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A stabilized finite volume method for Stokes equations using the lowest order $P_1 - P_0$ element pair

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Abstract We present a new stabilized finite volume method for Stokes problem using the lowest order $P_1 - P_0$ element pair. To offset the lack of the inf-sup condition, a simple jump term of discrete pressure is added to the continuity approximation equation. A discrete inf-sup condition is established for this stabilized scheme. The optimal error estimates are given in the H^1- and L_2-norms for velocity and in the L_2-norm for pressure, respectively.

Keywords Finite volume element · Stokes problem · Stabilized method · $P_1 - P_0$ element pair

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1 Introduction

The finite volume element (FVE) method is a discretization technique for solving partial differential equations. The important feature of FVE method is that it inherits

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some physical conservation laws of original problems locally, that is very desirable in practical applications, for example, in computational fluid mechanics and heat transfer problems. During the past decades, there have been many research works for FVE methods. We refer to monograph [28] for general presentation of this method and to [7–11, 20, 29, 32, 35, 36] and the references therein for details.

Numerical simulation of the incompressible flow motion has long been an important and challenging problem. In the conventional finite element methods solving the Stokes and Navier-Stokes equations, an elementary requirement for the velocity-pressure element pair is the so-called inf-sup condition. The importance of ensuring the inf-sup condition is wildly understood. Numerical experiments show that the violation of the inf-sup condition often leads to unphysical oscillations. From the computational viewpoint, the simple lower-order element pairs (for example, the $P_1 - P_0$, $Q_1 - Q_0$, $P_1 - P_1$ and $Q_1 - Q_1$ pairs) should be preferred in applications. But unfortunately, these element pairs do not satisfy the inf-sup condition. In order to circumvent the inf-sup condition, many stabilized finite element methods have been proposed for solving the Stokes and Navier-Stokes equations. For example, the penalty methods [23, 25, 30, 31], the consistently stabilized methods [2, 4, 19], the pressure macro-element methods [16, 22], the pressure gradient projection methods [5, 15, 24] and the local polynomial pressure projection methods [3, 6, 18], and so on. Recently, a local Gauss integration stabilized method is also constructed for the Stokes equations using the $P_1 - P_1$ element pair [26], and Li and Chen [27] further expand this stabilized technique from finite element method to the FVE method. To the authors' knowledge, except using the nonconforming P_1 velocity element or the P_0 pressure macro-element, almost no stabilized methods have been presented for the $P_1 - P_0$ element pair.

Many finite volume methods have been developed for solving the Stokes equations [12–14, 17, 33, 34]. But most of them use the stable velocity-pressure element pairs, which satisfy the inf-sup condition. In this paper, we will present a new stabilized FVE method for Stokes equations by using the lowest order $P_1 - P_0$ element pair. The key of our stabilized technique is to introduce a jump term of discrete pressure in the continuity approximation equation to offset the lack of the inf-sup condition. A similar stabilization term made of gradient jump had been introduced in [1] for the mixed element method solving biharmonic equation. Compared with some known stabilized methods, the advantage of our method is that it is free from stabilization parameters [2, 4, 23, 25, 30, 31], does not require any calculation of high-order derivatives [5, 15, 24] and pressure macro-element structures [16, 22], and no any projections need to be introduced [3, 6, 18, 26]. Moreover, the added stabilization term (the jump term) in our method is very simple and can be generated at the element level with very little computation cost. For this stabilized FVE scheme, we establish a discrete inf-sup condition which ensures the stability of the discrete solutions. Furthermore, the optimal error estimates are given in the H^1- and L_2-norms for the velocity approximation and in the L_2-norm for the pressure approximation, respectively. It should be pointed out that our stabilized method is presented for the FVE setting, while most known stabilized methods are for the finite element setting. And our FVE scheme includes the corresponding finite element scheme as a special case.

This paper is organized as follows. In Section 2, we introduce the stabilized FVE scheme and give the unique existence of the discrete solutions. Section 3 is devoted to the optimal error estimates for velocity and pressure approximations, respectively. In Section 4 we further establish a discrete inf-sup condition satisfied by this FVE scheme. Numerical example is presented in Section 5 to verify our theoretical analysis. In Section 6, some conclusions are given.

Throughout this paper, we adopt the notations $H^m(D)$ to indicate the usual Sobolev spaces on subdomain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,D}$ and semi-norm $|\cdot|_{m,D}$. When $D = \Omega$, we omit the index D. The inner product and norm in space $L_2(\Omega)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. We will use letter C to represent a generic positive constant, independent of the mesh size h.

2 The stabilized finite volume method

We consider the Stokes equations

$$-\nu\triangle\mathbf{u} + \nabla p = \mathbf{f},\ in\ \Omega, \tag{2.1}$$

$$\mathrm{div}\,\mathbf{u} = 0,\ in\ \Omega, \tag{2.2}$$

$$\mathbf{u} = 0,\ on\ \partial\Omega, \tag{2.3}$$

where $\Omega \subset R^2$ is a convex polygonal domain with boundary $\partial\Omega$, symbols $\triangle$, ∇ and div denote the Laplacian, gradient and divergence operators, respectively, and $\mathbf{u} = (u_1, u_2)$ represents the velocity, p the pressure and $\mathbf{f}$ the external volumetric force acting on the fluid. We assume the viscosity $\nu = 1$.

Let $T_h = \bigcup\{K\}$ be a regular triangulation of domain Ω so that $\overline{\Omega} = \bigcup_{K\in T_h}\{,\overline{K}\}$, where $h = \max\ h_K$, h_K is the diameter of element K. Associated with triangulation T_h, we introduce the velocity and pressure approximation spaces,

$$X_h = \{\mathbf{v}_h \in X:\ \mathbf{v}_h|_K \in [P_1(K)]^2,\ K \in T_h\ \}, \tag{2.4}$$

$$M_h = \{q_h \in M:\ q_h|_K \in P_0(K),\ K \in T_h\}, \tag{2.5}$$

where $P_k(K)$ is the set of all k-order polynomials on K and spaces

$$X = [H_0^1(\Omega)]^2,\ M = \{q \in L_2(\Omega) : \int_\Omega q\, dx = 0\}.$$

It is well known that the space pair $X_h \times M_h$ does not satisfy the inf-sup condition

$$\sup_{0\neq\mathbf{v}_h\in X_h} \frac{(\mathrm{div}\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1} \geq \beta\|q_h\|,\ q_h \in M_h, \tag{2.6}$$

where $\beta > 0$ is a constant independent of h, so the lowest order $P_1 - P_0$ element pair is not available in the conventional finite element framework.

In order to define the FVE method, we need a dual partition associated with the primal partition T_h. We construct the barycenter dual partition T_h^* by connecting the barycenter to the midpoints of edges of each $K \in T_h$ by straight lines. Thus, for each nodal point P in T_h, there exists a polygonal K_P^* surrounding P, $K_P^* \in T_h^*$ is called the dual element or the control volume at point P, see Fig. 1. Now we can introduce the test function space defined on T_h^*,

$$V_h = \{\mathbf{v}_h \in [L_2(\Omega)]^2 : \mathbf{v}_h|_{K_P^*} = constant,\ \forall P \in N_h, \mathbf{v}_h|_{K_P^*} = 0,\ \forall P \in \partial\Omega\},$$

where N_h is the set of all nodal points of T_h.

