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Acceleration of stabilized finite element discretizations for the Stokes eigenvalue problem

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Abstract The stabilized finite element method based on local projection stabilization is applied to discretize the Stokes eigenvalue problems, then the corresponding stability and convergence properties are given. Furthermore, we use a postprocessing technique to accelerate the convergence rate of the eigenpair approximations. The postprocessing strategy contains solving an additional Stokes source problem in an augmented finite element space which can be constructed either by refining the mesh, or increasing the order of finite element space. Numerical tests are also provided to confirm the theoretical results.

Keywords Stokes eigenvalue problem · Finite element method · Stabilization · Postprocessing

Mathematics Subject Classifications (2010) 65B99 · 65N25 · 65N30

1 Introduction

In this paper we apply the local projection stabilization (LPS) method to discretize the Stokes eigenvalue problems, and then use a postprocessing technique to accelerate the convergence rate of the eigenpair approximations. The LPS method in a

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two-level approach was proposed for the Stokes problem in [4]. This technique has been investigated to stabilize dominating advection for transport equations in [5] and extended to the Oseen equation in [6]. Application to equal-order interpolation discretization can be found in [16] for the Stokes problem, in [23] for Oseen problem, and in [24] for convection-diffusion problem. In this frame of the LPS method, the stabilization term is based on a projection $\pi_h : V_h \to \mathcal{D}_h$ of the finite element space V_h which approximates the solution into a discontinuous space $\mathcal{D}_h$. The standard Galerkin discretization is then stabilized by adding a term which gives L^2 control over the fluctuation $id - \pi_h$ of the gradient of the solution. For more details about LPS methods, please read the book [29].

On the other hand, many effective postprocessing methods have been proposed to improve the convergence rate for the approximations of eigenvalue problems by finite element methods (c.f. [1, 28, 37]). Xu and Zhou [37] introduced a two-grid discretization technique to improve the convergence rate of second order elliptic eigenvalue problems and integral eigenvalue problems (the idea of the two-grid comes from [35, 36] for nonsymmetric or indefinite problems and nonlinear elliptic equations), they also [38] developed local and parallel algorithms based on two-grid discretizations for eigenvalue problems. (Note that the idea of the two-grid discretization technique is related to that of the iterative Galerkin method, which was introduced by Sloan [32] and Lin and Xie [20]. But the two-grid method is based on two finite element spaces with different meshes.) Racheva and Andreev [28] applied a postprocessing method to improve the convergence rate for the numerical solution of $2m$-order self-adjoint eigenvalue problems. Using the ideas of [28], Andreev, Lazarov and Racheva [1] proposed a postprocessing procedure for the mixed finite element solutions of the biharmonic eigenvalue problem to enhance the accuracy. A similar method has been given in [11] for the Stokes eigenvalue problem by mixed finite element methods, where the postprocessing procedure was implemented as follows: (1) solving the Stokes eigenvalue problem in the original finite element space, (2) solving an additional Stokes source problem in an augmented space using the previous obtained eigenvalue multiplying the corresponding eigenfunction as the load vector. This procedure improves the convergence rate of the eigenpair approximations with relative inexpensive computation because the eigenvalue problem is replaced by an additional source problem in an augmented finite element space.

In this paper, we apply the LPS method developed in [16, 23] to discretize the Stokes eigenvalue problems. A postprocessing technique is then used to accelerate the convergence rate of the eigenpair approximations. The postprocessing strategy contains solving an additional Stokes source problem on an augmented finite element space which can be constructed either by refining the mesh or increasing the order of mixed finite element but on the same mesh.

An outline of the paper goes as follows. In Section 2, we apply the LPS method to the Stokes eigenvalue problem, and then analyze the stability and convergence properties. Section 3 is devoted to developing the acceleration technique. Section 4 focuses on the implementation of the acceleration technique proposed in Section 3. In Section 5, we give numerical tests to confirm the theoretical analysis. Some concluding remarks are given in the last section.

Throughout this paper, C (with or without subscripts) denotes a generic positive constant which may vary at different occurrences. For convenience, following Xu [34], the symbols $\lesssim$, $\gtrsim$ and $\approx$ will be used in this paper. That $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \approx y_3$, mean that $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$ for some constants C_1, c_2, c_3 and C_3 that are independent of mesh sizes.

The Stokes eigenvalue problem will be considered in a bounded polygonal domain $\Omega \subset \mathcal{R}^2$. We use the standard notations (c.f. [8, 9, 13]) for the Sobolev spaces $H^m(\Omega)$ and their associated inner products $(\cdot,\cdot)_m$, norms $\|\cdot\|_m$ and seminorms $|\cdot|_m$ for $m \geq 0$. The Sobolev space $H^0(\Omega)$ coincides with $L^2(\Omega)$, in which case the norm and inner product are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. This notation of inner products, norms and semi-norms is also used for the vector-valued case.

2 Stokes eigenvalue problem and its stabilized formulation

In this section, we first apply the LPS method to the Stokes eigenvalue problem, and then give the corresponding stability and convergence analysis.

