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Non-linear sampling recovery based on quasi-interpolant wavelet representations

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Abstract We investigate a problem of approximate non-linear sampling recovery of functions on the interval $\mathbb{I} := [0, 1]$ expressing the adaptive choice of n sampled values of a function to be recovered, and of n terms from a given family of functions Φ. More precisely, for each function f on $\mathbb{I}$, we choose a sequence $\xi = \{\xi^s\}_{s=1}^n$ of n points in $\mathbb{I}$, a sequence $a = \{a_s\}_{s=1}^n$ of n functions defined on $\mathbb{R}^n$ and a sequence $\Phi_n = \{\varphi_{k_s}\}_{s=1}^n$ of n functions from a given family Φ. By this choice we define a (non-linear) sampling recovery method so that f is approximately recovered from the n sampled values $f(\xi^1), f(\xi^2), ..., f(\xi^n)$, by the n-term linear combination

$$S(f) = S(\xi, \Phi_n, a, f) := \sum_{s=1}^{n} a_s(f(\xi^1), ..., f(\xi^n))\varphi_{k_s}.$$

In searching an optimal sampling method, we study the quantity

$$\nu_n(f, \Phi)_q := \inf_{\Phi_n, \xi, a} \|f - S(\xi, \Phi_n, a, f)\|_q,$$

where the infimum is taken over all sequences $\xi = \{\xi^s\}_{s=1}^n$ of n points, $a = \{a_s\}_{s=1}^n$ of n functions defined on $\mathbb{R}^n$, and $\Phi_n = \{\varphi_{k_s}\}_{s=1}^n$ of n functions from Φ. Let $U_{p,\theta}^\alpha$ be the unit ball in the Besov space $B_{p,\theta}^\alpha$, and **M** the set of centered B-spline wavelets

$$M_{k,s}(x) := N_r(2^k x + \rho - s),$$

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which do not vanish identically on $\mathbb{I}$, where N_r is the B-spline of even order $r = 2\rho \geq [\alpha] + 1$ with knots at the points $0, 1, ..., r$. For $1 \leq p, q \leq \infty,\ 0 < \theta \leq \infty$ and $\alpha > 1$, we proved the following asymptotic order

$$\nu_n\left(U^\alpha_{p,\theta}, \mathbf{M}\right)_q := \sup_{f \in U^\alpha_{p,\theta}} \nu_n(f, \mathbf{M})_q \asymp n^{-\alpha}.$$

An asymptotically optimal non-linear sampling recovery method S^* for $\nu_n(U^\alpha_{p,\theta}, \mathbf{M})_q$ is constructed by using a quasi-interpolant wavelet representation of functions in the Besov space in terms of the B-splines $M_{k,s}$ and the associated equivalent discrete quasi-norm of the Besov space. For $1 \leq p < q \leq \infty$, the asymptotic order of this asymptotically optimal sampling non-linear recovery method is better than the asymptotic order of any linear sampling recovery method or, more generally, of any non-linear sampling recovery method of the form $R(H, \xi, f) := H(f(\xi^1), ..., f(\xi^n))$ with a fixed mapping $H : \mathbb{R}^n \to C(\mathbb{I})$ and n fixed points $\xi = \{\xi^s\}_{s=1}^n$.

Keywords Non-linear sampling recovery · Quasi-interpolant wavelet representation · Adaptive choice · B-spline · Besov space

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1 Introduction

1.1

We begin with shortly considering some known problems of sampling recovery of functions defined on the interval $\mathbb{I} := [0, 1]$. Suppose that $\xi = \{\xi^k\}_{k=1}^n$ is a fixed sequence of n points in $\mathbb{I}$, and we want to approximately recover a function f on $\mathbb{I}$ from the sampled values $f(\xi^1), f(\xi^2), ..., f(\xi^n)$. Using this information we can approximately recover a continuous function f on $\mathbb{I}$, by the linear sampling recovery method L defined by

$$L(f) = L(\Phi, \xi, f) := \sum_{k=1}^n f(\xi^k)\varphi_k, \tag{1}$$

where $\Phi = \{\varphi_k\}_{k=1}^n$ is a fixed sequence of n functions $\mathbb{I}$. Denote by $L_q := L_q(\mathbb{I})$ the normed space of functions on $\mathbb{I}$ with the usual qth integral norm $\|\cdot\|_q$ for $1 \leq q < \infty$, and the normed space $C(\mathbb{I})$ of continuous functions on $\mathbb{I}$ with the max-norm $\|\cdot\|_\infty$ for $p = \infty$. We will measure the error of the approximate recovery (1) by $\|f - L(\Phi, \xi, f)\|_q$. For a subset $W \subset L_q$, the worst case error of the recovery of $f \in W$ by $L(f)$ can be represented by

$$\sup_{f \in W} \|f - L(\Phi, \xi, f)\|_q.$$

To study optimal sampling linear methods of the form (1) for recovering $f \in W$, we can use the quantity

$$\lambda_n(W)_q := \inf_{\Phi,\xi} \sup_{f \in W} \|f - L(\Phi, \xi, f)\|_q, \tag{2}$$

where the infimum is taken over all pairs (Φ, ξ) with $\xi = \{\xi^k\}_{k=1}^n$ and $\Phi = \{\varphi_k\}_{k=1}^n$.