Set the algebraic sum space

$$U(h) = [H^2(\Omega) \bigcap H_0^1(\Omega)]^2 \oplus X_h = \{\mathbf{v} + \mathbf{v_h} : \mathbf{v} \in [H^2(\Omega) \bigcap H_0^1(\Omega)]^2, \mathbf{v}_h \in X_h\}.$$

Define a mapping $\gamma : U(h) \to V_h$ by

$$\gamma \mathbf{u} = \sum_{P \in N_h} \mathbf{u}(P)\chi_P, \forall \mathbf{u} \in U(h), \tag{2.7}$$

where χ_P is the characteristic function of the dual element K_P^*. Obviously, γ is a one to one mapping from the trial space X_h onto the test space V_h.

We denote by $\Gamma_h = \bigcup\{e \subset \partial K : K \in T_h\}$ the union of all edges of elements of T_h, $\Gamma_h^0 = \Gamma_h \backslash \partial\Omega$ the union of all element edges $\{e\}$ that are not contained in $\partial\Omega$. Let $e = \partial K_1 \bigcap \partial K_2$ be the edge shared by two adjacent elements K_1 and K_2 of T_h, $v_i = v|_{e\cap\partial K_i}$ $(i = 1, 2)$ the trace of v on e from the interior of K_i, and $n_i = n|_{e\cap\partial K_i}$, where n is the unit normal vector external to the element boundary. For a piecewise smooth function v on T_h, we define the jump $[v]$ of v on $e \in \Gamma_h^0$ as follows:

$$[v] = v_1 n_1 + v_2 n_2,\ on\ e \in \Gamma_h^0.$$

Let $(\mathbf{u}, p)$ satisfy (2.1) and $\mathbf{v}_h \in V_h$. Then by using the Green formula, we have

$$-\int_{\partial K_P^*} \frac{\partial \mathbf{u}}{\partial n} \cdot \mathbf{v}_h ds + \int_{\partial K_P^*} p\mathbf{v}_h \cdot n ds = \int_{K_P^*} f\mathbf{v}_h,\ \forall\ \mathbf{v}_h \in V_h,\ K_P^* \in T_h^*,$$

or

$$-\int_{\partial K_P^*} \frac{\partial \mathbf{u}}{\partial n} \cdot \gamma\mathbf{v}_h ds + \int_{\partial K_P^*} p\gamma\mathbf{v}_h \cdot n ds = \int_{K_P^*} f\gamma\mathbf{v}_h,\ \forall\ \mathbf{v}_h \in X_h,\ K_P^* \in T_h^*. \tag{2.8}$$

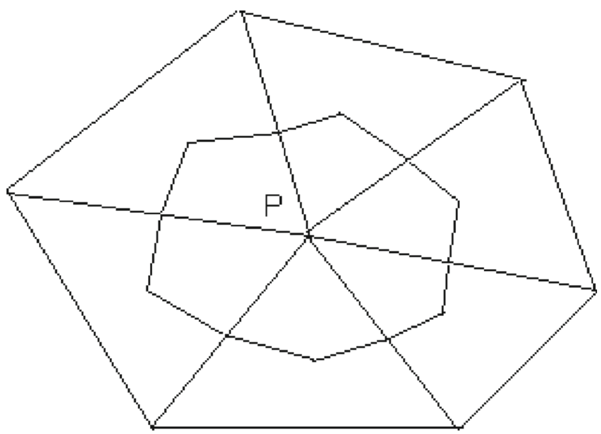

Fig. 1 Dual element K_P^* at node P

Motivated by this weak formula, we introduce the following bilinear forms

$$a(\mathbf{u},\mathbf{v}) = -\sum_{K_P^*\in T_h^*}\int_{\partial K_P^*}\frac{\partial \mathbf{u}}{\partial n}\cdot \mathbf{v}ds,\ \ b(p,\mathbf{v}) = \sum_{K_P^*\in T_h^*}\int_{\partial K_P^*} p\mathbf{v}\cdot nds, \tag{2.9}$$

$$c(\mathbf{u},q)=(\mathrm{div}\mathbf{u},q),\ G(p,q)=\int_{\Gamma_h^0} h_\mu[p]\cdot[q]ds \doteq \sum_{e\in\Gamma_h^0}\int_e h_\mu[p]\cdot[q]ds, \tag{2.10}$$

where $h_\mu|_e = \mu h_e$, $h_e = \mathrm{diam}(e)$, and parameter $\mu > 0$ can be properly chosen for the purpose of enhancing the stability of the method. In general case, we may take $\mu = 1$.

Now we define the stabilized FVE approximation of problem (2.1)∼(2.3) by finding $(\mathbf{u}_h, p_h) \in X_h \times M_h$ such that

$$a(\mathbf{u_h},\gamma\mathbf{v}_h) + b(p_h,\gamma\mathbf{v}_h) = (=f,\gamma\mathbf{v}_h),\ \forall\ \mathbf{v}_h \in X_h, \tag{2.11}$$

$$c(\mathbf{u_h},q_h) + G(p_h,q_h) = 0,\ \forall\ q_h \in M_h. \tag{2.12}$$

In the continuity approximation (2.12), the additional term $G(p_h, q_h)$ is the stabilization term which is introduced to offset the lack of the inf-sup condition. If removing $G(p_h, q_h)$, scheme (2.11)∼(2.12) will be the standard FVE scheme used in the case that the discrete velocity and pressure spaces satisfy the inf-sup condition [33]. It is easy to see that bilinear form $G(p_h, q_h)$ is symmetric and positive definite on $M_h \times M_h$ and it can be generated on local set K with little computation cost, noting that $[p_h]_e = constant$ for $p_h \in M_h$.

Let $(\mathbf{u}, p) \in X\times M$ be the solution of problem (2.1)∼(2.3) and $p \in H^1(\Omega)$. By the Sobolev trace theory, we have $[p]_e = 0$ if $p \in H^1(\Omega)$, which implies $G(p, q_h) = 0$. Then, from equations (2.8) and (2.10) we see that FVE scheme (2.11)∼(2.12) is consistent and the following error equations hold.

$$a(\mathbf{u}-\mathbf{u_h},\gamma\mathbf{v}_h) + b(p-p_h,\gamma\mathbf{v}_h) = 0,\ \forall\ \mathbf{v}_h \in X_h, \tag{2.13}$$

$$c(\mathbf{u}-\mathbf{u_h},q_h) + G(p-p_h,q_h) = 0,\ \forall\ q_h \in M_h. \tag{2.14}$$

Let $\Pi_h u \in X_h$ be the usual linear interpolation approximation of continuous function u. In our analysis, the following approximation property and trace inequality will be used frequently,

$$\|u-\Pi_h u\|_{m,K} \le Ch_K^{2-m}\|u\|_{2,K},\ \ K \in T_h, 0 \le m \le 2, \tag{2.15}$$

$$\|u\|_{0,\partial K} \le Ch_K^{-\frac{1}{2}}\left(\|u\|_{0,K} + h_K\|\nabla u\|_{0,K}\right),\ u \in H^1(K). \tag{2.16}$$

Furthermore, for operator γ, we have the following lemma.

Lemma 2.1 [28, 36]*Let $K \in T_h$, $e \subset \partial K$ be an edge of K. Then, for $\mathbf{v}_h \in X_h$, we have*

$$\int_K(\mathbf{v}_h-\gamma\mathbf{v}_h) = 0,\quad \int_e(\mathbf{v}_h-\gamma\mathbf{v}_h)ds = 0, \tag{2.17}$$

$$\|\mathbf{v}_h-\gamma\mathbf{v}_h\|_{0,K} \le Ch_K\|\mathbf{v}_h\|_{1,K}, \tag{2.18}$$

$$\|\mathbf{v}_h-\gamma\mathbf{v}_h\|_{0,\partial K} \le Ch_K^{\frac{1}{2}}\|\mathbf{v}_h\|_{1,K}. \tag{2.19}$$

The following two lemmas are very useful in our analysis.

