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Numerical integration in Galerkin meshless methods, applied to elliptic Neumann problem with non-constant coefficients

Qinghui Zhang · Uday Banerjee

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Abstract In this paper, we explore the effect of numerical integration on the Galerkin meshless method used to approximate the solution of an elliptic partial differential equation with non-constant coefficients with Neumann boundary conditions. We considered Galerkin meshless methods with shape functions that reproduce polynomials of degree $k \geq 1$. We have obtained an estimate for the energy norm of the error in the approximate solution under the presence of numerical integration. This result has been established under the assumption that the numerical integration rule satisfies a certain discrete Green's formula, which is not problem dependent, i.e., does not depend on the non-constant coefficients of the problem. We have also derived numerical integration rules satisfying the discrete Green's formula.

Keywords PDE with non-constant coefficients · Galerkin methods · Meshless methods · Quadrature · Numerical integration · Error estimates

Mathematics Subject Classifications (2010) 65N15 · 65N30 · 41A30

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Q. Zhang
Department of Scientific Computing and Computer Applications, Sun Yat-Sen University, Guangzhou, 510275, People's Republic of China
e-mail: zhangqh6@mail.sysu.edu.cn

U. Banerjee (✉)
Department of Mathematics, Syracuse University, 215 Carnegie, Syracuse, NY 13244, USA
e-mail: banerjee@syr.edu

1 Introduction

For the last 20 years, a lot of progress has been made in the development of the Meshless Methods (MM) and it has been applied to solve complicated engineering problems (see e.g., [1–3, 9, 19, 22, 25]). There are many classes of MM used in practice, e.g., meshless collocation methods, MM based on Radial Basis Functions, Galerkin MM, etc (see [19, 23]). In this paper we address the Galerkin MM, where the shape functions *reproduce polynomials* [3, 8, 9, 13, 25]. We note that this method is referred to in the literature as the Galerkin meshfree method, the element free Galerkin method, the method of spheres, the meshfree method, Galerkin MM, MM, etc. Throughout this paper, we will refer to this method as Galerkin Meshless Method (GMM). In contrast to the FEM, the construction of shape functions used in GMM does not require a mesh, however, the shape functions are not piecewise polynomials. This feature poses a serious challenge in the use of numerical integration to compute the elements of the stiffness/mass matrices and the load vector.

The challenge of numerical integration has been recognized from the very beginning of the development of the GMM, and it has been addressed in various engineering papers [5, 7, 8, 11–14, 16–18, 20]. Several approaches to implement numerical integration have been proposed in the literature; we refer to Section 3 of [6] for a brief review of these approaches. A mathematical analysis of the effect of numerical integration was first reported in [5], where it was shown that the approximate solution obtained from the GMM, using standard numerical integration, may not converge. It was also shown that if the stiffness matrix (numerically computed with quadrature) satisfies a row sum condition then the error in the approximate solution (in energy norm) is $O(h+\eta)$, where h is the discretization parameter and the parameter η indicates the accuracy of the underlying numerical integration. Thus with $\eta = O(h)$, the GMM with numerical integration yields the optimal order of convergence. However, the analysis presented in [5] was restricted to the shape functions of the GMM that reproduce polynomials of degree $k = 1$ and could not be extended for $k > 1$.

Another analysis of the effect of numerical integration on the GMM was presented later in [6], where the quadrature is required to satisfy a discrete Green's formula. This analysis is valid for the GMM, where the shape functions reproduce polynomials of degree $k \geq 1$. It was shown that the energy-norm of the error in the approximate solution obtained from the GMM is $O(h^{k-1}(h+\eta))$, and optimal order of convergence is obtained with $\eta = O(h)$. However in [6], the GMM was used to approximate the solution of a Neumann problem with constant coefficients and with no lower order term. We further note that a direct application of the ideas in [6] to the situation, where the GMM is applied to a problem with non-constant coefficient, requires the quadrature to be problem dependent, e.g., dependent on the non-constant coefficients of the problem.

In this paper we extend the analysis in [6] to study the effect of numerical integration, when the GMM is used to approximate the solution of a Neumann problem with non-constant coefficients including the lower order term. We require the quadrature to satisfy a certain version of the discrete Green's formula, which is not problem dependent (but is slightly stronger than the condition used in [6]). We show that the energy norm of the error in the approximate solution obtained from the GMM with quadrature is $O(h^{k-1}(h+\eta))$. For a Neumann problem with no lower order term, we mention that the condition on the quadrature required in this paper is the same as the condition required in [6] for $k=1$. However for $k>1$, the situation is different; a quadrature satisfying the condition proposed in this paper automatically satisfies the condition required in [6], but not vice versa. In this paper, we have also investigated the possibility of using different numerical integration rules to compute the elements of the stiffness matrix, the mass matrix, and the load vector, which was not done in [6]. Moreover, we have derived numerical integration rules, satisfying the extended discrete Greene's formula, in two dimensions for $k=1$ and $k=2$ in this paper; numerical integration rules only for $k=1$ in one dimension was presented in [6].

The outline of the paper is as follows: In Section 2, we present the notations and the elliptic Neumann model problem with non-constant coefficients. The GMM and the various properties of the associated finite dimensional space are given in Section 3. In Section 4, we define the GMM with numerical integration and list the assumptions imposed on the numerical integration rule. The effect of numerical integration on the energy norm of the error in the approximate solution, obtained from the GMM, has been investigated in Section 5. Our main analytical result, Theorem 5.5, has also been presented in this section. In Section 6, we derive numerical integration rules, in 2-dimensions, that satisfy the main assumption given in Section 4. Finally, we present some numerical examples in Section 7 that shows the effect of the numerical integration on the energy norm of the error in the approximate solution.

2 Preliminaries and model problem

Let $\mathbb{N}$ be the set of all positive integers. For a domain $D \subset \mathbb{R}^d$, an integer $m \in \mathbb{N} \cup \{0\}$, and $p \in \mathbb{N} \cup \{\infty\}$, we denote the usual Sobolev space by $W^{m,p}(D)$ with the norm $\|\cdot\|_{W^{m,p}(D)}$ and semi-norm $|\cdot|_{W^{m,p}(D)}$. We will only consider $p=2,\infty$ in this paper. The Sobolev space $W^{m,p}(D)$ will be represented by $H^m(D)$ in the case $p=2$ and by $L_p(D)$ in the case $m=0$. Likewise, for a hypersurface ∂D in $\mathbb{R}^d$, we will use the space $L_p(\partial D)$ equipped with the norm $\|\cdot\|_{L_p(\partial D)}$.

Let V be a normed linear space. We define $\widetilde{V}$ to be the product space V^d, where $\widetilde{v} = [v_i]_{i=1}^d \in \widetilde{V}$ is a vector-valued function with its components $v_i \in V$, $i = 1, 2, \cdots, d$. When $V = W^{m,p}(D)$ or $L_p(\partial D)$, the associated norm of $\widetilde{V}$ is defined by $\|\widetilde{v}\|_V = \left(\sum_{i=1}^d \|v_i\|_V^p\right)^{\frac{1}{p}}$ in the case $1 \le p < \infty$ and $\|\widetilde{v}\|_V = \max\{\|v_i\|_V : \; i = 1, 2, \cdots, d\}$ in the case $p=\infty$; the semi-norm $|\widetilde{v}|_V$ (for $V = W^{m,p}(D)$) is defined by using $|v_i|_V$ instead of $\|v_i\|_V$ in the above definitions.

A domain D is star-shaped with respect to a ball $B \subset D$ if, for all $x \in D$, the closed convex hull of $\{x\} \cup B$ is a subset of D. Let $\rho_{\max} = \sup\{\rho :$ D is star-shaped with respect to a ball of radius $\rho\}$, then the *chunkiness parameter* of D is defined by

$$\gamma_D = \frac{\text{diam}(D)}{\rho_{\max}}.$$

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz continuous boundary $\Gamma = \partial\Omega$. For the model problem, we consider the Neumann problem

$$\begin{aligned} \mathcal{L}u \equiv -\nabla \cdot \left(A\,\nabla u\right) + cu = f, \qquad & \text{in } \Omega \\ A\,\nabla u \cdot \vec{n} = g, \qquad & \text{on } \Gamma \end{aligned} \tag{2.1}$$

where $A(x) = \{a_{ij}(x)\}_{1 \le i,j \le d}$ is a symmetric matrix, $a_{ij} \in C^k(\overline{\Omega})$, $c \in C(\overline{\Omega})$, $f \in L_2(\Omega)$, $g \in L_2(\Gamma)$ and $\vec{n}$ is the outward unit normal vector to Γ. We assume that there is a constant $\beta > 0$ such that

$$\sum_{i,j=1}^{d} u_i a_{ij}(x) u_j \ge \beta \sum_{i=1}^{d} u_i^2,\ \forall\, u \in \mathbb{R}^d \text{ and } c(x) \ge \beta, \quad \forall\, x \in \Omega. \tag{2.2}$$

We note that for $v \in H^1(\Omega)$, $a\nabla v$ is a vector-valued function, but for simplicity of notation, we do not put a tilde over it.

The associated variational formulation of (2.1) is given by

$$\begin{aligned} &\text{Find } u \in H^1(\Omega) \text{ such that} \\ &B(u, v) = L(v), \quad \forall\, v \in H^1(\Omega) \end{aligned} \tag{2.3}$$

where

$$B(u, v) \equiv B_1(u, v) + B_0(u, v), \qquad L(v) \equiv \int_\Omega f\, v\, dx + \int_\Gamma g\, v\, ds$$

and

$$B_1(u, v) \equiv \int_\Omega A\,\nabla u \cdot \nabla v\, dx, \qquad B_0(u, v) \equiv \int_\Omega c\, u\, v\, dx$$

The bilinear form $B(\cdot,\cdot)$ is continuous and coercive (using (2.2)) on $H^1(\Omega) \times H^1(\Omega)$, and it is well known [10] that the variational problem (2.3) has a unique solution.

3 Galerkin meshless methods

The GMM to approximate the solution of the variational problem (2.3) is a Galerkin method, where the construction of the underlying finite dimensional subspace either does not depend on a mesh, or uses a mesh only minimally.

To this end, we consider a one-parameter family of finite dimensional spaces $V_h \subset H^1(\Omega)$, given by

$$V_h = \text{span}\left\{\phi_i^h \in C(\Omega) : i \in N_h\right\}; \quad N_h \text{ is an index set.}$$

The functions $\{\phi_i^h\}_{i\in N_h}$ are referred to as *shape functions* and their construction either does not depend on a mesh or depends only minimally. Each ϕ_i^h has compact support and we let ω_i^h denote the interior of supp ϕ_i^h with $h_i = \text{diam}\, \omega_i^h$. We assume that each ω_i^h is star-shaped with respect to a ball $o_i^h \subset \omega_i^h$ and their chunkiness parameters satisfy $\gamma_{\omega_i^h} \leq C, \forall\, i \in N_h$.

Often the shape functions $\{\phi_i^h\}_{i\in N_h}$ are constructed relative to a set of *particles* $X_h = \{x_i^h : i \in N_h\} \subset \mathbb{R}^d$ and each ϕ_i^h is associated with a particle x_i^h. When $\overline{\omega}_i^h \cap \Gamma = \emptyset$, then the associated particle $x_i^h \in \omega_i^h$. But when $\overline{\omega}_i^h \cap \Gamma \neq \emptyset$, then the associated particle x_i^h could be outside Ω. We divide the index set N_h into two disjoint parts, N_h' and N_h'', where,

$$N_h' = \{i \in N_h : \partial\omega_i \cap \Gamma \neq \emptyset\} \text{ and } N_h'' = \{i \in N_h : \bar{\omega}_i \subset \Omega\}.$$

Now, we make several assumptions on the space V_h.

A1 For $i \in N_h$, let $S_i \equiv \{j \in N_h : \omega_i^h \cap \omega_j^h \neq \emptyset\}$. We assume that there is a constant κ, independent of i, j, and h, such that

$$\text{card}\, S_i \leq \kappa, \quad \forall\, i \in N_h.$$

Remark 3.1 This property is referred to as the *finite overlap property*. If we let $S_x = \{j \in N_h : x \in \omega_j^h\}$, then the finite overlap property implies

$$\text{card}\, S_x \leq \kappa, \ \forall\, x \in \Omega. \tag{3.1}$$

A2 There exist positive constants C_2 and C_2, independent of h and i, such that

$$C_1 \leq \frac{h_i}{h} \leq C_2, \ \ C_1 h^d \leq |\omega_i| \leq C_2 h^d, \text{ and } C_1 h^{d-1} \leq |\bar{\omega}_i \cap \Gamma| \leq C_2 h^{d-1}, \tag{3.2}$$

where $|\omega_i|$ is the "area" of ω_i in $\mathbb{R}^d$ and $|\overline{\omega}_i \cap \Gamma|$ is the "length" of $\overline{\omega}_i \cap \Gamma$ in $\mathbb{R}^{d-1}$.

A3 The shape functions *reproduce polynomials of degree k*, i.e.,

$$\sum_{i\in N_h} p\left(x_i^h\right)\phi_i^h(x) = p(x), \quad \forall p \in \mathcal{P}^k \text{ and } x \in \Omega. \tag{3.3}$$

A4 There exists a positive constant C, independent of i and h, such that

$$\left\| D^\alpha \phi_i^h \right\|_{L_\infty(\Omega)} \leq C h_i^{-|\alpha|} \text{ for } |\alpha| \leq q \text{ for some } q \geq 1, \tag{3.4}$$

where α is a multi-index. In this paper, we will assume $q = k + 1$.

A5 There exist positive constants C_1, C_2, independent of h and i, such that for any $i \in N_h$,

$$C_1 \|v\|^2_{L_2(\omega_i)} \leq h_i^d \sum_{j \in S_i} v_j^2 \leq C_2 \|v\|^2_{L_2(\omega_i)}, \tag{3.5}$$

$$C_1 \|v\|^2_{L_2(\partial\omega_i \cap \Gamma)} \leq h_i^{d-1} \sum_{j \in S_i'} v_j^2 \leq C_2 \|v\|^2_{L_2(\partial\omega_i \cap \Gamma)}, \tag{3.6}$$

$$C_1 |v|^2_{H_1(\omega_i)} \leq h_i^{d-2} \sum_{j \in S_i} (v_j - v_i)^2 \leq C_2 |v|^2_{H_1(\omega_i)} \tag{3.7}$$

where $v \in V_h$ is of the form $v = \sum_{i \in N_h} v_i \phi_i^h$ and $S_i' \equiv \{j \in N_h' : \partial\omega_j \cap (\partial\omega_i \cap \Gamma) \neq \emptyset\} \subset N_h$.

Remark 3.2 The finite dimensional space V_h could be viewed as a generalization of the standard piecewise linear finite element space based on quasi-uniform mesh. In the finite element setting, the shape functions ϕ_i^h are hat functions, the particles x_i^h are the finite element nodes, and the supports $\overline{\omega}_i^h$ are the finite element "stars". The quasi-uniform finite elements satisfy the assumptions A1 and A2, whereas the hat functions satisfy A3 and A4 with $k = 1$. The inequalities (3.5)–(3.7) are also true for piecewise linear finite elements. But finite elements are piecewise polynomials and their construction requires a mesh.

Remark 3.3 Many approaches to construct shape functions for GMM are available primarily in the engineering literature; we refer to [8, 9, 16, 21, 23, 24] for details. In all these approaches, the shape functions are not piecewise polynomials and are not available in terms of explicit mathematical formulas that could be easily evaluated. This is the price one pays for avoiding a mesh. For example, in the reproducing kernel particle (RKP) technique, one starts with a weight function $w(x)$ with compact support such that the origin is in the interior of the support. The shape function ϕ_i^h is sought in the form

$$\phi_i^h(x) = w_i^h(x) \sum_{|\alpha| \leq k} \left(x - x_i^h\right)^\alpha b_\alpha^h(x), \quad (\alpha \text{ is a multi-index})$$

where $w_i^h(x) = w(\frac{x - x_i}{h_i})$. For each $x \in \Omega$, $b_\alpha^h(x)$ are chosen so that (3.3) is satisfied, which requires solving a linear system. For details, we refer the reader to [21, 25]. We also note that the shape functions $\phi_i^h(x)$, constructed using these approaches, do not satisfy the *Kronecker delta property*, i.e., $\phi_i^h(x_j^h) \neq \delta_{ij}$. We further note that if the weight function $w \in C^q(\Omega)$, then the shape function $\phi_i^h \in C^q(\Omega)$. Thus it is easy to construct smooth shape functions, in particular with $q = k + 1$, as assumed in assumption A4.

Remark 3.4 The inequality (3.5) in assumption A5 implies the local linear independence of the shape functions $\{\phi_j : j \in S_i\}$ on ω_i. The constants C_1 and

C_2 appearing in A5 may depend on the geometry of ω_i and ω_j with $j \in S_i$, but are independent of i and h.

