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An adaptive residual local projection finite element method for the Navier–Stokes equations

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Abstract This work proposes and analyses an adaptive finite element scheme for the fully non-linear incompressible Navier-Stokes equations. A residual a posteriori error estimator is shown to be effective and reliable. The error estimator relies on a Residual Local Projection (RELP) finite element method for which we prove well-posedness under mild conditions. Several well-established numerical tests assess the theoretical results.

Keywords Navier–Stokes equations · Stabilized finite element methods · A posteriori error estimates

Mathematics Subject Classifications (2010) 65N12 · 65N15 · 65N30

1 Introduction

A posteriori error analysis for adaptive finite element methods has been a very active and successful subject of research since the pioneering work of Babuska and

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Rheinboldt in [7]. In the context of fluid flow problems, researchers have been focused on improving numerical precision while making the computational cost affordable. For the Stokes problem we cite the relevant works by Verfürth [30], Bank and Welfert [8] and Ainsworth and Oden [1]. Regarding the Navier-Stokes equations it is worth mentioning the residual-based estimators proposed in [6, 12, 16, 21], the goal-oriented scheme in [11], and the hierarchical a posteriori error estimator in [5] and the ones based on local problem solutions in [20, 23] (see also [2, 32] for an overview).

Stabilized finite element methods for Navier-Stokes equations use equal-order pairs of interpolation spaces for the velocity and pressure. Well-balanced numerical diffusion may be also incorporated into such methods through the stabilization parameter. This is a crucial point when it comes to numerically solving advection dominated (high Reynolds number) flows (see [13, 17, 28], for instance). The association of stabilized methods with a posteriori error estimators greatly improves the quality of the numerical solutions while keeping the computational cost relatively low (see [3]). Such a feature is particularly attractive if one approximates solutions with multiple scales, as in the case of the non-linear Navier-Stokes equations.

Residual Local Projection (RELP) stabilized methods add new stabilization to the Galerkin method as a result of a space enriching strategy. First proposed in [9, 10] for the Stokes operator, and further extended to the fully non-linear Navier-Stokes equations in [4], these methods rely on the solution of element-wise problems. Such a local solution designs the stabilization parameter with the right dose of numerical diffusion and stabilizes the equal order and the simplest elements. In this work, we develop a new residual-based a posteriori error estimator for the non-linear incompressible Navier-Stokes equations. To this end, we consider a variation of the RELP method proposed in [4] for which we prove the existence and the uniqueness of the solution. Also, we prove that the new estimator is effective and reliable following closely the theory presented in Verfürth [31]. This variant of the RELP method keeps the good precision of the original method and turns out to be more suitable to build residual a posteriori error estimators. However, and unlike the original RELP method, its relationship with an enriching space strategy remains an open question.

The paper is organized as follows: Section 2 states the problem and introduces preliminary results. Section 3 presents the RELP method and a study of the existence and uniqueness of the discrete solution. The residual a posteriori error estimator is presented and analyzed in Section 4, followed by numerical experiments in Section 5.

2 Model problem and preliminary results

The steady incompressible Navier–Stokes problem consists of finding the velocity u and the pressure p solution of

$$
\text{(NS)} \qquad \begin{cases} -\nu\,\Delta u \;+\; (\nabla u)\,u \;+\; \nabla p = f & \text{in } \Omega, \\ \nabla\cdot u = 0 & \text{in } \Omega, \\ u = \mathbf{0} & \text{on } \partial\Omega, \end{cases}
$$

where $\Omega \subseteq \mathbb{R}^2$ is a polygonal open domain, $\nu \in \mathbb{R}^+$ is the fluid viscosity and $f \in L^2(\Omega)^2$ is a given function. We set $V := H_0^1(\Omega)^2$ and $Q := L_0^2(\Omega)$ and introduce the weak form of (NS): *Find* $(u, p) \in V \times Q$ *such that*

$$\nu\,(\nabla u, \nabla v) + ((\nabla u)\,u, v) - (p, \nabla \cdot v) - (q, \nabla \cdot u) = (f, v), \tag{1}$$

for all $(v, q) \in V \times Q$. Here $(\cdot, \cdot)$ stands for the $L^2(\Omega)$-inner product, where we use the same notation for vector, or tensor, valued functions.

Problem (1) may be rewritten in a more convenient form in view of analysis. To this end, consider the operator $F : V \times Q \longrightarrow (V \times Q)'$ defined by

$$\langle F(u, p), (v, q)\rangle := \nu\,(\nabla u, \nabla v) + ((\nabla u)\,u, v) - (p, \nabla \cdot v) - (q, \nabla \cdot u) - (f, v),$$

where $\langle\cdot,\cdot\rangle$ is the duality product in $(V \times Q)' \times (V \times Q)$. Note that (1) is equivalent to: *Find* $(u, p) \in V \times Q$ *such that*

$$\langle F(u, p), (v, q)\rangle = 0 \qquad \forall (v, q) \in V \times Q. \tag{2}$$

To present the discrete version of (2) and the numerical analysis of it, we need some notations and also some standard technical results. We denote the derivative of F at $(v, q) \in V \times Q$ by $DF(v, q) \in \mathscr{L}((V \times Q), (V \times Q)')$, where $\mathscr{L}((V \times Q), (V \times Q)')$ stands for the space of bounded linear mappings acting on elements of $V \times Q$ with values in $(V \times Q)'$ and equipped with the norm $\|\cdot\|_{\mathscr{L}((V\times Q),(V\times Q)')}$ with its usual meaning.

We assume that problem (2) has a regular solution (u, p) in the sense that $DF(u, p)$ is an isomorphism from $V \times Q$ onto $(V \times Q)'$ (see Chapter IV in [19]). Also, we assume that there is a constant $R_0 > 0$ such that (u, p) is unique in the ball $\mathbb{B}((u, p), R_0)$ (see Section IV.3.2 in [19]). Thereby, the differential operator $DF(u, p)$ is Lipschitz continuous at (u, p), i.e.,

$$\gamma := \sup_{(v,q)\in\mathbb{B}((u,p),R_0)} \frac{\|DF(v, q) - DF(u, p)\|_{\mathscr{L}((V\times Q),(V\times Q)')}}{\|(v - u, q - p)\|_{V\times Q}} < \infty.$$

We assume that $\{\mathscr{T}_h\}_{h>0}$ is a regular family of triangulations of Ω into triangles K with boundary ∂K and diameter $h_K := diam(K)$, and $h := \max\{h_K : K \in \mathscr{T}_h\}$. The set of internal edges F reads $\mathscr{E}_h$ and we define $h_F := |F|$. We denote by n the outward normal vector on ∂K; by $[\![v]\!]_F$ we mean the jump of v over F. Given $K \in \mathscr{T}_h$ and $F \in \mathscr{E}_h$, we denote by $\mathscr{N}(K)$ the set of nodes of K, $\mathscr{N}(F)$ the set of nodes of F, and $\mathscr{E}(K)$ the set of edges of K. Also, we define the following neighborhoods:

$$\widetilde{\omega}_K := \bigcup_{\mathscr{N}(K)\cap\mathscr{N}(K')\neq\emptyset} K', \quad , \quad \omega_F := \bigcup_{F\in\mathscr{E}(K')} K' \quad , \quad \widetilde{\omega}_F := \bigcup_{\mathscr{N}(F)\cap\mathscr{N}(K')\neq\emptyset} K'.$$

We denote by $\Pi_S(q)$ the average of a function $q \in L^2(S)$, over the domain $S \subset \mathbb{R}^2$, i.e.

$$\Pi_S(q) := \frac{1}{|S|} \int_S q\, dx.$$

The approximate velocity space V_h is composed of vector–valued piecewise linear continuous functions with zero trace on $\partial\Omega$. For the pressure, the approximate space Q_h is spanned by piecewise polynomial functions of degree l, $(l = 0, 1)$ with zero mean value on Ω. On such spaces, we use the Clément interpolation operator $\mathscr{I}_h : V \longrightarrow V_h$ and the operator $\mathscr{J}_h : Q \longrightarrow Q_h$, where $\mathscr{J}_h$ means either the modified Clément interpolation operator in the continuous pressure case $(l = 1)$ or the L^2 orthogonal projection onto the constant space $(l = 0)$. Such operators have the following approximability properties (see [14, 15] for details):

$$|v - \mathscr{I}_h v|_{m,K} \le C\, h_K^{l-m}\, |v|_{l,\widetilde{\omega}_K} \qquad \forall v \in H^l(\widetilde{\omega}_K)^2, \tag{3}$$

$$|\mathscr{I}_h v|_{1,K} \le C\, |v|_{1,\widetilde{\omega}_K} \qquad \forall v \in H^1(\widetilde{\omega}_K)^2, \tag{4}$$

$$\|v - \mathscr{I}_h v\|_{0,F} \le C\, h_F^{l-1/2}\, \|v\|_{l,\widetilde{\omega}_F} \qquad \forall v \in H^l(\widetilde{\omega}_F)^2, \tag{5}$$

$$|p - \mathscr{J}_h p|_{i,K} \le C h_K^{j-i} |p|_{j,\widetilde{\omega}_F} \qquad \forall p \in H^j(\widetilde{\omega}_K), \tag{6}$$

$$\|p - \mathscr{J}_h p\|_{0,F} \le C h_F^{j-1/2} \|p\|_{j,\widetilde{\omega}_F} \qquad \forall p \in H^j(\widetilde{\omega}_F), \tag{7}$$

where $0 \le m \le 2$, $\max\{m, 1\} \le l \le k + 1$, and $0 \le i \le 1$, $1 \le j \le k$. Hereafter, the positive constants C are independent of h but can assume different values in each occurrence.

We equip the space $V \times Q$ with the following product norm

$$\|(v, q)\| := \left\{ \nu\, |v|_{1,\Omega}^2 + \frac{1}{\nu}\, \|q\|_{0,\Omega}^2 \right\}^{1/2}.$$

Next, we recall some standard results which will be extensively used in the sequel.