Lemma 2.2 *For any* $\mathbf{w} \in U(h)$, $\mathbf{v} \in X_h$, *we have*

$$a(\mathbf{w}, \gamma\mathbf{v}) - \sum_{K\in T_h} (\nabla\mathbf{w}, \nabla\mathbf{v})_K = \sum_{K\in T_h} \int_{\partial K} \frac{\partial \mathbf{w}}{\partial n} \cdot (\gamma\mathbf{v} - \mathbf{v}) ds + \sum_{K\in T_h} (\triangle\mathbf{w}, \mathbf{v} - \gamma\mathbf{v})_K. \tag{2.20}$$

Particularly, when $\mathbf{w}, \mathbf{v} \in X_h$, we have

$$a(\mathbf{w}, \gamma\mathbf{v}) = (\nabla\mathbf{w}, \nabla\mathbf{v}), \ \mathbf{w}, \mathbf{v} \in X_h. \tag{2.21}$$

Proof By using the Green formula, we have

$$\sum_{K\in T_h} \int_K \nabla\mathbf{w} \cdot \nabla\mathbf{v} = -\sum_{K\in T_h} \int_K \triangle\mathbf{w} \cdot \mathbf{v} + \sum_{K\in T_h} \int_{\partial K} \frac{\partial \mathbf{w}}{\partial n} \cdot \mathbf{v} ds,$$

and (see Fig. 1)

$$\sum_{K\in T_h} \int_K \triangle\mathbf{w} \cdot \gamma\mathbf{v} = \sum_{K\in T_h} \sum_{K_P^*\in T_h^*} \int_{K\cap K_P^*} \triangle\mathbf{w} \cdot \gamma\mathbf{v} = \sum_{K\in T_h} \int_{\partial K} \frac{\partial \mathbf{w}}{\partial n} \cdot \gamma\mathbf{v} ds + \sum_{K_P^*\in T_h^*} \int_{\partial K_P^*} \frac{\partial \mathbf{w}}{\partial n} \cdot \gamma\mathbf{v} ds.$$

Combining this two identities with the definition of $a(\mathbf{w}, \gamma\mathbf{v})$, (2.20) is derived. The equality (2.21) follows from equations (2.20) and (2.17), noting that $\triangle\mathbf{w}|_K = 0$, $\nabla\mathbf{w} \cdot n|_e = constant$ if $\mathbf{w} \in X_h$. □

Lemma 2.3 *For any* $(\mathbf{w}, q) \in U(h) \times M$, *we have*

$$b(q, \gamma\mathbf{w}) = -(div\mathbf{w}, q) + \sum_{K\in T_h} (\nabla q, \gamma\mathbf{w} - \mathbf{w})_K + \sum_{K\in T_h} \int_{\partial K} q(\mathbf{w} - \gamma\mathbf{w}) \cdot n ds. \tag{2.22}$$

Particularly, when $(\mathbf{w}, q) \in X_h \times M_h$, we have

$$b(q, \gamma\mathbf{w}) = -c(\mathbf{w}, q), \ \forall\, (\mathbf{w}, q) \in X_h \times M_h. \tag{2.23}$$

Proof By using the divergence formula

$$\int_D \mathrm{div}\mathbf{w} q = -\int_D \nabla q \cdot \mathbf{w} + \int_{\partial D} q\mathbf{w} \cdot n ds,$$

we have (see Fig. 1)

$$
\begin{aligned}
b(q,\gamma\mathbf{w}) &= \sum_{K_p^*\in T_h^*}\int_{\partial K_p^*} q\gamma\mathbf{w}\cdot nds \\
&= \sum_{K\in T_h}\sum_{K_P^*\in T_h^*}\int_{\partial(K\cap K_P^*)} q\gamma\mathbf{w}\cdot nds - \sum_{K\in T_h}\int_{\partial K} q\gamma\mathbf{w}\cdot nds \\
&= \sum_{K\in T_h}\sum_{K_P^*\in T_h^*}\int_{K\cap K_P^*}\nabla q\cdot\gamma\mathbf{w} + \sum_{K\in T_h}\int_{\partial K} q(\mathbf{w}-\gamma\mathbf{w})\cdot nds - \sum_{K\in T_h}\int_{\partial K} q\mathbf{w}\cdot nds \\
&= \sum_{K\in T_h}(\nabla q,\gamma\mathbf{w})_K + \sum_{K\in T_h}\int_{\partial K} q(\mathbf{w}-\gamma\mathbf{w})\cdot nds - \sum_{K\in T_h}(\nabla q,\mathbf{w})_K - \sum_{K\in T_h}(\mathrm{div}\mathbf{w},q)_K.
\end{aligned}
$$

This gives (2.22). When $(\mathbf{w},q)\in X_h\times M_h$, noting that $q|_K = constant$, we obtain (2.23) from equations (2.22) and (2.17). □

Introduce the norm notation

$$|||(\mathbf{u},p)|||^2 = \|\nabla\mathbf{u}\|^2 + \int_{\Gamma_h^0} h_\mu |[p]|^2 ds. \tag{2.24}$$

Let $(\mathbf{u}_h,p_h)\in X_h\times M_h$. When $|||(\mathbf{u}_h,p_h)||| = 0$ we have $\|\nabla\mathbf{u}_h\| = 0$, $[p_h]_e = 0$, $e\in\Gamma_h^0$, this results in $\mathbf{u}_h = 0$ and $p_h|_\Omega = constant$. Further $\int_\Omega p_h dx = 0$ implies $p_h = 0$. So $|||(\mathbf{u}_h,p_h)|||$ is a norm on $X_h\times M_h$. Introduce the two-fold bilinear form

$$B((\mathbf{u},p),(\mathbf{v},q)) = a(\mathbf{u},\gamma\mathbf{v}) + b(p,\gamma\mathbf{v}) + c(\mathbf{u},q) + G(p,q). \tag{2.25}$$

Then the FVE scheme (2.11)∼(2.12) can be rewritten as finding $(\mathbf{u}_h,p_h)\in X_h\times M_h$ such that

$$B((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)) = (\mathbf{f},\gamma\mathbf{v}_h),\ \forall\,(\mathbf{v}_h,q_h)\in X_h\times M_h. \tag{2.26}$$

Taking $(\mathbf{v}_h,q_h) = (\mathbf{u}_h,p_h)$ in equation (2.26), and using (2.21) and (2.23)∼(2.25), we have

$$|||(\mathbf{u}_h,p_h)|||^2 = B((\mathbf{u}_h,p_h),(\mathbf{u}_h,p_h)),\ \forall\,(\mathbf{u}_h,p_h)\in X_h\times M_h. \tag{2.27}$$

Now we can give the stability theorem.

Theorem 2.1 *The FVE solution $(\mathbf{u}_h,p_h)\in X_h\times M_h$ exists uniquely and satisfies the following stability estimate*

$$|||(\mathbf{u}_h,p_h)||| \le C\|\mathbf{f}\|. \tag{2.28}$$

Proof For the linear system (2.26), we only need to prove the stability estimate (2.28). Taking $(\mathbf{v}_h,q_h) = (\mathbf{u}_h,p_h)$ in equation (2.26) and using (2.27), we have

$$|||(\mathbf{u}_h,p_h)|||^2 = B((\mathbf{u}_h,p_h),(\mathbf{u}_h,p_h)) \le \|\mathbf{f}\|\,\|\gamma\mathbf{u}_h\|. \tag{2.29}$$

It follows from equation (2.18) that

$$\|\gamma\mathbf{u}_h\| = \|\gamma\mathbf{u}_h - \mathbf{u}_h\| + \|\mathbf{u}_h\| \le Ch\|\mathbf{u}_h\|_1 + \|\mathbf{u}_h\| \le C\|\mathbf{u}_h\|_1 \le C|||(\mathbf{u},p)|||.$$

Combining this with (2.29), the proof is completed. □

3 Optimal error estimates in the H^1- and L_2-norms

In this section, we will give the optimal error estimates for velocity approximation in the H^1-norm and L_2-norm. Moreover, the optimal error estimate for pressure approximation in the L_2-norm is also established.

