) to calculate the derivative

$$\frac{d}{dy}Q_n(y) = \frac{n+1}{y^2-1}\left(Q_n(y)-yQ_n(y)\right). \tag{4.59}$$

Calculating the second derivative leads to

$$F.P.\int_{-1}^{1}\frac{P_n(x)}{(x-y)^3}dx = \frac{2(n+1)}{(y^2-1)^2}\Big(\Big(1+(2+n)y^2\Big)Q_n(y) -(5+2n)yQ_{n+1}(y)+(2+n)Q_{n+2}(y)\Big). \tag{4.60}$$

By substituting this equation into Eq. 4.51, we can find weights for the supersingular integrals and then calculate them using the quadrature formula (4.50).

In Section 4.5, we will present further results on calculating divergent integrals using the regularized formula (4.50) with weights (4.53), (4.54), and (4.56) and compare that approach with other approaches.

4.5 Calculation of divergent integrals with classical and generalized-functions approaches

In previous sections, we have presented three approaches and three types of formulas for calculating the main types of divergent integrals that appear in BEM, namely weakly singular, singular, and hypersingular. We used *Mathematica* to verify the obtained formulas and compare the calculations of divergent integrals for different types of regular functions $f(x)$.

To verify our regularization formulas and computer codes, we have calculated analytically, in the sense of generalized functions, weakly singular, singular, and hypersingular integrals for the function $f(x) = x^3$. The analytical formulas have the form

$$W.S.\int_{-1}^{1} x^3 \ln\frac{1}{|y-x|}dx = \frac{1}{12}\left(2\left(y+3y^3\right) - 3\left(-1+y^4\right)\ln\left|\frac{1}{1-y}\right| + 3\left(-1+y^4\right)\ln\left|\frac{1}{1+y}\right|\right), \tag{4.61}$$

$$P.V.\int_{-1}^{1} \frac{x^3}{y-x}dx = y^3\left(\ln|1+x| - \ln|x-1|\right) - 2y^2 - \frac{2}{3}, \tag{4.62}$$

$$F.P.\int_{-1}^{1} \frac{x^3}{(y-x)^2}dx = \frac{y}{y^2-1}\left(3y\left(y^2+2y-1\right)\left(\ln|1-y| - \ln|1+y| - 4\right)\right). \tag{4.63}$$

We compare the results obtained using these analytical formulas and those obtained numerically using the classical approach (4.15), (4.19), and (4.23); the generalized-functions approach (4.44), (4.46), and (4.48); and the approach based on the regularization of Legendre polynomials (4.50) with weights (4.53), (4.54), and (4.56) respectively. Calculations using the analytical formulas (4.61)–(4.63) for the point $y = 0.5$ give us the following results: for the weakly singular integral, (0.403321); for the singular integral, (–1.02934); and for the hypersingular integral, (0.842707). We also used the analytical expressions (4.55) and (4.57) to verify the regularization formulas for the singular and hypersingular integrals for the case $f(x) = P_n(x)$. For calculating weakly singular integrals in this case, we used *Mathematica*'s built-in function **NIntegrate**[ ] , which can evaluate weakly singular integrals numerically. For the point $y = 0.5$, the results are as follows: for weakly singular, (0. 138307); for singular, (0.284024); and for hypersingular, (–7.7452). The results of our calculations using the regularized formulas are presented in Table 1. The results obtained using the classical approach are marked (**I**), those obtained using the generalized-functions approach are marked (**II**), and those obtained using the approach based on the regularization of Legendre polynomials are marked (**III**). For the function $f(x) = x^3$, all the formulas give a very good correlation with the analytical solutions. In Table 1, we also present calculations of divergent integrals for some other types of functions $f(x)$ specified in the table. All calculations were carried out for the point $y = 0.5$

Table 1 Calculation of the $1-D$ divergent integrals for the point $y=0.5$

		x^3	$P_6(x)$	$x^3\left(1+x^2\right)^{1/2}$	$x^3\left(1+x^2\right)^{-1/2}$	$\cos(x)$	$\sin(x)$	e^{-x^2}
I	WS	0.41268	0.14462	0.51758	0.33118	1.51975	0.86066	1.36713
	PV	–1.02934	0.28402	–1.31601	–0.80683	1.44029	–1.28453	1.61326
	FP	0.84271	–7.7452	1.37197	0.45211	–2.72533	–2.5622	–2.35505
II	WS	0.39643	0.14346	0.49988	0.31989	1.52289	0.85236	1.37028
	PV	–1.02909	0.28504	–1.31322	–0.79895	1.44594	–1.28582	1.60969
	FP	0.84274	–7.7461	1.37131	0.44951	–2.72162	–2.5691	–2.35828
III	WS	0.40332	0.14462	0.50693	0.32296	1.52266	0.846709	1.37883
	PV	–1.02934	0.28402	–1.31582	–0.80748	1.44029	–1.28453	1.61375
	FP	0.84271	–7.7452	1.37110	0.45486	–2.72533	–2.56225	–2.35636

and for $N=8$, the Gaussian quadrature interpolation formula (4.5) of order 8. Note that the regularized formula (4.23) for hypersingular integrals obtained using the classical approach contains a first-order derivative, and the regularized formulas (4.44), (4.46), and (4.48) obtained using the generalized-functions approach contain derivatives up to third order. The formulas based on the regularization of Legendre polynomials are free from derivatives. To be sure, the presence of derivatives is a shortcoming, but on the other hand, the formulas based on the generalized-functions approach are more stable with respect to a change in coordinate of the collocation point y than those obtained using the classical approach, which for some values of y give inaccurate results, and to improve accuracy, it is necessary to increase the number of nodes in the quadrature formula. The formulas based on regularization of Legendre polynomials give very good results for polynomial and similar functions, and for polynomials of degree not greater than N, the results are exact.

Timings of the computations show that in all considered cases, the time required for calculations using formulas based on the classical approach is a little less than that required using formulas based on the generalized-functions approach. The time of calculation using formulas based on the regularization of Legendre polynomials is almost twenty times greater. Of course, the time of calculation can be speeded up if the weights (4.51) have been calculated before the calculation of the divergent integrals. But one has to take into account that the weights depend on the parameter y, and has to be calculated separately for each collocation point. Our conclusion is that all the regularization formulas presented here are valid and can be used for calculating weakly singular, singular, and hypersingular integrals.

In order to show the dependence of the divergent integrals on the position of the collocation point y and once again compare the results of calculations using the three considered approaches, we used *Mathematica*'s built-in function **Plot**[] to generate plots of the divergent integrals for the functions $f(x)=\cos(x)$ and $f(x)=\sin(x)$ using the classical, generalized-functions, and regularization of Legendre polynomials approaches. The results of our calculations and plotting are presented in Figs. 5, 6 and 7. From these graphs, it follows that within the interval $y\in[-1,1]$, all the graphs coincide, but outside the interval, the Legendre polynomials approach does

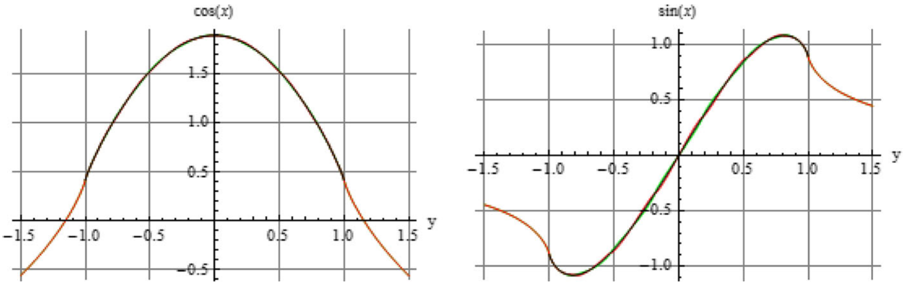

Fig. 5 Values of weakly singular integrals versus the parameter y

not work. It cannot be used for $|y| > 1$, because the polynomials $Q_n(y)$ are defined only inside the interval $y \in [-1, 1]$. At the points $y = \pm 1$, the singular and hypersingular integrals are undefined. But they can be easily calculated using the classical or generalized-functions approaches; see [9, 74, 115] for details.

5 Calculation of two-dimensional integrals

The structure of this section is the same as that of the previous one, but here we will consider two-dimensional divergent integrals. Our focus will be on the computational aspects of the problem. The theoretical aspects of the problem can be found in [18, 32, 100]. We assume that all functions under consideration are well defined, possess all necessary properties of continuity and differentiability in the classical or generalized sense to ensure that all actions to be performed are valid.

We consider a function of two variables $f(\mathbf{x})$with singularities concentrated in the region $\mathbf{x} \in V$. We ask how the integral

$$I_0 = \int\limits_{\Omega} f(\mathbf{x}) d\Omega \tag{5.1}$$

of a singular function should be defined and how it should be calculated. Of course, in the case of two- and higher-dimensional integrals, the situation becomes much more complicated in general even for the case of regular functions. Let us for simplicity

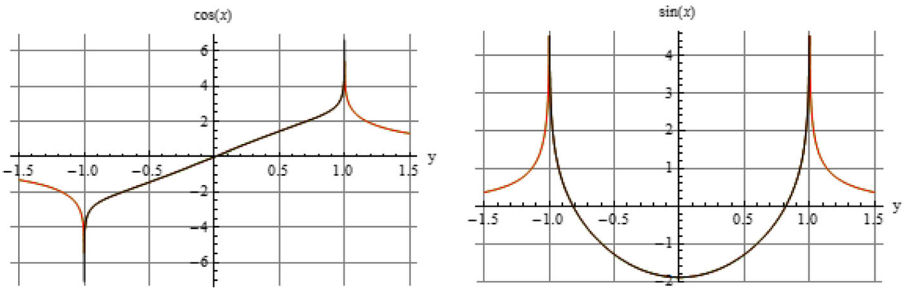

Fig. 6 Values of singular integrals versus the parameter y

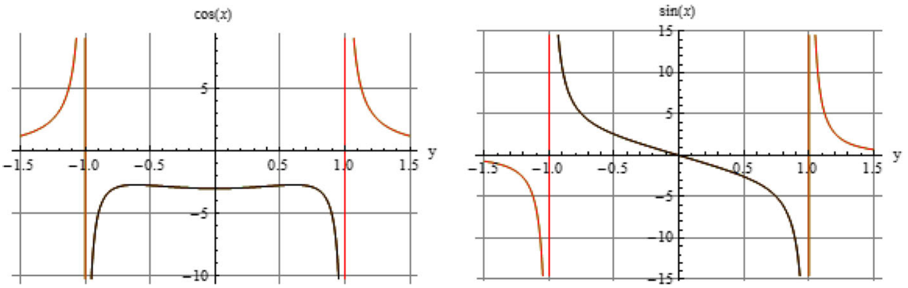

Fig. 7 Values of hypersingular integrals versus the parameter y

consider a rectangular domain $V = [-a_1, b_1] \times [-a_2, b_2]$. In the same way as in the one-dimensional case, for the regular function $f(\mathbf{x})$ we have at least two approaches. First, if there exists a function$g(\mathbf{x})$ such that

$$f(\mathbf{x}) = \frac{d^2 g(\mathbf{x})}{dx^1 dx^2}, \tag{5.2}$$

we can use the generalized Newton–Leibniz formula and define the integral (5.1) analytically as

$$I_0 = \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x_1 x_2) dx_1 dx_2 = \left. g(x_1, x_2)|_{a_1}^{b_1} \right|_{a_2}^{b_2} = g(b_1, b_2) - g(a_1, a_2). \tag{5.3}$$