Remark 3.5 Summing the terms of the inequality (3.5) over $i \in N_h$ and using the assumptions A1 and A2, we can obtain

$$C_1 \|v\|^2_{L_2(\Omega)} \leq h^d \sum_{i \in N_h} v_i^2 \leq C_2 \|v\|^2_{L_2(\Omega)}, \;\; \forall \; v = \sum_{i \in N_h} v_i \phi_i \in V_h \tag{3.8}$$

Similarly, we obtain

$$C_1 \|v\|^2_{L_2(\Gamma)} \leq h^{d-1} \sum_{i \in N'_h} v_i^2 \leq C_2 \|v\|^2_{L_2(\Gamma)}, \;\; \forall \; v = \sum_{i \in N'_h} v_i \phi_i \in V_h. \tag{3.9}$$

In particular, substituting $v = 1$ in these two inequalities, we get

$$\begin{aligned} C_1 h^{-d} \leq |N_h| \leq C_2 h^{-d}, \;\; C_2 h^{-d} \leq \left|N''_h\right| \leq C_2 h^{-d} \\ C_1 h^{-(d-1)} \leq \left|N'_h\right| \leq C_2 h^{-(d-1)} \end{aligned} \tag{3.10}$$

These estimates will be used later in the paper.

In the rest of the paper, we will suppress the parameter h for notational clarity and write ϕ_i, x_i, ω_i, and o_i for ϕ_i^h, x_i^h, ω_i^h, and o_i^h, respectively, with the understanding that they depend on h.

Based on the finite dimensional space $V_h \subset H^1(\Omega)$, as described above, *the Galerkin meshless method* to approximate the solution of (2.3) is given by

$$\begin{aligned} &\text{Find } u_h \in V_h \text{ such that} \\ &B(u_h, v_h) = L(v_h), \quad \forall \; v_h \in V_h \end{aligned} \tag{3.11}$$

The approximation of the exact solution $u \in H^1(\Omega)$ by the solution $u_h \in V_h$ of (3.11) depends on the approximation property of the space V_h, which has been studied in [21, 25]. But in these studies, the set of particles, X_h, has been assumed to be in Ω, which may give rise to boundary layer in the error as indicated in [4]. This is precisely the reason that some of the particles have been allowed to be outside Ω in this paper, as well as in [3, 5, 6]. But the approximation result for V_h remains the same as in [21, 25], even when some of the particles are allowed to outside Ω; only the analysis requires slight modification based on an extension result. For completeness, we present the modified analysis in this paper.

For a function $u \in W^{k+1,\infty}(\mathbb{R}^d)$, we define its V_h-*"interpolant"* on Ω by $\mathcal{I}_h u$, given by

$$\mathcal{I}_h u(x) = \sum_{i \in N_h} u(x_i) \phi_i(x), \;\; x \in \Omega. \tag{3.12}$$

It is clear from the reproducing property (3.3) that $\mathcal{I}_h p(x) = p(x)$, for $x \in \Omega$ and $p \in \mathcal{P}^k$. Strictly speaking, $\mathcal{I}_h$ is not an interpolation operator since $\mathcal{I}_h u(x_j) \neq u(x_j)$ for $x_j \in X_h$; $\mathcal{I}_h$ is a *quasi-interpolation* operator. We will use the terms interpolation and interpolant throughout this paper, with an understanding that they are quasi-interpolation and quasi-interpolant, respectively.

When u is defined only on Ω, the interpolant $\mathcal{I}_h u$ is undefined as some of the particles x_i may be outside Ω. To address this issue, we use the well-known extension theorem (see [10, 26]), which provides us with an extension operator

$$E : L_2(\Omega) \rightarrow L_2(\mathbb{R}^d),$$
$$u \mapsto \bar{u} \equiv Eu$$

such that

$$\bar{u}(x) = u(x), \text{ for } x \in \Omega \text{ and } \|Eu\|_{W^{k+1,\infty}(\mathbb{R}^d)} \leq C\|u\|_{W^{k+1,\infty}(\Omega)} \tag{3.13}$$

where constant C is independent of $u \in L_2(\Omega)$. We now define the V_h-interpolant of $u \in W^{k+1,\infty}(\Omega)$ by $\mathcal{I}_h u(x) \equiv \mathcal{I}_h \bar{u}(x)$, for $x \in \Omega$. We now present an interpolation result that indicates the approximation property of V_h.

Theorem 3.1 *Let $u \in W^{k+1,\infty}(\Omega)$ and $\mathcal{I}_h u$ be the V_h-interpolant of u. Then there is a positive constant C, independent of h, such that*

$$\|u - \mathcal{I}_h u\|_{W^{l,p}(\Omega)} \leq C h^{k+1-l} \|u\|_{W^{k+1,\infty}(\Omega)} \ \forall\ 0 \leq l \leq k+1 \text{ and } p \geq 1. \tag{3.14}$$

Proof For $i \in N_h$, let $\hat{\omega}_i$ be the smallest ball containing the set $\cup_{j \in S_i} \omega_j$. Consider $Q_i^{k+1} Eu(x)$, the Taylor polynomial of degree k (i.e., of order $k+1$) of Eu averaged over the ball $\hat{\omega}_i$ (see the Definition 4.1.3 in [10]). Then from the Lemma 4.3.8 of [10] and assumption A1, we have

$$\|Eu - Q_i^{k+1} Eu\|_{W^{l,\infty}(\hat{\omega}_i)} \leq C h^{k+1-l} |Eu|_{W^{k+1,\infty}(\hat{\omega}_i)} \ \forall\ 0 \leq l \leq k+1. \tag{3.15}$$

The constant C depends on κ and the chunkiness parameter of $\hat{\omega}_i$, which is 1, and thus C is independent of i.

For $x \in \omega_i$, we note that $\phi_j(x) = 0$ for $j \notin S_i$. Therefore, for $x \in \omega_i$, we have

$$\begin{aligned} u(x) - \mathcal{I}_h u(x) &= Eu(x) - Q_i^{k+1} Eu(x) + Q_i^{k+1} Eu(x) \\ &\quad - \sum_{j \in S_i} \left[Eu(x_j) - Q_i^{k+1} Eu(x_j) \right] \phi_j(x) - \sum_{j \in S_i} \left[Q_i^{k+1} Eu(x_j) \right] \phi_j(x) \\ &= Eu(x) - Q_i^{k+1} Eu(x) - \sum_{j \in S_i} \left[Eu(x_j) - Q_i^{k+1} Eu(x_j) \right] \phi_j(x), \end{aligned}$$

where we used (3.3) with $p(x) = Q_i^{k+1} Eu(x)$. Therefore, from (3.15) and assumptions A4, A1, A2 we get

$$\begin{aligned}\|u - \mathcal{I}_h u\|_{W^{l,\infty}(\omega_i)} &\leq \left\| Eu - Q_i^{k+1} Eu \right\|_{W^{l,\infty}(\hat{\omega}_i)} \\ &\quad + \sum_{j \in S_i} \left\| Eu - Q_i^{k+1} Eu \right\|_{W^{0,\infty}(\hat{\omega}_i)} \|\phi_j\|_{W^{l,\infty}(\hat{\omega}_i)} \\ &\leq Ch^{k+1-l} \| Eu\|_{W^{k+1,\infty}(\hat{\omega}_i)} + \sum_{j \in S_i} Ch^{k+1} \| Eu\|_{W^{k+1,\infty}(\hat{\omega}_i)} h^{-l} \\ &\leq Ch^{k+1-l} \| Eu\|_{W^{k+1,\infty}(\hat{\omega}_i)}, \end{aligned} \tag{3.16}$$

where C may depend on κ. Thus we immediately get

$$\|u - \mathcal{I}_h u\|_{W^{l,p}(\omega_i)} \leq Ch^{\frac{d}{p}} h^{k+1-l} \| Eu\|_{W^{k+1,\infty}(\hat{\omega}_i)}. \tag{3.17}$$

Finally, using (3.16), (3.17), the assumption A1, and (3.10), we get

$$\begin{aligned}\|u - \mathcal{I}_h u\|_{W^{l,\infty}(\Omega)} &\leq \sup_{i \in N_h} \|u - \mathcal{I}_h u\|_{W^{l,\infty}(\omega_i)} \leq Ch^{k+1-l} \sup_{i \in N_h} \| Eu\|_{W^{k+1,\infty}(\hat{\omega}_i)} \\ &\leq Ch^{k+1-l} \| Eu\|_{W^{k+1,\infty}(\mathbb{R}^d)} \leq Ch^{k+1-l} \|u\|_{W^{k+1,\infty}(\Omega)}\end{aligned}$$

and

$$\begin{aligned}\|u - \mathcal{I}_h u\|_{W^{l,p}(\Omega)} &\leq \left[\sum_{i \in N_h} \|u - \mathcal{I}_h u\|^p_{W^{l,p}(\omega_i)} \right]^{\frac{1}{p}} \\ &\leq Ch^{\frac{d}{p}} h^{k+1-l} \|\bar{u}\|_{W^{k+1,\infty}(\mathbb{R}^d)} |N_h|^{\frac{1}{p}} \\ &\leq Ch^{k+1-l} \|u\|_{W^{k+1,\infty}(\Omega)},\end{aligned}$$

which gives the desired result. □

Remark 3.6 We note that Theorem 3.1 holds for $u \in W^{k+1,p}(\Omega)$, $1 < p < \infty$, provided $k+1 > d/p$ ($k+1 \geq d$ when $p = 1$). Also for a given l, we only need $q = l$ in assumption A4 (instead of $q = k+1$).

Now, with Lax–Milgram Theorem, Céa' Theorem [10] and (3.14), the following approximation result for the GMM with exact integration is immediate:

Theorem 3.2 *Let $u \in W^{k+1,\infty}(\Omega)$. Then there is a unique solution $u_h \in V_h$ of the variational problem* (3.11) *satisfying*

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch^k \|u\|_{W^{k+1,\infty}(\Omega)}, \tag{3.18}$$

where C is independent of h.

Another consequence of the error estimate (3.14) in Theorem 3.1 is

$$\begin{aligned}\|\mathcal{I}_h u\|_{W^{k+1,\infty}(\Omega)} &\le \|u\|_{W^{k+1,\infty}(\Omega)} + \|u - \mathcal{I}_h u\|_{W^{k+1,\infty}(\Omega)} \\ &\le C\|u\|_{W^{k+1,\infty}(\Omega)},\end{aligned} \tag{3.19}$$

which will be used later in this paper. This is the reason that we required $q = k + 1$ in assumption A4.

4 The Galerkin meshless method with numerical integration

In this section, we will present the GMM with numerical integration (also referred to as quadrature). We will also state the assumptions imposed on the underlying numerical integration rule and discuss them.

To motivate the quadrature in the GMM, we write the solution u_h of (3.11) as $u_h = \sum_{j\in N_h} c_j\phi_j$. Then the coefficients $\{c_j\}_{j\in N_h}$ can be determined uniquely from the linear system

$$\sum_{j\in N_h} \big(\gamma_{ij} + \sigma_{ij}\big)c_j = l_i,\ \ \forall\ i \in N_h,$$

where

$$\gamma_{ij} \equiv B_1(\phi_j, \phi_i) = \int_\Omega A\nabla\phi_j \cdot \nabla\phi_i\, dx = \int_{\omega_i} A\nabla\phi_j \cdot \nabla\phi_i\, dx,$$

$$\sigma_{ij} \equiv B_0(\phi_j, \phi_i) = \int_\Omega c\, \phi_j\phi_i\, dx = \int_{\omega_i} c\, \phi_j\phi_i\, dx,$$

and

$$l_i \equiv L(\phi_i) = \int_\Omega f\phi_i\, dx + \int_\Gamma g\phi_i\, ds = \int_{\omega_i} f\phi_i\, dx + \int_{\partial\omega_i\cap\Gamma} g\phi_i\, ds;$$

we recall that the shape function ϕ_i has compact support $\overline{\omega}_i$. We mention that $\omega_i \cap \omega_j$ can also be used as the domain of integration in the definition of γ_{ij} and σ_{ij}, since the shape function ϕ_j has compact $\overline{\omega}_j$. Consequently, the matrices $\{\gamma_{ij}\}$ and $\{\sigma_{ij}\}$ are symmetric. We have used ω_i (instead of $\omega_i \cap \omega_j$) in the definition of γ_{ij} and σ_{ij} to motivate the numerical integration scheme in this paper. The integrals $\gamma_{ij}, \sigma_{ij}, \int_{\omega_i} f\phi_i\, dx$ and $\int_{\partial\omega_i\cap\Gamma} g\phi_i\, ds$ have to be computed numerically using numerical integration formulas on $\omega_i,\ i \in N_h$ and on $\partial\omega_i \cap \Gamma$, $i \in N_h'$. Let

$$\gamma^*_{ij} \equiv B^*_1(\phi_j, \phi_i) \equiv \int\!\!\!\!\!\!-_{\omega_i}^{s} A\nabla\phi_j \cdot \nabla\phi_i\, dx,\ \ \sigma^*_{ij} \equiv B^*_0(\phi_j, \phi_i) \equiv \int\!\!\!\!\!\!-_{\omega_i}^{m} c\phi_j\phi_i\, dx,$$

and

$$l^*_i \equiv \int\!\!\!\!\!\!-_{\omega_i}^{l} f\phi_i\, dx + \int\!\!\!\!\!\!-_{\partial\omega_i\cap\Gamma} g\phi_i\, ds,$$

where $⨍^s_{\omega_i}$, $⨍^m_{\omega_i}$, and $⨍^l_{\omega_i}$ denote the numerical integration rules, defined on ω_i, to approximate the entries of the stiffness matrix, mass matrix, and the load vector (only the volume integrals), respectively; $⨍_{\partial\omega_i\cap\Gamma}$ is the numerical integration rule to approximate the "boundary integral" in the elements of the load vector.

We note that for a given $i \in N_h$, we use the same quadrature rule $⨍^s_{\omega_i}$ to compute γ^*_{ij} for each $j \in S_i$ (recall the definition of S_i in assumption A1 in Section 3); similarly, the same quadrature rule $⨍^m_{\omega_i}$ is used to compute σ^*_{ij} for $j \in S_i$. But the quadrature rules $⨍^s_{\omega_i}$ and $⨍^m_{\omega_i}$ could possibly be different, i.e., different quadrature rules could be used to approximate the integrals in the stiffness matrix and the mass matrix. The idea of using possibly different quadrature rules to compute the stiffness and mass matrix was not considered in [6].

Remark 4.1 It is easy the check that

$$\sum_{j\in N_h} \gamma_{ij} = 0, \tag{4.1}$$

namely, matrix $\{\gamma_{ij}\}_{i,j\in N_h}$ satisfies "zero row-sum" condition. The same is true for the matrix $\{\gamma^*_{ij}\}_{i,j\in N_h}$. Suppose $(y_{l,i}, v_{l,i})_{l=1}^M$ be the set of integration points and corresponding weights of an M-point quadrature rule $⨍^s_{\omega_i}$. Then

$$\begin{aligned}
\sum_{j\in N_h} \gamma^*_{ij} &= \sum_{j\in N_h} ⨍^s_{\omega_i} A\nabla\phi_j \cdot \nabla\phi_i \, dx \\
&= \sum_{j\in N_h}\sum_{l=1}^{M} A(y_{l,i})\nabla\phi_j(y_{l,i}) \cdot \nabla\phi_i(y_{l,i})\, v_{l,i} \\
&= \sum_{l=1}^{M} A(y_{l,i})\nabla\left[\sum_{j\in N_h}\phi_j(y_{l,i})\right] \cdot \nabla\phi_i(y_{l,i})\, v_{l,i} \\
&= \sum_{l=1}^{M} A(y_{l,i})\nabla 1 \cdot \nabla\phi_i(y_{l,i})\, v_{l,i} = 0.
\end{aligned} \tag{4.2}$$

We note that (4.2) was an assumption on the quadrature rule in [5], where γ^*_{ij} was defined by using quadrature on $\omega_i \cap \omega_j$. This is one of the reasons that we defined γ^*_{ij} by numerically integrating over ω_i in this paper (also in [6]) so that (4.2) is automatically satisfied.