Let $\rho_h : M \to M_h$ be the projection operator defined by

$$\rho_h p|_K = \frac{1}{|K|}\int_K p dx, \ \forall\, K \in T_h.$$

Then $\rho_h p$ has the approximation property

$$\|p - \rho_h p\|_{0,K} \le C h_K \|p\|_{1,K}, \ \ K \in T_h. \tag{3.1}$$

Theorem 3.1 *Let* $(\mathbf{u}, p) \in X \times M$ *and* $(\mathbf{u}_h, p_h) \in X_h \times M_h$ *be the solutions of problems* (2.1) $\sim$ (2.3) *and* (2.11)$\sim$(2.12)*, respectively, and* $(\mathbf{u}, p) \in [H^2(\Omega)]^2 \times H^1(\Omega)$. *Then we have*

$$|||(\mathbf{u} - \mathbf{u}_h, p - p_h)||| \le Ch(\|\mathbf{u}\|_2 + \|p\|_1). \tag{3.2}$$

Proof From equations (2.13), (2.14) and (2.25), we first have the error equation

$$B((\mathbf{u} - \mathbf{u}_u, p - p_h), (\mathbf{v}_h, q_h)) = 0, \ \forall\, (\mathbf{v}_h, q_h) \in X_h \times M_h. \tag{3.3}$$

Denote the error functions:

$$e_{\mathbf{u}} = \mathbf{u}_h - \Pi_h \mathbf{u}, \ e_p = p_h - \rho_h p, \ \ (e_{\mathbf{u}}, e_p) \in X_h \times M_h.$$

Then, it follows from equations (2.27) and (3.3) that

$$\begin{aligned}
|||(e_{\mathbf{u}}, e_p)|||^2 &= B((e_{\mathbf{u}}, e_p), (e_{\mathbf{u}}, e_p)) = B((\mathbf{u} - \Pi_h \mathbf{u}, p - \rho_h p), (e_{\mathbf{u}}, e_p)) \\
&= a(\mathbf{u} - \Pi_h \mathbf{u}, \gamma e_{\mathbf{u}}) + b(p - \rho_h p, \gamma e_{\mathbf{u}}) + c(\mathbf{u} - \Pi_h \mathbf{u}, e_p) + G(p - \rho_h p, e_p) \\
&= E_1 + E_2 + E_3 + E_4.
\end{aligned} \tag{3.4}$$

Below we need to estimate $E_1 \sim E_4$. From Lemma 2.2, (2.15) and (2.18), we have

$$\begin{aligned}
E_1 &= (\nabla(\mathbf{u} - \Pi_h \mathbf{u}), \nabla e_{\mathbf{u}}) + \sum_{K\in T_h}\int_{\partial K}\frac{\partial(\mathbf{u} - \Pi_h \mathbf{u})}{\partial n}\cdot(\gamma e_{\mathbf{u}} - e_{\mathbf{u}})ds \\
&+ \sum_{K\in T_h}(\triangle \mathbf{u}, e_{\mathbf{u}} - \gamma e_{\mathbf{u}})_K \\
&\le Ch\|\mathbf{u}\|_2\|e_{\mathbf{u}}\|_1 + \sum_{K\in T_h}\int_{\partial K}\frac{\partial(\mathbf{u} - \Pi_h \mathbf{u})}{\partial n}\cdot(\gamma e_{\mathbf{u}} - e_{\mathbf{u}})ds \le Ch\|\mathbf{u}\|_2\|e_{\mathbf{u}}\|_1,
\end{aligned}$$

where we have used (2.16) and (2.19) to estimate

$$\begin{aligned}
&\sum_{K\in T_h}\int_{\partial K}\frac{\partial(\mathbf{u} - \Pi_h \mathbf{u})}{\partial n}\cdot(\gamma e_{\mathbf{u}} - e_{\mathbf{u}})ds \\
&\le \sum_{K\in T_h} Ch_K^{-\frac{1}{2}}\left(\|\mathbf{u} - \Pi_h \mathbf{u}\|_{1,K} + h_K\|\mathbf{u}\|_{2,K}\right)\|\gamma e_{\mathbf{u}} - e_{\mathbf{u}}\|_{0,\partial K} \le Ch\|\mathbf{u}\|_2\|e_{\mathbf{u}}\|_1.
\end{aligned}$$

Next, by using Lemma 2.3, (3.1), (2.18)–(2.19) and the trace inequality, we have (noting that $\rho_h p|_K = constant$)

$$\begin{aligned} E_2 &= -(\mathrm{div} e_{\mathbf{u}}, p - \rho_h p) + \sum_{K \in T_h} (\nabla(p - \rho_h p), e_{\mathbf{u}} - \gamma e_{\mathbf{u}})_K \\ &\quad + \sum_{K \in T_h} \int_{\partial K} (p - \rho_h p)(e_{\mathbf{u}} - \gamma e_{\mathbf{u}}) \cdot n ds \\ &\le Ch\|e_{\mathbf{u}}\|_1 \|p\|_1 + \sum_{K \in T_h} Ch_K^{-\frac{1}{2}} \left(\|p - \rho_h p\|_{0,K} + h_K \|\nabla p\|_{0,K}\right) \|\gamma e_{\mathbf{u}} - e_{\mathbf{u}}\|_{0,\partial K} \\ &\le Ch\|e_{\mathbf{u}}\|_1 \|p\|_1. \end{aligned}$$

Since $\mathbf{u} - \Pi_h \mathbf{u} \in [H_0^1(\Omega)]^2$ and $e_p|_K = constant$, then

$$\begin{aligned} E_3 &= (\mathrm{div}(\mathbf{u} - \Pi_h \mathbf{u}), e_p) = \sum_{K \in T_h} \int_{\partial K} (\mathbf{u} - \Pi_h \mathbf{u}) \cdot n\, e_p\, ds \\ &= \sum_{e \in \Gamma_h^0} \int_e (\mathbf{u} - \Pi_h \mathbf{u})[e_p] ds \le \left(\int_{\Gamma_h^0} h_\mu^{-1} |\mathbf{u} - \Pi_h \mathbf{u}|^2 ds \right)^{\frac{1}{2}} \left(\int_{\Gamma_h^0} h_\mu |[e_p]|^2 ds \right)^{\frac{1}{2}} \\ &\le Ch\|\mathbf{u}\|_2 \left(\int_{\Gamma_h^0} h_\mu |[e_p]|^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Furthermore

$$E_4 \le \left(\int_{\Gamma_h^0} h_\mu |[p - \rho_h p]|^2 ds \right)^{\frac{1}{2}} \left(\int_{\Gamma_h^0} h_\mu |[e_p]|^2 ds \right)^{\frac{1}{2}} \le Ch\|p\|_1 \left(\int_{\Gamma_h^0} h_\mu |[e_p]|^2 ds \right)^{\frac{1}{2}}.$$

Substituting estimates $E_1 \sim E_4$ into (3.4), it yields

$$|||(e_{\mathbf{u}}, e_p)|||^2 \le Ch(\|\mathbf{u}\|_2 + \|p\|_1)|||(e_{\mathbf{u}}, e_p)|||.$$