2.1 Weak formulation

We consider the following Stokes eigenvalue problem

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \lambda \mathbf{u} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \\ \int_\Omega \mathbf{u}^2 d\Omega = 1, \end{cases} \tag{2.1}$$

where $\mathbf{u}$ and p denote the velocity and pressure fields, respectively. By introducing the spaces $\mathbf{V} := (H_0^1(\Omega))^2$ and $W =: L_0^2(\Omega)$, a weak formulation of Eq. 2.1 reads:

Find $(\lambda, \mathbf{u}, p) \in \mathcal{R} \times \mathbf{V} \times W$ such that $r(\mathbf{u},\mathbf{u}) = 1$ and

$$\begin{cases} a(\mathbf{u},\mathbf{v}) - b(\mathbf{v}, p) = \lambda r(\mathbf{u},\mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) = 0 & \forall q \in W, \end{cases} \tag{2.2}$$

where

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u}\nabla\mathbf{v} d\Omega, \quad b(\mathbf{v}, p) = \int_\Omega \nabla\cdot\mathbf{v} p d\Omega, \quad r(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u}\mathbf{v} d\Omega.$$

It is known that the eigenvalue problem (2.2) has an eigenvalue sequence $\{\lambda_j\}$ (c.f. [3]):

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and corresponding eigenfunctions

$$(\mathbf{u}_1, p_1), (\mathbf{u}_2, p_2), \cdots, (\mathbf{u}_k, p_k), \cdots,$$

which can be assumed to satisfy $r(\mathbf{u}_i, \mathbf{u}_j) = \delta_{ij}$. To state our method clearly, we only consider the case of simple eigenvalues in this paper, however, the corresponding results can be extended to the multiple case.

From [5, 17] we know that the following properties hold:

$$\sup_{0\neq\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_1} \gtrsim \|q\|_0, \quad \|\mathbf{u}\|_1 + \|p\|_0 \lesssim \sup_{0\neq(\mathbf{v},q)\in\mathbf{V}\times W} \frac{A\big((\mathbf{u}, p); (\mathbf{v}, q)\big)}{\|\mathbf{v}\|_1 + \|q\|_0},$$

where

$$A\big((\mathbf{u}, p); (\mathbf{v}, q)\big) = a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) + b(\mathbf{u}, q). \tag{2.3}$$

It is easily seen from Eq. 2.2 that the following Rayleigh quotient expression

$$\lambda = \frac{a(\mathbf{u}, \mathbf{u})}{r(\mathbf{u}, \mathbf{u})} \tag{2.4}$$

holds for each eigenvalue λ and corresponding eigenfunction $\mathbf{u}$.

2.2 Local projection stabilized formulation

There are several works (see, e.g., [2, 3, 10, 21, 25, 27]) in which numerical methods for eigenvalue problems are discussed. In this paper, we consider LPS formulation for the Stokes eigenvalue problems. To be specific, we use equal order interpolations stabilized by the local projection method in its one-level variant as developed in [16, 23]. (For the two-level approach we refer to [4, 6, 26].)

We are given a family $\mathcal{T}_h$ of shape-regular decompositions of Ω into triangles. The diameter of an triangle K is denoted by h_K. The mesh parameter h describes the maximum diameter of the $K \in \mathcal{T}_h$. Denote by V_h a scalar finite element space of continuous, piecewise k-th order polynomials over $\mathcal{T}_h$. The spaces for approximating velocity and pressure are given by $\mathbf{V}_h := V_h^2 \cap \mathbf{V}$ and $W_h := V_h \cap W$, respectively. The standard Galerkin discretization reads:

Find $(\lambda_h, \mathbf{u}_h, p_h) \in \mathcal{R} \times \mathbf{V}_h \times W_h$ such that $r(\mathbf{u}_h, \mathbf{u}_h) = 1$ and

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) - b(\mathbf{v}_h, p_h) = \lambda_h r(\mathbf{u}_h, \mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) = 0 & \forall q_h \in W_h. \end{cases} \tag{2.5}$$

It is known that, in general, equal order interpolations do not satisfy the Babuška-Brezzi condition ([9])

$$\exists \beta_0 > 0, \quad \forall h : \inf_{q_h\in W_h} \sup_{\mathbf{v}_h\in\mathbf{V}_h} \frac{(q_h, \nabla\cdot\mathbf{v}_h)}{\|\mathbf{v}_h\|_1 \, \|q_h\|_0} \geq \beta_0. \tag{2.6}$$

Therefore, we add to Eq. 2.5 a stabilizing term based on local projection, which leads to the stabilized discretization for Stokes eigenvalue problem (2.2):

Find $(\lambda_h, \mathbf{u}_h, p_h) \in \mathcal{R} \times \mathbf{V}_h \times W_h$ such that $r(\mathbf{u}_h, \mathbf{u}_h) = 1$ and