Let $v_h = \sum_{i\in N_h} v_i\phi_i$ and $w_h = \sum_{i\in N_h} w_i\phi_i$ be arbitrary elements in V_h. Then

$$B_1(v_h, w_h) = \sum_{i,j\in N_h} v_i\gamma_{ji}w_j, \quad B_0(v_h, w_h) = \sum_{i,j\in N_h} v_i\sigma_{ji}w_j,$$

$$B(v_h, w_h) = \sum_{i,j\in N_h} v_i\big(\gamma_{ji} + \sigma_{ji}\big)w_j, \quad \text{and} \ \ L(v_h) = \sum_{i\in N_h} v_i l_i.$$

Therefore, we naturally define

$$B_1^*(v_h, w_h) \equiv \sum_{i,j\in N_h} v_i\gamma_{ji}^* w_j, \quad B_0^*(v_h, w_h) \equiv \sum_{i,j\in N_h} v_i\sigma_{ji}^* w_j, \tag{4.3}$$

$$B^*(v_h, w_h) \equiv \sum_{i,j\in N_h} v_i\big(\gamma_{ji}^* + \sigma_{ji}^*\big)w_j, \quad \text{and} \ \ L^*(v_h) \equiv \sum_{i\in N_h} v_i l_i^*. \tag{4.4}$$

From this definition, the functional $L^*(\cdot)$ is linear on V_h and the forms $B_1^*(\cdot,\cdot)$, $B_0^*(\cdot,\cdot)$, $B^*(\cdot,\cdot)$ are bilinear on $V_h \times V_h$. Also from (4.2) and (4.1), it is clear that

$$B_1^*(1, \phi_i) = 0 = B_1(1, \phi_i), \ \ \forall\, i \in N_h. \tag{4.5}$$

But it is important to note that the matrix $\{\gamma_{ij}^*\}_{i,j\in N_h}$ may not be symmetric (in contrast to $\{\gamma_{ij}\}_{i,j\in N_h}$), since

$$\gamma_{ij}^* = \int\!\!\!\!\!\!-^{s}_{\omega_i} A\nabla\phi_j \cdot \nabla\phi_i\, dx \neq \int\!\!\!\!\!\!-^{s}_{\omega_j} A\nabla\phi_i \cdot \nabla\phi_j\, dx = \gamma_{ji}^*.$$

Therefore, $B_1^*(\phi_i, 1)$, $\forall\, i \in N_h$ may not be zero. Similarly, we can show that the matrix $\{\sigma_{ij}^*\}_{i,j\in N_h}$ may not be symmetric, and consequently, the matrix $\{\gamma_{ij}^* + \sigma_{ij}^*\}_{i,j\in N_h}$ may not be symmetric.

The GMM with numerical quadrature to approximate the solution of (2.3) is given by

$$\begin{aligned} &\text{Find } u_h^* \in V_h \text{ such that} \\ &B^*(u_h^*, v_h) = L^*(v_h), \quad \forall\, v_h \in V_h, \end{aligned} \tag{4.6}$$

where $B^*(\cdot,\cdot)$ and $L^*(\cdot)$ is defined in (4.4). We note that the bilinear form $B^*(\cdot,\cdot)$ is not symmetric.

Next, we state certain assumptions on the quadrature used in the GMM. Some of these assumptions were given in [6]. We include these assumptions also in this paper for completeness.

QA 4.1 There exist positive constants η and τ, small enough and independent of i and h, such that

$$\left| \int^{t}_{\omega_i} \varrho\, dx - \int\!\!\!\!\!\!-^{t}_{\omega_i} \varrho\, dx \right| \leq \eta\, |\omega_i|\, \|\varrho\|_{L_\infty(\omega_i)}, \ \ t = s, m, l, \tag{4.7}$$

and

$$\left| \int_{\partial\omega_i \cap \Gamma} \vartheta \, ds - \fint_{\partial\omega_i \cap \Gamma} \vartheta \, ds \right| \leq \tau \, |\partial\omega_i \cap \Gamma| \, \|\vartheta\|_{L_\infty(\partial\omega_i \cap \Gamma)} \tag{4.8}$$

for a class of functions $\varrho \in W^{m_1,\infty}(\omega_i)$ and $\vartheta \in W^{m_2,\infty}(\partial\omega_i \cap \Gamma)$ satisfying

$$\|D^\alpha \varrho\|_{L_\infty(\omega_i)} \leq C(h_i)^{-|\alpha|} \|\varrho\|_{L_\infty(\omega_i)}, \ \ |\alpha| \leq m_1 \tag{4.9}$$

and

$$\|D^\alpha \vartheta\|_{L_\infty(\partial\omega_i \cap \Gamma)} \leq C(h_i)^{-|\alpha|} \|\vartheta\|_{L_\infty(\partial\omega_i \cap \Gamma)}, \ \ |\alpha| \leq m_2 \tag{4.10}$$

where $C > 0$ is independent of $i \in N_h$ and $m_1, \ m_2 > 1$ may depend on the numerical integration rules and the assumption A4 in Section 2.

Remark 4.2 The constants η and τ are associated with the numerical integration rules. It is possible to choose numerical integration rules (e.g., by taking more integration points) such that η and τ are small enough. We refer to Remark 3.3 of [6] for specific examples. We mention that in all the numerically approximated integrals in this paper, the integrands satisfy the conditions (4.9) and (4.10).

QA 4.2 For each $i \in N_h$, let $G_i^* : \widetilde{C^1}(\bar{\omega}_i) \to \mathbb{R}$ be a linear functional given by

$$G_i^*(\widetilde{v}) = \fint_{\omega_i}^{s} \widetilde{v} \cdot \nabla\phi_i \, dx + \fint_{\omega_i}^{l} \nabla \cdot \widetilde{v} \, \phi_i \, dx - \fint_{\partial\omega_i \cap \Gamma} \widetilde{v} \cdot \vec{n} \, \phi_i \, ds \tag{4.11}$$

where $\vec{n}$ is the outward normal to $\partial\omega_i \cap \Gamma$. We assume that

$$G_i^*(\widetilde{p}) = 0, \quad \forall \widetilde{p} \in \widetilde{\mathcal{P}}^{k-1} \tag{4.12}$$

where $\mathcal{P}^{k-1}$ is the space of polynomials of degree $k-1$.

Remark 4.3 For each $i \in N_h$, we consider linear functional $G_i : \widetilde{H}^1(\omega_i) \to \mathbb{R}$ given by

$$G_i(\widetilde{v}) = \int_{\omega_i} \widetilde{v} \cdot \nabla\phi_i \, dx + \int_{\omega_i} \nabla \cdot \widetilde{v} \, \phi_i \, dx - \int_{\partial\omega_i \cap \Gamma} \widetilde{v} \cdot \vec{n} \, \phi_i \, ds \tag{4.13}$$

It is clear from the Green Theorem that

$$G_i(\widetilde{p}) = 0, \quad \forall \widetilde{p} \in \widetilde{\mathcal{P}}^{k-1} \tag{4.14}$$

Hence, the assumption (4.12) mimics (4.14) and could be viewed as a discrete version of the Green Theorem on a particular class of functions $\widetilde{\mathcal{P}}^{k-1}$. We will show how to construct the quadrature rules satisfying (4.12) later.

Remark 4.4 We note that the assumption QA 4.2, in particular (4.12), is slightly stronger than a similar assumption QA3 used in [6]. We mention however, that for problems with non-constant coefficients $A(x)$, a direct use of the ideas presented in [6] will require the underlying numerical integration rule to satisfy a modified version of the assumption QA3 of [6] involving $A(x)$.

Numerical integration rules, satisfying this modified assumption, will depend on $A(x)$, i.e., will be problem dependent. The assumption QA 4.2 in this paper does not require the quadrature rules to depend on $A(x)$.

Remark 4.5 Using $\widetilde{v} \in \widetilde{\mathcal{P}}^0$ (i.e., $k = 1$) in (4.11), we have for each $i \in N_h$,

$$\fint^s_{\omega_i} \frac{\partial \phi_i}{\partial x_j}\, dx = \fint_{\partial\omega_i \cap \Gamma} n_j \phi_i\, ds. \quad j = 1, 2, \cdots, d. \tag{4.15}$$

This is the Integration Constraint in the SCNI method described in [13]. SCNI uses nodal integration and a strain smoothing technique so that (4.15), or (4.11) with $k = 1$ holds. In Sections 6 and 7, we will construct quadrature rule on ω_i such that (4.11) is satisfied for $1 \le k \le 2$.

QA 4.3 For each $i \in N_h$, we assume $\fint^m_{\omega_i} = \fint^l_{\omega_i}$, i.e., the elements of the mass matrix and the volume integrals in the elements the load vector are computed using the same integration rule.

We note that the integration rules $\fint^s_{\omega_i}$ and $\fint^l_{\omega_i}$ could be different.

QA 4.4 There is a constant $C > 0$ such that for η small enough,

$$\left|B_1^*(w_h, v_h)\right| \le C\|w_h\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)}, \ \forall\, w_h, v_h \in V_h, \tag{4.16}$$

and

$$B_1^*(v_h, v_h) \ge C\|v_h\|^2_{H^1(\Omega)}, \ \forall\, v_h \in V_h. \tag{4.17}$$

Lemma 4.1 *Suppose the quadrature satisfy the assumptions* QA 4.1 *and* QA 4.4*. Then for η, small enough, there are constants C_1 and C_2, independent of h, such that*

$$\left|B^*(w_h, v_h)\right| \le C_1\|w_h\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)} \ \text{ and } \ B^*(v_h, v_h) \ge C_2\|v_h\|^2_{H^1(\Omega)}$$

for any $w_h, v_h \in V_h$.

Proof Let $w_h = \sum_{i\in N_h} w_i\phi_i$ and $v_h = \sum_{i\in N_h} v_i\phi_i$ be in V_h. We first estimate $|B_0(w_h, v_h) - B_0^*(w_h, v_h)|$. For any $i \in N_h$, using (4.7), the assumption A1, and (3.5), we have

$$\begin{aligned}
\left|B_0(w_h, \phi_i) - B_0^*(w_h, \phi_i)\right| &= \left|\int^m_{\omega_i} c\, w_h\, \phi_i\, dx - \fint^m_{\omega_i} c\, w_h\, \phi_i\, dx\right| \\
&\le C\sum_{j\in S_i}|w_j|\left|\int^m_{\omega_i} c\,\phi_j\,\phi_i\, dx - \fint^m_{\omega_i} c\,\phi_j\,\phi_i\, dx\right| \\
&\le C\eta\,|\omega_i|\,\|c\,\phi_j\,\phi_i\|_{L_\infty(\omega_i)}\left(\sum_{j\in S_i}|w_j|^2\right)^{\frac{1}{2}}\sqrt{\kappa} \\
&\le C\eta\, h^d\, h^{-\frac{d}{2}}\|w_h\|_{L_2(\omega_i)}\sqrt{\kappa}.
\end{aligned} \tag{4.18}$$

Therefore, squaring both sides of the above inequality and summing over all $i \in N_h$, we get

$$
\begin{aligned}
\left|B_0(w_h, v_h) - B_0^*(w_h, v_h)\right| &\le \left(\sum_{i\in N_h}\left[B_0(w_h,\phi_i) - B_0^*(w_h,\phi_i)\right]^2\right)^{\frac{1}{2}} \left(\sum_{i\in N_h} v_i^2\right)^{\frac{1}{2}} \\
&\le C\eta h^{\frac{d}{2}} \left(\sum_{i\in N_h}\|w_h\|^2_{L_2(\omega_i)}\right)^{\frac{1}{2}} C h^{-\frac{d}{2}}\|v_h\|_{L_2(\Omega)} \\
&\le C\eta\|w_h\|_{L_2(\Omega)}\|v_h\|_{L_2(\Omega)},
\end{aligned} \tag{4.19}
$$

where the second and the last inequalities were obtained from (3.8) and the assumption A1, respectively.

Finally, from the assumption (4.4) and (4.19), we get

$$
\begin{aligned}
\left|B^*(w_h, v_h)\right| &\le \left|B_1^*(w_h, v_h)\right| + |B_0(w_h, v_h)| + \left|B_0(w_h, v_h) - B_0^*(w_h, v_h)\right| \\
&\le C\|w_h\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)} + C(1+\eta)\|w_h\|_{L_2(\Omega)}\|v_h\|_{L_2(\Omega)} \\
&\le C(1+\eta)\|w_h\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)}
\end{aligned}
$$

and from (2.2)

$$
\begin{aligned}
\left|B^*(v_h, v_h)\right| &\ge B_1^*(v_h, v_h) + B_0(v_h, v_h) - \left|B_0(v_h, v_h) - B_0^*(v_h, v_h)\right| \\
&\ge C\|v_h\|^2_{H^1(\Omega)} + \beta\|v_h\|^2_{L_2(\Omega)} - C\eta\|v_h\|^2_{L_2(\Omega)} \\
&\ge \min\{C, \beta - C\eta\}\|v_h\|^2_{H^1(\Omega)}.
\end{aligned}
$$

We get the desired result by considering $\eta < \beta/C$. □

It is clear from Lemma 4.1 that the bilinear form $B^*(\cdot,\cdot)$ is bounded and coercive, and therefore from the Lax–Milgram lemma we conclude that the problem (4.6) has a unique solution $u_h^* \in V_h$.

Remark 4.6 We note that Assumption QA 4.4 is not needed if we put a restriction on η, namely, $\eta \le Ch$. Under this restrictive condition on η, we can prove (4.16), (4.17), and incorporate it into the proof of Lemma 4.1 (as in Lemma 3.1 of [6] for $A = I$). However, from our computational experience we have noticed that (4.16), (4.17) hold without the condition $\eta \le Ch$, i.e., the condition is not necessary. Precisely for this reason we assume (4.16), (4.17) under QA 4.4, and do not use $\eta \le Ch$.

Remark 4.7 It is instructive to illustrate the assumption QA 4.2, i.e., (4.12) in simpler situations. Let $\Omega \subset \mathbb{R}^2$ and $k = 1$, then $\widetilde{\mathcal{P}}^{k-1} = \widetilde{\mathcal{P}}^0 = \text{span}\{[1,0],[0,1]\}$ Considering $\widetilde{p}(x_1, x_2) = [1, 0]$ in (4.12), we get

$$
G_i^*([1,0]) = \int_{\omega_i}^{s} \frac{\partial\phi_i}{\partial x_1}\,dx - \int_{\partial\omega_i\cap\Gamma} n_1\phi_i\,ds = 0, \quad i \in N_h, \tag{4.20}
$$

where $\vec{n} = [n_1, n_2]$. Similarly, considering $\widetilde{p}(x_1, x_2) = [0, 1]$ in (4.12), we get

$$G_i^*([0,1]) = \int_{\omega_i}^{s} \frac{\partial \phi_i}{\partial x_2}\, dx - \int_{\partial\omega_i \cap \Gamma} n_2 \phi_i\, ds = 0, \quad i \in N_h \tag{4.21}$$

Thus for $k = 1$, the quadrature must satisfy the two conditions (4.20) and (4.21) for each $i \in N_h$. In particular, the quadrature must satisfy

$$\int_{\omega_i}^{s} \nabla \phi_i\, dx = 0, \quad \forall i \in N_h''. \tag{4.22}$$

We now illustrate (4.12) for $k = 2$. In this case, we know that

$$\widetilde{\mathcal{P}}^{k-1} = \widetilde{\mathcal{P}}^1 = \text{span}\{[1,0], [0,1], [x_1, 0], [x_2, 0], [0, x_1], [0, x_2]\}.$$

Considering $\widetilde{p}(x_1, x_2) = [x_1, 0]$ in (4.12), we get

$$G_i([x_1, 0]) = \int_{\omega_i}^{s} x_1 \frac{\partial \phi_i}{\partial x_1}\, dx + \int_{\omega_i}^{l} \phi_i\, dx - \int_{\partial\omega_i \cap \Gamma} x_1\, n_1\, \phi_i\, ds = 0, \quad \forall i \in N_h \tag{4.23}$$

Similarly, considering $\widetilde{p}(x_1, x_2) = [x_2, 0]$, $p(x_1, x_2) = [0, x_1]$, and $\widetilde{p}(x_1, x_2) = [0, x_2]$ in (4.12), we get

$$G_i([x_2, 0]) = \int_{\omega_i}^{s} x_2 \frac{\partial \phi_i}{\partial x_1}\, dx - \int_{\partial\omega_i \cap \Gamma} x_2\, n_1\, \phi_i\, ds = 0, \quad \forall i \in N_h, \tag{4.24}$$

$$G_i([0, x_1]) = \int_{\omega_i}^{s} x_1 \frac{\partial \phi_i}{\partial x_2}\, dx - \int_{\partial\omega_i \cap \Gamma} x_1\, n_2\, \phi_i\, ds = 0, \quad \forall i \in N_h, \tag{4.25}$$

and

$$G_i([0, x_2]) = \int_{\omega_i}^{s} x_2 \frac{\partial \phi_i}{\partial x_2}\, dx + \int_{\omega_i}^{l} \phi_i\, dx - \int_{\partial\omega_i \cap \Gamma} x_2\, n_2\, \phi_i\, ds = 0, \quad \forall i \in N_h \tag{4.26}$$

Thus, for $k = 2$, the quadrature must satisfy (4.23)–(4.26) in addition to the assumptions (4.20) and (4.21).