The proof is completed by using the triangle inequality. □

Theorem 3.1 gives the optimal error estimate in the H^1-norm for velocity approximation. In order to obtain the L_2-error estimate for pressure approximation, we introduce the following auxiliary problem: For $p - p_h \in M$, there exists $\mathbf{w} \in X$ such that (see [21, Chapter 1, Lemma 3.2])

$$\mathrm{div}\mathbf{w} = p - p_h, \quad \|\mathbf{w}\|_1 \le C\|p - p_h\|. \tag{3.5}$$

Let $R_h \mathbf{w} \in X_h$ be the finite element elliptic projection of $\mathbf{w} \in X$,

$$(\nabla(\mathbf{w} - R_h \mathbf{w}), \nabla \mathbf{v}_h) = 0, \ \forall\ \mathbf{v}_h \in X_h. \tag{3.6}$$

It is well-known that $R_h \mathbf{w}$ satisfies

$$\|R_h \mathbf{w}\|_1 \le C\|\mathbf{w}\|_1, \quad \|\mathbf{w} - R_h \mathbf{w}\| + h\|\mathbf{w} - R_h \mathbf{w}\|_1 \le Ch\|\mathbf{w}\|_1. \tag{3.7}$$

Theorem 3.2 *Let* $(\mathbf{u}, p) \in X \times M$ *and* $(\mathbf{u}_h, p_h) \in X_h \times M_h$ *be the solutions of problems* (2.1) $\sim$ (2.3) *and* (2.11)$\sim$(2.12)*, respectively, and* $(\mathbf{u}, p) \in [H^2(\Omega)]^2 \times H^1(\Omega)$*. Then, we have*

$$\|p - p_h\| \leq Ch(\|\mathbf{u}\|_2 + \|p\|_1). \tag{3.8}$$

Proof Using problem (3.5), we have from Lemma 2.3 and error equation (2.13) that

$$\begin{aligned}
&\|p - p_h\|^2 = (\mathrm{div}\mathbf{w}, p - p_h) = (\mathrm{div}(\mathbf{w} - R_h\mathbf{w}), p - p_h) + (\mathrm{div}R_h\mathbf{w}, p - p_h)\\
&= (\mathrm{div}(\mathbf{w} - R_h\mathbf{w}), p - p_h) - b(p - p_h, \gamma R_h\mathbf{w})\\
&\quad + \sum_{K\in T_h}(\nabla(p - p_h), \gamma R_h\mathbf{w} - R_h\mathbf{w})_K + \sum_{K\in T_h}\int_{\partial K}(p - p_h)(R_h\mathbf{w} - \gamma R_h\mathbf{w})\cdot n ds\\
&= (\mathrm{div}(\mathbf{w} - R_h\mathbf{w}), p - p_h) + a(\mathbf{u} - \mathbf{u}_h, \gamma R_h\mathbf{w})\\
&\quad + \sum_{K\in T_h}(\nabla p, \gamma R_h\mathbf{w} - R_h\mathbf{w})_K + \sum_{K\in T_h}\int_{\partial K}(p - p_h)(R_h\mathbf{w} - \gamma R_h\mathbf{w})\cdot n ds\\
&= F_1 + F_2 + F_3 + F_4.
\end{aligned} \tag{3.9}$$

Below we estimate $F_1 \sim F_4$. First, by using the divergence formula, we obtain

$$\begin{aligned}
F_1 &= -\sum_{K\in T_h}(\mathbf{w} - R_h\mathbf{w}), \nabla(p - p_h))_K\\
&\quad + \sum_{K\in T_h}\int_{\partial K}(\mathbf{w} - R_h\mathbf{w})\cdot n(p - p_h)ds = F_{11} + F_{12}.
\end{aligned}$$

For F_{11}, we have from equations (3.5) and (3.7) that

$$F_{11} = (\mathbf{w} - R_h\mathbf{w}, \nabla p) \leq Ch\|\mathbf{w}\|_1\|p\|_1 \leq Ch\|p - p_h\|\,\|p\|_1.$$

Since $\mathbf{w} - R_h\mathbf{w} \in [H_0^1(\Omega)]^2$, we obtain from equations (2.16), (3.5), (3.7) and (3.2) that

$$\begin{aligned}
F_{12} &= \int_{\Gamma_h^0}(\mathbf{w} - R_h\mathbf{w})[p - p_h]ds\\
&\leq \left(\int_{\Gamma_h^0} h_\mu^{-1}|\mathbf{w} - R_h\mathbf{w}|^2 ds\right)^{\frac{1}{2}}\left(\int_{\Gamma_h^0} h_\mu|[p - p_h]|^2 ds\right)^{\frac{1}{2}}\\
&\leq Ch^{-1}(\|\mathbf{w} - R_h\mathbf{w}\| + h\|\nabla(\mathbf{w} - R_h\mathbf{w})\|)|||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||\\
&\leq C(\|\mathbf{w}\|_1 + \|R_h\mathbf{w}\|_1)h(\|\mathbf{u}\|_2 + \|p\|_1) \leq Ch\|p - p_h\|(\|\mathbf{u}\|_2 + \|p\|_1).
\end{aligned}$$

Combining estimate F_{11} with F_{12}, we obtain

$$F_1 \leq Ch(\|\mathbf{u}\|_2 + \|p\|_1)\|p - p_h\|.$$

Next, similar to the estimate of $E_1 = a(\mathbf{u} - R_h\mathbf{u}, \gamma e_{\mathbf{u}})$ in Theorem 3.1 and using the known result: $\|\mathbf{u} - \mathbf{u}_h\|_1 \leq Ch(\|\mathbf{u}\|_2 + \|p\|_1)$, we obtain

$$F_2 \leq Ch(\|\mathbf{u}\|_2 + \|p\|_1)\|R_h\mathbf{w}\|_1 \leq Ch(\|\mathbf{u}\|_2 + \|p\|_1)\|p - p_h\|.$$

Now, it follows from equations (2.18), (3.5) and (3.7) that

$$F_3 \leq Ch\|p\|_1\|R_h\mathbf{w}\|_1 \leq Ch\|p\|_1\|p - p_h\|.$$

For F_4, using equations (2.17), (2.19) and the trace inequality to obtain

$$\begin{aligned}
F_4 &= \sum_{K\in T_h}\int_{\partial K} p(R_h\mathbf{w}-\gamma R_h\mathbf{w})\cdot n ds\\
&= \sum_{K\in T_h}\int_{\partial K} (p-\rho_h p)(R_h\mathbf{w}-\gamma R_h\mathbf{w})\cdot n ds\\
&\leq \sum_{K\in T_h} h_K^{-\frac{1}{2}}(\|p-\rho_h p\|_{0,K}+h_K\|p\|_{1,K})\|R_h\mathbf{w}-\gamma R_h\mathbf{w}\|_{0,\partial K}\\
&\leq Ch\|p\|_1\|R_h\mathbf{w}\|_1\leq Ch\|p\|_1\|\mathbf{w}\|_1\leq Ch\|p\|_1\|p-p_h\|.
\end{aligned}$$

Substituting estimates $F_1\sim F_4$ into (3.9), the proof is completed. □

Now we are in the position to derive the L_2-error estimate for velocity approximation. For this purpose, we need the following lemma.