In a linear sampling recovery method (1) we use the information of the sampled values of f at n fixed points $\xi = \{\xi^k\}_{k=1}^n$. Restricted ourselves by the same information, we can consider some non-linear sampling recovery methods. One of them is defined by

$$G(\Phi, \xi, a, f) := \sum_{k=1}^{n} a_k(f(\xi^1), ..., f(\xi^n))\varphi_k, \tag{3}$$

where $a = \{a_k\}_{k=1}^n$ is a given sequence of n functions on $\mathbb{R}^n$. Similarly to (2), to study optimal linear methods of the form (3) for recovering $f \in W$, we can use the quantity

$$\gamma_n(W)_q \ := \inf_{\Phi,\xi,a} \ \sup_{f\in W} \|f - G(\Phi, \xi, a, f)\|_q,$$

where the infimum is taken over all triples (Φ, ξ, a) with $\xi = \{\xi^k\}_{k=1}^n$, $a = \{a_k\}_{k=1}^n$ and $\Phi = \{\varphi_k\}_{k=1}^n$. Another is the sampling method R given by

$$R(H, \xi, f) := H(f(\xi^1), ..., f(\xi^n)) \tag{4}$$

where H is a mapping from $\mathbb{R}^n$ into L_q. To study optimal sampling methods of recovery for $f \in W$ from n their values, we can use the quantity

$$\varrho_n(W)_q \ := \inf_{H,\xi} \ \sup_{f\in W} \|f - R(H, \xi, f)\|_q,$$

where the infimum is taken over all sequences $\xi = \{\xi^k\}_{k=1}^n$ and all mappings H from $\mathbb{R}^n$ into L_q.

We use the notations: $x_+ := \max\{0, x\}$ for $x \in \mathbb{R}$; $A_n(f) \ll B_n(f)$ if $A_n(f) \le CB_n(f)$ with C an absolute constant not depending on n and/or $f \in W$, and $A_n(f) \asymp B_n(f)$ if $A_n(f) \ll B_n(f)$ and $B_n(f) \ll A_n(f)$.

Denote by $U^\alpha_{p,\theta}$ the unit ball of the Besov space $B^\alpha_{p,\theta}$ of functions on $\mathbb{I}$. The following results are known (see [13, 17, 19, 20, 23] and references there).

Theorem 1 *Let* $1 \le p, q \le \infty,\ 0 < \theta \le \infty$ *and* $\alpha > 1/p$. *Then there are the asymptotic equivalent relations*

$$\varrho_n\left(U^\alpha_{p,\theta}\right)_q \asymp \lambda_n\left(U^\alpha_{p,\theta}\right)_q \asymp \gamma_n\left(U^\alpha_{p,\theta}\right)_q \asymp n^{-\alpha+(1/p-1/q)_+}.$$

Moreover, we can explicitly construct an asymptotically optimal linear sampling recovery method L^* *of the form* (1)*, that is,*

$$\sup_{f\in U^\alpha_{p,\theta}} \|f - L^*(f)\|_q \asymp n^{-\alpha+(1/p-1/q)_+}.$$

1.2

In a sampling recovery method of the forms (1), (3) and (4) the points $\xi = \{\xi^k\}_{k=1}^n$ at which the sampled values are taken, and the mappings L, G, R which can be linear or non-linear are the same for all functions, i. e., the information and recovery method are non-adaptive. Let us introduce a new setting of non-linear sampling recovery with adaptive information and recovery methods. Namely, we will let the choice of points $\{\xi^k\}_{k=1}^n$ and a recovery approximant constructed from the sampled values at these points depend on a concrete function.

Let $W \subset L_q$ and $\Phi = \{\varphi_k\}_{k\in K}$ be a family of functions in L_q. Let us have the freedom to choose n terms φ_k from Φ and n sampled values for constructing an approximate recovery. More precisely, given a function $f \in W$, we choose a sequence $\xi = \{\xi^k\}_{k=1}^n$ of n points in $\mathbb{I}$, a sequence $a = \{a_k\}_{k=1}^n$ of n functions defined on $\mathbb{R}^n$ and a sequence $\Phi_n = \{\varphi_{k_s}\}_{s=1}^n$ of n functions from Φ. This choice defines an sampling recovery method given by

$$S(f) \;=\; S(\Phi_n, a, \xi, f) \;:=\; \sum_{s=1}^{n} a_s(f(\xi^1), ..., f(\xi^n))\varphi_{k_s}. \tag{5}$$

Then we consider the approximate recovery of f from its values $f(\xi^s)$, $s = 1, 2, ..., n$, by $S(f)$. Clearly, an efficient choice essentially depends on f, and this dependence is non-linear. Unlike sampling recovery methods of the forms (1), (3) and (4), for each function f we will first search an optimal sampling recovery method with regard to Φ

$$\nu_n(f, \Phi)_q := \inf_{\Phi_n, a, \xi} \|f - S(\Phi_n, a, \xi, f)\|_q,$$

where the infimum is taken over all sequences $\xi = \{\xi^k\}_{k=1}^n$ of n points in $\mathbb{I}$, $a = \{a_k\}_{k=1}^n$ of n functions defined on $\mathbb{R}^n$, and $\Phi_n = \{\varphi_{k_s}\}_{s=1}^n$ of n functions from Φ. Then we want to know the worst case of non-linear sampling recovery with regard to Φ for $f \in W$ by considering the quantity

$$\nu_n(W, \Phi)_q \;:=\; \sup_{f\in W} \nu_n(f, \Phi)_q.$$

The idea of non-linear sampling recovery in terms of the quantity $\nu_n(W, \Phi)_q$ naturally comes from the non-linear n-term approximation. The reader can find in [10, 24] surveys on various aspects of this approximation and its applications.

For a given even natural number $r = 2\rho$, let N_r be the B-spline of order r with knots at the points $0, 1, ..., r$, and

$$M_r := N_r(\cdot + \rho)$$

be the centered B-spline. Denote by $\mathbf{M}$ the set of all such B-spline wavelets

$$M_{k,s}(x) := \; M_r(2^k x - s),$$

which do not vanish identically on $\mathbb{I}$.

The main result of the present paper is the following theorem.

Theorem 2 *Let $1 \le p, q \le \infty$, $0 < \theta \le \infty$, and $1 < \alpha < r$. Then for the unit ball $U^\alpha_{p,\theta}$ of the Besov space, there is the following asymptotic order*

$$\nu_n\left(U^\alpha_{p,\theta}, \mathbf{M}\right)_q \;\asymp\; n^{-\alpha}. \tag{6}$$

For $1 \le p < q \le \infty$, the asymptotic order of optimal non-linear sampling recovery method for $\nu_n(U^\alpha_{p,\theta}, \mathbf{M})_q$ is better than the asymptotic order of any linear sampling recovery method of the form (1) and of any non-linear sampling recovery method of the form (3) or (4). Namely, the asymptotic orders of λ_n, γ_n and ϱ_n are $n^{-\alpha+1/p-1/q}$, while the asymptotic order of ν_n is $n^{-\alpha}$.