This formula can be generalized to a more complicated region using the Gauss–Ostrogradsky formula. If there exists a function$g(\mathbf{x})$ such that

$$f(\mathbf{x}) = \Delta g(\mathbf{x}), \tag{5.4}$$

then the integral (5.1) can be transformed in the following way:

$$\int_{\Omega} f(\mathbf{x}) dS = \int_{\Omega} \Delta g(\mathbf{x}) dS = \int_{\Omega} \nabla \cdot \nabla g(\mathbf{x}) dS = \int_{\partial\Omega} \mathbf{n}(\mathbf{x}) \cdot \nabla g(\mathbf{x}) dS = \int_{\partial\Omega} \partial_n g(\mathbf{x}) dS, \tag{5.5}$$

where $\nabla = \frac{\partial^2}{\partial x_1} + \frac{\partial^2}{\partial x_2}$ and $\Delta = \nabla \cdot \nabla = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ are the Hamilton and Laplace operators, respectively.

Formulas (5.3) and (5.5) are commonly used in mathematical analysis for calculating regular multidimensional integrals.

In the more general case in which such a function $g(\mathbf{x})$ does not exist, formulas (5.3) and (5.5) cannot help in calculating multidimensional integrals. In that case, we can use the definition of the integral as a limit of finite sums and use a finite number of terms in that sum for numerical calculation of the multidimensional integral of the regular function $f(x)$. In the simplest form, this approach takes the form

$$I_0 = \int_{\Omega} f(\mathbf{x}) dS = \lim_{\Delta\Omega_i \to 0} \sum f(\mathbf{x}_i) \Delta\Omega_i \approx \sum_{i=1}^{N} f(\mathbf{x}_i) \Delta\Omega_i. \tag{5.6}$$

For a singular function $f(\mathbf{x})$, neither approach can be applied directly. The first approach does not work because for a singular function, the derivative does not exist in the classical sense, and formulas (5.2) and (5.4) cannot be used. The second approach fails for a singular function because the sum in (5.6) contains singular terms and is therefore infinite.

There are at least two approaches to overcoming these difficulties. One is based on the theory of generalized functions. It consists in considering the generalized derivative instead of the classical one in Eqs. (5.2) and (5.4) and in extending the formulas of multivariate calculus to the case of singular functions. Another approach is based on the definition of the integral as the limit of finite sums and consists in changing definition (5.6) in such way that the limit in definition (5.6) is finite.

We consider here both approaches to the regularization of divergent integrals, compare formulas based on them, and calculate divergent integrals with different singularities using the obtained formulas. In order to study divergent integrals, following [73], we present here slightly modified definitions and classifications of integrals with various types of singularities.

Definition. Let us consider two points with coordinates $\mathbf{x},\ \mathbf{y} \in \Re^2$ and a region Ω with smooth boundary $\partial\Omega$ of class$C^{0,1}$. An integral over the domain Ω of type

$$\int\limits_{\Omega} \frac{f(\mathbf{x})}{r(\mathbf{x}-\mathbf{y})^{\alpha}} \phi(\mathbf{x}) d\Omega\ , \quad \alpha > 0, \tag{5.7}$$

where $f(\mathbf{x})$is a smooth enough function in the domain Ω, is *weakly singular* for $\alpha = 1$, *strongly singular* for $\alpha = 2$, and *hypersingular* for $\alpha = 3$.

Because the regularized formulas obtained in this section contain regular integrals that have to be calculated numerically, we discuss here briefly the numerical calculation of two-dimensional regular integrals.

5.1 Quadrature for the calculation two-dimensional regular integrals

There are many approaches and formulas for the numerical calculation of two-dimensional integrals; see [62, 65] for references. Because the aim of this paper is the regularization of divergent integrals with emphasis on BEM applications, we will not consider numerical calculation of two-dimensional integrals in detail. Instead, we will focus on generalizing the Gaussian quadrature interpolation formulas based on Legendre polynomials to the case of rectangular and triangular domains.

For rectangular domains, the generalization is straightforward. An arbitrary domain $[a_1, b_1] \times [a_2, b_2]$ is transformed to the domain $[-1, 1] \times [-1, 1]$, and the integral (5.1) is approximately represented using the two-dimensional the Gaussian quadrature interpolation formula as

$$I_0 = \int\limits_{a_2}^{b_2} \int\limits_{a_1}^{b_1} f(x_1, x_2) dx_1 dx_2 \approx \sum_{j}^{N_2} \sum_{i=1}^{N_1} w_{1i} w_{2i} f(\xi_{1i}, \xi_{2j}), \tag{5.8}$$

where the points $\xi_{1i}, \xi_{2i} \in [-1, 1]$ and the coefficients $w_{1i}, w_{2i} \in R^1$ are referred to as the *nodes* and *weights* of the quadrature, respectively.

The nodes $\xi_{\alpha 1}, \xi_{\alpha 1}, ..., \xi_{\alpha N_\alpha}$ are the roots of the Nth Legendre polynomial $P_{N_\alpha}(\xi)$, and the weights $w_{\alpha 1}, w_{\alpha 1}, ..., w_{N_\alpha}$ are calculated by formula (4.7) using *Mathematica* or some other mathematically oriented software.

In the case of a triangular domain, following [49], we use the transformation formulas

$$\begin{aligned} x_1 &= x_1^1 + \tfrac{1}{2}\left(x_2^1 - x_1^1\right)(1+\xi_1) + \tfrac{1}{4}\left(x_1^3 - x_1^1\right)(1-\xi_1)(1+\xi_2), \\ x_2 &= x_2^1 + \tfrac{1}{2}\left(x_2^1 - x_2^1\right)(1+\xi_1) + \tfrac{1}{4}\left(x_2^3 - x_2^1\right)(1-\xi_1)(1+\xi_2), \end{aligned} \tag{5.9}$$

which transform an arbitrary triangle with vertices at $\left(x_1^k, x_2^k\right), \quad k = 1, 2, 3$, to the rectangular domain $[-1, 1] \times [-1, 1]$. Then the integral over the triangular domain can be approximately presented using the two-dimensional Gaussian quadrature interpolation formula (5.8) as

$$I_0 = \int_\Omega f(x_1(\xi_1, \xi_2), x_2(\xi_1, \xi_2))d\Omega \approx Area \sum_{j}^{N_2} \sum_{i=1}^{N_1} w_{1i} w_{2i} f(x_1(\xi_{1i}, \xi_{2j}), x_2(\xi_{1i}, \xi_{2j})). \tag{5.10}$$

Here $Area$ is the area of the triangle, ξ_{1i}, ξ_{2i}, and w_{1i}, w_{2i} are the same as in the case of a rectangular domain.

Formulas (5.8) and (5.10) have been tested using *Mathematica*'s built-in function **NIntegrate**[ ] to calculate some two-dimensional integrals over rectangular and triangular domins analytically. The tested formulas give very good results for regular integrals and therefore will be applied to the calculation of regular integrals in regularized formulas. We note that for verification and error estimation, we shall on occasion compare the obtained results with results given by **NIntegrate**[]

5.2 The classical approach to the regularization of two-dimensional divergent integrals

As in the one-dimensional case, there in the two-dimensional case, many more or less different definitions of the divergent integral; see the above-mentioned references. We will extend the approach developed in the previous section using the idea introduced by Hadamard [44] of the finite part of an integral to two-dimensional divergent integrals.

5.2.1 Regularization of weakly singular integrals

The definition of the weakly singular integral is based on consideration of the limit

$$W.S. \int_\Omega \frac{1}{r(\mathbf{x}-\mathbf{y})} d\Omega = \lim_{\varepsilon \to 0} \int_{\Omega \backslash \Omega_\varepsilon} \frac{1}{r(\mathbf{x}-\mathbf{y})} d\Omega. \tag{5.11}$$

Because the limit here is finite, it can be considered the definition of the weakly singular integral. Of course, we can calculate the weakly singular integral in Eq. 5.11 analytically by calculating the limit, but a more elegant solution comes from the generalized-functions approach.

As in the one-dimensional case, a direct application of the Gaussian quadrature formulas for numerical calculation of weakly singular integrals gives incorrect results. Many formulas have been proposed for calculating weakly singular integrals that contain a sufficiently smooth function $f(x)$. We use here the following one:

$$W.S.\int_{\Omega}\frac{f(\mathbf{x})}{r(\mathbf{x}-\mathbf{y})}d\Omega=\int_{\Omega}\frac{f(\mathbf{x})-f(\mathbf{y})}{r(\mathbf{x}-\mathbf{y})}d\Omega+f(\mathbf{y})W.S.\int_{\Omega}\frac{1}{r(\mathbf{x}-\mathbf{y})}d\Omega. \quad (5.12)$$

In Section 5.4, we will present further results on calculating weakly singular integrals using the regularized formula (5.8) and compare that approach with other approaches.

5.2.2 Regularization of singular integrals

For the definition of the singular integral, we consider the limit

$$F.P.\int_{\Omega}\frac{1}{r(\mathbf{x}-\mathbf{y})^2}d\Omega=\lim_{\varepsilon\to 0}\left(\int_{\Omega\setminus\Omega(r<\varepsilon)}\frac{1}{r(\mathbf{x}-\mathbf{y})^2}d\Omega-\frac{\Omega(r<\varepsilon)}{\varepsilon}\right). \quad (5.13)$$

This limit can be considered the definition of the singular integral. Here we have the same situation: we can calculate the singular integral in Eq. 5.9 analytically by calculating the limit, but a more elegant solution comes from the generalized-functions approach.