Lemma 3.1 *Let* $(\mathbf{u},p)\in X\times M$ *and* $(\mathbf{u}_h,p_h)\in X_h\times M_h$ *be the solutions of problems* (2.1) $\sim$ (2.3) *and* (2.11) $\sim$ (2.12)*, respectively, and* $(\mathbf{u},p)\in[H^2(\Omega)]^2\times H^1(\Omega)$, $\mathbf{f}\in[H^1(\Omega)]^2$. *Further let function* $\mathbf{v}\in[H^2(\Omega)\bigcap H_0^1(\Omega)]^2$ *and* $div\mathbf{v}=0$. *Then we have*

$$|(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla\mathbf{v})|\leq Ch^2(\|\mathbf{u}\|_2+\|p\|_1+\|\mathbf{f}\|_1)\|\mathbf{v}\|_2. \tag{3.10}$$

Proof By Lemma 2.2 we have

$$\begin{aligned}
&(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla\mathbf{v})=(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla(\mathbf{v}-\Pi_h\mathbf{v}))+(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla\Pi_h\mathbf{v})\\
&=(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla(\mathbf{v}-\Pi_h\mathbf{v}))+a(\mathbf{u}-\mathbf{u}_h,\gamma\Pi_h\mathbf{v})\\
&-\sum_{K\in T_h}\int_{\partial K}\frac{\partial(\mathbf{u}-\mathbf{u}_h)}{\partial n}\cdot(\gamma\Pi_h\mathbf{v}-\Pi_h\mathbf{v})ds-\sum_{K\in T_h}(\triangle\mathbf{u},\Pi_h\mathbf{v}-\gamma\Pi_h\mathbf{v})_K.
\end{aligned}$$

It implies from the error equation (2.13) and $-\triangle\mathbf{u}+\nabla p=\mathbf{f}$ that

$$\begin{aligned}
&(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla\mathbf{v})=(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla(\mathbf{v}-\Pi_h\mathbf{v}))-b(p-p_h,\gamma\Pi_h\mathbf{v})\\
&-\sum_{K\in T_h}\int_{\partial K}\frac{\partial(\mathbf{u}-\mathbf{u}_h)}{\partial n}\cdot(\gamma\Pi_h\mathbf{v}-\Pi_h\mathbf{v})ds+\sum_{K\in T_h}(\mathbf{f}-\nabla p,\Pi_h\mathbf{v}-\gamma\Pi_h\mathbf{v})_K\\
&=(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla(\mathbf{v}-\Pi_h\mathbf{v}))-\sum_{K\in T_h}\int_{\partial K}\frac{\partial(\mathbf{u}-\mathbf{u}_h)}{\partial n}\cdot(\gamma\Pi_h\mathbf{v}-\Pi_h\mathbf{v})ds\\
&+\sum_{K\in T_h}(\mathbf{f},\Pi_h\mathbf{v}-\gamma\Pi_h\mathbf{v})_K+\left[-b(p-p_h,\gamma\Pi_h\mathbf{v})-\sum_{K\in T_h}(\nabla(p-p_h),\Pi_h\mathbf{v}-\gamma\Pi_h\mathbf{v})_K\right]\\
&=S_1+S_2+S_3+S_4.
\end{aligned}\tag{3.11}$$

We need to estimate $S_1\sim S_4$. First, from Theorem 3.1 we have

$$S_1\leq\|\nabla(\mathbf{u}-\mathbf{u}_h)\|\|\nabla(\mathbf{v}-\Pi_h\mathbf{v})\|\leq Ch^2(\|\mathbf{u}\|_2+\|p\|_1)\|\mathbf{v}\|_2.$$

Next, it follows from equation (2.17) that

$$S_2 = -\sum_{K\in T_h}\int_{\partial K}\frac{\partial(\mathbf{u}-\mathbf{u}_h)}{\partial n}\cdot(\gamma\Pi_h\mathbf{v}-\Pi_h\mathbf{v})ds = -\sum_{K\in T_h}\int_{\partial K}\frac{\partial\mathbf{u}}{\partial n}\cdot(\gamma\Pi_h\mathbf{v}-\Pi_h\mathbf{v})ds.$$

Observing that $\gamma\Pi_h\mathbf{v}-\Pi_h\mathbf{v}$ is continuous across the element edge (except at the midpoint of the edge), $(\gamma\Pi_h\mathbf{v}-\Pi_h\mathbf{v})|_{\partial\Omega}=0$, $\nabla\mathbf{u}\in[H^1(\Omega)]^2$, and any edge $e\in\Gamma_h^0$ is a common edge of two adjacent elements with opposite unit normal vectors on the edge, we deduce that

$$S_2 = -\sum_{K\in T_h}\int_{\partial K}\frac{\partial\mathbf{u}}{\partial n}\cdot(\gamma\Pi_h\mathbf{v}-\Pi_h\mathbf{v})ds = 0.$$

Again using equation (2.17) to obtain

$$S_3 = \sum_{K\in T_h}(\mathbf{f}-\mathbf{f}_h,\Pi_h\mathbf{v}-\gamma\Pi_h\mathbf{v})_K \le Ch^2\|\mathbf{f}\|_1\|\Pi_h\mathbf{v}\|_1 \le Ch^2\|\mathbf{f}\|_1\|\mathbf{v}\|_2,$$

where $\mathbf{f}_h$ is the piecewise constant approximation of function $\mathbf{f}$ on T_h. For S_4, we have from Lemma 2.3, $\mathrm{div}\mathbf{v}=0$ and (2.17) that

$$\begin{aligned} S_4 &= (\mathrm{div}\Pi_h\mathbf{v}, p-p_h) - \sum_{K\in T_h}\int_{\partial K}(p-p_h)(\Pi_h\mathbf{v}-\gamma\Pi_h\mathbf{v})\cdot n ds \\ &= (\mathrm{div}(\Pi_h\mathbf{v}-\mathbf{v}), p-p_h) - \sum_{K\in T_h}\int_{\partial K}p(\Pi_h\mathbf{v}-\gamma\Pi_h\mathbf{v})\cdot n ds. \end{aligned}$$

Similar to the argument for S_2, it is easy to see that for $p\in H^1(\Omega)$,

$$\sum_{K\in T_h}\int_{\partial K}p(\Pi_h\mathbf{v}-\gamma\Pi_h\mathbf{v})\cdot n ds = 0.$$

So it follows from Theorem 3.2 that

$$S_4 \le \|\Pi_h\mathbf{v}-\mathbf{v}\|_1\|p-p_h\| \le Ch^2(\|\mathbf{u}\|_2+\|p\|_1)\|\mathbf{v}\|_2.$$

Substituting estimates $S_1\sim S_4$ into (3.11), the proof is completed. □

Now we can give the optimal error estimate for $\mathbf{u}-\mathbf{u}_h$ in the L_2-norm. Introduce the following auxiliary problem [21, Chapter 1, Theorem 5.2]: $(\mathbf{w},q)\in[H^2(\Omega)]^2\times(H^1(\Omega)\bigcap M)$ such that

$$\begin{cases} -\triangle\mathbf{w}+\nabla q = \mathbf{u}-\mathbf{u}_h, \ in \ \ \Omega, \\ \mathrm{div}\mathbf{w}=0, in \ \ \Omega, \ \ \mathbf{w}=0, \ \ on \ \ \partial\Omega, \end{cases} \tag{3.12}$$

and

$$\|\mathbf{w}\|_2+\|q\|_1 \le C\|\mathbf{u}-\mathbf{u}_h\|. \tag{3.13}$$

Theorem 3.3 *Let* $(\mathbf{u},p)\in X\times M$ *and* $(\mathbf{u}_h,p_h)\in X_h\times M_h$ *be the solutions of problems (2.1)* $\sim$ *(2.3) and (2.11)* $\sim$ *(2.12), respectively, and* $(\mathbf{u},p)\in[H^2(\Omega)]^2\times H^1(\Omega)$, $\mathbf{f}\in[H^1(\Omega)]^2$. *Then we have*

$$\|\mathbf{u}-\mathbf{u}_h\| \le Ch^2(\|\mathbf{u}\|_2+\|p\|_1+\|\mathbf{f}\|_1). \tag{3.14}$$