1.3

To construct an asymptotically optimal non-linear sampling recovery method S^* for $\nu_n(U^\alpha_{p,\theta}, \mathbf{M})_q$ which gives the upper bound of (6) we used a quasi-interpolant wavelet representation of functions in the Besov space in terms of the B-splines $M_{k,s}$. It is well known that a function on $\mathbb{I}$ has a B-spline wavelet representation:

$$f(x) \;=\; \sum_{k=0}^{\infty} \sum_{s \in J(k)} \lambda_{k,s}(f) M_{k,s}(x) \tag{7}$$

where $J(k)$ is the set of s for which $M_{k,s}$ do not vanish identically on $\mathbb{I}$, and $\lambda_{k,s}$ are appropriate coefficient functionals.

There are many ways to define the functionals $\lambda_{k,s}$ (see [9, 10, 21] and references there). For construction of an asymptotically optimal sampling method for $\nu_n(U^\alpha_{p,\theta}, \mathbf{M})_q$ we need coefficient functionals of a special form $\lambda_{k,s}(f)$ which are functions of a finite number of values of f. It is important that this number should not depend on neither k, s nor f. Such a representation can be constructed by using a quasi-interpolant of the form

$$Q(f, x) := \sum_{k=-\infty}^{\infty} \Lambda(f, k) M(x-k), \tag{8}$$

defined for functions on $\mathbb{R}$, where

$$\Lambda(f, s) = \sum_{|j| \le J} \lambda_j f(s-j)) \tag{9}$$

and $\Lambda = \{\lambda_j\}_{|j| \le J}$ a given finite even sequence. We can see later that the B-spline wavelet representation (7) based on a quasi-interpolant (8)–(9) has the coefficients $\lambda_{k,s}(f)$ as functions of no more than $2J + r$ values of f, with J any fixed number not smaller than $r/2$.

An asymptotically optimal non-linear sampling recovery method S^* is constructed as the sum of a linear quasi-interpolant operator $Q_{\bar{k}(n)}$ and non-linear operator G^*_n. The linear part $Q_{\bar{k}(n)}(f)$ with an appropriate $\bar{k}(n)$ gives the same approximation order $n^{-\alpha + (1/p - 1/q)_+}$ as of $\lambda_n(U^\alpha_{p,\theta})_q$ and $\gamma_n(U^\alpha_{p,\theta})_q$ (see Corollary 2) while the "additional" non-linear part $G^*_n(f)$ which is the sum of greedy algorithms at some B-spline dyadic scales improves the approximation order for the case $1 \le p < q \le \infty$.

We restrict ourselves to consider the sampling recovery as an approximation problem, not concerning the computation aspect. It is interesting to investigate the cost of non-linear sampling recovery methods (algorithms) and complexity of our problem. Notice that in the non-linear sampling recovery in terms the quantity ν_n of the cost to compute the non-linear part of the approximant is mostly too expensive (see [7, 8] for details).

The main results of the present paper were announced in [16].

We give a brief description of the remaining sections. In Section 2 we construct a quasi-interpolant wavelet representation in terms of the B-splines $M_{k,s} \in \mathbf{M}$ for Besov spaces and prove some quasi-norm equivalences based on this representation, in particular, a discrete quasi-norm in terms of the coefficient functionals. In Section 3 we will discuss linear and non-linear sampling recovery methods using quasi-interpolant wavelet representations, and give a Proof of Theorem 2.

2 Quasi-interpolant wavelet representations

2.1

Let

$$S(\varphi) := \mathrm{span}\{\,\varphi(\cdot - s)\,\}_{s\in\mathbb{Z}}$$

be the space spanned by the integer translates of a B-spline φ. A B-spline quasi-inerpolant for $S(\varphi)$ is a linear map

$$Q_\varphi(f) := \sum_{k\in\mathbb{Z}} \lambda(f,k)\varphi(\cdot - k)$$

from a normed space of functions f on $\mathbb{R}$ into $S(\varphi)$ which is local, bounded and reproduces some nontrivial polynomial space [9, p. 63]. For construction of sampling methods of recovery we will consider some special types of discrete quasi-interpolants for which the coefficient functionals $\lambda(f,k)$ are linear combinations of values of a function f or its derivatives at a finite number of points.

Denote by N_r the B-spline of order r with knots at the points $0, 1, ..., r$. The B-spline N_1 can be defined as the characteristic function of the interval $[0, 1)$. For $r \geq 2$, N_r can be defined recursively by convolution:

$$N_r(x) := \int_{-\infty}^{\infty} N_{r-1}(x-y)N_1(y)dy.$$

Notice that the support of N_r is $[0, r]$ and N_r satisfies the refinement equation:

$$N_r(x) := 2^{-r+1}\sum_{s=0}^{r}\binom{r}{s}N_r(2x-s). \tag{10}$$

Let

$$M_r := N_r(\cdot + r/2)$$

be the centered B-spline. Denote by $\mathbf{S}_r$ and $\mathbf{S}_r^*$ the span of $N_r(\cdot - s),\ s\in\mathbb{Z}$, and $M_r(\cdot - s),\ s\in\mathbb{Z}$, respectively.