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) - b(\mathbf{v}_h, p_h) = \lambda_h r(\mathbf{u}_h, \mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) + S_h(p_h, q_h) = 0 & \forall q_h \in W_h, \end{cases} \tag{2.7}$$

where the stabilization term with user-chosen parameters α_K is given by

$$S_h(p,q) = \sum_{K\in\mathcal{T}_h} \alpha_K (\kappa_h \nabla p, \kappa_h \nabla q)_K. \tag{2.8}$$

As in [16], the fluctuation operator $\kappa_h : L^2(\Omega) \to L^2(\Omega)$ acting componentwise is defined as follows. Let $\mathcal{P}_k(K)$, $k = 0, 1, \cdots$, denote the set of all polynomials of degree less than or equal to k and $\mathcal{D}_h(K)$ be a finite dimensional space on the cell $K \in \mathcal{T}_h$ with $\mathcal{P}_k(K) \subset \mathcal{D}_h(K)$. The definition is extended by allowing $\mathcal{D}_h(K) = \{0\}$, together with $\mathcal{P}_{-1}(K) = \mathcal{D}_h(K)$. We then define the associated global space of discontinuous finite elements

$$\mathcal{D}_h := \bigoplus_{K\in\mathcal{T}_h} \mathcal{D}_h(K)$$

and the local $L^2(K)$-projection $\pi_K : L^2(K) \to \mathcal{D}_h(K)$, which leads to the global projection $\pi_h : L^2(\Omega) \to \mathcal{D}_h$ by

$$(\pi_h w)\big|_K := \pi_K(w|_K) \qquad \forall K \in \mathcal{T}_h,\ \forall w \in L^2(\Omega).$$

The fluctuation operator $\kappa_h : L^2(\Omega) \to L^2(\Omega)$ used in Eq. 2.8 is then given by $\kappa_h := id - \pi_h$ where $id : L^2(\Omega) \to L^2(\Omega)$ is the identity on $L^2(\Omega)$. However, $\kappa_h \nabla p$ has to be understood as acting on each component of ∇p separately.

Similar to the continuous case (2.4), the following Rayleigh quotient for λ_h holds

$$\lambda_h = \frac{a(\mathbf{u}_h, \mathbf{u}_h) + S_h(p_h, p_h)}{r(\mathbf{u}_h, \mathbf{u}_h)}. \tag{2.9}$$

It is also known that the Stokes eigenvalue problem (2.7) has eigenvalues (c.f. [3])

$$0 < \lambda_{1,h} \le \lambda_{2,h} \le \cdots \le \lambda_{k,h} \le \cdots \le \lambda_{N_h,h},$$

and the corresponding eigenfunctions

$$(\mathbf{u}_{1,h}, p_{1,h}), (\mathbf{u}_{2,h}, p_{2,h}), \cdots, (\mathbf{u}_{k,h}, p_{k,h}), \cdots, (\mathbf{u}_{N_h,h}, p_{N_h,h}),$$

where $r(\mathbf{u}_{i,h}, \mathbf{u}_{j,h}) = \delta_{ij}$, $1 \le i, j \le N_h := \dim(\mathbf{V}_h \times W_h)$.

In order to study the properties of Eq. 2.7 on the product space $\mathbf{V}_h \times W_h$, we introduce the bilinear form

$$A_h\left((\mathbf{u}, p); (\mathbf{v}, q)\right) = a\left(\mathbf{u}, \mathbf{v}\right) - b(\mathbf{v}, p) + b(\mathbf{u}, q) + S_h(p, q), \tag{2.10}$$

and the mesh-dependent norm

$$|||(\mathbf{v}, q)|||_A := \Big(|\mathbf{v}|_1^2 + \|q\|_0^2 + \sum_{K\in\mathcal{T}_h} \alpha_K \|\kappa_h \nabla q\|_{0,K}^2\Big)^{1/2}. \tag{2.11}$$

2.3 Stability and convergence

The existence and uniqueness for discrete solutions of Stokes problem have been studied in [16, 23] for different pairs $(V_h, \mathcal{D}_h)$ of approximation and projection spaces, respectively. Based on these results, the existence and uniqueness of eigenvalue problem (2.7) can be given similarly.

To prove the stability and convergence properties of the LPS method, we need the following assumptions (c.f. [23, 29]).

Assumption A1 There is an interpolation operator $i_h : H^1(\Omega) \to V_h$ such that

$$\|v - i_h v\|_{0,K} + h_K |v - i_h v|_{1,K} \leq C h_k^{\ell} \|v\|_{\ell,\omega(K)}, \tag{2.12}$$

for all $K \in \mathcal{T}_h$, $v \in H^{\ell}(\omega(K))$ and $1 \leq \ell \leq k+1$, where $\omega(K)$ denotes a certain local neighborhood of K which appears in the definition of these interpolation operators for non-smooth functions, see [14, 31] for more details.

Assumption A2 There exists a constant $\beta_1 > 0$ such that for all $h > 0$

$$\inf_{q_h \in \mathcal{D}_h(K)} \sup_{v_h \in V_h(K)} \frac{(v_h, q_h)_K}{\|v_h\|_{0,K} \ \|q_h\|_{0,K}} \geq \beta_1 > 0 \tag{2.13}$$

is satisfied where $V_h(K) = \{v_h|_K : v_h \in V_h,\ v_h = 0 \text{ in } \Omega \backslash K\}$.