Lemma 1 *Given $v \in H^1(K)^2$ it holds,*

$$\|v\|_{0,\partial K}^2 \le C \left\{ h_K^{-1}\, \|v\|_{0,K}^2 + h_K\, |v|_{1,K}^2 \right\}. \tag{8}$$

Proof See [27] for details. □

Lemma 2 *Given $v_h \in V_h$ and $p_h \in Q_h$, it holds*

$$\|v_h\|_{\infty,K} \le C\, h_K^{-1}\, \|v_h\|_{0,K}, \tag{9}$$

$$\|[\![p_h]\!]\|_{0,F} \le C\, h_F^{-1/2} \|p_h\|_{0,\omega_F}, \tag{10}$$

$$h_K\, |v_h|_{1,K} \le C\, \|v_h\|_{0,K}. \tag{11}$$

Proof Results (9) and (11) follow from Lemma 1.138 in [15], and (10) follows from the mesh regularity and Lemma 1. □

Lemma 3.

$$\|v - \Pi_K v\|_{0,K} \leq Ch_K |v|_{1,K} \qquad \forall v \in H^1(K), \tag{12}$$

$$\|\Pi_K v\|_{0,K} \leq C\|v\|_{0,K} \qquad \forall v \in L^2(K), \tag{13}$$

$$\|\Pi_K v\|_{\infty,K} \leq Ch_K^{-1}\|v\|_{0,K} \qquad \forall v \in L^2(K). \tag{14}$$

Proof Estimates (13) and (14) follow from Lemma 1.131 and Proposition 1.134 in [15], respectively. Estimate (14) is a consequence of mesh regularity, Cauchy-Schwarz's inequality and the definition of Π_K. □

Hereafter, we will use intensively the *fluctuation* operator χ_h defined by $\chi_h = \mathrm{I} - \Pi_K$, where I is the identity operator. Observe that, from Lemma 3, it holds

$$\|\chi_h(x \cdot \Pi_K v)\|_{0,K} \leq Ch_K \|v\|_{0,K} \qquad \forall v \in L^2(K)^2. \tag{15}$$

Now, we define functions with support on a triangle or on an edge which will be used to prove the local efficiency of the a posteriori error estimator. Given $K \in \mathscr{T}_h$, we introduce the elementary bubble function, b_K, by

$$b_K := 27 \prod_{x \in \mathscr{N}(K)} \lambda_x,$$

where λ_x denotes the barycentric coordinates associated to the vertex x. To define an edge bubble function, we denote by $\hat{K}$ the unitary reference triangle element and we set

$$b_{\hat{F}} := 4\hat{\lambda}_3\hat{\lambda}_1 \quad \text{on} \quad \hat{K},$$

where $\hat{F} := \{(t,0) \in \mathbb{R}^2 : 0 \leq t \leq 1\}$. Next, given $F \in \mathscr{E}_h$ and assuming that $\omega_F = K_1 \cup K_2$, let $G_{F,i}$ be the (orientation preserving) affine transformation (see Fig. 1) such that $G_{F,i}(\hat{K}) = K_i$ and $G_{F,i}(\hat{F}) = F$, $i = 1, 2$. Thus the bubble function associated to an edge F reads

$$b_F := \begin{cases} b_{\hat{F}} \circ G_{F,i}^{-1} & \text{on } K_i,\ i = 1, 2, \\ 0 & \text{on } \Omega \setminus \omega_F. \end{cases}$$

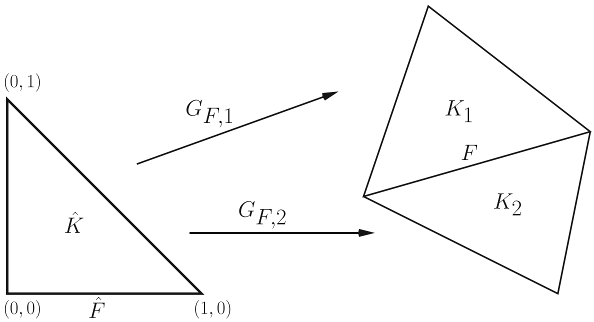

Fig. 1 Affine transformation $G_{F,i}$, $i = 1, 2$

Let $\hat{\Pi} := \{(x, 0) : x \in \mathbb{R}\}$ and $\hat{Q} : \mathbb{R}^2 \to \hat{\Pi}$ be the orthogonal projection. We introduce a lifting operator acting on functions defined on the reference element as follows $\hat{P}_{\hat{F}} : \mathbb{P}_k(\hat{F}) \to \mathbb{P}_k(\hat{K})$

$$\hat{s} \longmapsto \hat{P}_{\hat{F}}(\hat{s}) = \hat{s} \circ \hat{Q}.$$

Next, we propose the lifting operator on the real element $K_i \subseteq \omega_F$, $P_{F,K_i} : \mathbb{P}_k(F) \to \mathbb{P}_k(K_i)$, given by

$$P_{F,K_i}(s) = \hat{P}_{\hat{F}}(s \circ G_{F,i}) \circ G_{F,i}^{-1},$$

from which we define $P_F : \mathbb{P}_k(F) \to \mathbb{P}_k(\omega_F)$ by

$$s \in \mathbb{P}_k(F) \longmapsto P_F(s) := \begin{cases} P_{F,K_1}(s) & \text{in } K_1, \\ P_{F,K_2}(s) & \text{in } K_2. \end{cases}$$

If $\boldsymbol{s} := (s_1, s_2) \in \mathbb{P}_k(F)^2$, then we define $\mathscr{P}_F : \mathbb{P}_k^2(F) \to \mathbb{P}_k^2(\omega_F)$ by

$$\mathscr{P}_F(\boldsymbol{s}) = (P_F(s_1), P_F(s_2)).$$

From the previous definitions and using standard scaling arguments, the following equivalences hold.

Lemma 4 *Let $K \in \mathscr{T}_h$ and $F \in \mathscr{E}_h$. Given $v \in \mathbb{P}_k(K)$ and $s \in \mathbb{P}_l(F)$ with $k, l \geq 0$, the following estimates hold*

$$C\, \|v\|_{0,K} \leq \sup_{\substack{w \in \mathbb{P}_k(K) \\ w \neq 0}} \frac{(v, b_K w)}{\|w\|_{0,K}} \leq \|v\|_{0,K}, \tag{16}$$

$$C\, \|s\|_{0,F} \leq \sup_{\substack{r \in \mathbb{P}_k(F) \\ r \neq 0}} \frac{(s, b_F r)}{\|r\|_{0,F}} \leq \|s\|_{0,F}, \tag{17}$$

$$C h_K^{-1} \|b_K v\|_{0,K} \leq |b_K v|_{1,K} \leq C h_K^{-1} \|b_K v\|_{0,K}, \tag{18}$$

$$C h_K^{-1} \|b_F P_F(s)\|_{0,K} \leq |b_F P_F(s)|_{1,K} \leq C h_K^{-1} \|b_F P_F(s)\|_{0,K}, \tag{19}$$

$$\|b_F P_F(s)\|_{0,K} \leq C\, h_K^{1/2} \|s\|_{0,F}. \tag{20}$$

Proof See Lemma 5.1 in [31]. □

3 The residual local projection method

The stabilized finite element proposed in this section is a variant of the RELP method introduced in [4], being the boundary stabilization term the only difference. Observe that such a simple modification leads the method to be fully residual-based which makes it more appropriate to develop a residual error estimator. The RELP method in this work (written in a consistent form) reads: *Find $(u_h, p_h) \in V_h \times Q_h$ such that*

$$B((u_h, p_h), (v_h, q_h)) = F(v_h, q_h), \tag{21}$$

for all $(v_h, q_h) \in V_h \times Q_h$, where the form $B(\cdot,\cdot)$ is given by

$$
\begin{aligned}
B((u_h, p_h),(v_h,q_h)) := \;& \nu(\nabla u_h, \nabla v_h) + ((\nabla u_h)u_h, v_h) - (p_h, \nabla\cdot v_h) - (q_h, \nabla\cdot u_h)\\
&- \sum_{K\in\mathscr{T}_h} \frac{\alpha_K}{\nu}\,(\chi_h(x\cdot \Pi_K(-\nu\Delta u_h + (\nabla u_h)u_h + \nabla p_h))\,,\\
&\qquad \chi_h(x\cdot \Pi_K(-(\nabla v_h)u_h + \nabla q_h)))_K\\
&+ \sum_{K\in\mathscr{T}_h} \frac{\gamma_K}{\nu}\,(\chi_h(x\,\nabla\cdot u_h), \chi_h(x\,\nabla\cdot v_h))_K\\
&- \sum_{F\in\mathscr{E}_h} \tau_F\,\big([\![-\nu\,\partial_n u_h + p_h\,n]\!], [\![\nu\,\partial_n v_h + q_h\,n]\!]\big)_F\,,
\end{aligned}
$$

and $F(\cdot)$ by

$$
F(v_h,q_h) := (f, v_h) - \sum_{K\in\mathscr{T}_h} \frac{\alpha_K}{\nu}\,(\chi_h(x\cdot\Pi_K f), \chi_h(x\cdot\Pi_K(-(\nabla v_h)u_h + \nabla q_h)))_K\,.
$$

The element–wise stabilization parameters α_K and γ_K are given by

$$
\alpha_K := \frac{1}{\max\{1, Pe_K\}} \qquad \text{and} \qquad \gamma_K := \frac{1}{\max\left\{1, \dfrac{Pe_K}{24}\right\}},
$$

where

$$
Pe_K := \frac{|u_h|_K h_K}{18\,\nu} \quad \text{with} \quad |u_h|_K := \frac{\|u_h\|_{0,K}}{|K|^{\frac{1}{2}}}.
$$

Also, the edge–wise parameter τ_F is defined by

$$
\tau_F := \begin{cases} \dfrac{h_F}{12\,\nu} & \text{if } |u_h|_F = 0\,,\\[2ex] \dfrac{1}{2\,|u_h|_F} - \dfrac{1}{|u_h|_F\,(1-\exp(Pe_F))}\Big(1 + \dfrac{1}{Pe_F}(1-\exp(Pe_F)\Big) & \text{otherwise}\,. \end{cases}
$$

Here

$$
Pe_F := \frac{|u_h|_F h_F}{\nu} \quad \text{with} \quad |u_h|_F := \frac{\|u_h\|_{0,F}}{h_F^{1/2}}.
$$

We note that τ_F satisfies (see Lemma 2 in [10])

$$
\tau_F \le C\,\frac{h_F}{\nu}, \tag{22}
$$

for all $F \in \mathscr{E}_h$ with a positive constant C which is independent of h and ν.

Mimicking what was done in the continuos case, method (21) is reformulate using the operator $F_h : V_h \times Q_h \longrightarrow (V_h \times Q_h)'$ which is defined by

$$\begin{aligned}\langle F_h(u_h, p_h), (v_h, q_h)\rangle \;:=\; & \langle F(u_h, p_h), (v_h, q_h)\rangle \\ & -\frac{1}{\nu}\sum_{K\in\mathscr{T}_h}\Big[\alpha_K\,(\chi_h(x\cdot\Pi_K(-\nu\,\Delta u_h + (\nabla u_h)u_h + \nabla p_h - f)), \\ & \qquad \chi_h(x\cdot\Pi_K(\nabla q_h - (\nabla v_h)u_h)))_K + \gamma_K\,(\chi_h(x\nabla\cdot u_h), \\ & \qquad \chi_h(x\nabla\cdot v_h))_K\Big] \\ & -\sum_{F\in\mathscr{E}_h}\tau_F\,\big([\![-\nu\,\partial_n u_h + p_h n]\!], [\![\nu\,\partial_n v_h + q_h n]\!]\big)_F .\end{aligned}$$

As a result, (21) can be rewritten as follows: *Find* $(u_h, p_h) \in V_h \times Q_h$ *such that*

$$\langle F_h(u_h, p_h), (v_h, q_h)\rangle = 0 \qquad \forall (v_h, q_h) \in V_h \times Q_h.$$

3.1 Existence of the discrete solution

Before heading to the proof of the existence and the uniqueness of a solution for RELP method (21), we need some auxiliary results. In what follows, we shall use that $\Delta u_h = 0$ in each K.