Let $\Lambda = \{\lambda_j\}_{|j|\leq J}$ be a finite even sequence, i.e., $\lambda_{-j} = \lambda_j$. We define the operator Q by

$$Q(f,x) := \sum_{s=-\infty}^{\infty} \Lambda(f,s)M_r(x-s) \tag{11}$$

for a function f defined on $\mathbb{R}$, where

$$\Lambda(f,s) := \sum_{|j|\leq J} \lambda_j f(s-j). \tag{12}$$

It is easy to verify that Q is a bounded linear operator in $C(\mathbb{R})$ and

$$\|Q(f)\|_{C(\mathbb{R})} \leq \|\Lambda\|\|f\|_{C(\mathbb{R})}$$

for each $f \in C(\mathbb{R})$, where

$$\|\Lambda\| = \sum_{|k| \le J} |\lambda_k|.$$

Moreover, Q is local in the following sense. There exists a positive number $\delta > 0$ such that for any $f \in C(\mathbb{R})$, and $x \in \mathbb{R}$, $Q(f, x)$ depends only on the value $f(y)$ at a finite number of points y with $|y - x| \le \delta$. In the present paper, we will require it to reproduce the space $\mathcal{P}_{r-1}$ of polynomials of order at most $r - 1$, that is,

$$Q(p) \ = \ p, \ \ p \in \mathcal{P}_{r-1}.$$

Then, such an operator Q will be a quasi-interpolant for $\mathbf{S}_r^*$ in the normed space $C(\mathbb{R})$. A method of construction of such a quasi-interpolant via Neumann series was suggested in [5] (see also [4, p. 100–109]), is as follows.

Let the Laurent polynomials $\widetilde{M}_r$ and $\widetilde{D}_r$ be defined by

$$\widetilde{M}_r(z) := \sum_k M_r(k) z^k,$$

$$\widetilde{D}_r(z) := \ 1 - \widetilde{M}_r(z).$$

Further, for a given non-negative integer ν we define $\Lambda^{(\nu)} = \{\lambda_k\}$ in terms of the finite Neumann series:

$$\widetilde{\Lambda}^{(\nu)}(z) := \sum_k \lambda_k z^k \ = \ 1 + \widetilde{D}(z) + \cdots + \widetilde{D}^{\nu}(z).$$

Clearly, $\Lambda^{(\nu)}$ is a finite even sequence. The operator Q in (11)–(12) associated with $\Lambda^{(\nu)}$, reproduces $\mathcal{P}_{r-1}$ and therefore, is a quasi-interpolant [5].

For an even $r = 2\rho$ and $J \ge \rho$, general solutions for the construction of quasi-interpolants of the form (11)–(12) with optimal approximation order were given in [2, 3] intiated by a work of Schoenberg [22]. Such quasi-interpolants with near minimal norm $\|\Lambda\|$ which may be useful for numerical applications have been recently constructed. See [21] for a survey on this direction.

We will need a quasi-interpolant for $\mathbf{S}_r$ in the norm of $C^{r-1}(\mathbb{R})$ introduced in [6]. This quasi-interpolant is based on the values of derivatives and defined as follows. For $f \in C^{r-1}(\mathbb{R})$, we let

$$P(f, x) := \sum_{k=-\infty}^{\infty} \alpha(f, k) N_r(x - k), \tag{13}$$

where

$$\alpha(f, k) \ := \sum_{j<r} w_{k,j} f^{(j)}(\xi_k), \tag{14}$$

with any point $\xi_k \in (k, k + r)$ and

$$\begin{aligned} w_{k,j} \ &:= \ (-1)^{r-1-j} \psi_k^{(r-1-j)}(\xi_k), \qquad j < r, \\ \psi_k(x) \ &:= (k + 1 - x) \cdots (k + r - 1 - x). \end{aligned}$$

Then P is a quasi-interpolant which is a local bounded linear B-spline operator mapping $C^{r-1}(\mathbb{R})$ to $\mathbf{S}_r$, and reproduces $\mathcal{P}_{r-1}$ [6].

2.2

In the present paper, we will consider sampling methods of recovering functions on the interval $\mathbb{I}$ which possess a certain smoothness. Let us introduce Sobolev and Besov spaces of smooth functions and give necessary knowledge of them. The reader can read this and more details of Sobolev and Besov spaces in the books [1, 11, 18].

Let $\mathbb{G} = [a, b]$ be an interval in $\mathbb{R}$. Denote by $L_p(\mathbb{G})$ the normed space of functions on $\mathbb{G}$ with the usual pth norm $\|\cdot\|_{p,\mathbb{G}}$ for $1 \le p < \infty$, and the normed space $C(\mathbb{G})$ of continuous functions on $\mathbb{G}$ with the max-norm $\|\cdot\|_{\infty,\mathbb{G}}$ for $p = \infty$. For $1 \le p \le \infty$ and natural number α, the Sobolev space $W_p^\alpha(\mathbb{G})$ is the set of functions $f \in L_p(\mathbb{G})$ for which $f^{(\alpha-1)}$ is absolutely continuous on $\mathbb{G}$ and $f^{(\alpha)} \in L_p(\mathbb{G})$. The Sobolev semi-norm and norm of $W_p^\alpha(\mathbb{G})$ are

$$|f|_{W_p^\alpha(\mathbb{G})} := \|f^{(\alpha)}\|_{p,\mathbb{G}}, \qquad \|f\|_{W_p^\alpha(\mathbb{G})} := \|f\|_{p,\mathbb{G}} + |f|_{W_p^\alpha(\mathbb{G})}.$$

Let

$$\omega_l(f, t)_{p,\mathbb{G}} := \sup_{|h|<t} \left\| \Delta_h^l f \right\|_{p,\mathbb{G}_{lh}}$$

be the lth modulus of smoothness of f where $\mathbb{G}_{lh} := [a + lh, b - lh]$, and the lth difference $\Delta_h^l f$ is defined by

$$\Delta_h^l f := \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} f(x + jh).$$

Let $1 \le p \le \infty$, $0 < \theta \le \infty$ and $0 < \alpha < l$. The Besov space $B_{p,\theta}^\alpha(\mathbb{G})$ is the set of functions $f \in L_p(\mathbb{G})$ for which the Besov quasi-semi-norm $|f|_{B_{p,\theta}^\alpha}(\mathbb{G})$ is finite. The Besov quasi-semi-norm $|f|_{B_{p,\theta}^\alpha(\mathbb{G})}$ is given by

$$|f|_{B_{p,\theta}^\alpha(\mathbb{G})} := \begin{cases} \left(\int_0^\infty \{t^{-\alpha} \omega_l(f, t)_{p,\mathbb{G}}\}^\theta dt/t \right)^{1/\theta}, & \theta < \infty, \\ \sup_{t>0} t^{-\alpha} \omega_l(f, t)_{p,\mathbb{G}}, & \theta = \infty. \end{cases} \tag{15}$$

The Besov quasi-norm is defined by

$$B(f) = \|f\|_{B_{p,\theta}^\alpha(\mathbb{G})} := \|f\|_{p,\mathbb{G}} + |f|_{B_{p,\theta}^\alpha(\mathbb{G})}. \tag{16}$$

The definition of $B_{p,\theta}^\alpha(\mathbb{G})$ does not depend on l, i. e., for a given α, (15)–(16) determine equivalent quasi-norms for all l such that $\alpha < l$.