The assumptions A1 and A2 guarantee the existence of an interpolation operator $j_h : H^1(\Omega) \to V_h$ satisfies the orthogonality property

$$(v - j_h v, q_h) = 0 \quad \forall q_h \in \mathcal{D}_h, \ \ \forall v \in H^1(\Omega), \tag{2.14}$$

and the approximation property

$$\|v - j_h v\|_{0,K} + h_K |v - j_h v|_{1,K} \leq C h_k^{\ell} \|v\|_{\ell,\omega(K)}.$$

for all $K \in \mathcal{T}_h$, $v \in H^{\ell}(\omega(K))$ and $1 \leq \ell \leq k+1$, where $\omega(K)$ denotes a certain local neighborhood of K.

In order to guarantee A2, V_h is required to be enriched by suitable bubble functions. For more details, please read the papers [23, 24] and the book [29].

Lemma 2.1 *([16]) Let the assumptions A1, A2, and $h_K^2 \lesssim \alpha_K$ be fulfilled. Then, there is a positive constant β_A independent of h such that*

$$\inf_{(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times W_h} \sup_{(\mathbf{w}_h, r_h) \in \mathbf{V}_h \times W_h} \frac{A_h\big((\mathbf{v}_h, q_h); (\mathbf{w}_h, r_h)\big)}{|||(\mathbf{v}_h, q_h)|||_A \ |||(\mathbf{w}_h, r_h)|||_A} \geq \beta_A > 0 \tag{2.15}$$

holds.

Based on Lemma 2.1, the discrete Stokes eigenvalue problem (2.7) is consistent with the continuous problem (2.2) (c.f. [16]). Combining the abstract spectral approximation theory [3] and the convergence results of LPS method for the Stokes problems in [16], we can easily give the convergence property of LPS method for the Stokes eigenvalue problem.

We first define the compact operator $T : (L^2(\Omega))^2 \to (H^1(\Omega))^2$ and the operator $K : (L^2(\Omega))^2 \to L^2_0(\Omega)$ by

$$A((T\mathbf{f}, K\mathbf{f}); (\mathbf{v}, q)) = r(\mathbf{f}, \mathbf{v}) \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times W. \tag{2.16}$$

Hence the eigenvalue problem (2.2) can be written as

$$\lambda T\mathbf{u} = \mathbf{u}. \tag{2.17}$$

Then, we introduce the eigenfunction set corresponding to the eigenvalue λ by

$$M(\lambda) = \Big\{(\mathbf{w}, \psi) \in (H^1_0(\Omega))^2 \times L^2_0(\Omega) : \ (\mathbf{w}, \psi) \text{ is an eigenfunction of (2.2) corresponding to } \lambda \text{ and} r(\mathbf{w}, \mathbf{w}) = 1\Big\}.$$

Theorem 2.1 *([9, 17, 25, 27]) Under the conditions of Lemma 2.1, there exists an exact eigenpair* $(\lambda, \mathbf{u}, p)$ *of* Eq. 2.2 *such that the discrete eigenpair* $(\lambda_h, \boldsymbol{u}_h, p_h)$ *of* Eq. 2.7 *has the following bounds*

$$\|\mathbf{u}_h - \mathbf{u}\|_1 + \|p - p_h\|_0 \lesssim \delta_h(\lambda), \tag{2.18}$$

$$\|\mathbf{u}_h - \mathbf{u}\|_{-1} \lesssim \eta_A(h)\delta_h(\lambda), \tag{2.19}$$

$$|\lambda_h - \lambda| \lesssim \delta_h^2(\lambda), \tag{2.20}$$

where $\delta_h(\lambda)$ *and* $\eta_A(h)$ *are defined by*

$$\delta_h(\lambda) := \sup_{(\mathbf{w},\psi)\in M(\lambda)} \left(\inf_{(\mathbf{v}_h,q_h)\in \mathbf{V}_h\times W_h} |||(\mathbf{w}-\mathbf{v}_h, \psi - q_h)|||_A + S_h^{1/2}(\psi, \psi)\right) \tag{2.21}$$

and

$$\eta_A(h) = \sup_{\|\mathbf{g}\|_1=1} \inf_{(\mathbf{v}_h,q_h)\in \mathbf{V}_h\times W_h} \left(|||(T\mathbf{g}-\mathbf{v}_h, K\mathbf{g} - q_h)|||_A + S_h^{1/2}(K\mathbf{g}, K\mathbf{g})\right) \tag{2.22}$$

respectively.

3 Acceleration technique

In this section, we present a postprocessing technique to improve the eigenpair approximation, which contains solving the Stokes eigenvalue problem in the original finite element space and one additional Stokes source problem in an augmented finite element space. We first introduce the error expansion of the eigenvalues by the Rayleigh quotient formula.