We define the operator $\mathscr{P} : V_h \longrightarrow Q_h$ by

$$\begin{aligned}&\sum_{K\in\mathscr{T}_h}\frac{\alpha_K}{\nu}\big(\chi_h(x\cdot\Pi_K(\nabla\mathscr{P}(u_h))), \chi_h(x\cdot\Pi_K(\nabla q_h))\big)_K + \sum_{F\in\mathscr{E}_h}\tau_F\,\big([\![\mathscr{P}(u_h)]\!], [\![q_h]\!]\big)_F \\ &\quad = -(q_h, \nabla\cdot u_h) - \sum_{K\in\mathscr{T}_h}\frac{\alpha_K}{\nu}\,(\chi_h(x\cdot\Pi_K((\nabla u_h)u_h - f)), \chi_h(x\cdot\Pi_K(\nabla q_h)))_K \\ &\qquad - \sum_{F\in\mathscr{E}_h}\tau_F\,\big([\![-\nu\partial_n u_h]\!], [\![q_h\, n]\!]\big)_F ,\end{aligned} \tag{23}$$

for all $u_h \in V_h$, $q_h \in Q_h$. Observe that $\mathscr{P}$ is well-defined from Lax–Milgram's Theorem with the norm

$$\|q_h\|_* := \left\{\sum_{K\in\mathscr{T}_h}\frac{\alpha_K}{\nu}\|\chi_h(x\cdot\Pi_K(\nabla q_h))\|_{0,K}^2 + \sum_{F\in\mathscr{E}_h}\tau_F\,\|[\![q_h]\!]\|_{0,F}^2\right\}^{1/2}.$$

Also, define the mapping $\mathscr{N} : V_h \longrightarrow V_h$ by

$$\begin{aligned}(\mathscr{N}(u_h), v_h) \;=\; & \nu(\nabla u_h, \nabla v_h) + ((\nabla u_h)u_h, v_h) - (\mathscr{P}(u_h), \nabla\cdot v_h) - (f, v_h) \\ & + \sum_{K\in\mathscr{T}_h}\frac{\gamma_K}{\nu}\,(\chi_h(x\,\nabla\cdot u_h), \chi_h(x\,\nabla\cdot v_h))_K \\ & - \sum_{K\in\mathscr{T}_h}\frac{\alpha_K}{\nu}\,(\chi_h(x\cdot\Pi_K((\nabla u_h)u_h - f + \nabla\mathscr{P}(u_h)))\,, \\ & \qquad \chi_h(-x\cdot\Pi_K((\nabla v_h)u_h)))_K \\ & - \sum_{F\in\mathscr{E}_h}\tau_F\,\big([\![-\nu\partial_n u_h + \mathscr{P}(u_h)n]\!], [\![\nu\partial_n v_h]\!]\big)_F \quad \forall u_h, v_h \in V_h.\end{aligned}$$

The next result provides a characterization of the solution of problem (21) with respect to the operators $\mathscr{P}$ and $\mathscr{N}$.

Lemma 5 *The pair $(u_h, p_h) \in V_h \times Q_h$ is a solution of problem (21) if and only if $\mathscr{N}(u_h) = \mathbf{0}$ and $p_h = \mathscr{P}(u_h)$.*

Proof See Lemma 3.5 in [4]. □

We are now ready to prove the well-posedness of problem (21). The proof follows closely the arguments presented in [4].

Theorem 1 *There is a positive constant $\tilde{C}$, which is independent of h and ν, such that problem (21) admits at least one solution (u_h, p_h) provided*

$$\frac{h^{1-\kappa}}{\nu^{3/2}} \left\{ \frac{1}{\nu} \|f\|^2_{-1,\Omega} + \sum_{K\in\mathscr{T}_h} \frac{\alpha_K}{\nu} \|\chi_h(x \cdot \Pi_K f)\|^2_{0,K} \right\}^{1/2} \leq \tilde{C}\,, \tag{24}$$

where $0 < \kappa < 1$.

Proof Let $u_h \in V_h$, with $|u_h|_{1,\Omega} = R$, where R is a positive number that will be choose later. Denote

$$\begin{aligned}
a_1 &:= \left\{ \sum_{K\in\mathscr{T}_h} \frac{\alpha_K}{\nu} \|\chi_h(x \cdot \Pi_K((\nabla u_h)u_h + \nabla\mathscr{P}(u_h)))\|^2_{0,K} \right\}^{1/2}, \\
a_2 &:= \left\{ \sum_{F\in\mathscr{E}_h} \tau_F \,\left\| [\![-\nu\partial_n u_h + \mathscr{P}(u_h)n]\!] \right\|^2_{0,F} \right\}^{1/2}, \\
a_3 &:= \left\{ \frac{1}{\nu}\|f\|^2_{-1,\Omega} + \sum_{K\in\mathscr{T}_h} \frac{\alpha_K}{\nu} \|\chi_h(x \cdot \Pi_K f)\|^2_{0,K} \right\}^{1/2}, \\
a_4 &:= \left\{ \sum_{K\in\mathscr{T}_h} \frac{\gamma_K}{\nu} \|\chi_h(x\, \nabla \cdot u_h)\|^2_{0,K} \right\}^{1/2}.
\end{aligned}$$

Taking $q_h = \mathscr{P}(u_h)$ in Eq. 23 give us

$$\begin{aligned}
-(\mathscr{P}(u_h), \nabla\cdot u_h) = &\sum_{K\in\mathscr{T}_h} \frac{\alpha_K}{\nu} (\chi_h(x \cdot \Pi_K((\nabla u_h)u_h \\
&- f + \nabla\mathscr{P}(u_h))), \chi_h(x \cdot \Pi_K(\nabla\mathscr{P}(u_h))))_K \\
&+ \sum_{F\in\mathscr{E}_h} \tau_F \left([\![-\nu\partial_n u_h + \mathscr{P}(u_h)n]\!], [\![\mathscr{P}(u_h)n]\!] \right)_F.
\end{aligned}$$

From Cauchy-Schwarz's inequality and the identity $((\nabla u_h)u_h, u_h) = -\frac{1}{2}(\nabla \cdot u_h, u_h \cdot u_h)$, we get

$$\begin{aligned}
(\mathscr{N}(u_h), u_h) &= \nu |u_h|_{1,\Omega}^2 + ((\nabla u_h)u_h, u_h) - (f, u_h) \\
&\quad + \sum_{K \in \mathscr{T}_h} \frac{\alpha_K}{\nu} (\chi_h(x \cdot \Pi_K((\nabla u_h)u_h - f \\
&\quad + \nabla \mathscr{P}(u_h))), \chi_h(x \cdot \Pi_K((\nabla u_h)u_h + \nabla \mathscr{P}(u_h))))_K \\
&\quad + \sum_{K \in \mathscr{T}_h} \frac{\gamma_K}{\nu} \|\chi_h(x\, \nabla \cdot u_h)\|_{0,K}^2 \\
&\quad + \sum_{F \in \mathscr{E}_h} \tau_F \, \|[\![-\nu \partial_n u_h + \mathscr{P}(u_h) n]\!]\|_{0,F}^2 \\
&\geq \frac{\nu}{2} R^2 + \frac{1}{2} a_1^2 + a_2^2 + a_4^2 - \frac{1}{2} a_3^2 - \frac{1}{2} (\nabla \cdot u_h, u_h \cdot u_h). \qquad (25)
\end{aligned}$$

Now, if we take $q_h = \mathscr{J}_h(u_h \cdot u_h)$ in (23), use Cauchy–Schwarz's inequality, Lemma 3, the fact that $\alpha_K \leq 1$, (6), (15), (22), (7), mesh regularity, and following closely what was done in [4] (see page 10), we get

$$|(\nabla \cdot u_h, u_h \cdot u_h)| \leq \frac{C}{\sqrt{\nu}} \{\sqrt{\nu} R + a_1 + a_2 + a_3\} \left\{ \sum_{K \in \mathscr{T}_h} h_K^2 |u_h \cdot u_h|_{1,K}^2 \right\}^{1/2}. \quad (26)$$

Moreover, using the local inverse inequality $\|v_h\|_{\infty,K} \leq C h_K^{-\frac{2}{q}} \|v_h\|_{0,q,K}$ for all $1 \leq q \leq \infty$ (see [15]) and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for all $2 \leq q \leq \infty$, we obtain

$$|u_h \cdot u_h|_{1,K} \leq C |u_h|_{1,K} \, \|u_h\|_{\infty,K} \leq C h_K^{-\frac{2}{q}} |u_h|_{1,K} \, \|u_h\|_{q,K} \leq C h_K^{-\frac{2}{q}} |u_h|_{1,K} \, |u_h|_{1,\Omega} \,,$$

and then from Eqs. 25 and 26, it holds

$$\begin{aligned}
(\mathscr{N}(u_h), u_h) &\geq \frac{\nu}{2} R^2 + \frac{1}{2} a_1^2 + a_4^2 + a_2^2 - \frac{1}{2} a_3^2 - \frac{1}{2} (\nabla \cdot u_h, u_h \cdot u_h) \\
&\geq \frac{\nu}{2} R^2 + \frac{1}{2} a_1^2 + a_4^2 + a_2^2 - \frac{1}{2} a_3^2 - \frac{C}{2\sqrt{\nu}} \{\sqrt{\nu} R + a_1 + a_2 + a_3\} \\
&\quad \times \left\{ \sum_{K \in \mathscr{T}_h} h_K^{2-2\kappa} |u_h|_{1,K}^2 \right\}^{\frac{1}{2}} |u_h|_{1,\Omega} \\
&\geq \frac{\nu}{2} R^2 + \frac{1}{2} a_1^2 + a_4^2 + a_2^2 - \frac{1}{2} a_3^2 - \frac{C}{\sqrt{\nu}} h^{1-\kappa} \{\sqrt{\nu} R + a_1 + a_2 + a_3\} R^2 \\
&\geq \frac{\nu}{2} R^2 + \frac{1}{2} a_1^2 + a_4^2 + a_2^2 - \frac{1}{2} a_3^2 - C h^{1-\kappa} R^3
\end{aligned}$$