In what follows, we will drop $\mathbb{I}$ in a notation if $\mathbb{G} = \mathbb{I}$, in particular, we will use the abbreviations: $L_p := L_p(\mathbb{I})$; $W_p^\alpha := W_p^\alpha(\mathbb{I})$; $B_{p,\theta}^\alpha := B_{p,\theta}^\alpha(\mathbb{I})$. We will assume that continuous functions to be recovered are from the Sobolev space W_p^α or the Besov space $B_{p,\theta}^\alpha$ with the restriction $\alpha > 1/p$ which is a sufficient condition of the compact embedding of these spaces into $C(\mathbb{I})$.

2.3

Let a quasi-interpolant Q of the form (11)–(12) be given. For $h > 0$ and a function f on $\mathbb{R}$, we define the operator Q^h by

$$Q^h(f) = \sigma_h \circ Q \circ \sigma_{1/h}(f),$$

where

$$\sigma_h(f,x) = f(x/h).$$

By definition it is easy to see that

$$Q^h(f,x) = \sum_k \Lambda^h(f,k) M_r(h^{-1}x - k),$$

where

$$\Lambda^h(f,k) := \sum_j \lambda_{k-j} f(hj).$$

If a function f is defined on $\mathbb{R}$ and possesses a smoothness α in a neighborhood of $\mathbb{I}$, then the approximation by means of Q^h has the asymptotic order [9, p. 63–65]

$$\|f - Q^h f\|_\infty = O(h^\alpha).$$

However, we consider only functions which are defined in $\mathbb{I}$. The quasi-interpolant Q^h is not defined for a function f on $\mathbb{I}$, and therefore, not appropriate for an approximate sampling recovery of f from its sampled values at points in $\mathbb{I}$. An approach to construct a quasi-interpolant for a function on $\mathbb{I}$ is to extend it by interpolation Lagrange polynomials.

For a non-negative integer k, we put $x_j = j2^{-k}$, $j \in \mathbb{Z}$. If f is a function on $\mathbb{I}$, let

$$U_k(f,x) := f(x_0) + \sum_{s=1}^{r-1} \frac{2^{sk}\Delta^s_{2^{-k}} f(x_0)}{s!} \prod_{j=0}^{s-1}(x - x_j),$$

$$V_k(f,x) := f(x_{2^k-r+1}) + \sum_{s=1}^{r-1} \frac{2^{sk}\Delta^s_{2^{-k}} f(x_{2^k-r+1})}{s!} \prod_{j=0}^{s-1}(x - x_{2^k-r+1+j}) \tag{17}$$

be the $(r-1)$th Lagrange polynomials interpolating f at the left end points $x_0, x_1, ..., x_{r-1}$, and right end points $x_{2^k-r+1}, x_{2^k-r+3}, ..., x_{2^k}$, of the interval $\mathbb{I}$, respectively. We define the function $\bar{f}$ as an extension of f on $\mathbb{R}$ by the formula

$$\bar{f}(x) := \begin{cases} U_k(f,x), & x < 0 \\ f(x), & 0 \le x \le 1 \\ V_k(f,x), & x > 1. \end{cases} \tag{18}$$

Obviously, $\bar{f}$ is a continuous function on $\mathbb{R}$. We introduce the operator Q_k by

$$Q_k(f) = Q^{2^{-k}}(\bar{f}).$$

We have

$$Q_k(f,x) = \sum_{s \in J(k)} a_{k,s}(f) M_{k,s}(x), \quad \forall x \in \mathbb{I}, \tag{19}$$

where $J(k) := \{s \in \mathbb{Z} : -r/2 < s < 2^k + r/2\}$ is the set of s for which $M_{k,s}$ do not vanish identically on $\mathbb{I}$, and

$$a_{k,s}(f) := \Lambda^{2^{-k}}(\bar{f}, s) = \sum_{|j| \le J} \lambda_j \bar{f}(2^{-k}(s-j)). \tag{20}$$

Notice that the number of the terms in $Q_k(f)$ is of the size $\asymp 2^k$.

An important property of Q_k is that the function $Q_k(f)$ is completely determined from the values of f at the points $x_0, x_1, ..., x_{2^k}$ which are in $\mathbb{I}$. For each pair k, s the coefficient $a_{k,s}(f)$ is a linear combination of the values $f(2^{-k}(s-j)),\ |j| \le J$, and maybe, $f(2^{-k}j)$ with $j = 0, 1, ..., r-1$ or $j = 2^k - r + 1, 2^k - r + 3, ..., 2^k$, if the point $2^{-k}s$ is near to the ends 0 or 1 of the interval $\mathbb{I}$, respectively. Thus, the number of these values does not exceed the $2J + r$ and not depend on neither functions f and nor k, s. The operator Q_k also has properties similar to the properties of the quasi-interpolants Q and Q^h. Namely, it is a local bounded linear mapping in $C(\mathbb{I})$ and reproducing $\mathcal{P}_{r-1}$, more precisely,

$$Q_k(p^*) \ = \ p, \ \ p \in \mathcal{P}_{r-1}, \tag{21}$$

where p^* is the restriction of p on $\mathbb{I}$. We will call Q_k a *quasi-interpolant for* $C(\mathbb{I})$.