Lemma 3.1 *([39]) Assume* $(\lambda, \mathbf{u}, p)$ *be the true solution of the Stokes eigenvalue problem* (2.2), $0 \neq \mathbf{w} \in (H_0^1(\Omega))^2$, $\psi \in L_0^2(\Omega)$ *and define*

$$\widehat{\lambda} = \frac{a(\mathbf{w}, \mathbf{w}) - 2b(\mathbf{w}, \psi)}{r(\mathbf{w}, \mathbf{w})}. \tag{3.1}$$

Then, the following relationship holds

$$\widehat{\lambda} - \lambda = \frac{a(\mathbf{w}-\mathbf{u}, \mathbf{w}-\mathbf{u}) - \lambda r(\mathbf{w}-\mathbf{u}, \mathbf{w}-\mathbf{u}) + 2b(\mathbf{w}-\mathbf{u}, p-\psi)}{r(\mathbf{w}, \mathbf{w})}. \tag{3.2}$$

Proof From Eq. 2.2 and Eq. 3.1, we have

$$\begin{aligned}
\widehat{\lambda} - \lambda &= \frac{a(\mathbf{w}, \mathbf{w}) - 2b(\mathbf{w}, \psi) - \lambda r(\mathbf{w}, \mathbf{w})}{r(\mathbf{w}, \mathbf{w})} \\
&= \frac{a(\mathbf{w}-\mathbf{u}, \mathbf{w}-\mathbf{u}) + 2a(\mathbf{w}, \mathbf{u}) - a(\mathbf{u}, \mathbf{u}) - 2b(\mathbf{w}, \psi) - \lambda r(\mathbf{w}, \mathbf{w})}{r(\mathbf{w}, \mathbf{w})} \\
&= \frac{a(\mathbf{w}-\mathbf{u}, \mathbf{w}-\mathbf{u}) + 2\lambda r(\mathbf{w}, \mathbf{u}) + 2b(\mathbf{w}, p-\psi) - \lambda r(\mathbf{u}, \mathbf{u}) - \lambda r(\mathbf{w}, \mathbf{w})}{r(\mathbf{w}, \mathbf{w})} \\
&= \frac{a(\mathbf{w}-\mathbf{u}, \mathbf{w}-\mathbf{u}) - \lambda r(\mathbf{w}-\mathbf{u}, \mathbf{w}-\mathbf{u}) + 2b(\mathbf{w}-\mathbf{u}, p-\psi)}{r(\mathbf{w}, \mathbf{w})}.
\end{aligned}$$

This is the desired result and the proof is completed. □

Algorithm 1 Acceleration Scheme

1. Solve the Stokes eigenvalue problem:
   Find $(\lambda_{j,H}, \mathbf{u}_{j,H}, p_{j,H}) \in \mathcal{R} \times \mathbf{V}_H \times W_H$ such that $r(\mathbf{u}_{j,H}, \mathbf{u}_{j,H}) = 1$ and

$$\begin{cases} a(\mathbf{u}_{j,H}, \mathbf{v}_H) - b(\mathbf{v}_H, p_{j,H}) = \lambda_{j,H} r(\mathbf{u}_{j,H}, \mathbf{v}_H) & \forall \mathbf{v}_H \in \mathbf{V}_H, \\ b(\mathbf{u}_{j,H}, q_H) + S_H(p_{j,H}, q_H) = 0 & \forall q_H \in W_H, \end{cases} \tag{3.3}$$

   where $j = 1, 2, \cdots, N_H$.
2. Define the following auxiliary source problem:
   Find $(\widehat{\mathbf{u}}_{j,h}, \widehat{p}_{j,h}) \in \mathbf{V}_h \times W_h$ such that

$$\begin{cases} a(\widehat{\mathbf{u}}_{j,h}, \mathbf{v}_h) - b(\mathbf{v}_h, \widehat{p}_{j,h}) = \lambda_{j,H} r(\mathbf{u}_{j,H}, \mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\widehat{\mathbf{u}}_{j,h}, q_h) + S_h(\widehat{p}_{j,h}, q_h) = 0 & \forall q_h \in W_h. \end{cases} \tag{3.4}$$

   Solve this Stokes problem approximately to obtain a new eigenfunction approximation $(\mathbf{u}_{j,h}, p_{j,h}) \in \mathbf{V}_h \times W_h$ such that

$$\|\mathbf{u}_{j,h} - \widehat{\mathbf{u}}_{j,h}\|_1 + \|p_{j,h} - \widehat{p}_{j,h}\|_0 \lesssim \delta_h(\lambda_j). \tag{3.5}$$

3. Compute the Rayleigh quotient for $(\mathbf{u}_{j,h}, p_{j,h})$:

$$\lambda_{j,h}^{\text{post}} = \frac{a(\mathbf{u}_{j,h}, \mathbf{u}_{j,h}) - 2b(\mathbf{u}_{j,h}, p_{j,h})}{r(\mathbf{u}_{j,h}, \mathbf{u}_{j,h})}. \tag{3.6}$$

Finally, we obtain a new eigenpair approximation $(\lambda_{j,h}^{\text{post}}, \mathbf{u}_{j,h}, p_{j,h})$.