For calculating singular integrals that contain a sufficiently smooth function $f(\mathbf{x})$, we use here a formula similar to Eq. 5.8. For singular integrals, it has the form

$$P.V.\int_{\Omega}\frac{f(\mathbf{x})}{r(\mathbf{x}-\mathbf{y})^2}d\Omega=\int_{\Omega}\frac{f(\mathbf{x})-f(\mathbf{y})}{r(\mathbf{x}-\mathbf{y})^2}d\Omega+f(\mathbf{y})W.S.\int_{\Omega}\frac{1}{r(\mathbf{x}-\mathbf{y})^2}d\Omega. \quad (5.14)$$

In Section 4.5, we will present further results on calculating weakly singular integrals using the regularized formula (5.14) and compare that approach with other approaches.

5.2.3 Regularization of hypersingular integrals

For the definition of the hypersingular integral, we consider the limit

$$F.P.\int_{\Omega}\frac{1}{r(\mathbf{x}-\mathbf{y})^3}d\Omega=\lim_{\varepsilon\to 0}\left(\int_{\Omega\setminus\Omega(r<\varepsilon)}\frac{1}{r(\mathbf{x}-\mathbf{y})^3}d\Omega-\frac{\Omega(r<\varepsilon)}{\varepsilon^2}\right). \quad (5.15)$$

This limit can be considered the definition of the singular integral. We will calculate analytically the hypersingular integral (5.13) using the generalized-functions approach.

For calculating hypersingular integrals that contain a sufficiently smooth function $f(\mathbf{x})$, we use here a formula similar to Eqs. 4.15 and 4.19. For hypersingular integrals, it has the form

$$P.V.\int\limits_{\Omega}\frac{f(\mathbf{x})}{r(\mathbf{x}-\mathbf{y})^2}d\Omega=\int\limits_{\Omega}\frac{f(\mathbf{x})-f(\mathbf{y})-\frac{\partial f(\mathbf{y})}{\partial y_\alpha}(x_\alpha-y_\alpha)}{r(\mathbf{x}-\mathbf{y})^2}d\Omega+f(\mathbf{y})F.P.\int\limits_{\Omega}\frac{1}{r(\mathbf{x}-\mathbf{y})^3}d\Omega$$
$$+\frac{\partial f(\mathbf{y})}{\partial y_\alpha}P.V.\int\limits_{\Omega}\frac{x_\alpha-y_\alpha}{r(\mathbf{x}-\mathbf{y})^3}d\Omega. \quad (5.16)$$

It should be noted that in [37, 38] and many other publications, similar formulas are presented in which polar coordinates are used for the regularization of singular and hypersingular integrals.

In Section 5.4, we will present further results of calculating weakly singular integrals using the regularized formula (5.16) and compare that approach with other approaches.

5.2.4 Generalization to divergent integrals with higher-order singularities

In the same way as in the one-dimensional case, for calculating divergent integrals with higher-order singularities that contain a sufficiently smooth function $f(\mathbf{x})$, one can use a generalization of formulas (5.12), (5.14), and (5.16). The corresponding regularization formula can be represented in the following form:

$$F.P.\int\limits_{\Omega}\frac{f(\mathbf{x})}{r(\mathbf{x}-\mathbf{y})^{n+1}}d\Omega=\int\limits_{\Omega}\frac{f(\mathbf{x})-\sum\limits_{k=0}^{r}\frac{\partial^k f(\mathbf{y})}{\partial y_\alpha^k}(x_\alpha-y_\alpha)^k/k!}{r(\mathbf{x}-\mathbf{y})^{n+1}}d\Omega$$
$$+\sum_{k=0}^{r}\frac{\partial^k f(\mathbf{y})}{\partial y_\alpha^k}\frac{1}{k!}P.F.\int\limits_{\Omega}\frac{(x_\alpha-y_\alpha)^k}{r(\mathbf{x}-\mathbf{y})^{n+1-k}}d\Omega. \quad (5.17)$$

Using this formula, one can calculate two-dimensional divergent integrals containing singularities of any order.

5.3 Generalized-functions approach to the regularization of two-dimensional divergent integrals

Let us consider again the integral (5.1) and study how the theory of generalized functions (distributions) allows us to deal with a singular function $f(\mathbf{x})$. But first, we extend some basic definitions of the theory of distributions introduced in Section 4 to the two-dimensional case.

A generalized function $f(\mathbf{x})$ in two dimensions can be defined for every $\varphi(\mathbf{x})\in C^\infty\left(R^2\right)$ as follows. We consider the functional

$$(f,\varphi)=\int\limits_{R^2}f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}, \quad (5.18)$$

which can be considered the definition of a generalized function.

The partial derivative of order r of the generalized function $f(\mathbf{x})$ is defined by the equation

$$D_x^r f(\mathbf{x}) = \left(\frac{\partial^r f(\mathbf{x})}{\partial x_1^{r_1} \partial x_2^{r_2}}, \varphi(\mathbf{x})\right) = (-1)^k \left(\frac{\partial^r \varphi(\mathbf{x})}{\partial x_1^{r_1} \partial x_2^{r_2}}, f(\mathbf{x})\right), \ \forall\varphi(\mathbf{x}) \in C^\infty\left(R^2\right), \quad (5.19)$$

where $r = r_1 + r_2$ is the order of the partial derivative.

For a regular function, the generalized derivative (5.19) is equal to its partial derivative. For regularizing divergent integrals, we do not need to study generalized functions in full generality. The situation become much simpler if we restrict our attention to generalized functions that can be represented in the form

$$f(\mathbf{x}) = \nabla g(\mathbf{x}) \text{ and} f(\mathbf{x}) = \Delta g(\mathbf{x}), \quad (5.20)$$

where $g(\mathbf{x})$is any continuous or merely piecewise continuous function, referred to here as a generating function.

Following [18, 32, 41, 112, 121], we consider definite integrals of generalized function in the two-dimensional case. Let us consider a function $f(\mathbf{x})$ defined in a finite region $\Omega \subset R^2$ such that all its singularities are concentrated in the subregion $\Omega^\varepsilon \subset \Omega$. In the region $\Omega \backslash \Omega^\varepsilon$ including the boundary, the function is regular and possesses all necessary derivatives.

Let as consider the definite integral

$$I_0 = \int_\Omega f(\mathbf{x}) d\mathbf{x} \quad (5.21)$$

over a finite region. In order to consider this integral in the sense of distributions, we introduce a function $g(\mathbf{x})$ such that

$$f(\mathbf{x}) = \Delta^k g(\mathbf{x}), \quad (5.22)$$

where $\Delta^k = \underbrace{\Delta \cdot \Delta \cdot \Delta \cdots \Delta}_{k}$ is called the k-dimensional Laplace operator.

This representation of the function $f(\mathbf{x})$ can be considered in the classical sense in the region Ω^0, but in the region Ω, it has to be considered in the sense of distributions. In the region $\Omega^0 = R^2/\Omega$, the function is regular and smooth up to the boundary, which means that $f(\mathbf{x}) \in C^k\left(\Omega^0\right)$. The boundary $\partial\Omega$ must satisfy the usual conditions of smoothness, which are discussed in every standard course in analysis.

We also introduce the test function $\phi(x) \in C^\infty\left(R^2\right)$ such that $\phi(\mathbf{x}) = 1, \forall \mathbf{x} \in \Omega$. The function $\phi(\mathbf{x})$ is finite and can be extended smoothly to the region Ω^0, in which case, its derivatives are equal to zero in the region Ω, including its boundary $\partial\Omega$, i.e.,

$$\Delta^k \phi(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega. \quad (5.23)$$

Let us consider the scalar product that defines the singular function $f(\mathbf{x})$ in the sense of distributions:

$$(f, \phi) = \int_{R^n} f(\mathbf{x}) \phi(\mathbf{x}) d\mathbf{x}. \quad (5.24)$$

$$\int_{R^n} \phi(\mathbf{x})\Delta^k g(\mathbf{x})d\mathbf{x} = \int_{\Omega\cup\Omega^0} \phi(\mathbf{x})\Delta^k g(\mathbf{x})d\Omega. \quad (5.25)$$

$$\int_{\Omega\cup\Omega^0} \phi(\mathbf{x})\Delta^k g(\mathbf{x})d\mathbf{x} = (-1)^k \int_{\Omega\cup\Omega^0} g(\mathbf{x})\Delta^k \phi(\mathbf{x})d\Omega = (-1)^k \int_{\Omega^0} g(\mathbf{x})\Delta^k \phi(\mathbf{x})dV. \quad (5.26)$$

Integration by parts in reverse order for the last integrals above leads to the result

$$\int_{\Omega^0} \phi(\mathbf{x})\Delta^k g(\mathbf{x})d\Omega = \sum_{i=0}^{k-1}(-1)^{i+1}\int_{\partial\Omega^0}\left[\phi(\mathbf{x})\partial_n\Delta^{k-i-1}g(\mathbf{x}) - g(\mathbf{x})\partial_n\Delta^{k-i-1}\phi(\mathbf{x})\right]dS$$
$$+(-1)^k\int_{\Omega^0} g(\mathbf{x})\Delta^k\phi(\mathbf{x})d\Omega. \quad (5.27)$$

Here, $\partial_n = n_i\partial_i$ is the normal derivative on the surface with respect to $\mathbf{x}$, and $n_i(\mathbf{x})$ is a unit normal to the surface.

Taking into account that

$$\int_{R^n} f(\mathbf{x})\phi(\mathbf{x})d\Omega = \int_{\Omega\cup\Omega^0} f(\mathbf{x})\phi(\mathbf{x})d\Omega - \int_{\Omega^0} f(\mathbf{x})\phi(\mathbf{x})d\Omega \quad (5.28)$$

and considering Eqs. 5.26 and 5.27, we obtain a formula for calculating divergent integrals involving functions of type (5.22) with any singularity. In particular, from these equations, it follows that for such classes of generalized functions, we have the Gauss–Ostrogradsky and Green's integral theorems. For example, using the Gauss–Ostrogradsky theorem, divergent integrals can be represented in the form

$$I_0 = F.P.\int_{\Omega} f(\mathbf{x})d\Omega = F.P.\int_{\Omega}\Delta g(\mathbf{x})d\Omega = \int_{\partial\Omega}\partial_n g(\mathbf{x})dS, \quad (5.29)$$

while with the first Green's theorem, we obtain the form

$$I_0 = F.P.\int_{\Omega}\varphi(\mathbf{x})f(\mathbf{x})d\Omega = \int_{\partial\Omega}\varphi(\mathbf{x})\partial_n g(\mathbf{x})dS - \int_{\Omega}\nabla g(\mathbf{x})\nabla\varphi(\mathbf{x})d\Omega, \quad (5.30)$$

and with the second Green's theorem, they can be represented in the form

$$I_0 = F.P.\int_{\Omega}\varphi(\mathbf{x})f(\mathbf{x})d\Omega = \int_{\partial\Omega}(\varphi(\mathbf{x})\partial_n g(\mathbf{x}) - g(\mathbf{x})\partial_n\varphi(\mathbf{x}))\,dS - \int_{\Omega} g(\mathbf{x})\Delta\varphi(\mathbf{x})d\Omega. \quad (5.31)$$

In order to illustrate the advantages of such an approach, let us consider divergent integrals of the form

$$F.P.\int_{\Omega}\frac{1}{r(\mathbf{x}-\mathbf{y})^k}d\Omega. \quad (5.32)$$

It can be shown that for $k > 0$ and $k \neq 2$, we have $g(\mathbf{x}) = \frac{1}{(k-2)^2 r(\mathbf{x}-\mathbf{y})^{k-2}}$ and

$$F.P. \int_{\Omega} \frac{d\Omega}{r(\mathbf{x}-\mathbf{y})^k} = -\frac{1}{(k-2)} \int_{\partial\Omega} \frac{r_n}{r(\mathbf{x}-\mathbf{y})^k} d\Omega. \tag{5.33}$$

Here $r_n = (x_\alpha - y_\alpha)n_\alpha$ and $\alpha = 1, 2$.