2.4

For approximation a function $f \in W^\alpha_p$, it is natural to use the quasi-interpolant Q_m. We will prove the following theorem.

Theorem 3 *Let* $1 \le p \le \infty,\ \alpha \le r$. *Then for each* $f \in W^\alpha_p$, *we have*

$$\|f - Q_m f\|_p \ \le C|f|_{W^\alpha_p} 2^{-\alpha m},$$

where C *is a constant depending on* J, r, α *and the norm* $\|\Lambda\|$ *only.*

Proof Let $I_s := [hs, h(s+1)] \cap \mathbb{I}$, where we use the abbreviation $h = 2^{-k}$. We have

$$\|f - Q_m f\|_p^p = \sum_s \int_{I_s} |f(x) - Q_m(f, x)|^p dx =: \sum_s \mathcal{I}_s. \tag{22}$$

Let T be the Taylor polynomial of order $\alpha - 1$ at a point $x_s \in I_s$ of f. For simplicity we use the same letter T to denote its restriction on $\mathbb{I}$. Then, for each $x \in \mathbb{I}$

$$F(x) := \ f(x) - T(x) = \int_{x_s}^x f^{(\alpha)}(t) \frac{(x-t)^{\alpha-1}}{(\alpha-1)!}\, dt. \tag{23}$$

By (21) we have $Q_m T = T$, and therefore,

$$f(x) - Q_m(f, x) = \ F(x) - Q_m(F, x). \tag{24}$$

Applying Hölder's inequality to the right side of (23) gives

$$\begin{aligned} |F(x)| \ &\le \frac{1}{(\alpha-1)!} |I_s|^{\alpha - 1/p} \|f^{(\alpha)}\|_{L_p(I_s)} \\ &\le \frac{h^{\alpha-1/p}}{(\alpha-1)!} \|f^{(\alpha)}\|_{L_p(I_s)}, \quad x \in I_s. \end{aligned} \tag{25}$$

Let us estimate the second term in (24). By definition it is easy to see that

$$Q_m(F, x) = \sum_{k \in J_s} \Lambda^h(\bar{F}, k) M_r(h^{-1}x - k), \quad x \in I_s, \tag{26}$$

where

$$\Lambda^h(\bar{F}, k) = \sum_j \lambda_{k-j} \bar{F}(hj)$$

and $J_s := \{k \in \mathbb{Z} : -r/2 - 1 < s - k < r/2\}$. Indeed, we have

$$Q_m(F, x) = \sum_k \Lambda^h(\bar{F}, k) M_r(h^{-1}x - k).$$

Further, $M_r(h^{-1}x - k) \neq 0$ for some $x \in I_s$ if and only if $0 < h^{-1}x + r/2 - k < r$. Hence, if $M_r(h^{-1}x - k) \neq 0$, then $0 < (s + 1) + r/2 - k$ and $s + r/2 - k < r$. Thus, if the inequalities

$$-r/2 - 1 < s - k < r/2$$

do not hold, then $M_r(h^{-1}x - k) = 0$ for all $x \in I_s$.

Using the inequalities $0 \le M_r(x) \le 1$ and $|J_s| \le r$, we obtain by (26)

$$\begin{aligned} |Q_m(F, x)| &\le \sum_{k \in J_s} |\Lambda^h(\bar{F}, k)| \\ &\le r \max_{k \in J_s} |\Lambda^h(\bar{F}, k)|, \qquad x \in I_s, \end{aligned} \tag{27}$$

We will estimate $\Lambda^h(\bar{F}, k)$ for $k \in J_s$. From the equation

$$\Lambda^h(\bar{F}, k) = \sum_{j \in Z_k} \lambda_{k-j} \bar{F}(hj),$$

with $Z_k := \{j \in \mathbb{Z} : |j - k| \le J\}$, we obtain

$$|\Lambda^h(\bar{F}, k)| \le \max_{j \in Z_k} |\bar{F}(hj)| \sum_{j \in Z_k} |\lambda_{k-j}| = \|\Lambda\| \max_{j \in Z_k} |\bar{F}(hj)|. \tag{28}$$

Notice that

$$\cup_{k \in J_s} Z_k \subset Z_s^* := \{k \in \mathbb{Z} : |k - s| \le J + r/2\}.$$

Let $j \in Z_s^*$. We first consider the case when $hj \in \mathbb{I}$. This means that $0 \le j \le 2^k$. As in (23), we have

$$\bar{F}(hj) = F(hj) = \int_{x_s}^{hj} f^{(\alpha)}(t) \frac{(hj - t)^{\alpha - 1}}{(\alpha - 1)!} \, dt. \tag{29}$$

Obviously, $s \in Z_s^*$ and $hj \in I_s^* := \cup_{k \in Z_s^*} I_k$. Similar to (25), from (29) we derive that

$$\begin{aligned} |\bar{F}(hj)| &\le \frac{1}{(\alpha - 1)!} |I_s^*|^{\alpha - 1/p} \| f^{(\alpha)} \|_{L_p(I_s^*)} \\ &\le \frac{(2J + r + 1) h^{\alpha - 1/p}}{(\alpha - 1)!} \| f^{(\alpha)} \|_{L_p(I_s^*)}. \end{aligned} \tag{30}$$

We next consider the case when $hj \notin \mathbb{I}$. Then either $-J - r/2 \le j < 0$ or $2^k < j2^k + J + r/2$. For the case when $-J - r/2 \le j < 0$, by (17) and (18) we have

$$\begin{aligned}\bar{F}(hj) &= \bar{F}(0) + \sum_{s=1}^{r-1} \frac{2^{sk}\Delta^s_{2^{-k}}\bar{F}(0)}{s!} \prod_{i=0}^{s-1}(hj - hi),\\ &= \bar{F}(0) + \sum_{s=1}^{r-1} \frac{\Delta^s_{2^{-k}}\bar{F}(0)}{s!} \prod_{i=0}^{s-1}(j - i).\end{aligned}$$

Hence, by a simple computation we obtain

$$|\bar{F}(hj)| \le C_1 \max_{0 \le i \le r-1} |\bar{F}(hi)|, \tag{31}$$

where C_1 is a constant depending on J, r only.