In Algorithm 1, in order to obtain the eigenpair approximation $(\lambda_{j,h}^{\text{post}}, \mathbf{u}_{j,h}, p_{j,h})$, we only need to solve a Stokes eigenvalue problem in the low dimension space $\mathbf{V}_H \times W_H$ and and a Stokes problem in the fine space $\mathbf{V}_h \times W_h$. As we know, there exist many efficient preconditioners for the solution of boundary value problems compared with the solution of eigenvalue problems. Thus the replacement of the eigenvalue problem solving by the boundary value problem solving improves the computational efficiency.

For the aim of error analysis, we define the finite element projection $(R_h(\mathbf{u}, p), G_h(\mathbf{u}, p))$ as the finite element solution of the following Stokes problem: Find $(R_h(\mathbf{u}, p), G_h(\mathbf{u}, p)) \in \mathbf{V}_h \times W_h$ such that

$$A_h\big((R_h(\mathbf{u}, p), G_h(\mathbf{u}, p)); (\mathbf{v}_h, q_h)\big) = A\big((\mathbf{u}, p); (\mathbf{v}_h, q_h)\big) \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times W_h. \tag{3.7}$$

Theorem 3.1 *Assume the conditions of Lemma 2.1 hold. After implementing Algorithm 1, there exists an exact eigenpair $(\lambda, \mathbf{u}, p)$ of* Eq. 2.2 *such that the resultant approximation $(\lambda_{j,h}^{post}, \mathbf{u}_{j,h}, p_{j,h}) \in \mathcal{R} \times \mathbf{V}_h \times W_h$ has the following error estimates*

$$\|\mathbf{u}_j - \mathbf{u}_{j,h}\|_1 + \|p_j - p_{j,h}\|_0 \lesssim \varepsilon_h(\lambda_j), \tag{3.8}$$

$$|\lambda_j - \lambda_{j,h}^{post}| \lesssim \varepsilon_h^2(\lambda_j), \tag{3.9}$$

where $\varepsilon_h(\lambda_j) := \eta_A(H)\delta_H(\lambda_j) + \delta_H^2(\lambda_j) + \delta_h(\lambda_j)$, $\delta_H(\lambda_j)$ and $\eta_A(H)$ are defined by Eq. 2.21 *and* Eq. 2.22, *respectively.*

Proof First, from Theorem 2.1 we know that there exist an exact eigenpair $(\lambda, \mathbf{u}, p)$ such that the eigenpair approximation $(\lambda_{j,H}, \mathbf{u}_{j,H}, p_{j,H})$ has the following error estimates

$$\|\mathbf{u}_{j,H} - \mathbf{u}_j\|_1 + \|p_j - p_{j,H}\|_0 \lesssim \delta_H(\lambda_j), \tag{3.10}$$

$$\|\mathbf{u}_{j,H} - \mathbf{u}_j\|_{-1} \lesssim \eta_A(H)\delta_H(\lambda_j), \tag{3.11}$$

$$|\lambda_{j,H} - \lambda_j| \lesssim \delta_H^2(\lambda_j). \tag{3.12}$$

Together with Eqs. 2.2, 3.4, 3.7, and 3.10–3.12, we obtain the following estimate

$$\begin{aligned}
&\|\widehat{\mathbf{u}}_{j,h} - R_h(\mathbf{u}_j, p_j)\|_1 + \|\widehat{p}_{j,h} - G_h(\mathbf{u}_j, p_j)\|_0 \\
&\lesssim \sup_{0\neq(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times W_h} \frac{A_h\big((\widehat{\mathbf{u}}_{j,h} - R_h(\mathbf{u}_j, p_j), \widehat{p}_{j,h} - G_h(\mathbf{u}_j, p_j)); (\mathbf{v}_h, q_h)\big)}{\|\mathbf{v}_h\|_1 + \|q_h\|_0} \\
&= \sup_{0\neq(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times W_h} \frac{\lambda_{j,H} r(\mathbf{u}_{j,H}, \mathbf{v}_h) - \lambda_j r(\mathbf{u}_j, \mathbf{v}_h)}{\|\mathbf{v}_h\|_1 + \|q_h\|_0} \\
&\lesssim \sup_{0\neq(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times W_h} \frac{\big(|\lambda_{j,H} - \lambda_j| \cdot \|\mathbf{u}_{j,H}\|_{-1} + \lambda\|\mathbf{u}_j - \mathbf{u}_{j,H}\|_{-1}\big)\|\mathbf{v}_h\|_1}{\|\mathbf{v}_h\|_1 + \|q_h\|_0} \\
&\lesssim \delta_H^2(\lambda_j) + \eta_A(H)\delta_H(\lambda_j),
\end{aligned}$$

which leads to

$$\|\mathbf{u}_j - \widehat{\mathbf{u}}_{j,h}\|_1 + \|p_j - \widehat{p}_{j,h}\|_0 \lesssim \delta_H^2(\lambda_j) + \eta_A(H)\delta_H(\lambda_j) + \delta_h(\lambda_j). \quad (3.13)$$