$$
\begin{aligned}
&-\frac{C}{\sqrt{\nu}} h^{1-\kappa}\{a_1+a_2+a_3\}\, R^2 \\
&\geq \frac{\nu}{2}R^2+\frac{1}{2}a_1^2+a_4^2+a_2^2-\frac{1}{2}a_3^2-Ch^{1-\kappa}R^3-\frac{1}{2}a_1^2-\frac{1}{2}a_2^2-\frac{1}{2}a_3^2 \\
&-\frac{3}{2}\frac{C^2}{\nu}h^{2(1-\kappa)}R^4 \\
&\geq \frac{\nu}{2}R^2+a_4^2+\frac{1}{2}a_2^2-a_3^2-Ch^{1-\kappa}R^3-\frac{3}{2}\frac{C^2}{\nu}h^{2(1-\kappa)}R^4,
\end{aligned}
$$

where $\kappa := \frac{2}{q}$. Now, set $R := \dfrac{\nu}{6Ch^{1-\kappa}}$ and $\tilde{C} = \dfrac{1}{12C}$. Note that assumption (24) leads to $a_3 \leq \dfrac{\sqrt{\nu}}{2}R$, thus

$$
\begin{aligned}
(\mathscr{N}(u_h), u_h) &\geq \left(\frac{1}{2}-\frac{1}{6}-\frac{3}{72}\right)\nu R^2 - a_3^2 + \frac{1}{2}a_2^2 + a_4^2 \\
&\geq \frac{\nu}{4}R^2 - a_3^2 + \frac{1}{2}a_2^2 + a_4^2 \geq \frac{1}{2}a_2^2 + a_4^2 \geq 0\,.
\end{aligned}
$$

Thus Brouwer's fixed point Theorem implies (see Corollary 1.1, Chapter IV, in [19]) the existence of $u_h \in V_h$ with $|u_h|_{1,\Omega} \leq R$ and $\mathscr{N}(u_h) = \mathbf{0}$. □

3.2 Uniqueness of the discrete solution

We prove a uniqueness result for method (21) under the diffusion dominated assumption (i.e. ν large enough). As such, we set $\alpha_K = \gamma_K = 1$ for all $K \in \mathscr{T}_h$, and assume that $\tau_F = \dfrac{h_F}{12\nu}$ for all $F \in \mathscr{E}_h$, since both expressions in Eq. 22 are equivalent in this regime (see Lemma 2 in [10] for details). Also, we use that $\Delta u_h = 0$ in each K. Thereby, under such simplifications, method (21) reads: *Find* $(u_h, p_h) \in V_h \times Q_h$ *such that*

$$
\begin{aligned}
&\nu\,(\nabla u_h, \nabla v_h) + ((\nabla u_h)u_h, v_h) - (p_h, \nabla\cdot v_h) - (q_h, \nabla\cdot u_h) \\
&\quad - \frac{1}{\nu}\sum_{K\in\mathscr{T}_h}(\chi_h(x\cdot\Pi_K((\nabla u_h)u_h+\nabla p_h)), \chi_h(x\cdot\Pi_K(-(\nabla v_h)u_h+\nabla q_h)))_K \\
&\quad + \frac{1}{\nu}\sum_{K\in\mathscr{T}_h}(\chi_h(x\,\nabla\cdot u_h), \chi_h(x\,\nabla\cdot v_h))_K \\
&\quad - \sum_{F\in\mathscr{E}_h}\frac{h_F}{12\,\nu}\left([\![-\nu\partial_n u_h + p_h n]\!], [\![\nu\partial_n v_h + q_h n]\!]\right)_F \\
&= (f, v_h) - \frac{1}{\nu}\sum_{K\in\mathscr{T}_h}(\chi_h(x\cdot\Pi_K f), \chi_h(x\cdot\Pi_K(-(\nabla v_h)u_h+\nabla q_h)))_K \\
&\quad \forall (v_h, q_h) \in V_h\times Q_h. \qquad (27)
\end{aligned}
$$

We first write (27) as a fixed point problem. To this end, we define $T_h : V' \times Q \longrightarrow V_h \times Q_h$, a discrete Stokes operator, which for each $(w, r) \in V' \times Q$, it associates the unique solution $(u_h, p_h) \in V_h \times Q_h$ of

$$A((u_h, p_h), (v_h, q_h)) = \langle w, v_h \rangle + (r, q_h), \tag{28}$$

for all $(v_h, q_h) \in V_h \times Q_h$, where $A(\cdot, \cdot)$ reads

$$\begin{aligned} A((u_h, p_h), (v_h, q_h)) := \; & \nu (\nabla u_h, \nabla v_h) - (p_h, \nabla \cdot v_h) - (q_h, \nabla \cdot u_h) \\ & -\frac{1}{\nu} \sum_{K \in \mathscr{T}_h} (\chi_h(x \cdot \Pi_K(\nabla p_h)), \chi_h(x \cdot \Pi_K(\nabla q_h)))_K \\ & +\frac{1}{\nu} \sum_{K \in \mathscr{T}_h} (\chi_h(x \, \nabla \cdot u_h), \chi_h(x \, \nabla \cdot v_h))_K \\ & - \sum_{F \in \mathscr{E}_h} \frac{h_F}{12\,\nu} \left([\![-\nu \partial_n u_h + p_h n]\!], [\![\nu \partial_n v_h + q_h n]\!] \right)_F . \end{aligned}$$

Also, we introduce the mapping $G_h : H^2(\mathscr{T}_h)^2 \times H^1(\mathscr{T}_h) \longrightarrow V_h \times Q_h$, where $(w_h, r_h) := G_h(z, t)$ solves

$$\begin{aligned} (w_h, v_h) + (r_h, q_h) = \; & -(f - (\nabla z) z, v_h) \\ & +\frac{1}{\nu} \sum_{K \in \mathscr{T}_h} (\chi_h(x \cdot \Pi_K(f - (\nabla z) z)), \chi_h(x \cdot \Pi_K(\nabla q_h)))_K \\ & -\frac{1}{\nu} \sum_{K \in \mathscr{T}_h} (\chi_h(x \cdot \Pi_K(f + \nu \Delta z - (\nabla z) z - \nabla t)), \\ & \chi_h(x \cdot \Pi_K((\nabla v_h) z)))_K , \end{aligned}$$

for all $(v_h, q_h) \in V_h \times Q_h$. Combining these operators, problem (27) is written as the following fixed point problem

$$- T_h G_h(u_h, p_h) = (u_h, p_h). \tag{29}$$

Before proving the uniqueness result for problem (21), we need to establish the well-posedness of problem (28). This result is presented in the next lemma.

Lemma 6 *The mapping T_h is well–defined.*

Proof Use (15) and the ideas of Lemma A.1 in [4]. □

Lemma 7 *The operator T_h is continuous. More precisely, there exists $C > 0$, independent of h and ν, such that*

$$\| T_h(w, r) \| \leq C \sqrt{\nu}\, (1 + h)^2 \, \| (w, r) \|_{(V_h \times Q_h)'} ,$$

for all $(w, r) \in (V \times Q)'$.

Proof Use the same arguments of Lemma A.2 of [4] and Eq. 15. □

We are ready to prove the uniqueness result. We recall that ν is assumed to be large enough so that (21) reduces to (27). Let $(u_h, p_h) \in V_h \times Q_h$ be a solution of (27), and observe that from (29), (u_h, p_h) corresponds to a fixed point of the operator $-T_h G_h$. The proof then reduces to prove that the operator $-T_h G_h$ is a strict contraction onto $B := \{(v_h, q_h) \in V_h \times Q_h : \|(v_h, q_h)\| \leq 1\}$ thus the result follows from Banach's fixed point Theorem.

Let $(u_h, p_h), (v_h, q_h) \in B$. Using Lemma 7, the definition of operators T_h and G_h and $\Delta u_h = 0$ in each K, it holds

$$
\begin{aligned}
&\|T_h G_h(u_h, p_h) - T_h G_h(v_h, q_h)\| = \|T_h(G_h(u_h, p_h) - G_h(v_h, q_h))\| \\
&\leq C\sqrt{\nu}(1+h)^2 \sup_{\|(w_h,t_h)\|\leq 1} \Big\{((\nabla u_h)u_h - (\nabla v_h)v_h, w_h) \\
&\quad - \frac{1}{\nu} \sum_{K\in\mathscr{T}_h} (\chi_h(x \cdot \Pi_K(f - (\nabla u_h)u_h - \nabla p_h)), \chi_h(x \cdot \Pi_K(\nabla w_h(u_h - v_h))))_K \\
&\quad - \frac{1}{\nu} \sum_{K\in\mathscr{T}_h} (\chi_h(-x \cdot \Pi_K((\nabla u_h)u_h - (\nabla v_h)v_h - \nabla(p_h - q_h)), \chi_h(x \cdot \Pi_K((\nabla w_h)v_h)))_K \\
&\quad - \frac{1}{\nu} \sum_{K\in\mathscr{T}_h} (\chi_h(x \cdot \Pi_K((\nabla v_h)v_h - (\nabla u_h)u_h)), \chi_h(x \cdot \Pi_K(\nabla t_h)))_K \Big\} \\
&= C\sqrt{\nu}(1+h)^2 \sup_{\|(w_h,t_h)\|\leq 1} \Big\{\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}\Big\}.
\end{aligned}
\tag{30}
$$

Regarding item I above, we use that $((\nabla u)w, v) \leq C\,|u|_{1,\Omega}|v|_{1,\Omega}|w|_{1,\Omega}$ for all $u, v, w \in V$, and the definition of the norm $\|\cdot\|$ to get

$$
\mathrm{I} \leq \frac{C}{\nu\sqrt{\nu}} \|(u_h, p_h) - (v_h, q_h)\| \|(w_h, t_h)\|. \tag{31}
$$

To bound items II, III and IV, we use the same arguments as in Appendix B of [5], and (15). Thus we obtain

$$
\mathrm{II} \leq \frac{C}{\nu^2} \left\{ h\|f\|_{0,\Omega} + \frac{1}{\nu} + \sqrt{\nu} \right\} \|(u_h, p_h) - (v_h, q_h)\| \|(w_h, t_h)\|, \tag{32}
$$

and

$$
\mathrm{III} \leq \frac{C}{\nu^2\sqrt{\nu}} \left\{ \frac{2}{\nu} + 1 \right\} \|(u_h, p_h) - (v_h, q_h)\| \|(w_h, t_h)\|, \tag{33}
$$

and

$$
\mathrm{IV} \leq \frac{C}{\nu^2} \|(u_h, p_h) - (v_h, q_h)\| \|(w_h, t_h)\|. \tag{34}
$$

Collecting the bounds (31), (32), (33) and (34), inequality (30) becomes

$$
\begin{aligned}
\|T_h G_h(u_h, p_h) - T_h G_h(v_h, q_h)\| \;\leq\; & \frac{C}{\nu} \left\{ 2 + \frac{5}{\sqrt{\nu}} + \frac{h}{\sqrt{\nu}} \|f\|_{0,\Omega} \right\} \\
& \times (1+h)^2 \, \|(u_h, p_h) - (v_h, q_h)\|,
\end{aligned}
$$

and, thus, the result follows under the assumption that ν is such that $\frac{C}{\nu} \left\{ 2 + \frac{5}{\sqrt{\nu}} + \frac{h}{\sqrt{\nu}} \right\} (1+h)^2 < 1$.