Let us apply these formulas to the definition and regularization of divergent integrals.

5.3.1 Regularization of weakly singular integrals

To define weakly singular integrals of type (5.11), we consider in Eq. 5.33 the functions $f(\mathbf{x}) = \frac{1}{r(\mathbf{x}-\mathbf{y})}$ and $g(\mathbf{x}) = r(\mathbf{x}-\mathbf{y})$. Then by (5.33), the corresponding *weakly singular* integral is

$$W.S. \int_{\Omega} \frac{d\Omega}{r(\mathbf{x}-\mathbf{y})} = \int_{\partial\Omega} \frac{r_n(\mathbf{x},\mathbf{y})}{r(\mathbf{x}-\mathbf{y})} dS. \tag{5.34}$$

In our previous publications [116, 119, 122, 123], we developed effective analytical and numerical methods for calculating integrals of type (5.33) and an even wider class of divergent integrals that occur in BEM applications in the theory of elasticity.

In Fig. 8, we present results of calculating weakly singular integrals directly using the analytical expression (5.34) and using the *Mathematica* function **NIntegrate**[ ] for a square and triangle with side length equal to 1. It should be noted that numerical calculation of these weakly singular integrals using the Gaussian quadrature interpolation formula gives us an incorrect result.

For calculation of weakly singular integrals that contain a sufficiently smooth function $f(\mathbf{x})$, we use the second Green's formula (5.31). Then the regularized representation for the weakly singular integral takes the form

$$W.S. \int_{\Omega} \frac{f(\mathbf{x})}{r(\mathbf{x}-\mathbf{y})} d\Omega = \int_{\partial\Omega} \left(f(\mathbf{x}) \frac{r_n(\mathbf{x},\mathbf{y})}{r(\mathbf{x}-\mathbf{y})} - r(\mathbf{x}-\mathbf{y}) \partial_n f(\mathbf{x}) \right) dS - \int_{\Omega} r(\mathbf{x}) \Delta f(\mathbf{x}) d\Omega. \tag{5.35}$$

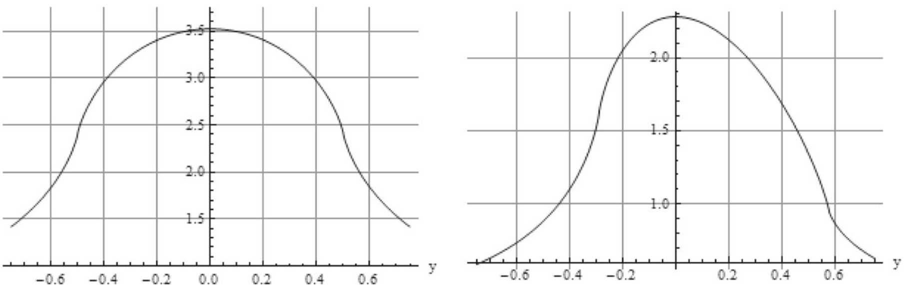

Fig. 8 Values of the weakly singular integral versus the parameter y_1 for the unit rectangle and triangle

In Section 5.4, we will present further results on calculating weakly singular integrals using the regularized formula (5.35) and compare that approach with other approaches. .

5.3.2 Regularization of singular integrals

In defining singular integrals of type (5.13), formula (5.33) cannot be applied. Therefore, we consider in (5.29) functions $f(\mathbf{x}) = \frac{1}{r(\mathbf{x}-\mathbf{y})^2}$ and $g(\mathbf{x}) = \frac{1}{2}\left(\ln \frac{1}{r(\mathbf{x}-\mathbf{y})}\right)^2$. Then from Eq. 5.29, the corresponding singular integral becomes

$$P.V. \int_{\Omega} \frac{d\Omega}{r(\mathbf{x}-\mathbf{y})^2} = \int_{\partial\Omega} \frac{r_n(\mathbf{x},\mathbf{y}) \ln r(\mathbf{x}-\mathbf{y})}{r(\mathbf{x}-\mathbf{y})^2} dS. \tag{5.36}$$

In Fig. 9, we present results of calculating singular integrals directly using the analytical expression (5.36) for a square and triangle with side length equal to 1. It should be noted that numerical calculation of these singular integrals using the Gaussian quadrature interpolation formula and the *Mathematica* function **NIntegrate**[ ] gives us incorrect results.

For calculating singular integrals that contain a sufficiently smooth function $f(\mathbf{x})$, we use the second Green's formula (5.31). The regularized representation for the singular integral then takes the form

$$P.V. \int_{\Omega} \frac{f(\mathbf{x})}{r(\mathbf{x}-\mathbf{y})} d\Omega = \int_{\partial\Omega} \left(f(\mathbf{x}) \frac{r_n(\mathbf{x},\mathbf{y}) \ln r(\mathbf{x}-\mathbf{y})}{r(\mathbf{x}-\mathbf{y})^2} - \frac{1}{2}\left(\ln \frac{1}{r(\mathbf{x}-\mathbf{y})}\right)^2 \partial_n f(\mathbf{x}) \right) dS$$
$$-\frac{1}{2} \int_{\Omega} \left(\ln \frac{1}{r(\mathbf{x}-\mathbf{y})}\right)^2 \Delta f(\mathbf{x}) d\Omega. \tag{5.37}$$

In Section 5.4, we will present further results on calculating weakly singular integrals using the regularized formula (5.37) and compare that approach with other approaches.

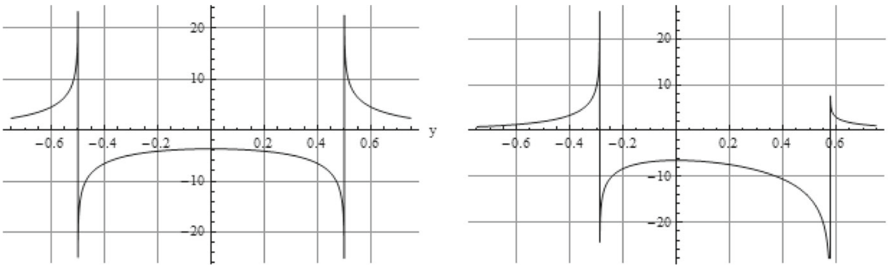

Fig. 9 Values of the singular integral versus the parameter y_1 for the unit rectangle and triangle

5.3.3 Regularization of hypersingular integrals

For defining hypersingular integrals of type (5.15), we consider in (5.33) the functions $f(\mathbf{x}) = \frac{1}{r(\mathbf{x}-\mathbf{y})^3}$ and $g(\mathbf{x}) = \frac{1}{r(\mathbf{x}-\mathbf{y})}$. Then from (5.33), the corresponding FP integral becomes

$$F.P. \int_{\Omega} \frac{d\Omega}{r(\mathbf{x}-\mathbf{y})^3} = -\int_{\partial\Omega} \frac{r_n(\mathbf{x},\mathbf{y})}{r(\mathbf{x}-\mathbf{y})^3} dS. \tag{5.38}$$

In Fig. 10, we present the results of calculating hypersingular integrals directly using the analytical expression (163) for the square and triangle of side length 1. It is obvious that numerical calculation of the hypersingular integrals using the Gaussian quadrature interpolation formula and the *Mathematica* function **NIntegrate**[ ] gives incorrect results.

For calculating hypersingular integrals that contain a sufficiently smooth function $f(\mathbf{x})$, we use the second Green's formula (5.31). After first application of the Green formula, we obtain equation that contain weakly singular integral over domain Ω. In order to avoid presence of that singular integral, we have to apply the Green formula (5.31) again. The regularized representation for the hypersingular integrals then takes the form

$$F.P. \int_{\Omega} \frac{f(\mathbf{x})}{r(\mathbf{x}-\mathbf{y})^3} d\Omega = \int_{\partial\Omega} \left(f(\mathbf{x}) \frac{r_n(\mathbf{x},\mathbf{y})}{r(\mathbf{x}-\mathbf{y})^3} - \frac{1}{r(\mathbf{x}-\mathbf{y})} \partial_n f(\mathbf{x}) - \Delta f(\mathbf{x}) \frac{r_n(\mathbf{x},\mathbf{y})}{r} (\mathbf{x}-\mathbf{y}) \right.$$
$$\left. + r(\mathbf{x}-\mathbf{y}) \partial_n \Delta f(\mathbf{x}) \right) dS + \int_{\Omega} r(\mathbf{x}) \Delta\Delta f(\mathbf{x}) d\Omega. \tag{5.39}$$

In Section 5.4, we will present further results on weakly singular integrals using the regularized formula (5.39) and compare that approach with other approaches.

5.3.4 Generalization to divergent integrals with higher-order singularities

As in the one-dimensional case, for calculating divergent integrals with higher-order singularities that contain a sufficiently smooth enough function $f(\mathbf{x})$, one can use a generalization of the formulas (5.35), (5.37), and (5.39). This case was analyzed

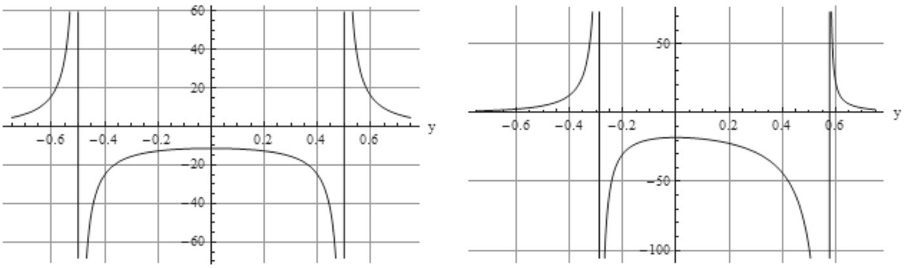

Fig. 10 Values of hypersingular integrals versus the parameter y_1 for the unit rectangle and triangle

using the generalized-functions approach in [22]. The corresponding regularization formula can be represented in the following form:

$$I_0 = F.P.\int\limits_V \frac{\varphi(\mathbf{x})}{r(\mathbf{x}-\mathbf{y})^m}dV = \sum_{i=0}^{k-1}(-1)^{i+1}\int\limits_{\partial V}\left[\Delta^{k-i-1}\varphi(\mathbf{x})\partial_n\frac{P_i}{r(\mathbf{x}-\mathbf{y})^{m-2i}}\right.$$
$$\left.-\frac{P_i}{r(\mathbf{x}-\mathbf{y})^{m-2i}}\partial_n\Delta^{k-i-1}\varphi(\mathbf{x})\right]dS+(-1)^k\int\limits_V\frac{1}{r(\mathbf{x}-\mathbf{y})^{m-2k}}\Delta^k\varphi(\mathbf{x})]dV, \quad (5.40)$$

where $P_k = (-1)^k \prod_{i=0}^{k-1} \frac{1}{(m+2i)^2}$ for $k, m > 1$.