Remark 4.8 It is clear from (4.20) and (4.21) that for $k = 1$, only the quadrature rule to compute the elements of the stiffness matrix , i.e., $\int_{\omega_i}^{s}$, and the elements of the load vector associated with the boundary, i.e., $\int_{\partial\omega_i \cap \Gamma}$ have to satisfy the assumption QA 4.2; the quadrature rule to compute the elements of the mass matrix and the volume integrals in the elements of the load vector, i.e., $\int_{\omega_i}^{m}$ (it is the same as $\int_{\omega_i}^{l}$), do not have to satisfy QA 4.2 and it could be any accurate rule satisfying assumption QA 4.1. We note that in SCNI method [13], $\int_{\partial\omega_i \cap \Gamma}$ needs to be consistent with the boundary integration of the smoothed gradient to satisfy condition (4.15), while in this paper, there is no constraint (other than accuracy) on $\int_{\partial\omega_i \cap \Gamma}$; $\int_{\omega_i}^{s}$ is carefully chosen such that (4.20), (4.21) hold. For $k = 2$, the conditions (4.23) and (4.26) indicate that $\int_{\omega_i}^{s}$ and $\int_{\omega_i}^{l}$ must be related. Even in this situation, $\int_{\omega_i}^{l}$ (which is same as $\int_{\omega_i}^{m}$)) could be any accurate quadrature rule, but $f^s_{\omega_i}$ has to satisfy (4.23) and (4.26). We will obtain $f^s_{\omega_i}$ later in the paper with this feature.

5 Effect of numerical integration

In this section, we will investigate the effect of numerical integration on the GMM; in particular, we will estimate the error $\|u - u_h^*\|_{H^1(\Omega)}$, where u is the solution of the problem (2.3) and u_h^* is the solution of the GMM (4.6) with numerical integration. We recall from Theorem 3.2 that $\|u - u_h\|_{H^1(\Omega)} = \mathcal{O}(h^k)$, where u_h is the solution of (3.11)—the GMM with exact integration. We will show in this section that $\|u - u_h^*\|_{H^1(\Omega)} \neq \mathcal{O}(h^k)$ in general, and the error depends on the quadrature parameters η and τ, defined in (4.7) and (4.8), respectively. We will assume in this section that the exact solution u of (2.3) is smooth, i.e., $u \in W^{k+1,\infty}(\Omega)$; this will enable us to focus only on numerical integration and will allow us to present the main ideas effectively.

It is well-known that Strang's Lemma [15, 27] is one of the main tools to study the perturbation in the solution of a Galerkin method due to variational crimes, e.g., numerical integration in a Galerkin method. We present a slight variation of the Strang's Lemma in the following result, which will provide us with an abstract framework to study the error $u - u_h^*$.

Lemma 5.1 *Suppose the quadrature rules satisfy the conditions in the lemma 4.1, and u and u_h^* are the solutions of the variational problems* (2.3) *and* (4.6), *respectively. Then there is a constant $C > 0$, independent of h, such that, for any $w_h \in V_h$,*

$$\|u - u_h^*\|_{H^1(\Omega)} \leq C\|u - w_h\|_{H^1(\Omega)} + \sup_{v_h \in V_h} \frac{\left|\left[B(w_h, v_h) - L(v_h)\right] - \left[B^*(w_h, v_h) - L^*(v_h)\right]\right|}{\|v_h\|_{H^1(\Omega)}}.$$

Proof Let $w_h \in V_h$ be arbitrary. Using the coercivity of the bilinear form $B^*(\cdot, \cdot)$ (see Lemma 4.1), we have

$$\begin{aligned}
C\|u_h^* - w_h\|^2_{H^1(\Omega)} &\leq B^*(u_h^* - w_h, u_h^* - w_h) \\
&= B(u - w_h, u_h^* - w_h) \\
&\quad + B(w_h, u_h^* - w_h) - B^*(w_h, u_h^* - w_h) \\
&\quad - B(u, u_h^* - w_h) + B^*(u_h^*, u_h^* - w_h) \\
&= B(u - w_h, u_h^* - w_h) \\
&\quad + \left[B(w_h, u_h^* - w_h) - L(u_h^* - w_h)\right] \\
&\quad - \left[B^*(w_h, u_h^* - w_h) - L^*(u_h^* - w_h)\right].
\end{aligned}$$

Therefore, dividing the above inequality by $\|u_h^* - w_h\|_{H^1(\Omega)}$ and using the boundedness of $B(\cdot, \cdot)$, we get

$$\|u_h^* - w_h\|_{H^1(\Omega)} \leq C\|u - w_h\|_{H^1(\Omega)} + \sup_{v_h \in V_h} \frac{\Big|\big[B(w_h, v_h) - L(v_h)\big] - \big[B^*(w_h, v_h) - L^*(v_h)\big]\Big|}{\|v_h\|_{H^1(\Omega)}}$$

Now, using the triangle inequality, we immediately get

$$\|u - u_h^*\|_{H^1(\Omega)} \leq (C+1)\|u - w_h\|_{H^1(\Omega)} + \sup_{v_h \in V_h} \frac{\Big|\big[B(w_h, v_h) - L(v_h)\big] - \big[B^*(w_h, v_h) - L^*(v_h)\big]\Big|}{\|v_h\|_{H^1(\Omega)}},$$

which is the desired result. □

Remark 5.1 It is clear from Lemma 5.1 that we need to estimate the consistency errors

$$\sup_{v_h \in V_h} \frac{\Big|\big[B(w_h, v_h) - L(v_h)\big] - \big[B^*(w_h, v_h) - L^*(v_h)\big]\Big|}{\|v_h\|_{H^1(\Omega)}} \tag{5.1}$$

to estimate the error $\|u - u_h^*\|_{H^1(\Omega)}$. We note that in the Strang's Lemma as presented in [15], this term is further divided into two terms

$$\sup_{v_h \in V_h} \frac{\Big|B(w_h, v_h) - B^*(w_h, v_h)\Big|}{\|v_h\|_{H^1(\Omega)}} \quad \text{and} \quad \sup_{v_h \in V_h} \frac{\Big|L(v_h) - L^*(v_h)\Big|}{\|v_h\|_{H^1(\Omega)}}.$$

Keeping the terms together, as in (5.1), is crucial for our analysis of the effect of numerical integration in the GMM.

We now present some notions and associated results that we will use later in this section. We first define a norm and semi-norm of the matrix-valued function $A(x)$; recall that we assumed $a_{ij}(x) \in C^k(\overline{\Omega})$, $\forall\ i,\ j = 1, 2, \cdots, d$. Suppose $D \subset \Omega$ be a domain and let

$$|A|_{W^{l,\infty}(D)} \equiv \max\left\{\sum_{j=1}^{d} \big|a_{ij}\big|_{W^{l,\infty}(D)} : 1 \leq i \leq d\right\}$$
$$\text{and } \|A\|_{W^{l,\infty}(D)} \equiv \max\left\{|A|_{W^{m,\infty}(D)} : 0 \leq m \leq l\right\},$$

for any non-negative integer $l \leq k$.

Lemma 5.2 *For $0 \leq l \leq k$, there exists a constant $C > 0$, depending only on l and d, such that*

$$\|A\,\widetilde{v}\|_{W^{l,\infty}(D)} \leq C\|A\|_{W^{l,\infty}(D)}\|\widetilde{v}\|_{W^{l,\infty}(D)}, \qquad \forall\, \widetilde{v} \in \widetilde{W}^{k,\infty}(D). \tag{5.2}$$

Proof Let $0 \leq \ell \leq l$ and suppose $\widetilde{v} = [v_j]_{j=1}^d$. Then using Leibnitz formula, we have

$$\begin{aligned}
|A\,\widetilde{v}|_{W^{\ell,\infty}(D)} &= \max_{1\leq i\leq d}\left\{ \Big|\sum_{j=1}^{d} a_{ij}v_j\Big|_{W^{\ell,\infty}(D)} \right\} \\
&\leq C \max_{1\leq i\leq d}\left\{ \sum_{j=1}^{d}\sum_{m=0}^{\ell} \big|a_{ij}\big|_{W^{\ell-m,\infty}(D)}\big|v_j\big|_{W^{m,\infty}(D)} \right\} \\
&\leq C \sum_{m=0}^{\ell} \big|\widetilde{v}\big|_{W^{m,\infty}(D)} \max_{1\leq i\leq d}\left\{ \sum_{j=1}^{d} \big|a_{ij}\big|_{W^{\ell-m,\infty}(D)} \right\} \\
&= C \sum_{m=0}^{\ell} \big|\widetilde{v}\big|_{W^{m,\infty}(D)} |A|_{W^{\ell-m,\infty}(D)} \\
&\leq C\|A\|_{W^{\ell,\infty}(D)}\|\widetilde{v}\|_{W^{\ell,\infty}(D)} \leq C\|A\|_{W^{l,\infty}(D)}\|\widetilde{v}\|_{W^{l,\infty}(D)},
\end{aligned}$$

where the constant C only depends on l and d. Therefore,

$$\|A\,\widetilde{v}\|_{W^{l,\infty}(D)} \leq C\|A\|_{W^{l,\infty}(D)}\|\widetilde{v}\|_{W^{l,\infty}(D)},$$

which is the desired result. □

We now present the next result. For a smooth function v and $i \in N_h$, let

$$T_i^{k-1}v(x) = \sum_{|\alpha|\leq k-1} \frac{D^\alpha v(\bar{x}_i)}{\alpha!}(x-\bar{x}_i)^\alpha$$

be the $(k-1)^{th}$ degree Taylor polynomial of v associated with the center $\bar{x}_i$ of the ball $o_i \subset \omega_i$ (recall in Section 2 that ω_i is star-shaped with respect to the ball o_i). It is well known that [10]

$$\Big|v - T_i^{k-1}v\Big|_{W^{j,\infty}(\omega_i)} \leq \frac{Ch^{k-j}}{(k-j)!}\|v\|_{W^{k,\infty}(\omega_i)}, \qquad j = 0, 1, \ldots, k \tag{5.3}$$

For a smooth vector-valued function $\widetilde{v} = [v_j]_{j=1}^d$ we define

$$\widetilde{T}_i^{k-1}\widetilde{v}(x) = \Big[T_i^{k-1}v_j\Big]_{j=1}^d.$$

$\widetilde{T}_i^{k-1}\widetilde{v}(x)$ is also a vector-valued function with its components being the $(k-1)^{th}$ degree Taylor polynomials of the corresponding components of $\widetilde{v}$, centered at $\bar{x}_i$. We will refer to $\widetilde{T}_i^{k-1}\widetilde{v}(x)$ as the Taylor polynomial of $\widetilde{v}$ associated with $\bar{x}_i$.

We define

$$\widetilde{R}_i \equiv A\nabla\mathcal{I}_h u - \widetilde{T}_i^{k-1}(A\nabla\mathcal{I}_h u), \tag{5.4}$$

where $\mathcal{I}_h u$ is the V_h-interpolant of u, defined in (3.12) through (3.13). Clearly $\widetilde{R}_i$ is the "remainder" of the Taylor polynomial $\widetilde{T}_i^{k-1}\widetilde{v}(x)$ with $\widetilde{v} = A\nabla\mathcal{I}_h u$.

Lemma 5.3 *Let $0 \le j \le k$. Then there exists a constant $C > 0$ such that*

$$\big|\widetilde{R}_i\big|_{W^{j,\infty}(\omega_i)} \le Ch^{k-j}\|A\|_{W^{k,\infty}(\Omega)}\|u\|_{W^{k+1,\infty}(\Omega)} \tag{5.5}$$

Proof Let $\widetilde{v} \in W^{k,\infty}(\omega_i)$. Then from the definition of norm of vector-valued functions and from (5.3), we immediately get

$$\left|\widetilde{v} - \widetilde{T}_i^{k-1}\widetilde{v}\right|_{W^{j,\infty}(\omega_i)} \le C\,h^{k-j}\|\widetilde{v}\|_{W^{k,\infty}(\omega_i)}, \quad j = 0, 1, \ldots, k$$

Now substituting $\widetilde{v} = A\nabla\mathcal{I}_h u$ in the above inequality, and using (5.2) and (3.19), we get, for $j = 0, 1, \ldots, k$,

$$\begin{aligned}\big|\widetilde{R}_i\big|_{W^{j,\infty}(\omega_i)} &\le Ch^{k-j}\|A\nabla\mathcal{I}_h u\|_{W^{k,\infty}(\omega_i)}\\ &\le Ch^{k-j}\|A\|_{W^{k,\infty}(\omega_i)}\|\nabla\mathcal{I}_h u\|_{W^{k,\infty}(\omega_i)}\\ &\le Ch^{k-j}\|A\|_{W^{k,\infty}(\Omega)}\|\nabla\mathcal{I}_h u\|_{W^{k,\infty}(\Omega)}\\ &\le Ch^{k-j}\|A\|_{W^{k,\infty}(\Omega)}\|u\|_{W^{k+1,\infty}(\Omega)},\end{aligned}$$

which is the desired result. □

The next lemma provides us with an estimate of the error in the numerical integration for a particular integrand, and it is an important ingredient in the proof of the main result of the paper. This result is a generalization of Lemma 4.2 in [6] in the context of variable coefficients $A(x) = [a_{ij}(x)]_{1\le i,j\le d}$; we mention that the matrix $A(x) = I$ was considered in [6].

Lemma 5.4 *For any $i \in N_h$, let $G_i(\cdot)$ and $G_i^*(\cdot)$ be functionals defined by (4.13), (4.11), respectively. Assume that the quadrature formulas satisfy the assumptions (4.7), (4.8), and (4.12). Then there exists a positive constant C, independent of h and i, such that, for $i \in N_h''$,*

$$\big|G_i(A\nabla\mathcal{I}_h u) - G_i^*(A\nabla\mathcal{I}_h u)\big| \le C\eta h^{k+d-1}\|A\|_{W^{k,\infty}(\Omega)}\|u\|_{W^{k+1,\infty}(\Omega)},$$

and, for $i \in N_h'$,

$$\big|G_i(A\nabla\mathcal{I}_h u) - G_i^*(A\nabla\mathcal{I}_h u)\big| \le C(\eta + \tau)h^{k+d-1}\|A\|_{W^{k,\infty}(\Omega)}\|u\|_{W^{k+1,\infty}(\Omega)}.$$

Proof For $i \in N_h$, let $\bar{x}_i$ be the center of the ball $o_i \subset \omega_i$. We expand the vector-valued function $A\nabla\mathcal{I}_h u$ with respect to $\bar{x}_i$ using (5.4) as

$$A\nabla\mathcal{I}_h u = \widetilde{T}_i^{k-1}(A\nabla\mathcal{I}_h u) + \widetilde{R}_i.$$

We note that $\widetilde{T}_i^{k-1}(A\nabla\mathcal{I}_h u) \in \widetilde{P}^{k-1}$. Therefore from the assumption on the quadrature (4.12) and the fact (4.14), we have

$$G_i^*\big(\widetilde{T}_i^{k-1}(A\nabla\mathcal{I}_h u)\big) = 0 \text{ and } G_i\big(\widetilde{T}_i^{k-1}(A\nabla\mathcal{I}_h u)\big) = 0.$$

Hence, for $i \in N_h'$,

$$\begin{aligned}\left|G_i(A\nabla\mathcal{I}_h u) - G_i^*(A\nabla\mathcal{I}_h u)\right| &= \left|G_i(\widetilde{R}_i) - G_i^*(\widetilde{R}_i)\right| \\ &\le \left|\int_{\omega_i} \widetilde{R}_i \cdot \nabla\phi_i \, dx - \fint_{\omega_i}^{s} \widetilde{R}_i \cdot \nabla\phi_i \, dx\right| \\ &\quad + \left|\int_{\omega_i} \nabla \cdot \widetilde{R}_i \, \phi_i \, dx - \fint_{\omega_i}^{l} \nabla \cdot \widetilde{R}_i \, \phi_i \, dx\right| \\ &\quad + \left|\int_{\partial\omega_i\cap\Gamma} \widetilde{R}_i \cdot \vec{n} \, \phi_i \, ds - \fint_{\partial\omega_i\cap\Gamma} \widetilde{R}_i \cdot \vec{n} \, \phi_i \, ds\right|. \end{aligned} \tag{5.6}$$

Also for $i \in N_h''$, recalling that $\phi_i = 0$ on $\partial\omega_i$, we get

$$\begin{aligned}\left|G_i(A\nabla\mathcal{I}_h u) - G_i^*(A\nabla\mathcal{I}_h u)\right| &= \left|G_i(\widetilde{R}_i) - G_i^*(\widetilde{R}_i)\right| \\ &\le \left|\int_{\omega_i} \widetilde{R}_i \cdot \nabla\phi_i \, dx - \fint_{\omega_i}^{s} \widetilde{R}_i \cdot \nabla\phi_i \, dx\right| \\ &\quad + \left|\int_{\omega_i} \nabla \cdot \widetilde{R}_i \, \phi_i \, dx - \fint_{\omega_i}^{l} \nabla \cdot \widetilde{R}_i \, \phi_i \, dx\right|. \end{aligned} \tag{5.7}$$

Now, from (5.6), the assumptions QA 4.1, A4, A2, and the remainder estimate (5.5), we obtain for $i \in N_h'$

$$\begin{aligned}\left|G_i(A\nabla\mathcal{I}_h u) - G_i^*(A\nabla\mathcal{I}_h u)\right| &= \eta \left|\widetilde{R}_i \cdot \nabla\phi_i\right|_{L_\infty(\omega_i)} |\omega_i| + \eta |\nabla \cdot \widetilde{R}_i \, \phi_i|_{L_\infty(\omega_i)} |\omega_i| \\ &\quad + \tau \left|\widetilde{R}_i \cdot \vec{n} \, \phi_i\right|_{L_\infty(\partial\omega_i\cap\Gamma)} |\partial\omega_i \cap \Gamma| \\ &\le 2C\eta h^{k-1+d} \|A\|_{W^{k,\infty}(\omega_i)} \|u\|_{W^{k+1,\infty}(\omega_i)} \\ &\quad + C\tau h^{k-1+d} \|A\|_{W^{k,\infty}(\omega_i)} \|u\|_{W^{k+1,\infty}(\omega_i)} \\ &\le C(\eta + \tau) h^{k-1+d} \|A\|_{W^{k,\infty}(\Omega)} \|u\|_{W^{k+1,\infty}(\Omega)}, \end{aligned}$$

which is the desired result for $i \in N_h'$. Also using (5.6) and similar arguments as above, we get for $i \in N_h''$

$$\left|G_i(A\nabla\mathcal{I}_h u) - G_i^*(A\nabla\mathcal{I}_h u)\right| \le C\eta h^{k-1+d} \|A\|_{W^{k,\infty}(\Omega)} \|u\|_{W^{k+1,\infty}(\Omega)},$$

which completes the proof. □

Now, we present our main result, where we estimate the energy norm of the error $u - u_h^*$; recall that u_h^* is the unique solution of the GMM (4.6) with numerical integration.