Similarly, for the case when $2^k < j2^k + J + r/2$, we have

$$|\bar{F}(hj)| \le C_2 \max_{2^k - r + 1 \le i \le 2^k} |\bar{F}(hi)|, \tag{32}$$

where C_2 is a constant depending on J, r only. The inequalities (30)–(32) yield

$$|\bar{F}(hj)| \le C_3 \| f^{(\alpha)}\|_{L_p(I^*_s)}, \quad j \in Z^*_s, \tag{33}$$

where C_3 is a constant depending on J, r, α only.

Combining (27), (28) and (33) gives

$$|Q_m(F, x)| \le C_4 h^{\alpha - 1/p}\| f^{(\alpha)}\|_{L_p(I^*_s)}, \quad x \in I_s, \tag{34}$$

where $C_4 = C_3\|\Lambda\|$.

Let us estimate $\mathcal{I}_s$. We have by (24), (25) and (34)

$$\begin{aligned}\mathcal{I}_s &= \int_{I_s} | f(x) - Q_m(f, x)|^p dx\\ &\le \int_{I_s} (|F(x)| + |Q_m(F, x)|)^p dx\\ &\le \int_{I_s} \left(\frac{h^{\alpha-1/p}}{(\alpha - 1)!}\| f^{(\alpha)}\|_{L_p(I_s)} + C_4 h^{\alpha-1/p}\| f^{(\alpha)}\|_{L_p(I^*_s)}\right)^p dx\\ &\le (2C_4 h^\alpha\| f^{(\alpha)}\|_{L_p(I^*_s)})^p.\end{aligned}$$

Hence, by (22) we get

$$\begin{aligned}\| f - Q_m f\|^p_p &= \sum_s \mathcal{I}_s\\ &\le \left(2C_4 h^\alpha\right)^p \sum_s \| f^{(\alpha)}\|^p_{L_p(I^*_s)}\\ &\le \left(2C_4 h^\alpha\right)^p \sum_s \sum_{k \in Z^*_s} \| f^{(\alpha)}\|^p_{L_p(I_k)}.\end{aligned}$$

If j is a natural number such that $I_j \neq \emptyset$, then there are no more than $2J + r + 1$ the term $\| f^{(\alpha)} \|^p_{L_p(I_j)}$ in the sum taken over $k \in Z^*_s$ in the last expression. Hence,

$$\begin{aligned} \| f - Q_m f \|_p^p &\le \left(2C_4 h^\alpha\right)^p (2J + r + 1) \sum_s \| f^{(\alpha)} \|^p_{L_p(I_s)} \\ &\le (Ch^\alpha)^p \| f^{(\alpha)} \|_p^p \\ &\le C^p | f |^p_{W^\alpha_p} 2^{-p\alpha m}, \end{aligned}$$

where C is a constant depending on J, r, α and $\|\Lambda\|$ only. □

2.5

If $\{ f_k \}_{k=0}^\infty$ is a sequence whose component functions are in $L_p(\mathbb{G})$, for $0 < \theta \le \infty$ and $\beta \ge 0$ we use the $l^\beta_\theta(L_p(\mathbb{G}))$ "quasi-norms"

$$\|\{ f_k \}\|_{l^\beta_\theta(L_p(\mathbb{G}))} := \left(\sum_{l=0}^{\infty} \{ 2^{\beta k} \| f_k \|_{p,\mathbb{G}} \}^\theta \right)^{1/\theta}$$

with the usual change to a supremum norm when $\theta = \infty$. When $\{ f_k \}_{k=0}^\infty$ is a sequence of real numbers, we replace $\| f_k \|_{p,\mathbb{G}}$ by $| f_k |$ and denote the corresponding norm by $\|\{ f_k \}\|_{l^\beta_\theta}$. We will need the following discrete Hardy inequality

$$\|\{b_k\}\|_{l^\beta_\theta} \ \le \ C\|\{a_k\}\|_{l^\beta_\theta} \tag{35}$$

which holds if

$$|b_k| \ \le \ C \left(\sum_{k=0}^{m} 2^{\lambda(k-m)} |a_k| \ + \sum_{k=m+1}^{\infty} |a_k| \right) \tag{36}$$

with $\lambda > \beta > 0$.

For the Besov space $B^\alpha_{p,\theta}(\mathbb{G})$, there is the following quasi-norm equivalence

$$B(f) \ \asymp \ B_1(f) \ := \ \left\|\left\{ \omega_l(f, 2^{-k})_p \right\}\right\|_{l^\alpha_\theta} \ + \ \| f \|_{p,\mathbb{G}}.$$

We let the B-splines $N_{k,s}$ be defined by

$$N_{k,s}(x) := N_r\left(2^k x - s\right), \quad k, s \in \mathbb{Z}.$$

Let $\mathbb{G} = [a, b]$ be an interval with integers a, b. Let $D(\mathbb{G}, k) := \{ s \in \mathbb{Z} : a2^k - r < s < b 2^k \}$ be the set of s for which $N_{k,s}$ do not vanish identically on $\mathbb{G}$, and let Σ_k be the span of the B-splines $N_{k,s}, \ s \in D(\mathbb{G}, k)$. For each $f \in L_p(\mathbb{G})$, the error of the approximation of f by the the B-splines from Σ_k is given by

$$E_k(f)_p \ := \inf_{\varphi \in \Sigma_k} \| f - \varphi \|_{p,\mathbb{G}}.$$

For a function $f \in C^{r-1}(\mathbb{G})$, we define the operator P_k by

$$P_k(f, x) := \sum_{s \in D(\mathbb{G},k)} \alpha_{k,s}(f) N_{k,s}(x),$$

where

$$\alpha_{k,s}(f) := \alpha(\sigma_{2^k}(f), s).$$

and $\alpha(\sigma_{2^k}(f), s)$ is given by (14) with ξ_s the center of an interval $(2^{-k}j, 2^{-k}(j+1))$, $j \in \mathbb{Z}$, contained in $\operatorname{supp}(N_{k,s}) \cap \mathbb{G}$. It was proven in [12] that the operator P_k can be extended to a bounded linear operator from $L_p(\mathbb{G})$ into Σ_k. We denote this extension again by P_k.