Using this formula, we can calculate two-dimensional divergent integrals containing singularities of any order.

5.4 Calculation of divergent integrals using the classical and generalized-functions approaches

In previous sections, we have presented regularized formulas for calculating the main types of divergent integrals—weakly singular, singular, and hypersingular—that result from the classical and generalized-functions approaches. We created computer codes in*Mathematica* to verify the formulas that we have obtained, calculate divergent integrals for different types of regular functions $f(\mathbf{x})$, and compare the results of the various approaches. The ideal way of verifying a numerical calculation is to compare it with the analytical solution. Relatively easy two-dimensional divergent integrals can be calculated analytically for a circular domain. Therefore, we consider weakly singular, singular, and hypersingular integrals over a circular domain for some relatively simple functions $f(\mathbf{x})$, which after transformation into polar coordinates can be represented in the form

$$\begin{aligned}
&\int\limits_\Omega \frac{\left(x_1^2+x_2^2\right)^2}{\left(x_1^1+x_2^2\right)^\alpha}dS = \pi\int\limits_{-R}^{R}\rho^k d\rho, \alpha = 1/2, 1, 3/2,\ k = 4, 3, 2,\\
&\int\limits_\Omega \frac{\sqrt{1+x_1^2+x_2^2}}{\left(x_1^1+x_2^2\right)^\alpha}dS = \pi\int\limits_{-R}^{R}\frac{\sqrt{1+\rho^2}}{\rho^k}d\rho, \alpha = 1/2, 1, 3/2, k = 0, 1, 2,\\
&\int\limits_\Omega \frac{1}{\left(x_1^1+x_2^2\right)^\alpha\sqrt{1+x_1^2+x_2^2}}dS = \pi\int\limits_{-R}^{R}\frac{1}{\rho^k\sqrt{1+\rho^2}}d\rho, \alpha = 1/2, 1, 3/2, k = 0, 1, 2, \quad (5.41)\\
&\int\limits_\Omega \frac{\cos\left(x_1^1+x_2^2\right)}{\left(x_1^1+x_2^2\right)^\alpha}dS = \pi\int\limits_{-R}^{R}\frac{\cos(\rho^2)}{\rho^k}d\rho, \alpha = 1/2, 1, 3/2, k = 0, 1, 2,\\
&\int\limits_\Omega \frac{\sin\left(x_1^1+x_2^2\right)}{\left(x_1^1+x_2^2\right)^\alpha}dS = \pi\int\limits_{-R}^{R}\frac{\sin(\rho^2)}{\rho^k}d\rho, \alpha = 1/2, 1, 3/2, k = 0, 1, 2,
\end{aligned}$$

where $\rho^2 = x_1^2 + x_2^2$.

The one-dimensional integrals in Eq. 5.41 can be calculated analytically using both the classical and generalized-functions approaches. For calculating these integrals using regularized formulas, we need to calculate the regular integrals over the circular domain numerically. We did those calculations in two ways: using the

Table 2 Calculations of divergent integrals over a circular domain for radius $R = 1$

		r^4	$(1+r^2)^{1/2}$	$(1+r^2)^{-1/2}$	$\cos(r^2)$	$\sin(r^2)$
An	WS	1.25664	7.21180	5.53783	5.68329	1.94947
	PV	1.57080	–	–	–	2.97221
	FP	2.09440	–3.34793	–8.88577	–7.2937	6.07947
CI	WS	1.29531	7.10922	5.65446	5.66707	1.76015
	PV	–	–	–	–	–
	FP	–	–2.84508	–10.5358	–5.8025	5.922195
CII	WS	1.25664	7.21180	5.53783	5.68329	1.94947
	PV	1.57080	1.41992	–1.18266	–0.75339	2.97221
	FP	2.09440	–3.34793	–8.88577	–7.2837	6.07947
GI	WS	1.25905	7.18741	5.56302	5.68212	1.90149
	PV	1.57080	1.24540	–1.00828	–0.75338	2.62314
	FP	2.04632	–4.488911	–7.75739	–7.2697	3.78433
GII	WS	1.25664	7.21180	5.53783	5.68329	1.94947
	PV	1.57080	1.41992	–1.18266	–0.75339	2.97221
	FP	2.09440	–3.34793	–8.88577	–7.2837	6.07947

quadrature formulas presented in [1] p. 892, where coordinates and weightts are calculated in the form

$$\begin{aligned}
&\xi_0 = 0, \quad \zeta_0 = 0, \quad w_0 = \tfrac{1}{9},\\
&\xi_k = \sqrt{\tfrac{6-\sqrt{6}}{10}} \cos\tfrac{2\pi k}{10}, \quad \zeta_k = \sqrt{\tfrac{6-\sqrt{6}}{10}} \cos\tfrac{2\pi k}{10}, \quad w_k = \tfrac{16+\sqrt{6}}{360},\\
&\xi_{10+k} = \sqrt{\tfrac{6+\sqrt{6}}{10}} \cos\tfrac{2\pi k}{10}, \quad \zeta_k = \sqrt{\tfrac{6+\sqrt{6}}{10}} \cos\tfrac{2\pi k}{10}, \quad w_k = \tfrac{16-\sqrt{6}}{360},
\end{aligned} \tag{5.42}$$

for $k = 1, \ldots, 10$ and using the *Mathematica* function **NIntegrate**[ ].

Table 3 Calculations of divergent integrals over a rectangular domain

		$x_1^4 + x_2^4$	$(1+x_1^4+x_2^4)^{1/2}$	$(1+x_1^4+x_2^4)^{-1/2}$	$\cos(r^2)$	$\sin(r^2)$
I	WS	0.051733	3.55109	3.50042	3.48654	0.37970
	PV	0.112737	–3.60801	–3.71871	–3.7467	0.89354
	FP	0.263781	–11.1828	–11.4426	–11.504	3.31186
II	WS	0.051699	3.55108	3.50044	3.38658	0.38103
	PV	0.114685	–3.60704	–3.71968	–3.7468	0.84107
	FP	0.287666	–11.1709	–11.4544	–11.518	3.41293

Table 4 Calculations of divergent integrals over a triangular domain

		$x_1^4 + x_2^4$	$(1 + x_1^4 + x_2^4)^{1/2}$	$(1 + x_1^4 + x_2^4)^{-1/2}$	$\cos(r^2)$	$\sin(r^2)$
I	WS	0.009479	2.28575	2.27638	2.27473	0.114859
	PV	0.026979	–6.48279	–6.50953	–6.51422	0.332212
	FP	0.095214	–17.9575	–18.0422	–18.0575	2.06240
II	WS	0.009473	2.28575	2.27638	2.27474	0.115366
	PV	0.275935	–6.48248	–6.50983	–6.51458	0.34442
	FP	0.094373	–17.9530	–18.0467	–18.0628	2.27902

We shall compare the results obtained using the analytical formulas (5.41) and the numerical formulas that correspond to the classical approach (5.12), (5.14), and (5.16) and to the generalized-functions approach (5.35), (5.37), and (5.39). Calculations at the point $y_1 = 0.0, \quad y_2 = 0.0$ for radius $R = 1$ using the analytical and regularized formulas are presented in Table 2. Here ***An*** corresponds to the analytical solutions obtained using formulas (5.41);**CI** and **CII** correspond to solutions obtained using the classical regularized formulas (5.12), (5.14), and (5.16); **GI** and **GII** correspond to solutions obtained using the regularized formulas based on the generalized-functions approach. The numbers **I** and **II** refer to calculations of the regular integrals by the quadrature formulas presented in [1] and using the *Mathematica* function **NItegrate**[ ], respectively. The symbol – corresponds to the case in which a solution does not exist or the calculations give an incorrect result.

Our numerical experiments show that for considered functions all regularized formulas based on classical and generalized function approaches give good correlated results when for regular integrals calculation *Mathematica* build-in function **NIntegrate**[ ] used, quadrature presented in [1] sometime give not accurate or even wrong results.

For BEM analysis, calculation of divergent integrals over a circular domain is not of great importance. Therefore, in Tables 3 and 4, we present calculations of divergent integrals over rectangular and triangular domains respectively. The calculations were performed for a square and for an isosceles triangle with side length equal to 1 for a point situated at the center of the square and triangle with coordinates $y_1 = 0.0, \ y_2 = 0.0$.

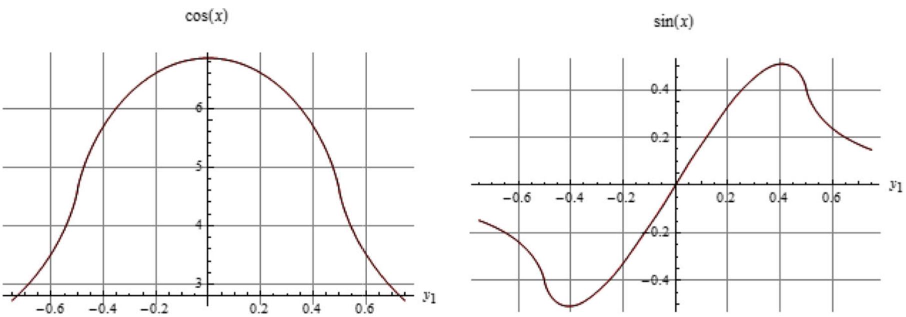

Fig. 11 Values of weakly singular integrals versus the parameter y_1 for a rectangular domain

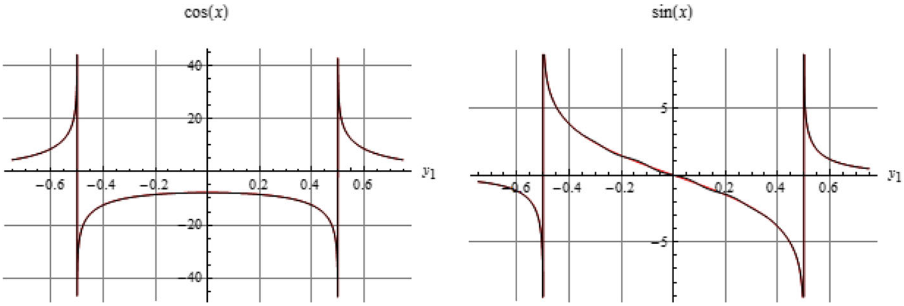

Fig. 12 Values of singular integrals versus the parameter y_1 for a rectangular domain

We have to mention that regularized formula (5.16) for hypersingular integrals obtained using classical approach contains first order partial derivative and regularized formulas (5.35), (5.37) and (5.39) obtained using generalized function approach contain partial derivatives up to fourth order. Of course presence of the derivatives is shortcoming, but from other hand formulas based on generalized function approach are more stable with respect to changing coordinates of collocation point y_1, y_2, in comparison with classical approach, which for some values of y_1, y_2 gives not accurate result and to improve the accuracy it is necessary to increase number of nodes in quadrature formula.