4 A residual error estimator

In this section, we propose a residual a posteriori error estimator for the method (21). The analysis follows mainly the ideas introduced by Verfürth in [31]. For sake of simplicity, we assume that

(F) f is a piecewise polynomial function, i.e., $f\Big|_K \in \mathbb{P}_l(K)^2$, $l \in \mathbb{N} \cup \{0\}$, $\forall K \in \mathscr{T}_h$.

It is worth mentioning that such an assumption may be relaxed. Indeed, if we only assume $f \in L^2(\Omega)^2$, for instance, estimates (38), (39) (see Theorem 3 below) will include a correction term of type $h_K \, \|f - \Pi_K f\|_{0,K}$, for $K \in \mathscr{T}_h$, which is in general a high order term.

To introduce the error estimator, we define for each $K \in \mathscr{T}_h$ and each $F \in \mathscr{E}_h$, the following residual quantities

$$\mathscr{R}_K := (f + \nu \Delta u_h - (\nabla u_h) u_h - \nabla p_h)_{|K} \quad \text{and} \quad \mathscr{R}_F := [\![-\nu \partial_n u_h + p_h n]\!]_F$$

Using these definitions, the residual–based error estimator reads

$$\eta := \left\{ \sum_{K \in \mathscr{T}_h} \eta_K^2 \right\}^{\frac{1}{2}}, \tag{35}$$

where

$$\eta_K^2 := \frac{h_K^2}{\nu} \|\mathscr{R}_K\|_{0,K}^2 + \nu \, \|\nabla \cdot u_h\|_{0,K}^2 + \frac{1}{2} \sum_{F \in \mathscr{E}(K) \cap \mathscr{E}_h} \frac{h_F}{\nu} \, \|\mathscr{R}_F\|_{0,F}^2.$$

The next result establishes a theoretical framework to develop the analysis of our a posteriori error estimator. Such a result is due to Verfürth (see Proposition 2.1 in [31]).

Theorem 2 *Let $(u, p) \in V \times Q$ be a regular solution of equation (2). Set*

$$R := \min \left\{ R_0, \gamma^{-1} \|\{DF(u,p)\}^{-1}\|_{\mathscr{L}((V\times Q)',(V\times Q))}^{-1}, 2\gamma^{-1} \|DF(u,p)\|_{\mathscr{L}((V\times Q),(V\times Q)')} \right\}.$$

Then, the following error estimates hold for all $(v_h, q_h) \in \mathbb{B}((u,p), R)$

$$\|(u - v_h, p - q_h)\| \le 2 \, \|\{DF(u,p)\}^{-1}\|_{\mathscr{L}((V\times Q)',(V\times Q))} \, \|F(v_h, q_h)\|_{(V\times Q)'}, \tag{36}$$

$$\|(u - v_h, p - q_h)\| \ge \frac{1}{2} \, \|DF(u,p)\|_{\mathscr{L}((V\times Q),(V\times Q)')}^{-1} \, \|F(v_h, q_h)\|_{(V\times Q)'}. \tag{37}$$

We are ready to present the main result of this section.

Theorem 3 *Let* (u, p) *be a regular solution of* (2) *and* (u_h, p_h) *be the solution of* (21)*. If we assume that* $(u_h, p_h) \in \mathbb{B}((u, p), R)$*, for* R *sufficiently small, then the following a posteriori error estimates hold*

$$\|(u - u_h, p - p_h)\| \le C_1 \max\left\{1, \frac{\|u_h\|_{0,\Omega}}{\nu}\right\} \eta_H, \tag{38}$$

$$\eta \le C_2 \|(u - u_h, p - p_h)\|, \tag{39}$$

where

$$\eta_H^2 := \sum_{K \in \mathcal{T}_h} \left[\eta_K^2 + \frac{h_K^4}{\nu^3} \|\nabla \cdot u_h\|_{0,K}^2\right],$$

and η *is defined in* (35)*. The positive constants* C_1 *and* C_2 *are independent of* h *and* ν*, but they may depend on* u *and* p*.*

Proof *Lower bound:* Define the finite–dimensional subspace $\mathcal{B}_h \subset V \times Q$ as follows

$$\mathcal{B}_h := \operatorname{span}\big\{(b_K v, 0), (b_F \mathcal{P}_F(\boldsymbol{s}), 0), (0, b_K r) : v \in \mathbb{P}_1(K)^2, \boldsymbol{s} \in \mathbb{P}_l(F)^2, r \in \mathbb{P}_0(K), \forall K \in \mathcal{T}_h, \forall F \in \mathcal{E}_h\big\},$$

with $l = 0, 1$. From Lemma 4, we get

$$\begin{aligned}\sqrt{\nu}\,\|\nabla \cdot u_h\|_{0,K} &\le C \sup_{r \in \mathbb{P}_0(K)\setminus\{0\}} \frac{(\nabla \cdot u_h, b_K r)_K}{\frac{1}{\sqrt{\nu}}\|r\|_{0,K}} \\ &\le C \sup_{r \in \mathbb{P}_0(K)\setminus\{0\}} \frac{\langle F(u_h, p_h), (0, b_K r)\rangle}{\frac{1}{\sqrt{\nu}}\|r\|_{0,K}} \\ &\le C \sup_{\substack{(v,q)\in \mathcal{B}_h|_K \\ \|(v,q)\|=1}} \langle F(u_h, p_h), (v, q)\rangle,\end{aligned} \tag{40}$$

and

$$\begin{aligned}\frac{1}{\sqrt{\nu}} h_K \|R_K\|_{0,K} &\le C \sup_{w \in \mathbb{P}_1(K)^2\setminus\{0\}} \frac{(R_K, b_K w)_K}{\sqrt{\nu}\,|b_K w|_{1,K}} \\ &\le C \sup_{w \in \mathbb{P}_1(K)^2\setminus\{0\}} \frac{\langle F(u_h, p_h), (b_K w, 0)\rangle}{\sqrt{\nu}\,|b_K w|_{1,K}} \\ &\le C \sup_{\substack{(v,q)\in \mathcal{B}_h|_K \\ \|(v,q)\|=1}} \langle F(u_h, p_h), (v, q)\rangle.\end{aligned} \tag{41}$$

In addition, using estimates (16)–(20) and (41), it holds

$$\frac{1}{\sqrt{\nu}} h_F^{1/2} \|R_F\|_{0,F}$$

$$
\begin{aligned}
&\leq C\, h_F^{1/2} \sup_{s \in \mathbb{P}_l(F)^2 \setminus \{0\}} \frac{(R_F, b_F s)_F}{\sqrt{\nu}\, \|s\|_{0,F}} \\
&\leq C\, h_F \sup_{s \in \mathbb{P}_l(F)^2 \setminus \{0\}} \frac{\left\{ \langle F(u_h, p_h), (b_F \mathscr{P}_F(s), 0) \rangle - \sum_{K \in \omega_F} (R_K, b_F \mathscr{P}_F(s))_K \right\}}{\sqrt{\nu}\, \|b_F \mathscr{P}_F(s)\|_{0,\omega_F}} \\
&\leq C \sup_{s \in \mathbb{P}_l(F)^2 \setminus \{0\}} \frac{\left\{ \langle F(u_h, p_h), (b_F \mathscr{P}_F(s), 0) \rangle - \sum_{K \in \omega_F} (R_K, b_F \mathscr{P}_F(s))_K \right\}}{\sqrt{\nu}\, |b_F \mathscr{P}_F(s)|_{1,\omega_F}} \\
&\leq C \sup_{\substack{(v,q) \in \mathscr{B}_h|_K \\ \|(v,q)\|=1}} \langle F(u_h, p_h), (v, q) \rangle + \frac{h_F}{\sqrt{\nu}} \sum_{K \in \omega_F} \|R_K\|_{0,K} \\
&\leq C \sup_{\substack{(v,q) \in \mathscr{B}_h|_{\omega_F} \\ \|(v,q)\|=1}} \langle F(u_h, p_h), (v, q) \rangle.
\end{aligned} \tag{42}
$$

Observe that inequalities (40)–(42) imply

$$
\eta_K \leq C \sup_{\substack{(v,q) \in \mathscr{B}_h|_{\omega_K} \\ \|(v,q)\|=1}} \langle F(u_h, p_h), (v, q) \rangle. \tag{43}
$$

Finally, as $\sum_{K \in \mathscr{T}_h} \eta_K^2 \leq \left[\sum_{K \in \mathscr{T}_h} \eta_K \right]^2$, we obtain from (43) that

$$
\left\{ \sum_{K \in \mathscr{T}_h} \eta_K^2 \right\}^{1/2} \leq \sum_{K \in \mathscr{T}_h} \eta_K \leq C\, \|F(u_h, p_h)\|_{\mathscr{B}_h'}, \tag{44}
$$

and from (37) the result follows.