Proof Using the auxiliary problem (3.12), we have from Lemma 3.1 and Theorem 3.1 that

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|^2 &= (\nabla\mathbf{w}, \nabla(\mathbf{u}-\mathbf{u}_h)) - (q, \operatorname{div}(\mathbf{u}-\mathbf{u}_h))\\ &= (\nabla(\mathbf{u}-\mathbf{u}_h), \nabla\mathbf{w}) - (q-\rho_h q, \operatorname{div}(\mathbf{u}-\mathbf{u}_h)) - (\rho_h q, \operatorname{div}(\mathbf{u}-\mathbf{u}_h))\\ &\le Ch^2(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{f}\|_1)(\|\mathbf{w}\|_2 + \|q\|_1) - (\rho_h q, \operatorname{div}(\mathbf{u}-\mathbf{u}_h)). \quad (3.15)\end{aligned}$$

From error equation (2.14) we obtain for $q \in H^1(\Omega)$ that

$$\begin{aligned}-(\rho_h q, \operatorname{div}(\mathbf{u}-\mathbf{u}_h)) &= \int_{\Gamma_h^0} h_\mu [p-p_h]\cdot[\rho_h q]ds = \int_{\Gamma_h^0} h_\mu [p-p_h]\cdot[\rho_h q - q]ds\\ &\le \left(\int_{\Gamma_h^0} h_\mu |[p-p_h]|^2 ds\right)^{\frac{1}{2}} \left(\int_{\Gamma_h^0} h_\mu |[q-\rho_h q]|^2 ds\right)^{\frac{1}{2}}\\ &\le Ch \left(\int_{\Gamma_h^0} h_\mu |[p-p_h]|^2 ds\right)^{\frac{1}{2}} \|q\|_1 \le Ch^2(\|\mathbf{u}\|_2 + \|p\|_1)\|q\|_1,\end{aligned}$$

where we have used Theorem 3.1. Substituting this estimate into (3.15), it yields

$$\|\mathbf{u}-\mathbf{u}_h\|^2 \le Ch^2(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{f}\|_1)(\|\mathbf{w}\|_2 + \|q\|_1).$$

The desired result is derived, noting that $\|\mathbf{w}\|_2 + \|q\|_1 \le C\|\mathbf{u}-\mathbf{u}_h\|$. □

Remark 3.1 The counterexamples in [9, 20] show that the assumption of $\mathbf{f} \in [H^1(\Omega)]^2$ in Theorem 3.3 is necessary for finite volume method in deriving the optimal order L_2-error estimate.

4 Further analysis on the stability

For finite element approximations to Stokes problems, an unstable scheme often leads to unphysical oscillation of the discrete solutions. In Theorem 2.1 we have given a stability estimate for the FVE solution $(\mathbf{u}_h, p_h)$ under the norm $|||(\mathbf{u}_h, p_h)|||$. From the definition of $|||(\mathbf{u}_h, p_h)|||$ (see equation (2.24)), we see that this stability estimate is weaker and h-dependent for pressure p_h. In this section, we will give a new and stronger stability estimate.

Definite the norm

$$|||(\mathbf{u}, p)|||_*^2 = \|\nabla\mathbf{u}\|^2 + \|p\|^2 + \int_{\Gamma_h^0} h_\mu |[p]|^2 ds. \quad (4.16)$$

Theorem 4.1 *There exists a constant $\beta > 0$ such that the following inf-sup condition holds.*

$$\beta|||(\mathbf{u}_h, p_h)|||_* \le \sup_{(\mathbf{v}_h, q_h)\in X_h\times M_h} \frac{B((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h))}{|||(\mathbf{v}_h, q_h)|||_*}, \quad (\mathbf{u}_h, p_h) \in X_h \times M_h. \quad (4.17)$$

Proof We first recall some known results. For any $p_h \in M_h (\subset M)$, there exists $\mathbf{w} \in X$ such that (see equation (3.5))

$$\mathrm{div}\mathbf{w} = p_h, \quad \|\mathbf{w}\|_1 \le C_0\|p_h\|. \tag{4.18}$$

Let $\mathbf{w}_h = R_h\mathbf{w} \in X_h$ be the finite element elliptic projection of function $\mathbf{w}$ (see equation (3.6)), which satisfies

$$\|\nabla\mathbf{w}_h\| \le \|\nabla\mathbf{w}\|, \quad \|\mathbf{w}-\mathbf{w}_h\| + h\|\mathbf{w}-\mathbf{w}_h\|_1 \le C_1 h\|\mathbf{w}\|_1. \tag{4.19}$$

Now, for any $0 < \alpha \le 1$, we have from equations (2.21), (2.23), 4.18 and 4.19 that

$$\begin{aligned}
&B((\mathbf{u}_h, p_h), (\mathbf{u}_h - \alpha\mathbf{w}_h, p_h))\\
&= a(\mathbf{u}_h, \gamma(\mathbf{u}_h - \alpha\mathbf{w}_h)) + b(p_h, \gamma(\mathbf{u}_h - \alpha\mathbf{w}_h)) + c(\mathbf{u}_h, p_h) + G(p_h, p_h)\\
&= (\nabla\mathbf{u}_h, \nabla(\mathbf{u}_h - \alpha\mathbf{w}_h)) + \alpha(\mathrm{div}\mathbf{w}_h, p_h) + G(p_h, p_h)\\
&= \|\nabla\mathbf{u}_h\|^2 - \alpha(\nabla\mathbf{u}_h, \nabla\mathbf{w}_h) + \alpha(\mathrm{div}(\mathbf{w}_h - \mathbf{w}), p_h) + \alpha\|p_h\|^2 + G(p_h, p_h)\\
&\ge (1 - \frac{1}{2}\alpha C_0^2)\|\nabla\mathbf{u}_h\|^2 + \frac{1}{2}\alpha\|p_h\|^2 + \alpha(\mathrm{div}(\mathbf{w}_h - \mathbf{w}), p_h) + \int_{\Gamma_h^0} h_\mu|[p_h]|^2 ds.
\end{aligned}$$

Next, by using the divergence formula and noting that $\mathbf{w}_h - \mathbf{w} \in [H_0^1(\Omega)]^2$, we have

$$\begin{aligned}
&\alpha(\mathrm{div}(\mathbf{w}_h - \mathbf{w}), p_h) = \alpha\sum_{K\in T_h}\int_{\partial K}(\mathbf{w}_h - \mathbf{w})\cdot n p_h ds\\
&= \alpha\sum_{e\in\Gamma_h^0}\int_e(\mathbf{w}_h - \mathbf{w})\cdot[p_h]ds \le \alpha\left(\int_{\Gamma_h^0} h_\mu^{-1}|\mathbf{w}_h - \mathbf{w}|^2\right)^{\frac{1}{2}}\left(\int_{\Gamma_h^0} h_\mu|[p_h]|^2\right)^{\frac{1}{2}}\\
&\le \alpha C h^{-1}(\|\mathbf{w}_h - \mathbf{w}\| + h\|\mathbf{w}_h - \mathbf{w}\|_1)\left(\int_{\Gamma_h^0} h_\mu|[p_h]|^2\right)^{\frac{1}{2}}\\
&\le \alpha C C_1\|\mathbf{w}\|_1\left(\int_{\Gamma_h^0} h_\mu|[p_h]|^2\right)^{\frac{1}{2}} \le \alpha C_2\|p_h\|\left(\int_{\Gamma_h^0} h_\mu|[p_h]|^2\right)^{\frac{1}{2}}\\
&\le \frac{1}{2}(\alpha C_2)^2\|p_h\|^2 + \frac{1}{2}\int_{\Gamma_h^0} h_\mu|[p_h]|^2,
\end{aligned}$$

where we have used the trace inequality, (4.18) and (4.19). Combining the above estimates we obtain

$$\begin{aligned}
&B((\mathbf{u}_h, p_h), (\mathbf{u}_h - \alpha\mathbf{w}_h, p_h))\\
&\ge (1 - \frac{1}{2}\alpha C_0^2)\|\nabla\mathbf{u}_h\|^2 + \frac{1}{2}\alpha(1 - \alpha C_2^2)\|p_h\|^2 + \frac{1}{2}\int_{\Gamma_h^0} h_\mu|[p_h]|^2 ds.
\end{aligned} \tag{4.20}$$