Timing of the computations showed that in all cases, the formulas based on the classical approach are a little faster than those based on the generalized-function sapproach. But the calculation time depends on a number of factors. For example, here we calculate all the derivatives in the regularized formulas directly in *Mathematica* using **Derivative**[ ], but if the derivative is calculated in advance, the calculation time is reduced significantly, and accuracy increases. From the above, it can be concluded that all the regularization formulas presented here are valid in that they give correct results and can be used for calculating various types of divergent integrals. In each case, one may choose the most suitable approach.

In order to show dependence of the divergent integrals on the position of collocation point (y_1, y_2) and once again to compare results of calculations using classical and generalized function approaches we generated with *Mathematica* build-in function **Plot**[] plots of the divergent integrals for functions $f(r) = \cos(r)$and $f(r) =$

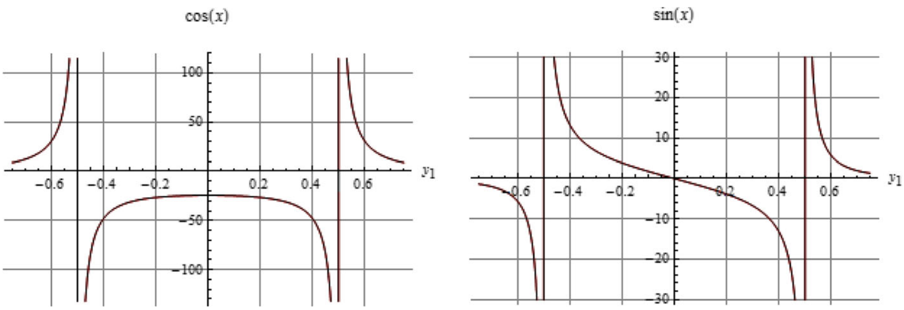

Fig. 13 Values of hypersingular integrals versus the parameter y_1 for a rectangular domain

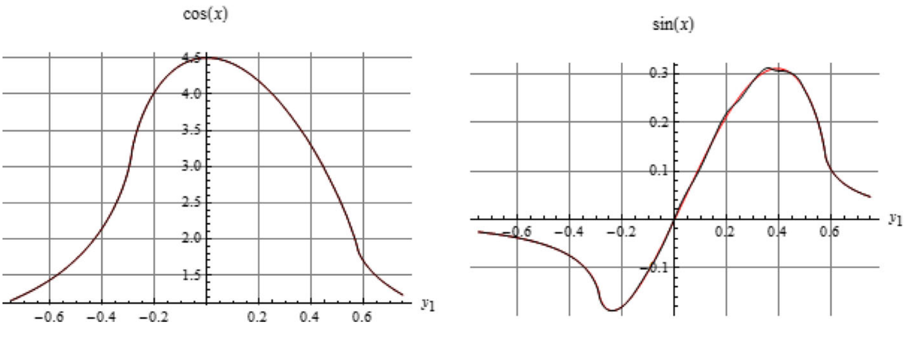

Fig. 14 Values of weakly singular integrals versus the parameter y_1 for a triangular domain

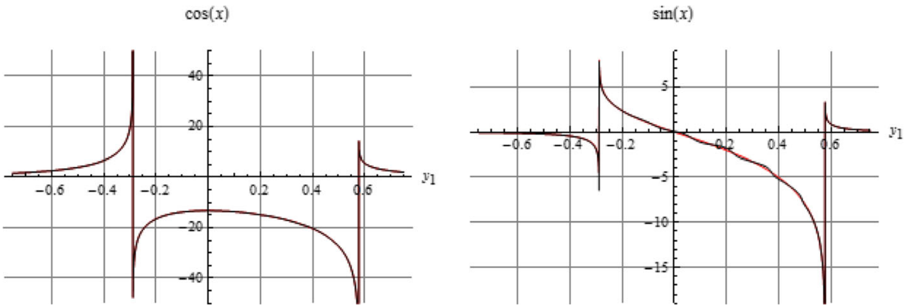

Fig. 15 Values of singular integrals versus the parameter y_1 for a triangular domain

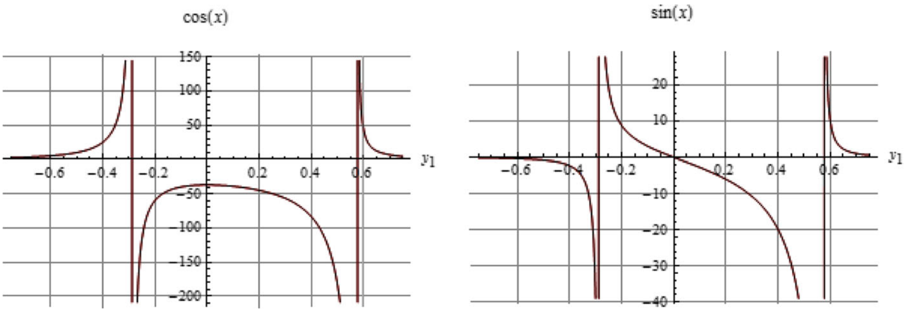

Fig. 16 Values of hypersingular integrals versus the parameter y_1 for a triangular domain

$\sin(r)$ calculated using classical, generalized function regularization approaches for unit square and unit equilateral triangle. The results calculations and plotting are presented on Figs. 11,12 and 13 for square and on Figs. 14, 15 and 16 for triangle. All graphs coincide within the investigated interval. At the points $y_1 = \pm\ 0.5$ for a rectangular domain and $y_1 = -\frac{1}{2\sqrt{3}},\quad y_1 = \frac{1}{\sqrt{3}}$ for a triangular domain, the singular and hypersingular integrals are undefined. But they can be easily calculated using the classical or generalized-functions approach; see [119] for details on the generalized-functions approach.

6 Conclusion

One of the main purposes of this article has been to show the scientific and engineering community that an approach to the regularization of divergent integrals based on generalized functions is not mere speculation but that is has a strong theoretical basis and can be used as an efficient computational tool. We have shown that using the theory of generalized functions, the main theorems of integral calculus can be easily and naturally extended to functions with singularities. For example, the Newton–Leibniz and integration by parts formulas in the one-dimensional case, and the Gauss–Ostrogradsky and Green's theorems in the two-dimensional case, can be applied correctly to singular functions only by making use of the theory of generalized functions. We have shown that methods for regularizing divergent integrals based on the theory of distributions not only possess a good mathematical foundation, but also are a very efficient tool for calculating such integrals. The distributions-based approach considers divergent integrals with various singularities as functionals defined in special functional spaces. In the one-dimensional case, using the Newton-Leibniz and integration by parts formulas, and in the two-dimensional case using the Gauss–Ostrogradsky and Green's theorems, we obtained regular formulas for calculating divergent integrals with any order of singularity over flat and curved domains. Using the formulas in the same way, one can calculate weakly singular, singular, and hypersingular integrals and also integrals with even higher-order singularities. In relatively simple cases, the regularized formulas contain only regular integrals over contours of the domain of integration, while in more complicated cases, regular integrals over the domain may also be present. The regularization formulas that we have obtained can be used also for calculating regular integrals. Such integrals appear when the collocation point moves to a BE that does not belong to the domain of integration.

We have also considered here traditional classical methods of regularizing divergent integrals and compared both approaches. To verify the equations that we have obtained and investigate their behavior for various collocation points, we used the computer algebra system *Mathematica*. We compared calculations obtained by both approaches with analytical calculations and with numerical calculations of regular integrals using Gaussian quadrature. In all cases, we observed good agreement between the classical and generalized-functions approaches. Our study shows that in some cases, the formulas based on the classical approach are somewhat faster than those based on the generalized-functions approach. However, the formulas based on the latter approach exhibit greater stability with respect to a change in the number of integration points and in the coordinates of the collocation point than do those based on the former. It must be taken into account that in the classical approach, we calculated the divergent parts of the regularized integrals using the generalized-functions approach.

From our research, it follows that both approaches may be applied to the regularization of the divergent integrals such as those that appear in BIE and BEM formulations of general elliptic boundary value problems. With both approaches, one can regularize and calculate one- and two-dimensional weakly singular, singular, and hypersingular divergent integrals with higher-order singularities over flat and curved boundary elements.

Acknowledgments This work was supported by the Comity of Science and Technology of Mexico (CONACYT), Project no. 000000000101415, and the Brazilian state foundation FAPERJ, Project E-26/110.011/2013, which are gratefully acknowledged.

The author is grateful to Professor Ney Dumont, of the Pontificia Universidade Católica do Rio de Janeiro, Brazil, for fruitful discussion and helpful advises.