Theorem 5.5 *Let $u \in W^{k+1,\infty}(\Omega)$, $a_{ij} \in C^k(\overline{\Omega})$, for $i, j = 1, 2, \cdots, d$ and $c \in C(\overline{\Omega})$. Suppose the subspace V_h satisfies assumptions* A1–A5 *and the quadrature schemes satisfy* QA1–QA4. *Then, for η small enough, there is a positive constant C, independent of u, η, τ, and h, such that*

$$\begin{aligned}
&\|u - u_h^*\|_{H^1(\Omega)} \\
&\quad \le Ch^k \|u\|_{W^{k+1,\infty}(\Omega)} \\
&\qquad + \left[C\eta\left(\|A\|_{W^{k,\infty}(\Omega)} + \|c\|_{L_\infty(\Omega)}h^2\right) + (\eta + \tau)\left(\|A\|_{W^{k,\infty}(\Omega)} + \|c\|_{L_\infty(\Omega)}h^2\right)h\right] \\
&\qquad \times\ h^{k-1}\|u\|_{W^{k+1,\infty}(\Omega)}.
\end{aligned} \tag{5.8}$$

Proof First, we substitute $w_h = \mathcal{I}_h u$ in the result of Lemma 5.1 and get

$$\begin{aligned}
\|u - u_h^*\|_{H^1(\Omega)} &\le C\|u - \mathcal{I}_h u\|_{H^1(\Omega)} \\
&\quad + \sup_{v_h \in V_h} \frac{\left|\left[B(\mathcal{I}_h u, v_h) - L(v_h)\right] - \left[B^*(\mathcal{I}_h u, v_h) - L^*(v_h)\right]\right|}{\|v_h\|_{H^1(\Omega)}}.
\end{aligned} \tag{5.9}$$

We will now estimate the second part of the RHS of (5.9) to prove (5.8). For any $v_h = \sum_{i \in N_h} v_i \phi_i \in V_h$, we have

$$\begin{aligned}
&\left[B(\mathcal{I}_h u, v_h) - L(v_h)\right] - \left[B^*(\mathcal{I}_h u, v_h) - L^*(v_h)\right] \\
&\quad = \sum_{i \in N_h} v_i\left(\left[B(\mathcal{I}_h u, \phi_i) - L(\phi_i)\right] - \left[B^*(\mathcal{I}_h u, \phi_i) - L^*(\phi_i)\right]\right).
\end{aligned} \tag{5.10}$$

For simplicity, in the rest of the proof we denote

$$E_i \equiv \left|\left[B(\mathcal{I}_h u, \phi_i) - L(\phi_i)\right] - \left[B^*(\mathcal{I}_h u, \phi_i) - L^*(\phi_i)\right]\right|.$$

Therefore, from (5.10), (3.9), (3.8), and (3.10), we get

$$\begin{aligned}
&\left|\left[B(\mathcal{I}_h u, v_h) - L(v_h)\right] - \left[B^*(\mathcal{I}_h u, v_h) - L^*(v_h)\right]\right| \\
&\quad \le \sup_{i \in N_h'} E_i \sum_{i \in N_h'} |v_i| + \sup_{i \in N_h''} E_i \sum_{i \in N_h''} |v_i| \\
&\quad \le C \sup_{i \in N_h'} E_i \left(\sum_{i \in N_h'} |v_i|^2\right)^{\frac{1}{2}} |N_h'|^{\frac{1}{2}} + C \sup_{i \in N_h''} E_i \left(\sum_{i \in N_h''} |v_i|^2\right)^{\frac{1}{2}} |N_h''|^{\frac{1}{2}} \\
&\quad \le C \sup_{i \in N_h'} E_i h^{-(d-1)} \|v_h\|_{L_2(\Gamma)} + C \sup_{i \in N_h''} E_i h^{-d} \|v_h\|_{L_2(\Omega)} \\
&\quad \le C \sup_{i \in N_h'} E_i h^{-(d-1)} \|v_h\|_{H^1(\Omega)} + C \sup_{i \in N_h''} E_i h^{-d} \|v_h\|_{L_2(\Omega)}
\end{aligned} \tag{5.11}$$

where the last inequality was obtained using the Trace theorem (see [10]). We will now estimate the terms $E_i, \ \forall \ i \in N_h$. For any $i \in N_h$, we have from the problem (2.1)

$$\begin{aligned}\int_{\omega_i} f\phi_i\,dx &= -\int_{\omega_i} \nabla\cdot\big(A\nabla u\big)\phi_i\,dx + \int_{\omega_i} cu\phi_i\,dx\\ &= -\int_{\omega_i} \nabla\cdot\big(A\nabla \mathcal{I}_h u\big)\phi_i\,dx\\ &\quad -\int_{\omega_i} \nabla\cdot\big[A\nabla(u-\mathcal{I}_h u)\big]\phi_i\,dx + \int_{\omega_i} cu\phi_i\,dx\end{aligned}$$

and

$$\begin{aligned}\int_{\partial\omega_i\cap\Gamma} g\phi_i\,ds &= \int_{\partial\omega_i\cap\Gamma} A\nabla u\cdot\vec{n}\phi_i\,ds\\ &= \int_{\partial\omega_i\cap\Gamma} A\nabla\mathcal{I}_h u\cdot\vec{n}\phi_i\,ds + \int_{\partial\omega_i\cap\Gamma} A\nabla\big(u-\mathcal{I}_h u\big)\cdot\vec{n}\phi_i\,ds.\end{aligned}$$

Now setting $e_I \equiv u - \mathcal{I}_h u$ and recalling the definition (4.13) of the functional G_i, we get

$$\begin{aligned}B(\mathcal{I}_h u,\phi_i) - L(\phi_i) &= B(\mathcal{I}_h u,\phi_i) - \int_{\omega_i} f\phi_i\,dx - \int_{\partial\omega_i\cap\Gamma} g\phi_i\,ds\\ &= \int_{\omega_i} A\nabla\mathcal{I}_h u\cdot\nabla\phi_i\,dx + \int_{\omega_i}\nabla\cdot\big(A\nabla\mathcal{I}_h u\big)\phi_i\,dx\\ &\quad -\int_{\partial\omega_i\cap\Gamma} A\nabla\mathcal{I}_h u\cdot\vec{n}\phi_i\,ds + \int_{\omega_i}\nabla\cdot\big[A\nabla(u-\mathcal{I}_h u)\big]\phi_i\,dx\\ &\quad -\int_{\partial\omega_i\cap\Gamma} A\nabla\big(u-\mathcal{I}_h u\big)\cdot\vec{n}\phi_i\,ds + \int_{\omega_i} c\,\mathcal{I}_h u\,\phi_i\,dx\\ &\quad -\int_{\omega_i} cu\phi_i\,dx\\ &= G_i(A\nabla\mathcal{I}_h u) + \int_{\omega_i}\nabla\cdot\big(A\nabla e_I\big)\phi_i\,dx\\ &\quad -\int_{\partial\omega_i\cap\Gamma} A\nabla e_I\cdot\vec{n}\phi_i\,ds - \int_{\omega_i} ce_I\phi_i\,dx.\end{aligned} \tag{5.12}$$

Similarly, again from the problem (2.1) and the quadratures developed in Section 4, we have

$$\begin{aligned}\fint_{\omega_i}^{l} f\phi_i\,dx &= -\fint_{\omega_i}^{l}\nabla\cdot\big(A\nabla u\big)\phi_i\,dx + \fint_{\omega_i}^{l} cu\phi_i\,dx\\ &= -\fint_{\omega_i}^{l}\nabla\cdot\big(A\nabla\mathcal{I}_h u\big)\phi_i\,dx - \fint_{\omega_i}^{l}\nabla\cdot\big[A\nabla(u-\mathcal{I}_h u)\big]\phi_i\,dx + \fint_{\omega_i}^{l} cu\phi_i\,dx\end{aligned}$$

and

$$\fint_{\partial\omega_i\cap\Gamma} g\phi_i\,ds = \fint_{\partial\omega_i\cap\Gamma} A\nabla u\cdot\vec{n}\phi_i\,ds$$
$$= \fint_{\partial\omega_i\cap\Gamma} A\nabla\mathcal{I}_h u\cdot\vec{n}\phi_i\,ds + \fint_{\partial\omega_i\cap\Gamma} A\nabla\big(u-\mathcal{I}_h u\big)\cdot\vec{n}\phi_i\,ds.$$

Then recalling the definition (4.11) of the functional G_i^*, we get

$$\begin{aligned}
B^*(\mathcal{I}_h u,\phi_i) - L^*(\phi_i) &= B^*(\mathcal{I}_h u,\phi_i) - \fint_{\omega_i}^{l} f\phi_i\,dx - \fint_{\partial\omega_i\cap\Gamma} g\phi_i\,ds\\
&= \fint_{\omega_i}^{s} A\nabla\mathcal{I}_h u\cdot\nabla\phi_i\,dx + \fint_{\omega_i}^{l} \nabla\cdot\big(A\nabla\mathcal{I}_h u\big)\phi_i\,dx\\
&\quad - \fint_{\partial\omega_i\cap\Gamma} A\nabla\mathcal{I}_h u\cdot\vec{n}\phi_i\,ds\\
&\quad + \fint_{\omega_i}^{l} \nabla\cdot\big[A\nabla(u-\mathcal{I}_h u)\big]\phi_i\,dx\\
&\quad - \fint_{\partial\omega_i\cap\Gamma} A\nabla\big(u-\mathcal{I}_h u\big)\cdot\vec{n}\phi_i\,ds\\
&\quad + \fint_{\omega_i}^{m} c\,\mathcal{I}_h u\,\phi_i\,dx - \fint_{\omega_i}^{l} cu\phi_i\,dx\\
&= G_i^*(A\nabla\mathcal{I}_h u) + \fint_{\omega_i}^{l} \nabla\cdot\big(A\nabla e_I\big)\phi_i\,dx\\
&\quad - \fint_{\partial\omega_i\cap\Gamma} A\nabla e_I\cdot\vec{n}\phi_i\,ds - \fint_{\omega_i}^{l} ce_I\phi_i\,dx,
\end{aligned} \tag{5.13}$$

where the last equality is due to the assumption $\fint_{\omega_i}^{l} = \fint_{\omega_i}^{m}$. Therefore, from (5.12), (5.13) and the assumptions (4.7), (4.8), we get the following estimates for $i\in N_h'$, namely,

$$\begin{aligned}
E_i &\le \big|G_i(A\nabla\mathcal{I}_h u) - G_i^*(A\nabla\mathcal{I}_h u)\big|\\
&\quad + \left|\int_{\omega_i} \nabla\cdot\big(A\nabla e_I\big)\phi_i\,dx - \fint_{\omega_i}^{l} \nabla\cdot\big(A\nabla e_I\big)\phi_i\,dx\right|\\
&\quad + \left|\int_{\partial\omega_i\cap\Gamma} A\nabla e_I\cdot\vec{n}\phi_i\,ds - \fint_{\partial\omega_i\cap\Gamma} A\nabla e_I\cdot\vec{n}\phi_i\,ds\right|\\
&\quad + \left|\int_{\omega_i} ce_I\phi_i\,dx - \fint_{\omega_i}^{l} ce_I\phi_i\,dx\right|\\
&\le \big|G_i(A\nabla\mathcal{I}_h u) - G_i^*(A\nabla\mathcal{I}_h u)\big| + \eta|\omega_i|\big|\big(A\nabla e_I\big)\phi_i\big|_{W^{1,\infty}(\omega_i)}\\
&\quad + \tau|\partial\omega_i\cap\Gamma|\|A\nabla e_I\cdot\vec{n}\phi_i\|_{L_\infty(\partial\omega_i\cap\Gamma)} + \eta|\omega_i|\|ce_I\phi_i\|_{L_\infty(\omega_i)},
\end{aligned} \tag{5.14}$$

and similarly, for $i \in N''_h$, recalling that $\phi_i = 0$ on $\partial\omega_i$, we have

$$E_i \leq \left|G_i(A\nabla\mathcal{I}_h u) - G_i^*(A\nabla\mathcal{I}_h u)\right| + \eta|\omega_i|\left|(A\nabla e_I)\phi_i\right|_{W^{1,\infty}(\omega_i)} + \eta|\omega_i|\|ce_I\phi_i\|_{L_\infty(\omega_i)}, \tag{5.15}$$

Now, from (5.2), the interpolation error (3.14), and the boundedness of ϕ_i, it immediately follows that for $i \in N_h$,

$$\left|(A\nabla e_I)\phi_i\right|_{W^{1,\infty}(\omega_i)} \leq C\|A\|_{W^{1,\infty}(\omega_i)} h^{k-1}\|u\|_{W^{k+1,\infty}(\Omega)}$$

$$\text{and} \quad \|ce_I\phi_i\|_{L_\infty(\omega_i)} \leq C\|c\|_{L_\infty(\omega_i)} h^{k+1}\|u\|_{W^{k+1,\infty}(\Omega)},$$

and for $i \in N'_h$,

$$\|A\nabla e_I \cdot \vec{n}\phi_i\|_{L_\infty(\partial\omega_i\cap\Gamma)} \leq C\|A\|_{L_\infty(\omega_i)} h^k \|u\|_{W^{k+1,\infty}(\Omega)}.$$

Therefore, from (5.14), (5.15), the assumption A2, and Lemma 5.4, we get

$$E_i \leq \begin{cases} C\big[(\eta+\tau)(\|A\|_{W^{k,\infty}(\Omega)} + \|c\|_{L_\infty(\Omega)}h^2)\big]h^{k+d-1}\|u\|_{W^{k+1,\infty}(\Omega)}, & i \in N'_h; \\ C\big[\eta(\|A\|_{W^{k,\infty}(\Omega)} + \|c\|_{L_\infty(\Omega)}h^2)\big]h^{k+d-1}\|u\|_{W^{k+1,\infty}(\Omega)}, & i \in N''_h. \end{cases} \tag{5.16}$$

Finally, from (5.9), the interpolation error (3.14), (5.11), and (5.16), we get

$$\begin{aligned}\|u-u_h^*\|_{H^1(\Omega)} \leq{}& Ch^k\|u\|_{W^{k+1,\infty}(\Omega)} \\ &+ \Big[C\eta(\|A\|_{W^{k,\infty}(\Omega)} + \|c\|_{L_\infty(\Omega)}h^2) \\ &\quad +(\eta+\tau)(\|A\|_{W^{k,\infty}(\Omega)} + \|c\|_{L_\infty(\Omega)}h^2)h\Big] \\ &\times h^{k-1}\|u\|_{W^{k+1,\infty}(\Omega)},\end{aligned}$$

which is the required result. □

Remark 5.2 The result (5.8) of Theorem 5.5 shows that $\|u-u_h^*\|_{H^1(\Omega)} = \mathcal{O}[h^k + (\eta+\tau)h^k + \eta h^{k-1}]$. Thus we do not have the optimal order of convergence (compare with (3.18)). But if we consider numerical integration such that $\eta = \mathcal{O}(h)$, i.e, we use more accurate integration scheme as we refine h, we get back the optimal order of convergence $\|u-u_h^*\|_{H^1(\Omega)} = \mathcal{O}(h^k)$. This feature of the GMM is very different from the standard FEM, where the same numerical integration can be used for all values of h to obtain the optimal order of convergence. We further note that (5.8) indicates that for larger values of h (i.e., in the pre-asymptotic range), the error $\|u-u_h^*\|_{H^1(\Omega)}$ may behave like $\mathcal{O}(h^k)$. But as h becomes smaller, we get $\|u-u_h^*\|_{H^1(\Omega)} = \mathcal{O}(h^{k-1})$. We will show this feature in our numerical experiments.