Together with Eqs. 3.5 and 3.13, we get the following estimate

$$\|\mathbf{u}_j - \mathbf{u}_{j,h}\|_1 + \|p - p_{j,h}\|_0 \lesssim \delta_H^2(\lambda_j) + \eta_A(H)\delta_H(\lambda_j) + \delta_h(\lambda_j).$$

This is the desired result (3.8), while (3.9) can be derived by Lemma 3.1 and (3.8). □

4 Practical acceleration algorithm

In this section, we adopt two different ways to construct the finer finite element space $\mathbf{V}_h \times W_h$. The first way is the "two-grid" method proposed by Xu and Zhou [37]. In this method, a finer mesh is used to get the approximation $(\lambda_{j,h}^{\text{post}}, \mathbf{u}_{j,h}, p_{j,h})$ of higher order accuracy. Since the approach uses the same finite element space as the original eigenvalue approximation, it does not require higher regularity of the exact eigenfunctions. The second way is proposed by Andreev and Racheva in [28], where the same finite element mesh but higher order finite element space is used. In this case, the regularity of the Stokes eigenvalue problem needs to be higher to ensure the accuracy for the eigenpair.

Way 1. ("Two-grid" method from [37]): In this case, $\mathbf{V}_h \times W_h$ is the same type of finite element space as $\mathbf{V}_H \times W_H$ on the finer mesh $\mathcal{T}_h$ with mesh size $h = H^\beta$ ($\beta > 1$). Here $\mathcal{T}_h$ is a finer mesh of Ω which can be generated by the regular refinement from the coarse mesh $\mathcal{T}_H$. Here, we assume the exact eigenfunction has the regularity of $\mathbf{u} \in (H^{1+\alpha}(\Omega))^2$ and $p \in H^\alpha(\Omega)$ with $\alpha > 0$. From Eqs. 2.21 and 2.22, we have

$$\delta_H(\lambda_j) \approx H^{s_1}, \qquad \eta_A(H) \approx H^\gamma, \quad (4.1)$$

where $s_1 = \min\{k, \alpha\}$ and $0 < \gamma \le s_1$ is a parameter depending on the largest interior angle of $\partial\Omega$. Then after Algorithm 1, the new eigenpair approximation $(\lambda_{j,h}^{\text{post}}, \mathbf{u}_{j,h}, p_{j,h})$ has the following error estimate

$$\varepsilon_h(\lambda_j) \approx H^{s_1+\gamma} + h^{s_1}. \quad (4.2)$$

If we choose $\beta = 1 + \gamma/s_1$, the eigenpair approximation $(\lambda_{j,h}^{\text{post}}, \mathbf{u}_{j,h}, p_{j,h})$ possesses the optimal convergence order h^{s_1}. Notice that we just need to solve the Stokes eigenvalue problem in the coarse space $\mathbf{V}_H \times W_H$ followed by a Stokes source problem in the space $\mathbf{V}_h \times W_h$, other than the Stokes eigenvalue problem in the finer space $\mathbf{V}_h \times W_h$. As we know, solving a Stokes problem is more efficient than the Stokes eigenvalue problem in the same scale. So Algorithm 1 improves the efficiency for solving the Stokes eigenvalue problem.

Way 2. ("Two-space" method from [28]): In this case, $\mathbf{V}_h \times W_h$ is defined on the same mesh $\mathcal{T}_H$ but with a higher order finite element space than $\mathbf{V}_H \times W_H$. Since the maximum regularity of the solution of the Stokes problem (3.4) is $(H^3(\Omega))^2 \times H^2(\Omega)$, we only use the first order finite element space, that is $k = 1$ to solve the original Stokes eigenvalue problem (3.3), and the Stokes source problem (3.4) in the second order finite element space. Here, we assume the exact the domain Ω is convex and the eigenfunction has the regularity of $\mathbf{u} \in (H^{1+\alpha}(\Omega))^2$ and $p \in H^{\alpha}(\Omega)$ with $1 < \alpha \leq 2$. From Eqs. 2.21 and 2.22, we have

$$\delta_H(\lambda_j) \approx H, \quad \eta_A(H) \approx H, \quad \delta_h(\lambda_j) \approx H^{\alpha}. \tag{4.3}$$

Then we have the following error estimate for $(\lambda_{j,h}^{\text{post}}, \mathbf{u}_{j,h}, p_{j,h})$

$$\varepsilon_h(\lambda_j) \approx H^{\alpha}. \tag{4.4}$$

This is also an obvious improvement of the efficiency for solving the Stokes eigenvalue problem.

5 Numerical results

In this section, we give two numerical tests to illustrate the efficiency of Algorithm 1. Consider the Stokes eigenvalue problem (2.1) on the domain $\Omega = (0, 1)^2$ and choose a sufficiently accurate first eigenvalue approximation $\lambda = 52.3446911$ as the first true one (c.f. [12, 33]) for our numerical tests.