Upper bound: From (3)–(5), we get

$$
\begin{aligned}
&\sup_{\substack{(v,q) \in V \times Q \\ \|(v,q)\|=1}} \langle F(u_h, p_h), (v - \mathscr{I}_h v, q) \rangle \\
&= \sup_{\substack{(v,q) \in V \times Q \\ \|(v,q)\|=1}} \Bigg\{ \sum_{K \in \mathscr{T}_h} [(-\nu\, \Delta u_h \\
&\quad + (\nabla u_h) u_h + \nabla p_h - f, v - \mathscr{I}_h v)_K - (\nabla \cdot u_h, q)_K] \\
&\quad + \sum_{F \in \mathscr{E}_h} ([\![-\nu\, \partial_n u_h + p_h n]\!], v - \mathscr{I}_h v)_F \Bigg\} \\
&\leq C \left\{ \sum_{K \in \mathscr{T}_h} \eta_K^2 \right\}^{1/2}.
\end{aligned} \tag{45}
$$

Next, using (44) and (45) we get

$$\sup_{\substack{(v,q)\in V\times Q\\ \|(v,q)\|=1}} \langle F(u_h,p_h),(v-\mathscr{I}_h v,q)\rangle \le C\,\|F(u_h,p_h)\|_{\mathscr{B}_h'}, \tag{46}$$

and by considering an arbitrary element $(v,q)\in V\times Q$ with $\|(v,q)\|=1$, we arrive at

$$\begin{aligned}
&\langle F(u_h,p_h),(v,q)\rangle\\
&\quad= \langle F(u_h,p_h),(v-\mathscr{I}_h v,q)\rangle + \langle F(u_h,p_h)-F_h(u_h,p_h),(\mathscr{I}_h v,0)\rangle\\
&\qquad\le \sup_{\substack{(v,q)\in V\times Q\\ \|(v,q)\|=1}} \langle F(u_h,p_h),(v-\mathscr{I}_h v,q)\rangle\\
&\qquad+ \|\mathscr{I}_h\|_{\mathscr{L}(V,V_h)}\|F(u_h,p_h)-F_h(u_h,p_h)\|_{(V_h\times Q_h)'}.
\end{aligned}$$

Thereby, from (46), we get

$$\begin{aligned}
&\|F(u_h,p_h)\|_{(V\times Q)'}\\
&\quad\le C\,\|F(u_h,p_h)\|_{\mathscr{B}_h'} + \|\mathscr{I}_h\|_{\mathscr{L}(V,V_h)}\,\|F(u_h,p_h)\\
&\qquad - F_h(u_h,p_h)\|_{(V_h\times Q_h)'}.
\end{aligned} \tag{47}$$

Now, given $(v_h,q_h)\in\mathscr{B}_h$ and integrating by parts, we obtain that

$$\begin{aligned}
&\langle F(u_h,p_h),(v_h,q_h)\rangle\\
&\quad= \sum_{K\in\mathscr{T}_h}\Big[(-\nu\,\Delta u_h + (\nabla u_h)u_h + \nabla p_h - f, v_h)_K - (\nabla\cdot u_h, q_h)_K\Big]\\
&\qquad+ \sum_{F\in\mathscr{E}_h}([\![-\nu\,\partial_n u_h + p_h n]\!], v_h)_F\\
&\quad= -\sum_{K\in\mathscr{T}_h}\Big[(\mathscr{R}_K,v_h)_K + (\nabla\cdot u_h,q_h)_K\Big] + \sum_{F\in\mathscr{E}_h}(\mathscr{R}_F,v_h)_F.
\end{aligned} \tag{48}$$

Next, we will estimate the terms on the right-hand side of (48). To this end, we use estimates (16)–(20) to get

$$\begin{aligned}
(\mathscr{R}_K, b_K w)_K &\le h_K\|\mathscr{R}_K\|_{0,K}|w|_{1,K},\\
(\nabla\cdot u_h, b_K r)_K &\le \|\nabla\cdot u_h\|_{0,K}\|r\|_{0,K},\\
(\mathscr{R}_F, b_F\mathscr{P}_F(s))_F &\le \|\mathscr{R}_F\|_{0,K}\|\mathscr{P}_F(s)\|_{0,F},
\end{aligned}$$

for all $r\in\mathbb{P}_0(K)$, $w\in\mathbb{P}_1(K)^2$ and $s\in\mathbb{P}_l(F)^2$, with $l=0,1$, thus we arrive at

$$\|F(u_h,p_h)\|_{\mathscr{B}_h'} \le C\left\{\sum_{K\in\mathscr{T}_h}\eta_K^2\right\}^{1/2}.$$

Using (9), (15) and method (21), we get

$$
\begin{aligned}
&\langle F(u_h, p_h) - F_h(u_h, p_h), (v_h, q_h)\rangle \\
&\le C\Bigg[\sum_{K\in\mathscr{T}_h} \frac{\alpha_K h_K^2}{\nu}\,\|f + \nu\,\Delta u_h - (\nabla u_h)u_h - \nabla p_h\|_{0,K}\|\nabla q_h - (\nabla v_h)u_h\|_{0,K} \\
&\qquad + \sum_{K\in\mathscr{T}_h} \frac{\gamma_K h_K^2}{\nu}\,\|\nabla\cdot u_h\|_{0,K}\|\nabla\cdot v_h\|_{0,K} \\
&\qquad + \sum_{F\in\mathscr{E}_h} \tau_F\,\|\mathscr{R}_F\|_{0,F}\|[\![\nu\,\partial_n v_h + q_h n]\!]\|_{0,F}\Bigg] \\
&\le C\Bigg[\sum_{K\in\mathscr{T}_h} \frac{\alpha_K h_K^2}{\nu}\,\|\mathscr{R}_K\|_{0,K}\big[|q_h|_{1,K} + \|u_h\|_{\infty,K}|v_h|_{1,K}\big] \\
&\quad + \sum_{K\in\mathscr{T}_h} \frac{\gamma_K h_K^2}{\nu}\,\|\nabla\cdot u_h\|_{0,K}|v_h|_{1,K} + \sum_{F\in\mathscr{E}_h} \tau_F\,\|\mathscr{R}_F\|_{0,F}\|[\![\nu\,\partial_n v_h + q_h n]\!]\|_{0,F}\Bigg] \\
&\le C\Bigg[\sum_{K\in\mathscr{T}_h} \frac{\alpha_K h_K}{\nu}\,\|\mathscr{R}_K\|_{0,K}\big[\|q_h\|_{0,K} + \|u_h\|_{0,K}|v_h|_{1,K}\big] \\
&\qquad + \sum_{K\in\mathscr{T}_h} \frac{\gamma_K h_K^2}{\nu}\,\|\nabla\cdot u_h\|_{0,K}|v_h|_{1,K} \\
&\qquad + \sum_{F\in\mathscr{E}_h} \tau_F\,\|\mathscr{R}_F\|_{0,F}\|[\![\nu\,\partial_n v_h + q_h n]\!]\|_{0,F}\Bigg] \\
&\le\; C\Lambda(\nu, u_h)\left\{\sum_{K\in\mathscr{T}_h} \frac{h_K^2}{\nu}\left[\|\mathscr{R}_K\|_{0,K}^2 + \frac{h_K^2}{\nu^2}\,\|\nabla\cdot u_h\|_{0,K}^2\right] + \sum_{F\in\mathscr{E}_h} \tau_F\,\|\mathscr{R}_F\|_{0,F}^2\right\}^{1/2} \\
&\quad \times\left\{\nu\,|v_h|_{1,\Omega}^2 + \frac{1}{\nu}\,\|q_h\|_{0,\Omega}^2 + \sum_{F\in\mathscr{E}_h} \tau_F\,\|[\![\nu\,\partial_n v_h + q_h n]\!]\|_{0,F}^2\right\}^{1/2} \\
&\le\; C\Lambda(\nu, u_h)\left\{\sum_{K\in\mathscr{T}_h} \frac{h_K^2}{\nu}\left[\|\mathscr{R}_K\|_{0,K}^2 + \frac{h_K^2}{\nu^2}\,\|\nabla\cdot u_h\|_{0,K}^2\right] + \sum_{F\in\mathscr{E}_h} \tau_F\,\|\mathscr{R}_F\|_{0,F}^2\right\}^{1/2} \\
&\quad \times\left\{\nu\,|v_h|_{1,\Omega}^2 + \frac{1}{\nu}\,\|q_h\|_{0,\Omega}^2 + \sum_{F\in\mathscr{E}_h} \tau_F \nu^2\,\|[\![\partial_n v_h]\!]\|_{0,F}^2 + \sum_{F\in\mathscr{E}_h} \tau_F\,\|[\![q_h]\!]\|_{0,F}^2\right\}^{1/2},
\end{aligned}
\tag{49}
$$

where $\Lambda(\nu, u_h) := \max\left\{1, \frac{\|u_h\|_{0,\Omega}}{\nu}\right\}$.

Applying mesh regularity, (10), (22), and local trace (8), we arrive at

$$
\sum_{F\in\mathscr{E}_h} \tau_F \nu^2\,\|[\![\partial_n v_h]\!]\|_{0,F}^2 + \sum_{F\in\mathscr{E}_h} \tau_F\,\|[\![q_h]\!]\|_{0,F}^2 \le C\,\|(v_h, q_h)\|^2. \tag{50}
$$

Combining (49) with (50), it holds

$$\|F(u_h, p_h) - F_h(u_h, p_h)\|_{(V_h \times Q_h)'} \leq C\,\Lambda(\nu, u_h)\,\eta_H, \tag{51}$$

thus from (49), (47) and (51) we obtain that

$$\|F(u_h, p_h)\|_{(V \times Q)'} \leq C\,\Lambda(\nu, u_h)\,\eta_H. \tag{52}$$

Finally, using (36) and (52) the result follows. □

5 Numerical examples

We solve RELP method (21) by a Newton–Picard scheme. The idea consists of starting with a solution u_h^0, p_h^0 and perform the following:

For $n = 1, 2, 3, \dots$

1. Compute δu_h^n and δp_h^n from the linear system

$$\begin{aligned}
&\nu\,(\nabla \delta u_h^n, \nabla v_h) - (\delta p_h^n, \nabla\cdot v_h) - (q_h, \nabla\cdot \delta u_h^n)\\
&\quad + ((\nabla \delta u_h^n) u_h^n, v_h) + ((\nabla u_h^n) \delta u_h^n, v_h)\\
&\quad - \sum_{K \in \mathscr{T}_h} \frac{\alpha_K^n}{\nu} \big(\chi_h(x \cdot \Pi_K((\nabla \delta u_h^n) u_h^n + \nabla \delta p_h^n)),\\
&\quad \chi_h(x \cdot \Pi_K(-(\nabla v_h) u_h^n + \nabla q_h))\big)_K\\
&\quad + \sum_{K \in \mathscr{T}_h} \frac{\gamma_K^n}{\nu} \Big(\chi_h(x\,\nabla \cdot \delta u_h^n), \chi_h(x\,\nabla \cdot v_h)\Big)_K\\
&\quad - \sum_{F \in \mathscr{E}_h} \tau_F^n \Big([\![-\nu \partial_n \delta u_h^n + \delta p_h^n n]\!], [\![\nu \partial_n v_h + q_h n]\!]\Big)_F\\
&= (f, v_h)_\Omega - \nu\,(\nabla u_h^n, \nabla v_h)_\Omega\\
&\quad + (p_h^n, \nabla\cdot v_h)_\Omega + (q_h, \nabla\cdot u_h^n)_\Omega - ((\nabla u_h^n) u_h^n, v_h)_\Omega\\
&\quad - \sum_{K \in \mathscr{T}_h} \frac{\alpha_K^n}{\nu} \big(\chi_h(x \cdot \Pi_K(f - (\nabla u_h^n) u_h^n - \nabla p_h^n)),\\
&\quad \chi_h(x \cdot \Pi_K(-(\nabla v_h) u_h^n + \nabla q_h))\big)_K\\
&\quad - \sum_{K \in \mathscr{T}_h} \frac{\gamma_K^n}{\nu} \Big(\chi_h(x\,\nabla \cdot u_h^n), \chi_h(x\,\nabla \cdot v_h)\Big)_K\\
&\quad + \sum_{F \in \mathscr{E}_h} \tau_F^n \Big([\![-\nu \partial_n u_h^n + p_h^n n]\!], [\![\nu \partial_n v_h + q_h n]\!]\Big)_F,
\end{aligned}$$

for all $(v_h, q_h) \in V_h \times Q_h$, where $\alpha_K^n := \alpha_K(u_h^n)$, $\gamma_K^n := \gamma_K(u_h^n)$ and $\tau_F^n := \tau_F(u_h^n)$.