References

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Volume 55 of Applied Mathematics Series, Department of Commerce, National Bureau of Standards (1972)
2. Aliabadi, M.H.: The boundary element method. Vol. 2. Application in Solids and Structures. John Wiley (2002)
3. Aliabadi, M.H., Hall, W.S.: The regularizing transformation integration method for BE. Comparison with Gauss quadrature. Engineering Analysis with Boundary Elements **6**, 66–71 (1986)
4. Ashour, S.A., Ahmed, H.M.: A Gauss quadrature rule for hypersingular integrals. Appl. Math. Comput. **186**, 1671–1682 (2007)
5. Boehme, T.K.: Operational calculus and the finite part of divergent integrals. Trans. Am. Math. Soc. **106**, 346–368 (1963)
6. Bonnet, M., Guiggiani, M.: Galerkin BEM with direct evaluation of hypersingular integrals. Electron. J. Bound. Elem. **1**(2), 95–111 (2003)
7. Bureau, F.J.: Divergent integrals and partial differential equations. Commun. Pur. Appl. Math. **8**(1), 143–202 (1955)
8. Caianiello, E.R.: Generalized integration procedure for divergent integrals. Il Nuovo cimento **15A**(2), 145–161 (1973)
9. Carley, M.: Numerical quadratures for singular and hypersingular integrals in boundary element methods. SIAM J. Sci. Comput. **29**(3), 1207–1216 (2007)
10. Carley, M.: Numerical quadratures for near-singular and near-hypersingular integrals in boundary element methods. Mathematical Proceedings of the Royal Irish Academy **109A**(1), 49(60) (2009)
11. Chan, Y.-S., Fannjiang, A.C., Paulino, G.H.: Integral equations with hypersingular kernels—-theory and applications to fracture mechanics. Int. J. Eng. Sci. **41**, 683–720 (2003)
12. Chen, G., Zhou, J.: Boundary Element Method, p. 646. Academic Press, London (1992)
13. Chen, J.T., Hong, H.-K.: Review of dual boundary element methods with emphasis on hypersingular integrals and divergent series. Appl. Mech. Rev. **52**(1), 17–33 (1999)
14. Chen, M.-C.: Application of finite-part integrals to three-dimensional fracture problems for piezoelectric media. Part II: Hypersingular integral equation and theoretical analysis. Int. J. Fract. **121**, 133–148 (2003)
15. Chen, M.-C.: Application of finite-part integrals to three-dimensional fracture problems for piezoelectric media. Part II: Numerical analysis. Int. J. Fract. **121**, 149–161 (2003)
16. Chen, Y.Z., Lin, X.Y., Peng, Z.Q.: Some particular solutions for penny-shaped crack problem by using hypersingular integral equation or differential-integral equation. Arch. Appl. Mech. **68**, 271–280 (1998)
17. Cheng, A.H.-D., Cheng, D.T.: Heritage and early history of the boundary element method. Engineering Analysis with Boundary Elements **29**, 268–302 (2005)
18. Courant, R., Hilbert, D.: Methods of Mathematical Physics, Vol. II. John Wiley, New York (1968)

19. Davey, K., Hinduja, S.: Analytical integration of linear three-dimensional triangular elements in BEM. Applied Mathematical Modeling **13**, 450–461 (1989)
20. Davis, P.J., Rabinowitz, P. Methods of Numerical Integration, 2nd edn. Academic Press, New York (1984)
21. de Klerk, J.H.: Hypersingular integral equations—past, present, future. Nonlinear Anal. **63**, 533–540 (2005)
22. Diligenti, M., Monegato, G.: Integral evaluation in the BEM solution of (hyper)singular integral equations. 2D problems on polygonal domains. J. Comput. Appl. Math. **81**, 29–57 (1997)
23. Dumont, N.A.: On the efficient numerical evaluation of integrals with complex singularity poles. Engineering Analysis with Boundary Elements **13**, 155–168 (1994)
24. Dumont, N.A.: Cauchy principal values, finite-part integrals and interval normalization: some basic considerations. In: Atluri, S.N., Yagawa, G., Cruse, T.A. (eds.): Computational Mechanics 95, pp. 2830–2835. Springer-Verlag, Berlin (1995)
25. Dumont, N.A., Noronha, M.A.M.: A simple, accurate scheme for the numerical evaluation of integrals with complex singularity poles. Comput. Mech. **22**, 42–49 (1998)
26. Erdogan, F.E., Gupta, G.D., Cook, T.S.: The Numerical Solutions of Singular Integral Equations. Methods of Analysis and Solutions of Crack Problems, pp. 368–425. Noordhoff Intern. Publ., Leyden (1973)
27. Fata, S.N.: Explicit expressions for 3D boundary integrals in potential theory. Int. J. Numer. Methods Eng. **78**, 32–47 (2009)
28. Frangi, A., Guiggiani, M.: Boundary element analysis of Kirchhoff plates with direct evaluation of hypersingular integrals. Int. J. Numer. Methods Eng. **46**(11), 1845–1863 (1999)
29. Frangi, A., Guiggiani, M.: A direct approach for boundary integral equations with high-order singularities. Int. J. Numer. Methods Eng. **49**, 871–898 (2000)
30. Gao, X.-W.: An effective method for numerical evaluation of general 2D and 3D high order singular boundary integrals. Comput. Methods Appl. Mech. Eng. **199**, 2856–2864 (2010)
31. Gautschi, W.: Orthogonal Polynomials.. Computation and Approximation, p. 312. Oxford University Press, Oxford (2004)
32. Gel'fand, I.M., Shilov, G.E.: Generalized Functions, vol. 1. Academic Press, New York (1964)
33. Gray, L.J., Glaeser, J., Kaplan, T.: Direct evaluation of hypersingular Galerkin surface integrals. SIAM J. Sci. Comput. **25**, 1534–1556 (2004)
34. Gray, L.J., Salvadori, A., Phan, A.-V., Mantic, V.: Direct Evaluation of Hypersingular Galerkin Surface Integrals. II. Electron. J. Bound. Elem. **4**(3), 105–130 (2006)
35. Guiggiani, M.: Computing principal value integrals in 3D BEM for time-harmonic elastodynamics. A direct approach. Communications in Applied Numerical Methods **8**, 141–149 (1992)
36. Guiggiani, M.: Formulation and numerical treatment of boundary integral equations with hypersingular kernels. In: Sladek, V., Sladek, J. (eds.): Singular Integrals in Boundary Element Methods. WIT Press, Southampton (1998)
37. Guiggiani, M., Gigante, A.: A general algorithm for multidimensional Cauchy principal value integrals in the boundary element method. ASME J. Appl. Mech. **57**, 906–915 (1990)
38. Guiggiani, M., Krishnasamy, G., Rudolphi, T.J., Rizzo, F.J.: A general algorithm for the numerical solution of hypersingular boundary integral equations. ASME J. Appl. Mech. **59**, 604–14 (1992)
39. Gunter, N.M.: Potential Theory And Its Application to Basic Problems of Mathematical Physics. Fredric Ungar, New York (1967)
40. Gurtin, M.E.: The linear theory of elasticity, In: Encyclopedia of Physics. In: Fluegge, S. (ed.): Vol. IVa-2. Springer, New York–Berlin (1972)
41. Guz, A.N., Zozulya, V.V.: Brittle Fracture of Constructive Materials Under Dynamic Loading. Kiev, Naukova Dumka (1993). (in Russian)
42. Guz, A.N., Zozulya, V.V.: Fracture dynamics with allowance for a crack edges contact interaction. International Journal of Nonlinear Sciences and Numerical Simulation **2**(3), 173–233 (2001)
43. Guz, A.N., Zozulya, V.V.: Elastodynamic unilateral contact problems with friction for bodies with cracks. Int. Appl. Mech. **38**(8), 3–45 (2002)
44. Hadamard, J.: Lectures on Cauchy's Problem in Linear Hyperbolic Differential Equations. Dover, New York (1953)
45. Hardy, G.H.: Divergent Series. Oxford At The Clarendon Press (1949)
46. Hörmander, L.: On the theory of general partial differential operators. Acta Mathematica **94**, 161–248 (1955)

47. Hong, H.-K., Chen, J.T.: Derivations of Integral Equations of Elasticity. ASCE J. Eng. Mech. **114**(6), 1028–1044 (1988)
48. Hsiao, G.C., Wendland, W.L.: Boundary Integral Equations. Springer-Verlag, Berlin Heidelberg (2008)
49. Hussain, F., Karim, M.S., Ahamad, R.: Appropriate Gaussian quadrature formulae for triangles. International Journal of Applied Mathematics and Computation **4**(1), 24–38 (2012)
50. Ioakimidis, N.I.: Application of finite-part integrals to the singular integral equations of crack problems in plane and three-dimensional elasticity. Acta Mech. **45**, 31–47 (1982)
51. Iovane, G.: Hypersingular integral equations, Kähler manifolds and Thurston mirroring effect in $\epsilon^{(\infty)}$ Cantorian spacetime. Chaos, Solitons and Fractals **31**, 1041–1053 (2007)
52. Iovane, G., Lifanov, I.K., Sumbatyan, M.A.: On direct numerical treatment of hypersingular integral equations arising in mechanics and acoustics. Acta Mech. **162**, 99–110 (2003)
53. Johnston, B.M., Johnston, P.R., Elliott, D.: A new method for the numerical evaluation of nearly singular integrals on triangular elements in the 3D boundary element method. J. Comput. Appl. Math. **245**, 148–161 (2013)
54. Kantor, B.J., Strel'nikova, E.A.: Hypersingular integral equations in the problems of solid mechanics. Kharkiv, Novoe slovo (2005). (in Russian)
55. Kantorovich, L.V.: On approximate calculation of some type of definite integrals and other applications of the method of singularities extraction. Matematicheskii Sbornik **41**(2), 235–245 (1934). (in Russian)
56. Karami, G., Derakhshan, D.: An efficient method to evaluate hypersingular and supersingular integrals in boundary integral equations analysis. Engineering Analysis with Boundary Elements **23**, 317–326 (1999)
57. Karasalo, I.: On evaluation of hypersingular integrals over smooth surfaces. Comput. Mech. **40**, 617–625 (2007)
58. Kaya, A.C., Erdogan, F.: On the solution of integral equations with strongly singular kernels. Q. Appl. Math. **45**, 105–122 (1987)
59. Kieser, R., Schwab, C., Wendland, W.L.: Numerical Evaluation of Singular and Finite-Part Integrals on Curved Surfaces Using Symbolic Manipulation. Computing **49**, 279–301 (1992)
60. Kim, P., Choi, U.J.: Two trigonometric quadrature formulae for evaluating hypersingular integrals. Int. J. Numer. Methods Eng. **56**, 469–486 (2003)
61. Kolm, P., Rokhlin, V.: Numerical quadratures for singular and hypersingular integrals. Computers and Mathematics with Applications **41**, 327–352 (2001)
62. Krylov, V.I.: Approximate Calculation of Integrals, p. 386. Dover Publications, Inc. (2005)
63. Kupradze, V.D.: Potential Methods in the Theory of Elasticity, p. 339. Daniel Davey & Co., Inc. (1965)
64. Kupradze, V.D., Gegelia, T.G., Basheleishvili, M.O., Burchuladze, T.V.: Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity, p. 929. North-Holland Publ. Comp., Amsterdam (1979)
65. Kythe, P.K., Schaferkotter, M.R.: Handbook of Computational Methods for Integration, p. 604. Chapman & Hall/CRC Press (2005)
66. Lebedev, L.P., Vorovich, I.I., Gladwell, G.K.: Functional Analysis. Applications in Mechanics and Inverse Problems. Springer, Netherlands (1996)
67. Lighthill, M.: Introduction to Fourier Analysis and Generalized Functions. Cambridge Univ. Press, Cambridge (1959)
68. Lifanov, I.K., Lifanov, P.I.: On some exact solutions of singular integral equations in the class of generalized functions and their numerical computation. Diff. Equat. **40**(12), 1770–1780 (2004)
69. Lifanov, I.K., Poltavskii, L.N., Vainikko, G.M.: Hypersingular Integral Equations and Their Applications. Chapman &amp Hall/CRC, London (2004)
70. Linkov, A.M.: Complex hypersingular integrals and integral equations in plane elasticity. Acta Mech. **105**, 18–205 (1994)
71. Martin, P.A., Rizzo, F.J.: Hypersingular integrals: How smooth must the density be? Int. J. Numer. Methods Eng. **39**, 687–704 (1996)
72. Michlin, S.G.: Singular integral equations. Uspehi Maematiceskih Nauk **3**(25), 29–112 (1948). (in Russian)
73. Michlin, S.G.: Multidimensional Singular Integrals and Integral Equations. Pergamon Press, Oxford (1965)