Corollary 5.6 *Suppose all the assumptions in Theorem* 5.5 *hold, except for the assumption* QA 4.2*, which is replaced as follows: For a non-negative integer $l < k$,*

$$G_i^*(\widetilde{p}) = 0, \quad \forall\, \widetilde{p} \in \widetilde{\mathcal{P}}^{l-1} \ \textit{and} \quad \forall\, i \in N_h; \tag{5.17}$$

for the case $l = 0$, we assume that the condition (5.17) is vacuous, namely, numerical integration rules satisfy only QA 4.1, QA 4.3, *and* QA 4.4. *Then, for η small enough, there is a positive constant C, independent of u, η, τ, and h, such that*

$$\|u - u_h^*\|_{H^1(\Omega)} \leq C\big[h^k + (\eta + \tau)h^l + \eta h^{l-1}\big]\|u\|_{W^{k+1,\infty}(\Omega)}.$$

Proof (Only a sketch) It can be easily shown by following the proof of the Lemma 5.4 that for $0 \leq l \leq k$,

$$|G_i(A\nabla\mathcal{I}_h u) - G_i^*(A\nabla\mathcal{I}_h u)| \leq \begin{cases} C(\eta + \tau)h^{l+d-1}\|A\|_{W^{k,\infty}(\Omega)}\|u\|_{W^{k+1,\infty}(\Omega)}, & i \in N_h' \\ C\eta h^{l+d-1}\|A\|_{W^{k,\infty}(\Omega)}\|u\|_{W^{k+1,\infty}(\Omega)}, & i \in N_h'' \end{cases}. \quad (5.18)$$

Moreover, for $l = 0$, we do not need to use the Taylor polynomial of $A\nabla\mathcal{I}_h u$ (as in the proof of Lemma 5.4) to get, (5.18). Now, instead of using the result of Lemma 5.4, we use (5.18) in the proof of Theorem 5.5 to get the desired result. □

Remark 5.3 The result in Corollary 5.6 shows that if the quadrature rules satisfy (4.12) of the assumption QA 4.2 with k replaced by l and $l < k$, then $\|u - u_h^*\|_{H^1(\Omega)} = \mathcal{O}(h^{l-1})$. Also, if the quadrature rules do not satisfy (4.12) (i.e., $l = 0$), then $\|u - u_h^*\|_{H^1(\Omega)} \leq Ch^{-1}$, which indicates that the error may increase as $h \to 0$.

We note that for the case $k = 1$, Theorem 5.5 yields $\|u - u_h^*\|_{H^1(\Omega)} = \mathcal{O}(h + \eta)$. In fact a similar result for $k = 1$ can be obtained using less restrictions on the numerical integration. We state the result in the following corollary.

Corollary 5.7 *Let $u \in C^2(\overline{\Omega})$, $a_{ij} \in C^1(\overline{\Omega})$, and $c \in C(\overline{\Omega})$. Suppose the subspace V_h, with $k = 1$, satisfies assumptions* A1–A5 *and the quadrature schemes satisfy* QA 4.1, QA 4.4, *and* (4.12) *only for $i \in N_h''$. Then, for η small enough, there is a positive constant C, independent of u, η, τ, and h, such that*

$$\|u - u_h^*\|_{H^1(\Omega)} \leq C(h + \eta + \tau)\|u\|_{W^{2,\infty}(\Omega)}.$$

The proof of this result can be obtained by slightly modifying the proofs of Lemma 5.4 and Theorem 5.5; we do not present the complete proof here.

6 Construction of numerical integration formula

In this section, we will derive numerical integration rules that satisfy the assumption QA 4.2, i.e., the condition (4.12) for $k = 1$ and $k = 2$ in two dimensions. We note that we have illustrated the conditions (4.12) in Remark 4.7.

To approximate the integral $\int_{\omega_i}^{s} \varrho(x)\,dx$, we seek a p-point quadrature rule $Q_c^i(\varrho)$ on ω_i, of the form

$$Q_c^i(\varrho) \equiv \sum_{l=1}^{p} \zeta_{c,l}^i \varrho(y_{c,l}^i), \quad \varrho \in C^0(\bar{\omega}_i) \text{ and } y_{c,l}^i \in \bar{\omega}_i, \tag{6.1}$$

that satisfies (4.12) with $\int_{\omega_i}^{s}$ replaced by Q_c^i.

The case $k = 1$ Recall that in this case, the shape functions $\{\phi_i\}_{i\in N_h}$ reproduce polynomials of degree $k = 1$. We will find the weights $\zeta_{c,l}^i$ and the integration points $y_{c,l}^i$ in (6.1) such that (4.12), i.e., (4.20) and (4.21), are satisfied with $\int_{\omega_i}^{s}$ replaced by $Q_c^i(\cdot)$. Suppose we have at our disposal a quadrature rule

$$Q_B^i(g) \equiv \int_{\partial\omega_i\cap\Gamma} g(s)\,ds, \quad g \in C^0(\bar{\omega}_i) \tag{6.2}$$

that accurately approximates the boundary integral $\int_{\partial\omega_i\cap\Gamma} g(s)\,ds$. We start with an accurate p-point quadrature rule $Q^i(\varrho)$ on ω_i of the form

$$Q^i(\varrho) \equiv \sum_{l=1}^{p} \varrho(y_l^i)\zeta_l^i. \tag{6.3}$$

We then define for $1 \le l \le p$,

$$\begin{aligned} y_{c,l}^i &= y_l^i \\ \zeta_{c,l}^i &= \zeta_l^i + \theta_1^i \zeta_l^i \frac{\partial \phi_i}{\partial x_1}\left(y_l^i\right) + \theta_2^i \zeta_l^i \frac{\partial \phi_i}{\partial x_2}\left(y_l^i\right), \end{aligned} \tag{6.4}$$

and choose θ_1^i and θ_2^i such that (4.20) and (4.21) are satisfied, i.e.,

$$\begin{aligned} Q_c^i\left(\frac{\partial \phi_i}{\partial x_1}\right) &= \int_{\partial\omega_i\cap\Gamma} n_1\phi_i\,ds = Q_B^i(n_1\phi_i) \\ Q_c^i\left(\frac{\partial \phi_i}{\partial x_2}\right) &= \int_{\partial\omega_i\cap\Gamma} n_2\phi_i\,ds = Q_B^i(n_2\phi_i). \end{aligned}$$

This yields the linear system

$$\begin{bmatrix} Q^i\left(\left(\frac{\partial\phi_i}{\partial x_1}\right)^2\right) & Q^i\left(\frac{\partial\phi_i}{\partial x_1}\frac{\partial\phi_i}{\partial x_2}\right) \\ Q^i\left(\frac{\partial\phi_i}{\partial x_2}\frac{\partial\phi_i}{\partial x_1}\right) & Q^i\left(\left(\frac{\partial\phi_i}{\partial x_2}\right)^2\right) \end{bmatrix} \begin{bmatrix} \theta_1^i \\ \theta_2^i \end{bmatrix} = \begin{bmatrix} Q_B^i\left(n_1\phi_i\right) - Q^i\left(\frac{\partial\phi_i}{\partial x_1}\right) \\ Q_B^i\left(n_2\phi_i\right) - Q^i\left(\frac{\partial\phi_i}{\partial x_2}\right) \end{bmatrix} \tag{6.5}$$

The components θ_1^i and θ_2^i of the solution of the above system are used in the definition of $\zeta_{c,l}^i$ (see (6.4)), and consequently, the resulting $Q_c^i(\varrho)$ satisfies the condition (4.12). We note that for $i \in N_h''$, the RHS of (6.5) does not contain the terms $Q_B^i(n_1\phi_i)$ and $Q_B^i(n_2\phi_i)$. We further note that $Q_c^i(\varrho)$ could be viewed as a corrected form of $Q^i(\varrho)$, such that $Q^i_c(\varrho)$ satisfies the condition (4.12); we will often refer to $Q^i_c(\varrho)$ as the *corrected numerical integration formula for* $k=1$. We note that $Q^i_c(\varrho)$ for $k=1$ in the one dimensional case was derived in [6].

Remark 6.1 To discuss the solvability of the system (6.5), we define a weighted inner product in $\mathbb{R}^p$ by

$$\langle u, v\rangle_w \equiv \sum_{l=1}^{p} u_l v_l \zeta^i_l, \ \forall\, u=(u_1,\cdots,u_p) \text{ and } v=(v_1,\cdots,v_p)\in\mathbb{R}^p$$

Let $V_1=\left(\frac{\partial\phi_i}{\partial x_1}(y^i_1),\cdots,\frac{\partial\phi_i}{\partial x_1}(y^i_p)\right)$ and $V_2=\left(\frac{\partial\phi_i}{\partial x_2}(y^i_1),\cdots,\frac{\partial\phi_i}{\partial x_2}(y^i_p)\right)$, then the coefficient matrix of the linear system (6.5) is

$$\begin{bmatrix} \langle V_1, V_1\rangle_w & \langle V_1, V_2\rangle_w \\ \langle V_2, V_1\rangle_w & \langle V_2, V_2\rangle_w \end{bmatrix}, \tag{6.6}$$

which is the Gramm matrix of the vectors V_1 and V_2 with respect to the inner product $\langle\cdot,\cdot\rangle_w$. This Gramm matrix is positive when V_1 and V_2 are linearly independent. Suppose $p\geq 2$ and let there be two integration points y^i_m and y^i_n in the set of integration points $\{y^i_l\}_{l=1}^p$ such that the vectors $\nabla\phi_i(y^i_m)$ and $\nabla\phi_i(y^i_n)$ are linearly independent, then it is easy to show that the vectors V_1 and V_2 are linearly independent.

The case $k=2$ We recall that in this case, the shape functions $\{\phi_i\}_{i\in N_h}$ reproduce polynomials of degree $k=2$. We will find $\zeta^i_{c,l}$ and $y^i_{c,l}$ in (6.1) such that (4.12), with $\int^s_{\omega_i}$ replaced by Q^i_c, is satisfied for $k=2$.

Suppose in addition to the quadrature rule $Q^i_B(g)$ (see (6.2)), we also have at our disposal a quadrature rule

$$Q^i_F(f)\equiv \int^l_{\omega_i} f(x)\,dx$$

that accurately approximates the integral $\int^l_{\omega_i} f(x)\,dx$. As in the case $k=1$, we start with an accurate p-point quadrature rule $Q^i(\varrho)$ (see (6.3)). We note that we could choose $Q^i(\cdot)$ to be the same as $Q^i_F(\cdot)$. Suppose $\mathcal{B}=\{\tilde{p}_m\}_{n=1}^6$ be a basis for $\widetilde{\mathcal{P}}^1$ (recall that dim $\widetilde{\mathcal{P}}^1=6$; a basis of $\widetilde{\mathcal{P}}^1$ is given in Remark 4.7). Then for $k=2$, the condition (4.12), with $\int^s_{\omega_i}$ replaced by $Q^i_c(\cdot)$, is equivalent to

$$Q^i_c\left(\tilde{p}_m\cdot\nabla\phi_i\right)=Q^i_B\left(\tilde{p}_m\cdot\vec{n}\phi_i\right)-Q^i_F\left(\nabla\cdot\tilde{p}_m\phi_i\right),\quad \text{for } 1\leq m\leq 6. \tag{6.7}$$

We now define, for $1\leq l\leq p$,

$$\begin{aligned} y^i_{c,l} &= y^i_l \\ \zeta^i_{c,l} &= \zeta^i_l+\sum_{n=1}^{6}\theta^i_n\zeta^i_l\left(\tilde{p}_n\cdot\nabla\phi_i\right)(y^i_l), \end{aligned} \tag{6.8}$$

where $\{\theta_n^i\}_{n=1}^6$ are chosen such that (6.7) is satisfied. We first note that from the definition of $Q_c^i(\cdot)$ in (6.1), with $y_{c,l}^i, \zeta_{c,l}^i$ as defined above, we get for $1 \le m \le 6$,

$$Q_c^i(\tilde{p}_m \cdot \nabla\phi_i) = Q^i\big(\tilde{p}_m \cdot \nabla\phi_i\big) + \sum_{l=1}^{p}\sum_{n=1}^{6} \theta_n^i \zeta_l^i \big(\tilde{p}_n \cdot \nabla\phi_i\big)(y_l^i)\,\big(\tilde{p}_m \cdot \nabla\phi_i\big)(y_l^i)$$
$$= Q^i\big(\tilde{p}_m \cdot \nabla\phi_i\big) + \sum_{n=1}^{6} \theta_n^i Q^i\Big((\tilde{p}_n \cdot \nabla\phi_i)\,(\tilde{p}_m \cdot \nabla\phi_i)\Big)$$

Therefore (6.7) is equivalent to the linear system

$$\sum_{n=1}^{6} \theta_n^i Q^i\Big((\tilde{p}_n \cdot \nabla\phi_i)\,(\tilde{p}_m \cdot \nabla\phi_i)\Big)$$
$$= Q_B^i\big(\tilde{p}_m \cdot \vec{n}\phi_i\big) - Q_F^i\big(\nabla \cdot \tilde{p}_m\phi_i\big) - Q^i\big(\tilde{p}_m \cdot \nabla\phi_i\big),$$
$$\text{for } 1 \le m \le 6. \tag{6.9}$$

We use the solution $\{\theta_n\}_{n=1}^6$ of the above linear system in the definition of $\zeta_{c,l}^i$ (see (6.8)), and consequently, $Q_c^i(\varrho)$ will satisfy the condition (4.12). We will often refer to $Q_c^i(\varrho)$ as the *corrected numerical integration formula for* $k = 2$. We note that solving the linear system (6.9) could be facilitated by considering the basis $\mathcal{B} = \{\tilde{p}_m\}_{m=1}^6 = \{(1, 0), (x_1 - x_{i1}, 0), (x_2 - x_{i2}, 0), (0, 1), (0, x_1 - x_{i1}), (0, x_2 - x_{i2})\}$, where $x_i = (x_{i1}, x_{i2})$ is the particle associated with ω_i.

Remark 6.2 To discuss the solvability of the system (6.9), we define a weighted inner product for $\mathbb{R}^p$ by

$$\langle u, v\rangle_w \equiv \sum_{l=1}^{p} u_l v_l \zeta_l^i, \;\; \forall\, u = (u_1, \cdots, u_p) \text{ and } v = (v_1, \cdots, v_p) \in \mathbb{R}^p$$

Let $V_n = \Big(\big[\tilde{p}_n \cdot \nabla\phi_i\big](y_1^i), \cdots, \big[\tilde{p}_n \cdot \nabla\phi_i\big](y_p^i)\Big),\ 1 \le n \le 6$, then the coefficient matrix of the linear system (6.9) is

$$\begin{bmatrix} \langle V_1, V_1\rangle_w & \langle V_1, V_2\rangle_w & \cdots & \langle V_1, V_6\rangle_w \\ \langle V_2, V_1\rangle_w & \langle V_2, V_2\rangle_w & \cdots & \langle V_2, V_6\rangle_w \\ \vdots & \vdots & \ddots & \vdots \\ \langle V_6, V_1\rangle_w & \langle V_6, V_2\rangle_w & \cdots & \langle V_6, V_6\rangle_w \end{bmatrix}$$

which is exactly the Gramm matrix of the vectors $V_n,\ 1 \le n \le 6$ with respect to the inner product $\langle \cdot, \cdot\rangle_w$. This Gramm matrix is positive if the vectors $\{V_l\}_{l=1}^6$ are linearly independent. We note that $p \ge 6$ is a necessary condition for the linear independence of the vectors $\{V_l\}_{l=1}^6$. We further mention that the positivity of the Gramm matrix is subtle. When ω_i is a square, our computations show that the Gramm matrix is positive when Q^i is the 4×4 Gauss rule on ω_i, but it has a zero eigenvalue when Q^i is the 3×3 Gauss rule.

Remark 6.3 The quadrature rule Q^i_c, which depends on Q^i, satisfies QA 4.1 provided Q^i is accurate enough. We give a brief sketch of the argument if $\omega_i \subset\subset \Omega$. Let Q^i be accurate and satisfy QA 4.1 with $\eta = \eta_{Q^i}$. For example, if ω_i is a square, Q^i could be an $n \times n$ Gauss rule; it is well known that $\eta_{Q^i} = O(p^{-m_1})$, where m_1 is as in (4.8) (see Remark 3.3 in [6]). Since $k = 1$, from (4.14) we have $\int_{\omega_i} \frac{\partial \phi_i}{\partial x_j} = 0$. Therefore the components of the vector in the RHS of (6.5) are extremely small, provided Q^i is accurate enough. Hence θ^i_1, θ^i_2 are small, $\zeta^i_{c,l} \approx \zeta^i_l$, and Q^i_c is close to Q_i. Since Q^i satisfies QA 4.1, one can show that Q^i_c also satisfy QA 4.1 with $\eta = \eta_{Q^i_c} \geq \eta_{Q^i}$. We note that if $Q^i(\partial\phi_i/\partial x_j) = 0$, then $Q^i_c = Q^i$, as shown in an example in Section 7 (see (7.2)). As mentioned before, Q^i_c will satisfy QA 4.4 under the restrictive condition $\eta_{Q^i_c} \leq Ch$. We have numerically checked that QA 4.4 holds for Q^i_c for various Q^i if p is large. For $k = 2$, the situation is similar.