We first test Algorithm 1 with the "two-grid" way, where the enriched space is constructed by refining the current mesh in the regular way. Here we use the finite element space $(V_h, \mathcal{D}_h) = (\mathcal{P}_1, \mathcal{P}_{-1}^{\text{disc}})$ with

$$\mathcal{P}_1 = \{v \in H^1(\Omega) : v|_K \in \mathcal{P}_1(K), \ \forall K \in \mathcal{T}_h\},$$
$$\mathcal{P}_{-1}^{\text{disc}} = \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_{-1}(K), \ \forall K \in \mathcal{T}_h\},$$

to solve the Stokes eigenvalue problem (3.3) and the Stokes source problem (3.4). The numerical results are shown in Fig. 1.

Then we give numerical results of Algorithm 1 with the "two-space" method. We first solve the Stokes eigenvalue problem (3.3) by the lowest order stabilization

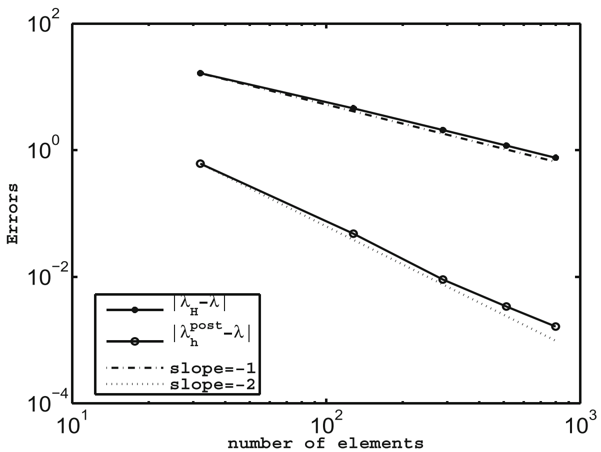

Fig. 1 Errors for the "two-grid" method with $\alpha_K = h_K^2/10$

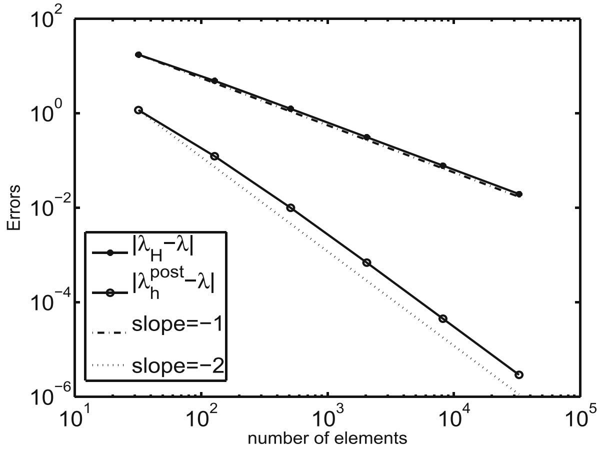

Fig. 2 Errors for the "two-space" method with $\alpha_K = h_K^2/10$

element $(V_h, \mathcal{D}_h) = (\mathcal{P}_1, \mathcal{P}_{-1}^{\text{disc}})$ and then the Stokes source problem (3.4) by the second order stabilization element $(V_h, \mathcal{D}_h) = (\mathcal{P}_2^+, \mathcal{P}_1^{\text{disc}})$ (c.f. [23]) with

$$\mathcal{P}_2^+ = \left\{v \in H^1(\Omega) : v|_K \in \mathcal{P}_2(K) \oplus \varphi_K \cdot \mathcal{P}_1(K),\ \forall K \in \mathcal{T}_h\right\},$$
$$\mathcal{P}_1^{\text{disc}} = \left\{v \in L^2(\Omega) : v|_K \in \mathcal{P}_1(K),\ \forall K \in \mathcal{T}_h\right\},$$

on the same triangular meshes, where the bubble function φ_K is defined by the barycenter coordinates $\lambda_{1,K}, \lambda_{2,K}$ and $\lambda_{3,K}$ on the element K with $\varphi_K := \lambda_{1,K}\lambda_{2,K}\lambda_{3,K}$. The numerical results are shown in Fig. 2.

Figures 1 and 2 show that Algorithm 1 improves the efficiency for solving the Stokes eigenvalue problem and this confirms the theoretical analysis.

6 Concluding remarks

In this paper, we apply the LPS method to discretize the Stokes eigenvalue problem, and then propose an acceleration scheme to improve the convergence order for the eigenpair approximation. The theoretical analysis is given and the corresponding numerical examples are also provided to confirm the analysis. The acceleration method proposed here can be coupled with the adaptive mesh refinement in the "two-grid" method. The application of LPS method makes the implementation of adaptive mesh refinement easier for solving Stokes eigenvalue problems especially on the meshes with hanging nodes (c.f. [30]).

In the future, we will extend our acceleration method to the nonsymmetric Stokes eigenvalue problems which is more general in the study of linearized stability for the Navier-Stokes equations. Furthermore, based on the acceleration method here, we will design a type of multilevel method for the Stokes eigenvalue problem by the stabilized finite element methods with the idea in [19].

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