Taking α small enough, it implies from (4.20) that

$$B((\mathbf{u}_h, p_h), (\mathbf{u}_h - \alpha\mathbf{w}_h, p_h)) \ge C_\alpha|||(\mathbf{u}_h, p_h)|||_*^2, \quad \forall\,(\mathbf{u}_h, p_h) \in X_h \times M_h. \tag{4.21}$$

On the other hand, from (4.18) and (4.19) we have

$$\|\nabla(\mathbf{u_h} - \alpha\mathbf{w}_h)\| \le \|\nabla\mathbf{u}_h\| + \|\nabla\mathbf{w}_h\| \le \|\nabla\mathbf{u}_h\| + C_0\|p_h\|.$$

Hence

$$|||(\mathbf{u}_h - \alpha \mathbf{w}_h, p_h)|||_* \le (1 + C_0)|||(\mathbf{u}_h, p_h)|||_*, \quad \forall\, (\mathbf{u}_h, p_h) \in X_h \times M_h. \tag{4.22}$$

Now, it follows from equations (4.21) and (4.22) that

$$\sup_{(\mathbf{v}_h, q_h) \in X_h \times M_h} \frac{B((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h))}{|||(\mathbf{v}_h, q_h)|||_*} \ge \frac{B((\mathbf{u}_h, p_h), (\mathbf{u}_h - \alpha \mathbf{w}_h, p_h))}{|||(\mathbf{u}_h - \alpha \mathbf{w}_h, p_h)|||_*}$$
$$\ge \frac{C_\alpha |||(\mathbf{u}_h, p_h)|||_*^2}{|||(\mathbf{u}_h - \alpha \mathbf{w}_h, p_h)|||_*} \ge \frac{C_\alpha |||(\mathbf{u}_h, p_h)|||_*}{(1 + C_0)}.$$

This gives the inf-sup condition (4.17) with $\beta = C_\alpha/(1 + C_0)$. □

From Theorem 4.1 and the FVE (2.26), we immediately obtain the stability result: there exists a positive constant $C > 0$ such that the FVE solution $(\mathbf{u}_h, p_h)$ satisfies

$$|||(\mathbf{u}_h, p_h)|||_* \le C\|\mathbf{f}\|.$$

5 Numerical experiment

In this section, we will present some numerical results to illustrate our theoretical analysis.

Let us consider problem (2.1)~(2.3) with the exact solution $(\mathbf{u}, p)$:

$$\mathbf{u}(x) = (u_1(x_1, x_2), u_2(x_1, x_2)), \quad u_1 = 2\pi \sin^2(\pi x_1) \sin(\pi x_2) \cos(\pi x_2), \tag{5.1}$$
$$u_2 = -2\pi \sin^2(\pi x_2) \sin(\pi x_1) \cos(\pi x_1), \quad p(x_1, x_2) = \cos(\pi x_1) \cos(\pi x_2), \tag{5.2}$$

and the corresponding volumetric force $\mathbf{f} = -\nu \Delta \mathbf{u} + \nabla p$, $\nu = 0.1$ and $\Omega = (0, 1)^2$.

First, we express the discrete system (2.11)~(2.12) as a linear algebraic systems in the following form

$$\begin{pmatrix} A & -B \\ B^T & G \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}, \tag{5.3}$$

where the matrices A, B and G are, respectively, deduced from the bilinear $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ and $G(\cdot, \cdot)$ in the usual manner, and F is the variation of the source term. In particular, the stabilized matrix $G = (g_{ij})_{M \times M}$ is computed by (see equation (2.10))

$$g_{ij} = G(\chi_i, \chi_j) = \int_{\Gamma_h^0} h_\mu [\chi_i] \cdot [\chi_j] ds = \sum_{e \in \Gamma_h^0} \mu h_e^2 [\chi_i]_e \cdot [\chi_j]_e, \; 1 \le i, j \le M, \tag{5.4}$$

where $\{\chi_j\}$ are the basis functions of space M_h. In general, the piecewise constant basis function χ_m (corresponding to element K_m) is local such that for edge $e = \partial K_i \cap \partial K_j$, $[\chi_m]_e = 0$ if $m \ne i, j$. Thus, we see that the stabilized matrix G can be generated easily with very little computation cost. In general, large parameter μ in equation (5.4) may enhance the stability of the discrete system (see equation (2.28)), but also may magnify the error bound (see the error estimates). Therefore, if not necessary, we usually take $\mu = 1$.

Table 1 Convergence rates of gradient and pressure approximations

	$\|\mathbf{u}-\mathbf{u}_h\|$		$\|p-p_h\|$	
mesh h	error	rate	error	rate
1/8	0.5443	–	0.8379	–
1/16	$1.465e-1$	1.893	$4.664e-1$	0.845
1/32	$3.751e-2$	1.966	$2.421e-1$	0.946
1/64	$0.968e-2$	1.954	$1.251e-1$	0.952
1/128	$0.248e-2$	1.965	$6.487e-2$	0.948

In the numerical experiments, we take the pressure-correction algorithm to solve the discrete system (5.3):

$$AU - BP = F,\ B^T U + GP = 0.$$

The procedure is as follows:

Step 1 Choose a prediction pressure P^*;
Step 2 Solve U^* from equation $AU^* = BP^* + F$;
Step 3 Solve the correction quantity P' from equation $GP' = -B^T U^*$, and set the correction pressure $P = P' + P^*$;
Step 4 Solve the correction velocity from equation $AU = BP + F$;
Step 5 If $\|U-U^*\|+\|P-P^*\| \le \varepsilon$ (tolerance of error), output (U, P); otherwise, set $P^* = P$ and return to Step 2.

We partition domain Ω into a uniform triangulation T_h made of triangle meshes. The refined meshes of T_h are obtained by connecting the midpoints of each edge of elements in T_h. Denote by e_h the computation error in the L_2-norm, the numerical convergence rate r is computed by using the formula $r = \ln(e_h/e_{\frac{h}{2}})/\ln 2$. Table 1 gives the numerical results with successively halved mesh size h. We see that the convergence rates for the velocity and pressure approximations are just about $O(h^2)$ and $O(h)$, respectively, as the theoretical prediction. In particular, no oscillation phenomenon is observed in the pressure computation.

6 Conclusion

We present a stabilized finite volume method for Stokes problem using the lowest order $P_1 - P_0$ element pair. The stabilized method is designed by adding the jump term of the discrete pressure to the continuity approximation equation. A discrete inf-sup condition is established for the stabilized finite volume scheme which assures the stability of the discrete solutions. The optimal error estimates are derived in the H^1- and L_2-norms for velocity and in the L_2-norm for pressure, respectively. Obviously, our stabilized method includes the corresponding finite element method as a simplified situation ($T_h^* = T_h,\ V_h = X_h, \gamma = I$ case). Another important element in favor of our stabilized method is that it can be extended to the Navier-Stokes equations.

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