2. Set $u_h^{n+1} = u_h^n + \delta u_h^n$.
3. Set $p_h^{n+1} = p_h^n + \delta p_h^n$.

4. If convergence `then exit`.

`End For`.

The adaptive procedure uses a quasi-uniform mesh to start the process. At each step, we compute the local error estimators $\eta_{H,K}$ for all K over the previous mesh $\mathscr{T}_h$, and refine those elements $K \in \mathscr{T}_h$ accordingly to

$$\eta_{H,K} \geq \theta \max\{\eta_{H,K} : K \in \mathscr{T}_h\},$$

where $\theta \in (0, 1)$ is a prescribed parameter.

For practical purposes, we used the software `Triangle` to generate adapted meshes, as it allowed us to create successively refined meshes based on a hybrid Delaunay refinement algorithm. This process provided a sequence of refined meshes that form a hierarchy of nodes rather than a hierarchy of elements (for details, see [26]).

We validate the stabilized method and the a posteriori error estimator. We first adopt a numerical test with an analytic solution, followed by some well-established benchmarks from the fluid dynamics literature. We measure the quality of the a posteriori error estimator through the so–called *effectivity index*, which is required to remain bounded as h goes to zero and is defined by

$$E := \frac{\eta_H}{\|(u - u_h, p - p_h)\|}.$$

Also interesting is to compare the accuracy of our residual-based estimator with a hierarchical approach. In fact, it is well-known that the latter yields more precise effectivity indexes than do residual estimators (in general), which tend to overestimate the true error [32]. We found, from the test case used to validate the hierarchical estimator in [5], that the effectivity index from our estimator is comparable to the one from the hierarchical estimator in [5] (the values are close to five in our case, and to one in [5]). It is worth recalling that, although more precise, hierarchical estimators are less cost effective since they demand the computation of auxiliary element-by-element problems.

5.1 Analytic solution

The domain is $\Omega := (0, 1) \times (0, 1)$ and $\nu = 1,\ 10^{-2}$, and f is chosen such that the exact solution is given by

$$u_1(x, y) := y - \frac{1 - e^{y/\nu}}{1 - e^{1/\nu}}, \quad u_2(x, y) := x - \frac{1 - e^{x/\nu}}{1 - e^{1/\nu}}, \quad p(x, y) := x - y.$$

Figures 2 and 3 show that method (21) remains precise when the viscosity coefficient is small. We notice that the method achieves optimal order of convergence for both pair of spaces $\mathbb{P}_1^2 \times \mathbb{P}_1$ and $\mathbb{P}_1^2 \times \mathbb{P}_0$. In Tables 1 and 2, we point out that the effectivity index stays bounded when h goes to zero for different values of ν.

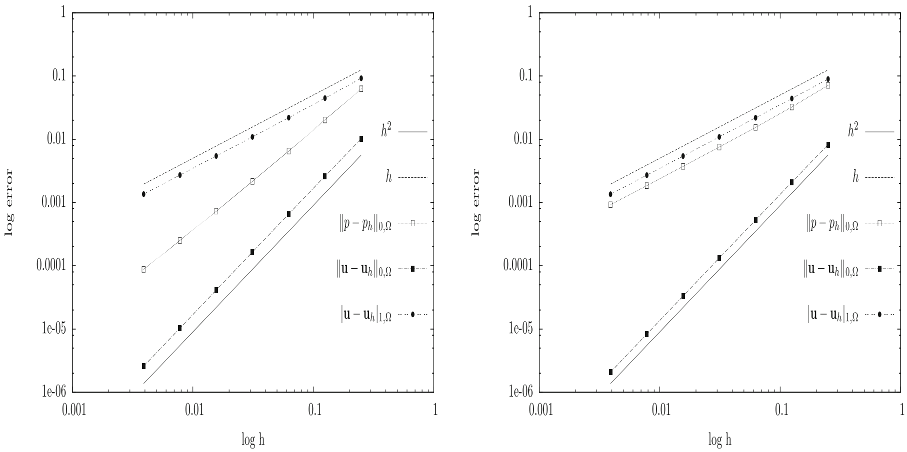

Fig. 2 Convergence history for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ (*left*) and $\mathbb{P}_1^2 \times \mathbb{P}_0$ (*right*) schemes, $\nu = 1$

5.2 Lid–driven cavity problem

The lid–driven cavity problem is a standard benchmark in computational fluid mechanics (see [18] and [29], for instance). The Reynolds number is given by $Re := 1/\nu$, and we perform the computation assuming $Re = 5000$. The final adapted mesh and the streamlines of the velocity on this mesh are depicted in Fig. 4. We observe that the mesh refinement concentrates inside the primary vortex which leads to an accurate approximation of the solution.

Finally, Table 3 shows that the location of the center of the primary vortex using RELP method (21) is in accordance with the one obtained from Ghia and Shin in [18], and from Medic and Mohammadi in [22].

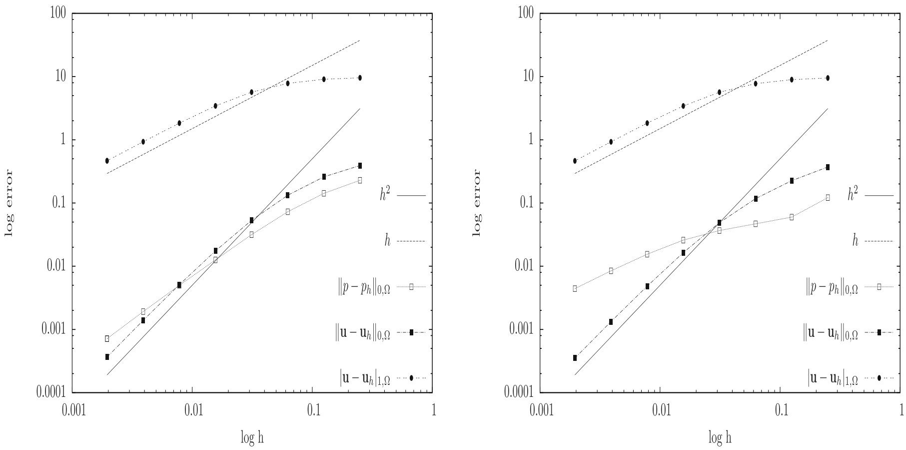

Fig. 3 Convergence history for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ (*left*) and $\mathbb{P}_1^2 \times \mathbb{P}_0$ (*right*) schemes, $\nu = 10^{-2}$

Table 1 Analytic solution with $\nu = 1$

h	$\mathbb{P}_1^2 \times \mathbb{P}_1$			$\mathbb{P}_1^2 \times \mathbb{P}_0$		
	$\|(u - u_h, p - p_h)\|$	η_H	E	$\|(u - u_h, p - p_h)\|$	η_H	E
0.125	0.49013E-01	0.22889E+00	4.6700	0.54635E-01	0.30763E+00	5.6306
0.0625	0.22923E-01	0.11701E+00	5.1044	0.26719E-01	0.15524E+00	5.8101
0.03125	0.11118E-01	0.59005E-01	5.3071	0.13223E-01	0.77965E-01	5.8961
0.015625	0.54837E-02	0.29606E-01	5.3990	0.65800E-02	0.39066E-01	5.9372
0.0078125	0.27247E-02	0.14826E-01	5.4414	0.32825E-02	0.19554E-01	5.9570
0.0039062	0.13583E-02	0.74182E-02	5.4615	0.16395E-02	0.97821E-02	5.9667

Exact error, a posteriori error estimator and effectivity index for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ and $\mathbb{P}_1^2 \times \mathbb{P}_0$ schemes, respectively

5.3 Circular cylinder problem

The domain and the boundary conditions are shown in Fig. 5. The inflow velocity field is $u_p = (1.2\, y\, (0.41 - y)/0.41^2, 0)^T = (U, 0)$ and the viscosity is set to $\nu = 10^{-3}$ (for further details, see [29]).

The *drag* and *lift* coefficients are useful to validate numerical schemes, and are defined by

$$C_D := \frac{2}{\bar{u}^2 D} \int_S \left(\nu \frac{\partial v_t}{\partial n} n_y - P n_x \right) dS, \quad C_L := -\frac{2}{\bar{u}^2 D} \int_S \left(\nu \frac{\partial v_t}{\partial n} n_x + P n_y \right) dS,$$

where we used the following notations: S corresponds to the boundary of the cylinder, $\boldsymbol{n} := (n_x, n_y)$ and $\boldsymbol{t} = (n_y, -n_x)$ are, respectively, the outward normal vector and the tangent vector on S and v_t is the tangential velocity on S. The diameter of the cylinder D is set to 0.1 and the mean velocity $\bar{u}$ is $\frac{2}{3} U(0, 0.205)$.