74. Monegato, G.: Numerical evaluation of hypersingular integrals. J. Comput. Appl. Math. **50**, 9–31 (1994)
75. Monegato, G.: Definitions, properties and applications of finite-part integrals. J. Comput. Appl. Math. **229**, 425–439 (2009)
76. Moore, M.N.J., Gray, L.J., Kaplan, T.: Evaluation of supersingular integrals: Second-order boundary derivatives. Int. J. Numer. Methods Eng. **69**, 1930–1947 (2007)
77. Mousavi, S.E., Xiao, H., Sukumar, N.: Generalized Gaussian quadrature rules on arbitrary polygons. Int. J. Numer. Methods Eng. **82**(1), 99–113 (2010)
78. Mukherjee, S.: CPV and HFP integrals and their applications in the boundary element method. Int. J. Solids Struct. **37**, 6623–6634 (2000)
79. Mukherjee, S.: Finite parts of singular and hypersingular integrals with irregular boundary source points. Engineering Analysis with Boundary Elements **24**, 767–776 (2000)
80. Mukherjeea, S., Mukherjee, Y.X., Wenjing, Y.: Cauchy principal values and finite parts of boundary integrals—revisited. Engineering Analysis with Boundary Elements **29**, 844–849 (2005)
81. Mukherjee, Y.X., Mukherjee, S.: Error analysis and adaptivity in three-dimensional linear elasticity by the usual and hypersingular boundary contour method. Int. J. Solids Struct. **38**, 161–178 (2001)
82. Mukherjee, S., Mukherjee, Y.X.: Boundary Methods: Elements, Contours, and Nodes. CRC/Taylor & Francis, Boca Raton (2005)
83. Muskhelishvili, N.I.: Singular Integral Equations, p. 452. P, Noordhoff Ltd. (1953)
84. Nagarajan, A., Lutz, E.D., Mukherjee, S.: A novel boundary element method for linear elasticity with no numerical integration for 2D and line integrals for 3D problems. J. Appl. Mech. **61**, 264–269 (1994)
85. Ninham, B.W.: Generalised Functions and Divergent Integrals. Numer. Math. **8**, 44–57 (1966)
86. Niu, Z., Wendland, W.L., Wang, X., Zhou, H.A.: semi-analytical algorithm for the evaluation of the nearly singular integrals in three-dimensional boundary element methods. Comput. Methods Appl. Mech. Engrg. **194**, 1057–1074 (2005)
87. Ortner, V.N.: Regularisierte Faltung von Distributionen. Teil l. Zur Berechnung von Fundamentallösungen. Z. Angew. Math. Phys. (ZAMP) **31**, 133–154 (1980)
88. Ortner, V.N.: Regularisierte Faltung von Distributionen. Teil 2. Eine Tabelle von Fundamentallösungen. Z. Angew. Math. Phys. (ZAMP) **31**, 155–173 (1980)
89. Polyzos, D., Tsepoura, K.G., Tsinopoulos, S.V., Beskos, D.E.: A boundary element method for solving 2-D and 3-D static gradient elastic problems. Part I: Integral formulation. Comput. Methods Appl. Mech. Eng. **192**, 2845–2873 (2003)
90. Qin, T.Y., Yu, Y.S., Noda, N.A.: Finite-part integral and boundary element method to solve three-dimensional crack problems in piezoelectric materials. Int. J. Solids Struct. **44**, 4770–4783 (2007)
91. Rathod, H.T., Karim, M.S.: An explicit integration scheme based on recursion for the curved triangular finite elements. Comput. Struct. **80**, 43–76 (2002)
92. Saez, A., Gallego, R., Dominguez, J.: Hypersingular quarter-point boundary elements for crack problems. Int. J. Numer. Methods Eng. **38**, 1681–1701 (1995)
93. Salvadori, A.: Analytical integrations in 2D BEM elasticity. Int. J. Numer. Methods Eng. **53**, 1695–1719 (2002)
94. Salvadori, A.: Analytical integration of hypersingular kernel in 3D BEM problems. Comput. Methods Appl. Mech. Eng. **190**, 3957–3975 (2001)
95. Salvadori, A.: Analytical integrations in 3D BEM for elliptic problems: evaluation and implementation. Int. J. Numer. Methods Eng. **84**, 505–542 (2010)
96. Salvadori, A., Temponi, A.: Analytical integrations for the approximation of 3D hyperbolic scalar boundary integral equations. Engineering Analysis with Boundary Elements **34**, 977–994 (2010)
97. Samko, S.G.: Hypersingular iIntegrals and Their Applications. Taylor & Francis (2002)
98. Sanz, J.A., Solis, M., Dominguez, J.: Hypersingular BEM for piezoelectric solids: formulation and applications for fracture mechanics. CMES: Comput. Model. Eng. Sci. **17**(3), 215–229 (2007)
99. Schwab, C., Wendland, W.L.: Kernel properties and representations of boundary integral operators. Mathematische Nachrichten **156**, 187–218 (1992)
100. Schwartz, L.: Theorie des distributions, Vols. I and II. Hermann, Paris (1957)
101. Sladek, V., Sladek, J. (eds.): Singular Integrals in Boundary Element Methods. WIT Press, Southampton (1998)

102. Sobolev, S.L.: Applications of Functional Analysis in Mathematical Physics, vol. 7 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, USA (1963)
103. Tanaka, M., Sladek, V., Sladek, J.: Regularization techniques applied to boundary element methods. Appl. Mech. Rev. **47**(10), 457–499 (1994)
104. Tomioka, S., Nishiyama, S.: Analytical regularization of hypersingular integral for Helmholtz equation in boundary element method. Engineering Analysis with Boundary Elements **34**, 393–404 (2010)
105. Tsepoura, K.G., Tsinopoulos, S., Polyzos, D., Beskos, D.E.: A boundary element method for solving 2-D and 3-D static gradient elastic problems. Part II: Numerical implementation. Comput. Methods Appl. Mech. Eng. **192**, 2875–2907 (2003)
106. Vainikko, G.: Multidimensional Weakly Singular Integral Equations, p. 169. Springer-Verlag, Berlin Heidelberg (1993)
107. Wendland, W.L., Stephan, E.P.: A hypersingular boundary integral method for two-dimensional screen and crack problems. Arch. Ration. Mech. Anal. **112**, 363–390 (1990)
108. Young, A.: Improved numerical method for the Traction BIE by application of Stokes' theorem. Int. J. Numer. Methods Eng. **40**, 3141–3161 (1997)
109. Zemanian, A.H.: Distribution Theory and Transform Analysis. Dover, New York (1987)
110. Zozulya, V.V.: Integrals of Hadamard type in dynamic problem of the crack theory. Doklady Academii Nauk. Ukrainskoy SSR, Ser. A. Physical Mathematical & Technical Sciences **2**, 19–22 (1991). (in Ukrainian)
111. Zozulya, V.V.: Contact interaction between the edges of a crack and an infinite plane under a harmonic loading. Int. Appl. Mech. **28**(1), 61–65 (1992)
112. Zozulya, V.V.: Regularization of the divergent integrals. I. General consideration. Electron. J. Bound. Elem. **4**(2), 49–57 (2006)
113. Zozulya, V.V.: Regularization of the divergent integrals. II. Application in Fracture Mechanics. Electron. J. Bound. Elem. **4**(2), 58–56 (2006)
114. Zozulya, V.V. In: Skerget, P., Brebbia, C.A. (eds.): Regularization of the hypersingular integrals in 3-D problems of fracture mechanics. In: Boundary Elements and Other Mesh Reduction Methods XXX, pp. 219–228. WIT Press, Southampton (2008)
115. Zozulya, V.V.: Divergent Integrals in Elastostatics. Regularization in 2-D case. Electron. J. Bound. Elem. **7**(2), 50–88 (2009)
116. Zozulya, V.V.: Regularization of hypersingular integrals in 3-D fracture mechanics: triangular BE, and piecewise-constant and piecewise-linear approximations. Engineering Analysis with Boundary Elements **34**(2), 105–113 (2010)
117. Zozulya, V.V. In: Constanda, C., Perez, M.E. (eds.): Regularization of the divergent integrals in boundary integral equations for elastostatics. In: Integral Methods in Science and Engineering. Vol. 1. Analytic Methods, pp. 333–347, Birkhäuser (2010)
118. Zozulya V.V. Regularization of the divergent integrals in boundary integral equations. In: Advances in Boundary Element Techniques. In: Zhang, Ch., Aliabadi, M.H., Schanz, M. (eds.), pp. 561–568. Published by EC, Ltd, UK (2010)
119. Zozulya, V.V.: Divergent integrals in elastostatics: regularization in 3-D case. CMES: Comput. Model. Eng. Sci. **70**(3), 253–349 (2010)
120. Zozulya, V.V.: Divergent integrals in elastostatics: general considerations. ISRN Applied Mathematics. Article ID **726402**, 25 (2011)
121. Zozulya, V.V.: An approach based on generalized functions to regularize divergent integrals. Engineering Analysis with Boundary Elements **40**, 162–180 (2014)
122. Zozulya, V.V., Gonzalez-Chi, P.I.: Weakly singular, singular and hypersingular integrals in elasticity and fracture mechanics. J. Chin. Inst. Eng. **22**(6), 763–775 (1999)
123. Zozulya, V.V., Lukin, A.N.: Solution of three-dimensional problems of fracture mechanics by the method of integral boundary equations. Int. Appl. Mech. **34**(6), 544–551 (1998)
124. Zozulya, V.V., Men'shikov, V.A.: Solution of three dimensional problems of the dynamic theory of elasticity for bodies with cracks using hypersingular integrals. Int. Appl. Mech. **36**(1), 74–81 (2000)