We now give a brief sketch of the derivation of the numerical integration rule $Q^i_c(\cdot)$, in 1 dimension (i.e., when $d = 1$) with $\omega_i = (\alpha_i, \beta_i)$, such that (4.12) for $k = 2$ is satisfied with $\int^s_{\omega_i}$ replaced by Q^i_c. We will use the one dimensional quadrature rule in our numerical examples in the next section.

As before, we start with the quadrature rules $Q^i_F(\cdot)$ and $Q^i(\cdot)$; we recall that both the rules could also be same. We first note that, for $d = 1$, the "boundary integral" term in (4.11) is $v\phi_i\big|^{\beta_i}_{\alpha_i}$; thus we do not need the quadrature rule $Q^i_B(\cdot)$. We further note that $\widetilde{\mathcal{P}}^1 = \mathcal{P}^1$ and therefore $m = \dim \mathcal{P}^1 = 2$. Thus (6.7), for $d = 1$, is written as

$$\begin{cases} Q^i_c(\phi'_i(x)) = \phi_i(x)\big|^{\beta_i}_{\alpha_i} \\ Q^i_c\big(\big[(x - x_i)\phi'_i(x)\big]\big) = (x - x_i)\phi_i(x)\big|^{\beta_i}_{\alpha_i} - Q^i_F(\phi_i) \end{cases} \tag{6.10}$$

We now define $y^i_{c,l}, \zeta^i_{c,l}$ (compare with (6.8)) as

$$\begin{cases} y^i_{c,l} = y^i_l \\ \zeta^i_{c,l} = \zeta^i_l + \theta^i_1\zeta^i_l\phi'_i(y^i_l) + \theta^i_2\zeta^i_l\big[(y^i_l - x_i)\phi'_i(y^i_l)\big], \end{cases} \tag{6.11}$$

where θ^i_1, θ^i_2 are chosen such that (6.10) is satisfied. Using $Q^i_c(\cdot)$, with $y^i_{c,l}$, $\zeta^i_{c,l}$ as defined above, in (6.10) yields the linear system for θ^i_1, θ^i_2, namely,

$$\begin{bmatrix} Q^i(\phi'^2_i(x)) & Q^i\big[(x - x_i)\phi'^2_i(x)\big] \\ Q^i\big[(x - x_i)\phi'^2_i(x)\big] & Q^i\big[(x - x_i)^2\phi'^2_i(x)\big] \end{bmatrix} \begin{bmatrix} \theta^i_1 \\ \theta^i_2 \end{bmatrix} = \begin{bmatrix} \phi_i(x)\big|^{\beta_i}_{\alpha_i} - Q^i(\phi'_i) \\ (x - x_i)\phi_i(x)\big|^{\beta_i}_{\alpha_i} - Q^i\big[(x - x_i)\phi'_i(x)\big] - Q^i_F(\phi_i)\big]. \end{bmatrix} \tag{6.12}$$

7 Numerical results

We present numerical examples to illuminate the results obtained in Section 5. Let $\Omega = (0, 1)$ and we consider the Neumann problem with non-constant coefficients, namely,

$$-(au')' + cu = f, \quad x \in \Omega$$
$$a(0)u'(0) = 1, \ a(1)u'(1) = 2e,$$

where $a(x) = 1 + x^3$, $c(x) = 1 + \sin^2 x$, and $f(x) = e^x(\sin^2 x - x^3 - 3x^2)$. The exact solution of the problem is $u(x) = e^x$.

To approximate the solution $u(x)$ of the above problem by the GMM (4.6), we first define the shape functions of the finite dimensional space V_h. For a given non-negative integer k and a positive real number R, let $\phi(x)$ be the basic RKP shape function with compact support $[-R, R]$ satisfying

$$\sum_{i \in \mathbb{Z}} i^l \phi(x - i) = x^l, \ \forall \ x \in \mathbb{R} \text{ and } l = 0, 1, \ldots, k. \tag{7.1}$$

We mention that there exists $\phi(x)$ satisfying (7.1) when $R \geq (k + 1)/2$ (see e.g., [3]). Consider a positive integer N and for $h = 1/N$, we consider the index set

$$N_h = \{-[R], \cdots, 0, \ 1, \cdots, N, \cdots, N + [R]\},$$

where $[R]$ is the integer part of R. For each $i \in N_h$, we define the RKP shape functions

$$\phi_i(x) \equiv \phi\left(\frac{x}{h} - i\right), \quad x \in \Omega.$$

Then $\operatorname{supp} \phi_i = [\alpha_i, \beta_i] = [ih - Rh, ih + Rh] \cap [0, 1]$. Defining the set of particles $X_h = \{x_i = ih, \ i \in N_h\}$, it can be easily shown that $\{\phi_i\}_{i=1}^{N_h}$ reproduce polynomials of degree k, i.e.,

$$\sum_{i \in N_h} x_i^l \phi_i(x) = x^l, \ \forall \ x \in \Omega \text{ and } l = 0, 1, \ldots, k.$$

Moreover, recalling the definitions of the index sets N_h' and N_h'', we have

$$N_h' = \{-[R], \cdots, [R], \ N - [R], \cdots, N + [R]\} \text{ and}$$
$$N_h'' = \{[R] + 1, \cdots, N - [R] - 1\}$$

We note that the function $\phi(x)$ has been constructed following the ideas mentioned in Remark 3.3 (using $h = 1$, $x_j = j \in \mathbb{Z}$, and $i = 0$, i.e., $\phi(x) = \phi_0^1(x)$), where we have used the cubic spline weight function for $w(x)$ with compact support $[-R, R]$; for the definition of cubic spline weight function, we refer to [3, 4]. We further note that the cubic spline weight function is symmetric in $[-R, R]$, and consequently the associated shape functions $\phi_i(x)$, $i \in N_h''$, are also symmetric in $[\alpha_i, \beta_i]$.

The case $k = 1$ The basic shape function $\phi(x)$ was constructed with $R = 1.8$. For $i \in N_h$, we consider the standard p-point Gaussian integration rule on $[\alpha_i, \beta_i]$, namely,

$$Q_g^i(f) \equiv \sum_{l=1}^{p} f(y_l^i)\zeta_l^i, \ \ \forall\ f \in C(\omega_i),$$

where $\{y_l^i : 1 \le l \le p\}$ are the Gaussian integration points in $[\alpha_i, \beta_i]$ and $\{\zeta_l^i : 1 \le l \le p\}$ are the associated weights. It is well known that the points y_l^i are symmetrically placed in the interval $[\alpha_i, \beta_i]$; also the weights ζ_l^i are symmetric, i.e., $\zeta_s^i = \zeta_{p+1-s}^i$, $s = 1, 2, \ldots, p$.

Recall that $\phi_i(x)$, for $i \in N_h''$, is symmetric, and consequently, $\phi_i'(x)$, $i \in N_h''$, is anti-symmetric in $[\alpha_i, \beta_i]$ about the mid-point. Therefore, we get

$$Q_g^i(\phi_i') = 0, \ \ \forall\ i \in N_h''. \tag{7.2}$$

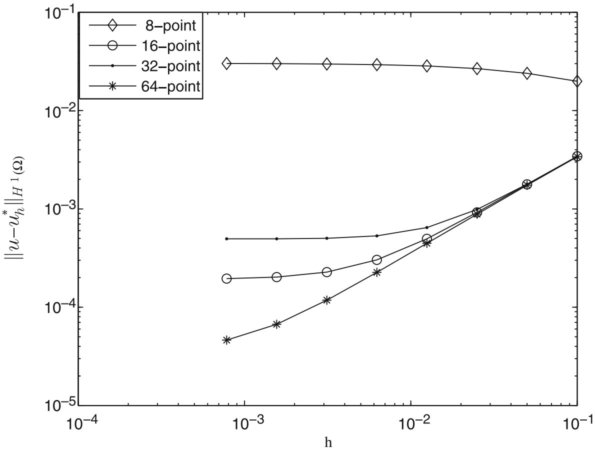
Thus the numerical integration rule Q_g^i satisfies the condition (4.12), i.e., the discrete Green's formula, for $i \in N_h''$ (see also (4.22)). We used Q_g^i, with $p = 8, 16, 32$, and 64 to compute γ_{ij}^*, σ_{ij}^*, and l_i^* in the variational problem (4.6). We note that in the one dimensional case, evaluation of the boundary integrals is trivial. We further note that $Q_c^i = Q_g^i$ satisfies QA 4.1 with $\eta = O(p^{-s})$, where s depends on the regularity of f (see Remark 3.3 in [6]). We have computed the solution u_h^* of (4.6) and have presented the error $\|u - u_h^*\|_{H^1(\Omega)}$ for various values of h in Table 1. We also presented the log-log graph of $\|u - u_h^*\|_{H^1(\Omega)}$ with respect to h in Fig. 1. It is clear that for $p = 16, 32$ and

Table 1 GMM with $k = 1$. Standard Gaussian integration rule

h	$\|u - u_h^*\|_{H^1(\Omega)}$			
	8 points	16 points	32 points	64 points
1/10	1.9883E−02	3.4312E−03	3.4040E−03	3.3908E−03
1/20	2.3933E−02	1.7751E−03	1.7851E−03	1.7427E−03
1/40	2.6763E−02	9.1993E−04	9.8673E−04	8.8411E−04
1/80	2.8425E−02	4.9672E−04	6.4523E−04	4.4625E−04
1/160	2.9324E−02	3.0333E−04	5.3111E−04	2.2612E−04
1/320	2.9791E−02	2.2761E−04	5.0200E−04	1.1769E−04
1/640	3.0029E−02	2.0272E−04	4.9631E−04	6.7007E−05
1/1280	3.0150E−02	1.9519E−04	4.9581E−04	4.6261E−05

The H^1 norm of the error, $\|u - u_h^*\|_{H^1(\Omega)}$, where $u = e^x$ and u_h^* is the solution of the GMM, employing shape functions that reproduce polynomials of degree $k = 1$. Standard p-point Gaussian integration rule, with $p = 8, 16, 32$ and 64, was used in the GMM. These rules satisfy the discrete Green's formula in the interior for $k = 1$, i.e., the assumption (7.2)

Fig. 1 The log-log plot of $\|u-u_h^*\|_{H^1(\Omega)}$ with respect to h, where $u=e^x$ and u_h^* is the solution of the GMM, employing shape functions that reproduce polynomials of degree $k=1$. Standard p-point Gaussian rule is used, which satisfies the assumption (7.2)



64, the error $\|u-u_h^*\|_{H^1(\Omega)}$ first decreases and then "levels off", which suggests that $\|u-u_h^*\|_{H^1(\Omega)}=\mathcal{O}(h+\eta)$. This illuminates the result of the Corollary 5.7.

We now show that the condition (4.12) on the underlying quadrature rule is a necessary condition for the result presented in Theorem 5.5, i.e., $\|u-u_h^*\|_{H^1(\Omega)}=\mathcal{O}(h+\eta)$. We consider a non-symmetric Gaussian integration rule that does not satisfy the condition (4.12), i.e., does not satisfy the discrete Green's formula. For $i\in N_h$, we consider the mapping $T_i:[\alpha_i,\beta_i]\to[\alpha_i,\beta_i]$ given by

$$z=T_i(y)=y+\frac{0.2}{\beta_i-\alpha_i}\left[\left(y-\frac{\alpha_i+\beta_i}{2}\right)^2-\left(\frac{\beta_i-\alpha_i}{2}\right)^2\right]$$

Therefore, for a smooth function f, we have

$$\int_{\alpha_i}^{\beta_i} f(z)dz=\int_{\alpha_i}^{\beta_i} f\left(T_i(y)\right)T_i'(y)dy.$$

The integral on the RHS of the above equality could be approximated by the Gaussian rule Q_g^i to obtain an integration rule on $[\alpha_i,\beta_i]$ to approximate the integral $\int_{\alpha_i}^{\beta_i} f(z)dz$, namely,

$$Q_{ng}^i(f)\equiv\sum_{l=1}^{p} f\left(y_{nl}^i\right)\zeta_{nl}^i, \tag{7.3}$$

where $y_{nl}^i=T_i(y_l^i)$ and $\zeta_{nl}^i=T_i'(y_l^i)\zeta_l^i$. We will refer to Q_{ng}^i as a *p-point non-symmetric Gaussian integration rule* on $[\alpha_i,\beta_i]$. It is well known that the algebraic precision of Q_g^i is $2p-1$; we can show that the algebraic precision of Q_{ng}^i is $p-1$.

Table 2 GMM with $k = 1$. Non-symmetric Gaussian integration rule

h	$\|u - u_h^*\|_{H^1(\Omega)}$			
	8 points	16 points	32 points	64 points
1/10	4.5373E−01	8.5286E−03	3.4098E−03	3.3908E−03
1/20	1.1694E+00	2.0281E−02	2.0558E−03	1.7425E−03
1/40	2.7436E+00	4.4534E−02	2.8166E−03	8.8444E−04
1/80	6.3125E+00	9.2759E−02	5.9275E−03	4.5245E−04
1160	1.4474E+01	1.8844E−01	1.2420E−02	2.7798E−04
1/320	3.2278E+01	3.7713E−01	2.5442E−02	3.5063E−04
1/640	6.9243E+01	7.4415E−01	5.1451E−02	6.6803E−04
1/1280	1.4414E+02	1.4379E+00	1.0329E−01	1.3328E−03

The H^1 norm of the error, $\|u - u_h^*\|_{H^1(\Omega)}$, where $u = e^x$ and u_h^* is the solution of the GMM, employing shape functions that reproduce polynomials of degree $k = 1$. Non-symmetric p-point Gaussian integration rule, with $p = 8$, 16, 32 and 64, was used in the GMM. These integration rules *do not* satisfy the discrete Green's formula for $k = 1$

We use the non-symmetric Gaussian integration rule Q_{ng}^i, with $p = 8$, 16, 32, and 64, to compute γ_{ij}^*, σ_{ij}^*, and l_i^* in the variational problem (4.6). We computed the solution u_h^* of (4.6) and presented the error $\|u - u_h^*\|_{H^1(\Omega)}$ for various values of h in Table 2. We also presented the log-log plot of $\|u - u_h^*\|_{H^1(\Omega)}$ with respect to h in Fig. 2. It is clear that $\|u - u_h^*\|_{H^1(\Omega)}$ increases as h decreases; for $p = 32$ and 64, the error first decreases and then increases. In all the cases, the error $\|u - u_h^*\|_{H^1(\Omega)}$ behaves like $\mathcal{O}(h^{-1})$, as indicated in Corollary 5.6 and Remark 5.3.

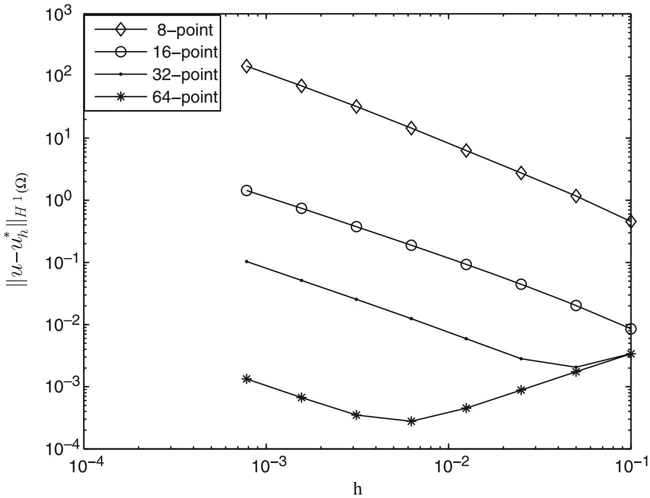


Fig. 2 The log-log plot of $\|u - u_h^*\|_{H^1(\Omega)}$ with respect to h, where $u = e^x$ and u_h^* is the solution of the GMM, employing shape functions that reproduce polynomials of degree $k = 1$. Non-symmetric p-point Gaussian integration rules were used, which *do not* satisfy the discrete Green's formula for $k = 1$

Now following the ideas presented in Section 6, we will correct the non-symmetric Gaussian integration rule $Q^i_{ng}(\cdot)$ (given in (7.3)), such that the corrected numerical integration rule (see (6.1))

$$Q^i_c(\varrho) \equiv Q^i_{ng,c}(\varrho) = \sum_{l=1}^{p} \varrho\left(y^i_{c,l}\right) \zeta^i_{c,l}$$

satisfies the condition (4.12). We note that for $d = 1$, the condition (4.12) for $k = 1$ is

$$Q^i_{ng,c}(\phi'_i) = \phi(\beta_i) - \phi(\alpha_i). \tag{7.4}$$

For $1 \leq i \leq p$, we consider

$$y^i_{c,l} = y^i_{nl} \text{ and } \zeta^i_{c,l} = \zeta^i_{nl} + \theta^i \zeta^i_{nl}\phi'_i(y^i_{nl}),$$

with

$$\theta^i = \frac{\phi_i(\beta_i) - \phi_i(\alpha_i) - \sum_{l=1}^{p} \phi'_i(y^i_{nl})\zeta^i_{nl}}{\sum_{l=1}^{p} \phi'^2_i(y^i_{nl})\zeta^i_{nl}}.$$

Then it can be shown, following the ideas in Section 6 for $d = 1$, that $Q^i_{ng,c}(\cdot)$ satisfies the condition (4.12), i.e., (7.4). However, we note that unlike the standard Gaussian integration rule $Q^i_g(\cdot)$, the integration points for the quadrature rule $Q^i_{ng,c}(\cdot)$ are not symmetrically placed in $[\alpha_i, \beta_i]$. The expression for the corrected numerical integration rule for $d = 1$ was also derived in [6] for a slightly different situation. We will refer to $Q^i_{ng,c}(\cdot)$ as the *corrected non-symmetric Gaussian integration rule for* $k = 1$. $Q^i_{ng,c}(\cdot)$ satisfies QA 4.1 with an η, which is close to η associated with Q^i_{ng} and is small for large p.