Table 2 Analytic solution with $\nu = 10^{-2}$

h	$\mathbb{P}_1^2 \times \mathbb{P}_1$			$\mathbb{P}_1^2 \times \mathbb{P}_0$		
	$\|(u - u_h, p - p_h)\|$	η_H	E	$\|(u - u_h, p - p_h)\|$	η_H	E
0.125	0.16857E+01	0.95797E+01	5.6830	0.10715E+01	0.10662E+02	9.9508
0.0625	0.10664E+01	0.52258E+01	4.9003	0.86888E+00	0.56398E+01	6.4909
0.03125	0.64970E+00	0.24258E+01	3.7337	0.59908E+00	0.25206E+01	4.2075
0.015625	0.36646E+00	0.11865E+01	3.2376	0.35385E+00	0.12207E+01	3.4496
0.0078125	0.19032E+00	0.65601E+00	3.4468	0.18757E+00	0.66840E+00	3.5635
0.0039062	0.95015E-01	0.35660E+00	3.7531	0.94635E-01	0.36193E+00	3.8245

Exact error, a posteriori error estimator and effectivity index for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ and $\mathbb{P}_1^2 \times \mathbb{P}_0$ schemes, respectively

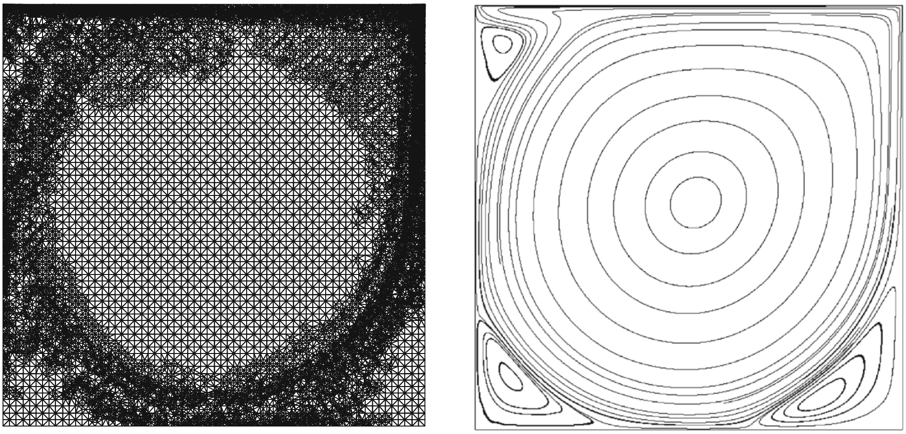

Fig. 4 Adapted mesh with the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element and streamlines of the velocity ($Re = 5000$)

The length of the recirculation and the difference of the pressure at points $(x_a, y_a) = (0.15, 0.2)$ and $(x_e, y_e) = (0.25, 0.2)$ are denoted by

$$L_r := x_r - x_e, \qquad \Delta p := P(x_a, y_a) - P(x_e, y_e),$$

where x_r is the x-coordinate of the end of the recirculation area. In Table 4, we compare these quantities using RELP method (21) to the ones obtained from [22] and [24]. Figure 6 depicts a zoom of the final adapted mesh with the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element. A zoom of the streamlines of the velocity and the isovalues of $|u_h|$ are presented in Fig. 7 for the adapted mesh.

5.4 The flat plate problem

Concerning a laminar flow over a flat plate, closed formulas for the friction coefficient and for the velocity profile are available to comparisons (see Blasius [25]). The statement of this problem follows [22] and consists of a rectangular domain $\Omega := (-0.2, 1) \times (0, 0.1)$ with prescribed velocity $u_p = (1, 0)^T$ at inflow boundary and viscosity $\nu = \frac{1}{33000}$ (i.e. $Re = 33000$), and $f = 0$. Since non-slip condition is imposed on the flat plate, a boundary layer starts at the “border of attack” and may be considered fully developed after a short distance.

Table 3 Position of the center of the primary vortex

Scheme	$Re = 5.000$
Ghia et al. [18]	$x = 0.5117$; $y = 0.5352$
Medic et. al. [22]	$x = 0.53$; $y = 0.53$
RELP $\mathbb{P}_1^2 \times \mathbb{P}_1$ (adapted mesh)	$x = 0.5157$; $y = 0.5350$
RELP $\mathbb{P}_1^2 \times \mathbb{P}_0$ (adapted mesh)	$x = 0.5205$; $y = 0.5309$

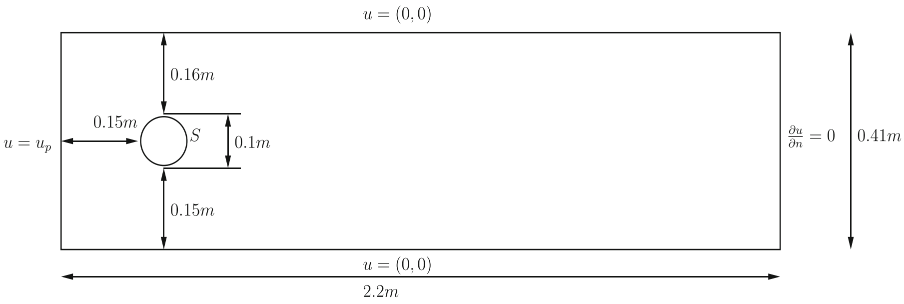

Fig. 5 Statement of the cylinder problem

Table 4 The circular cylinder problem with $\nu = 10^{-3}$

Scheme	C_D	C_L	Δp	L_r
Schäfer et. al. [24]	5.58	0.011	0.1175	0.085
Medic et. al. [22]	5.65	0.012	0.121	0.082
RELP $\mathbb{P}_1^2 \times \mathbb{P}_1$ (adapted mesh)	5.56	0.010	0.1170	0.084
RELP $\mathbb{P}_1^2 \times \mathbb{P}_0$ (adapted mesh)	5.54	0.012	0.1171	0.084

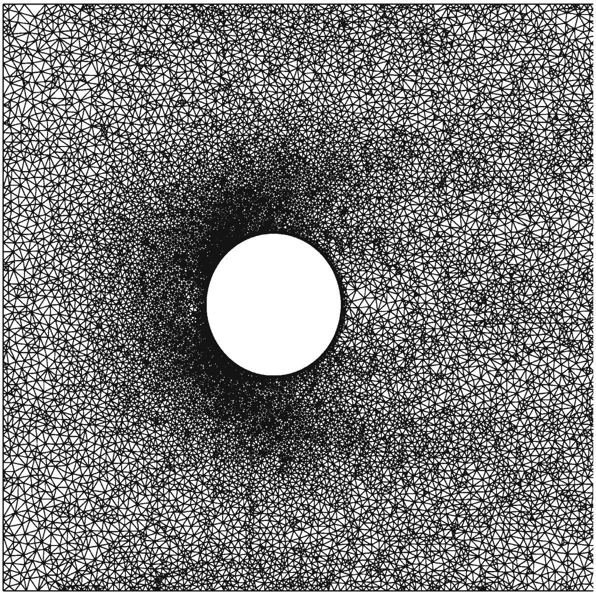

Fig. 6 The circular cylinder problem with $\nu = 10^{-3}$. Zoom adapted mesh for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element (60.593 elements)

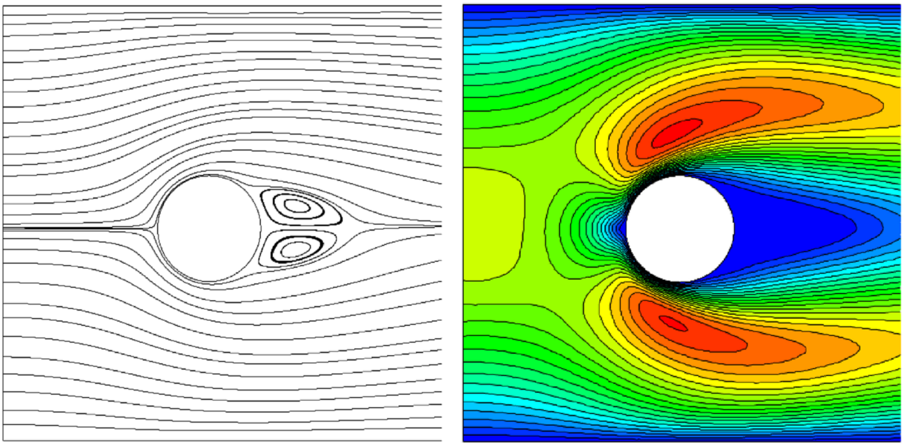

Fig. 7 The circular cylinder problem with $\nu = 10^{-3}$. Zoom of the streamlines and the isolines of $|u_h|$ in the final adapted mesh for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element (60.593 elements)

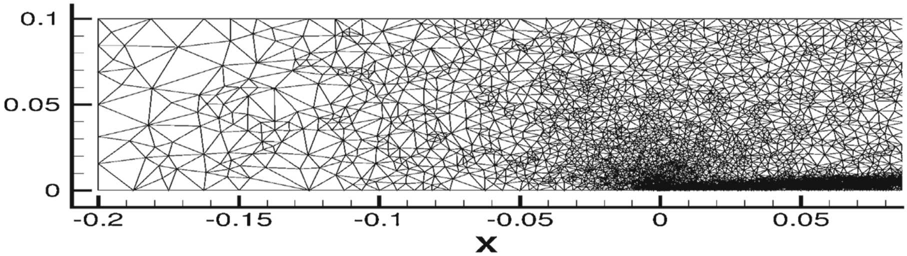

Fig. 8 The flat plate problem with $Re = 33000$. A zoom of the final adapted mesh for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ case (95.099 elements)

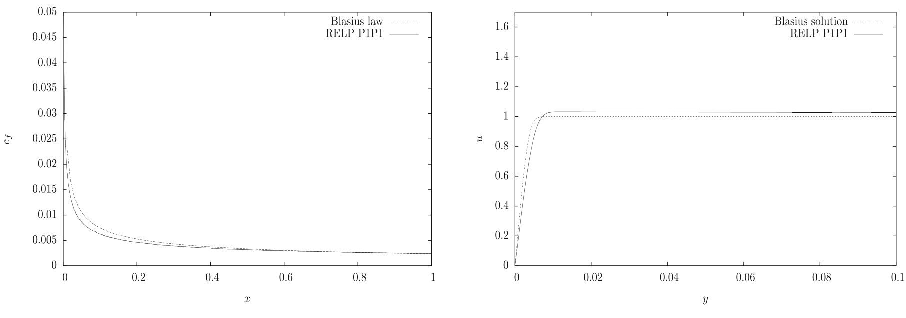

Fig. 9 Comparison of friction coefficient c_f on the plate (*left*) and a profile of the horizontal velocity at $x = 0.2$ (*right*) to Blasius solution

Fig. 10 The flat plate problem with $Re = 33000$. Isovalues of $|u_h|$ computed on the final adapted mesh for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element (95.099 elements)

Figure 8 depicts a zoom of the final adapted mesh using both pairs of interpolation spaces. As a result, we observe a dense concentration of elements inside the boundary layer region.

We compare the friction coefficient $c_f := \nu \dfrac{\partial u_h}{\partial n} \cdot t$ in Fig. 9, as well as the profile of the horizontal velocity at $x = 0.2$ with Blasius' solution. Here t is the unit tangent vector on the plate. Figure 10 shows the isovalues of $|u_h|$ and the isolines of the pressure for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element. We notice the absence of numerical spurious oscillations at the vicinity of the boundary layer which highlights the robustness of the approach.

6 Conclusions

We have presented a new a posteriori error estimator for the fully non-linear Navier-Stokes equations which efficiently drives mesh adaptation. We proved the estimator is equivalent to the approximation error in a special norm. Also, the stabilized method used to construct the estimator is proved to be well-posed using a fixed point theory. Extensive numerical experiments attested the accuracy of the methodology to handle high Reynolds number flows on a large variety of geometries.

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