We now use the corrected integration rule $Q^i_{ng,c}(\cdot)$ to compute γ^*_{ij} in problem (4.6); the terms σ^*_{ij} and l^*_i in (4.6) are computed using the integration rule Q^i_{ng} (uncorrected). We computed the solution u^*_h of (4.6) and have presented the error $\|u - u^*_h\|_{H^1(\Omega)}$, for various values of h in Table 3. We also presented the

Table 3 GMM with $k = 1$. Corrected non-symmetric Gaussian integration rule for $k = 1$

h	$\|u - u^*_h\|_{H^1(\Omega)}$			
	8 points	16 points	32 points	64 points
1/10	5.6887E−03	3.4243E−03	3.4004E−03	3.3907E−03
1/20	4.8897E−03	1.7538E−03	1.7735E−03	1.7424E−03
1/40	4.9036E−03	9.8324E−04	9.5897E−04	8.8351E−04
1/80	5.0486E−03	7.3643E−04	5.9529E−04	4.4492E−04
1/160	5.1580E−03	7.0595E−04	4.6444E−04	2.2331E−04
1/320	5.2221E−03	7.2101E−04	4.2812E−04	1.1201E−04
1/640	5.2564E−03	7.3597E−04	4.2001E−04	5.6271E−05
1/1280	5.2742E−03	7.4520E−04	4.1871E−04	2.8400E−05

The H^1 norm of the error, $\|u - u^*_h\|_{H^1(\Omega)}$, where $u = e^x$ and u^*_h is the solution of the GMM, employing shape functions that reproduce polynomials of degree $k = 1$. Corrected non-symmetric Gaussian integration rule for $k = 1$, with $p = 8, 16, 32$ and 64, was used in the GMM. The integration rules satisfy the discrete Green's formula for $k = 1$

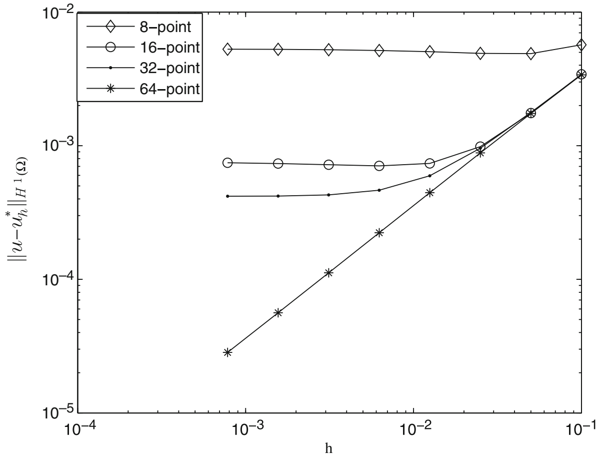


Fig. 3 The log-log plot of $\|u - u_h^*\|_{H^1(\Omega)}$ with respect to h, where $u = e^x$ and u_h^* is the solution of the GMM, employing shape functions that reproduce polynomials of degree $k = 1$. Corrected p-point non-symmetric Gaussian rules for $k = 1$ were used, which satisfy the discrete Green's formula for $k = 1$

log-log plot of $\|u - u_h^*\|_{H^1(\Omega)}$ with respect to h in Fig. 3. It is clear that $\|u - u_h^*\|_{H^1(\Omega)}$ levels off as h decreases; the error first decreases and then levels off for $p = 16$, 32, and 64. This suggests that $\|u - u_h^*\|_{H^1(\Omega)} = \mathcal{O}(h + \eta)$.

The case $k = 2$ The basic shape function $\phi(x)$, satisfying (7.1) with $k = 2$, was constructed with $R = 2.2$. Let $Q^i(\cdot) = Q^i_{ng}(\cdot)$ be the p-point non-symmetric Gaussian integration rule on $[\alpha_i, \beta_i]$, as given in (7.3). We consider the associated corrected non-symmetric Gaussian integration rule $Q^i_c(\cdot)$ for $k = 2$; $Q^i_c(\cdot)$ satisfies the discrete Green's formula (4.12) for $k = 2$, $d = 1$, i.e., it satisfies (6.10). The integration points $\{y^i_{c,l}\}_{l=1}^p$ and the associated weights $\{\zeta^i_{c,l}\}_{l=1}^p$ of $Q^i_c(\cdot)$ are given by (6.11) with $y^i_l = y^i_{n,l}$ and $\zeta^i_l = \zeta^i_{n,l}$ for $1 \le l \le p$. We mention that θ^i_1, θ^i_2 in (6.11) are obtained from the solution of (6.12), with $Q^i_F(\phi_i) =$

Table 4 GMM with $k = 2$. Corrected non-symmetric Gaussian rule for $k = 2$

h	$\|u - u_h^*\|_{H^1(\Omega)}$			
	8 points	16 points	32 points	64 points
1/10	6.8778E−03	1.3037E−03	7.0255E−04	6.8283E−04
1/20	4.4966E−03	8.3073E−04	2.1238E−04	1.7378E−04
1/40	2.5677E−03	4.8134E−04	8.4965E−05	4.4034E−05
1/80	1.3736E−03	2.5972E−04	4.1096E−05	1.1258E−05
1/160	7.1071E−04	1.3494E−04	2.0824E−05	3.0523E−06
1/320	3.6153E−04	6.8783E−05	1.0558E−05	9.6564E−07
1/640	1.8234E−04	3.4724E−05	5.3252E−06	3.8283E−07
1/1280	9.1564E−05	1.7446E−05	2.6751E−06	1.7719E−07

The H^1 norm of the error, $\|u - u_h^*\|_{H^1(\Omega)}$, where $u = e^x$ and u_h^* is the solution of the GMM, employing shape functions that reproduce polynomials of degree $k = 2$. Corrected p-point non-symmetric Gaussian integration rule for $k = 2$, with $p = 8, 16, 32$ and 64, was used in the GMM. The integration rules satisfy the discrete Green's formula for $k = 2$

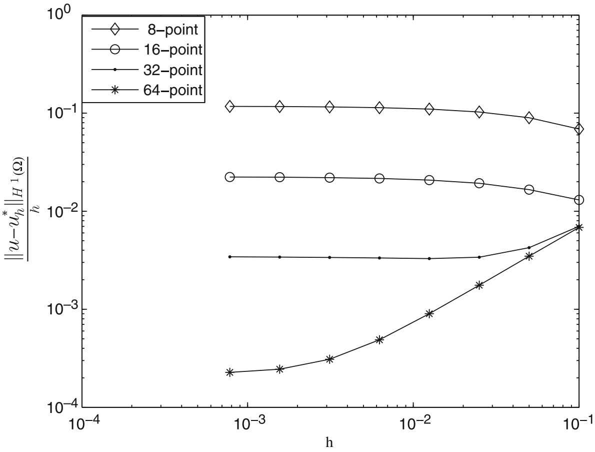


Fig. 4 The log-log plot of the ratio $\frac{\|u-u_h^*\|_{H^1(\Omega)}}{h}$ with respect to h, where $u = e^x$ and u_h^* is the solution of the GMM, employing shape functions that reproduce polynomials of degree $k = 2$. Corrected p-point non-symmetric Gaussian rules for $k = 2$ were used, which satisfy the discrete Green's formula for $k = 2$

$Q^i_{ng}(\phi_i)$. We used the corrected non-symmetric Gaussian integration rule $Q^i_c(\cdot)$ (for $k = 2$) to compute the terms γ^*_{ij} in the variational problem (4.6). The terms σ^*_{ij} and l^*_i were computed using the non-symmetric Gaussian integration rule (uncorrected) $Q^i_{ng}(\cdot)$. We computed the solution u^*_h of (4.6) and have presented the values of $\|u - u^*_h\|_{H^1(\Omega)}$, for various values of h in Table 4. We also presented the log-log plot of the ratio $\frac{\|u-u_h^*\|_{H^1(\Omega)}}{h}$ with respect to h in Fig. 4.

It is clear that u^*_h converges to u, the solution of (2.3), as h becomes smaller. The Fig. 4 also indicates that $\|u - u^*_h\|_{H^1(\Omega)} = O[h(h + \eta)]$, illuminating the result of Theorem 5.5 for $k = 2$.

We will now show the effect of quadrature on $\|u - u^*_h\|_{H^1(\Omega)}$, when the quadrature does not satisfy the condition (4.12) for $k = 2$. We first computed

Table 5 GMM with $k = 2$. Corrected non-symmetric Gaussian rule for $k = 1$

h	$\|u - u^*_h\|_{H^1(\Omega)}$			
	8 points	16 points	32 points	64 points
1/10	3.8161E−02	6.3683E−03	1.8719E−03	6.9042E−04
1/20	4.0997E−02	1.2019E−02	1.9499E−03	2.2130E−04
1/40	3.9586E−02	1.5615E−02	2.0005E−03	1.5928E−04
1/80	3.8040E−02	1.7666E−02	2.0156E−03	1.6104E−04
1/160	3.7035E−02	1.8762E−02	2.0196E−03	1.6434E−04
1/320	3.6472E−02	1.9330E−02	2.0205E−03	1.6610E−04
1/640	3.6174E−02	1.9619E−02	2.0208E−03	1.6699E−04
1/1280	3.6022E−02	1.9764E−02	2.0208E−03	1.6743E−04

The H^1 norm of the error, $\|u - u^*_h\|_{H^1(\Omega)}$, where $u = e^x$ and u^*_h is the solution of the GMM, employing shape functions that reproduce polynomials of degree $k = 2$. Corrected p-point non-symmetric Gaussian integration rules for $k = 1$ (not corrected for $k = 2$), with $p = 8, 16, 32$ and 64, were used in the GMM. The integration rules do not satisfy the discrete Green's formula for $k = 2$

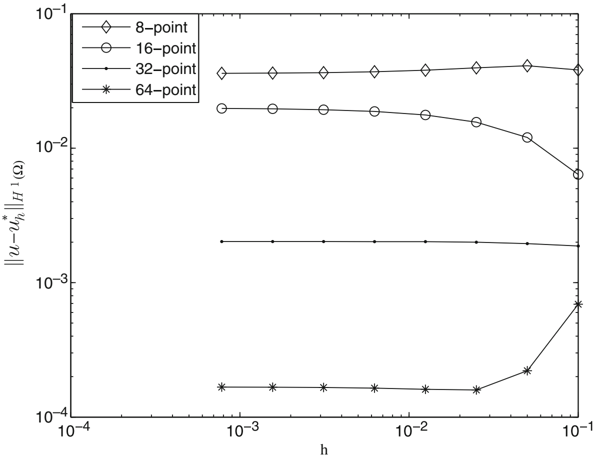


Fig. 5 The log-log plot of $\|u-u_h^*\|_{H^1(\Omega)}$ with respect to h, where $u=e^x$ and u_h^* is the solution of the GMM, employing shape functions that reproduce polynomials of degree $k=2$. Corrected p-point non-symmetric Gaussian rules for $k=1$ were used, which do not satisfy the discrete Green's formula for $k=2$

γ_{ij}^* in (4.6) using the corrected p-point non-symmetric integration rule for $k=1$ (see $Q^i_{ng,c}(\cdot)$ given before). We note that this quadrature rule satisfies only the first condition in (6.10). The terms σ_{ij}^* and l_i^* were computed using the p-point non-symmetric gaussian integration rule $Q^i_{ng}(\cdot)$. The error $\|u-u_h^*\|_{H^1(\Omega)}$ for various values of h and the associated log-log plot are given in Table 5 and Fig. 5, respectively. These results indicate that $\|u-u_h^*\|_{H^1(\Omega)}=\mathcal{O}(h+\eta)$ and $u_h^*\nrightarrow u$ as $h\to 0$.

Finally, we used $Q^i_{ng}(\cdot)$ to compute γ_{ij}^* in (4.6); $Q^i_{ng}(\cdot)$ does not satisfy any of the conditions in (6.10). The terms σ_{ij}^* and l_i^* were again computed using $Q^i_{ng}(\cdot)$. The error $\|u-u_h^*\|_{H^1(\Omega)}$ for various values of h and the associated log-log plot are given in Table 6 and Fig. 6, respectively. It is clear that the error

Table 6 Non-symmetric (no correction) Gaussian rule: $k=2$

h	$\|u-u_h^*\|_{H^1(\Omega)}$			
	8 points	16 points	32 points	64 points
1/10	9.4346E−01	4.6767E−02	7.5982E−03	7.6165E−04
1/20	2.0155E+00	4.5294E−02	2.1562E−02	1.0294E−03
1/40	4.3714E+00	9.2717E−02	5.0461E−02	2.3620E−03
1/80	9.7615E+00	2.8436E−01	1.0883E−01	5.0628E−03
1/160	2.1885E+01	7.1121E−01	2.2511E−01	1.0474E−02
1/320	4.7659E+01	1.6259E+00	4.5406E−01	2.1309E−02
1/640	1.0044E+02	3.6444E+00	8.9677E−01	4.3013E−02
1/1280	2.0680E+02	8.2709E+00	1.7231E+00	8.6554E−02

The H^1 norm of the error, $\|u-u_h^*\|_{H^1(\Omega)}$, where $u=e^x$ and u_h^* is the solution of the GMM, employing shape functions that reproduce polynomials of degree $k=2$. Non-symmetric p-point Gaussian integration rules, with $p=8$, 16, 32 and 64, were used in the GMM. The integration rules do not satisfy the discrete Green's formula for $k=2$

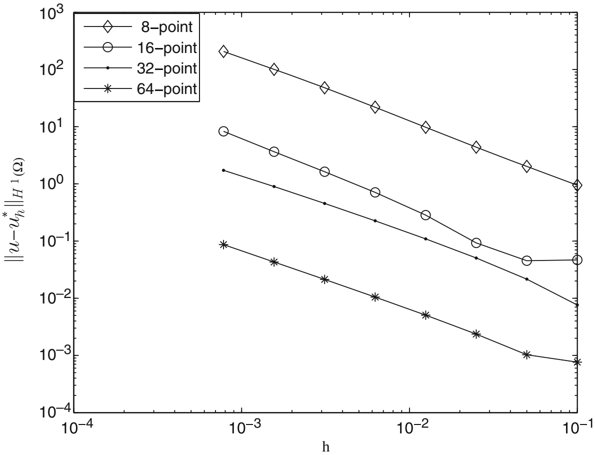


Fig. 6 The log-log plot of $\|u - u_h^*\|_{H^1(\Omega)}$ with respect to h, where $u = e^x$ and u_h^* is the solution of the GMM, employing shape functions that reproduce polynomials of degree $k = 2$. Non-symmetric p-point Gaussian integration rules were used, which do not satisfy the discrete Green's formula for $k = 2$

$\|u - u_h^*\|_{H^1(\Omega)}$ diverges as h decreases; in fact $\|u - u_h^*\|_{H^1(\Omega)}$ behaves like $\mathcal{O}(h^{-1})$ as suggested by Corollary 5.6 for $k = 2$.

8 Conclusion

In this paper, we have studied the effect of numerical integration on GMM to approximate the solution of a Neumann problem with non-constant coefficients and a lower order term. We have proposed a set of axioms on the quadrature rules used in the GMM and have studied the effect of the quadrature (satisfying these axioms) on the associated approximation error, when the shape functions of the GMM reproduce polynomials of degree k. The quadrature rules satisfying these axioms, particularly the axiom QA 4.2—a discrete Green's identity, do not depend on the non-constant coefficients of the Neumann problem. We also note that the Integration Constraint in [13] is precisely QA 4.2 for $k = 1$. Our analysis shows that the optimal order of convergence of the approximation error in energy norm, with respect to the discretization parameter h, can be achieved provided quadratures with increasing accuracy are used as $h \to 0$. We have outlined procedures to construct quadrature rules in 2-d for $k = 1, 2$ satisfying the axioms, in particular QA 4.2. Also the theoretical results have been illuminated with numerical experiment. We note however that problems with essential boundary conditions will require a different treatment and will be reported in future.

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