Let

$$p_k(f) := P_k(f) - P_{k-1}(f) \quad \text{with} \quad P_{-1}(f) = 0.$$

Theorem 4 *Let $1 \le p \le \infty$, $0 < \theta \le \infty$, $\alpha > 1/p$, $r > \alpha$. Let $\mathbb{G} = [a, b]$ be an interval with integers a, b. Then for the Besov space $B^\alpha_{p,\theta}(\mathbb{G})$, the following quasi-norms are equivalent to the Besov quasi-norm $B(f)$:*

$$\begin{aligned} B_2(f) &:= \|\{f - P_k(f)\}\|_{l^\alpha_\theta(L_p(\mathbb{G}))} + \|f\|_{p,\mathbb{G}}, \\ B_3(f) &:= \|\{p_k(f)\}\|_{l^\alpha_\theta(L_p(\mathbb{G}))} \\ B_4(f) &:= \|\{E_k(f)_{p,\mathbb{G}}\}\|_{l^\alpha_\theta} + \|f\|_{p,\mathbb{G}}. \end{aligned}$$

This theorem is known as a particular case of a more general result (see [12]).

It is easy to verify that there are constants C, C' such that for each linear combination

$$g = \sum_{s \in D(\mathbb{G},k)} a_s N_{k,s}, \tag{37}$$

from Σ_k, we have

$$C\,\|g\|_{p,\mathbb{G}} \le \left(2^{-k} \sum_{s \in D(\mathbb{G},k)} |a_s|^p\right)^{1/p} \le C'\|g\|_{p,\mathbb{G}}. \tag{38}$$

Let

$$q_k(f) := Q_k(f) - Q_{k-1}(f) \quad \text{with} \quad Q_{-1}(f) := 0.$$

Theorem 5 *Let $1 \le p \le \infty$, $0 < \theta \le \infty$, $1 < \alpha < r$. Then for the Besov space $B^\alpha_{p,\theta}$, the following quasi-norms are equivalent to $B(f)$:*

$$\begin{aligned} B_5(f) &:= \|\{f - Q_k(f)\}\|_{l^\alpha_\theta(L_p)} + \|f\|_p, \\ B_6(f) &:= \|\{q_k(f)\}\|_{l^\alpha_\theta(L_p)}. \end{aligned}$$

Proof We first prove the inequality

$$B_5(f) \ll B_3(f) \tag{39}$$

for any $f \in B^\alpha_{p,\theta}$.

Let $f \in B^\alpha_{p,\theta}$. According to Theorem 4, f can be decomposed into the series

$$f = \sum_{k=0}^{\infty} p_k(f)$$

converging in quasi-norm $B_3(f)$ and in L_p-norm. We will assume that the B-splines $N_{k,s}$ in the related quasi-interpolant $P_k(f)$ are of order $r' > r$. Since $p_k(f) \in \mathbf{S}_k$, we have

$$p_k(f, x) \;=\; \sum_{s \in D(\mathbb{I},k)} f_{k,s} N_{k,s}(x), \quad x \in \mathbb{I},$$

for some coefficient functionals $f_{k,s}$. Hence, for each $x \in \mathbb{I}$

$$f(x) \;= \sum_{k=0}^{\infty} \sum_{s \in D(\mathbb{I},k)} f_{k,s} N_{k,s}(x). \tag{40}$$

The series converges in the L_p-norm. By (37) and (38) we also have

$$\left(\sum_{s \in D(\mathbb{I},k)} |f_{k,s}|^p \right)^{1/p} \asymp\; 2^{k/p} \|p_k(f)\|_p. \tag{41}$$

The expression in the right-hand side of (40) can be considered as an extension of f to the whole $\mathbb{R}$, which we denote by F. For an integer m, we define the function G_m on $\mathbb{R}$ by

$$G_m(x) \;:=\; F(x) + (-1)^{r+1} \int_{-\infty}^{\infty} \Delta^r_{hu}(F, x) N_r(u) du,$$

where $h := 2^{-m}$. Let g_m be the restriction of G_m on $\mathbb{I}$. We will use the functions g_m and $Q_m(g_m)$ for mediate approximations of f, based on the identity

$$f - Q_m(f) \;=\; (f - g_m) + (g_m - Q_m(g_m)) + (Q_m(f - g_m)). \tag{42}$$

Let us first estimate the norms $\|f - g_m\|_p$ and $\|g_m - Q_m(g_m)\|_p$. Notice that $\operatorname{supp}(F) = [-r', 2r'] \subset \operatorname{supp}(G_m) = [-r' - r, 2r' + r] =: \mathbb{G}$. By a standard technique we derive that

$$\|f - g_m\|_p \;\le \|F - G_m\|_{p,\mathbb{G}} \le\; r^r \omega(2^{-m}), \tag{43}$$

and

$$2^{-rm} \|g_m^{(r)}\|_p \;\le 2^{-rm} \|G_m^{(r)}\|_{p,\mathbb{G}} \;\le\; 2^r \omega(2^{-m}),$$

where we use the abbreviation: $\omega(2^{-m}) := \omega_r(F, 2^{-m})_{p,\mathbb{G}}$. By Theorem 3 we have

$$\|g_m - Q_m(g_m)\|_p \;\le C 2^{-rm} \|g_m^{(r)}\|_p.$$

Hence,

$$\|g_m - Q_m(g_m)\|_p \;\le\; C 2^r \omega(2^{-m}). \tag{44}$$

Further, let us estimate the norm $\|Q_m(f - g_m)\|_p$. We put

$$\phi_m(x) :=\; f(x) - g_m(x) =\; (-1)^r \int_{-\infty}^{\infty} \Delta^r_{hu}(F, x) N_r(u) du, \; x \in \mathbb{I}. \tag{45}$$

By definition we have

$$Q_m(\phi_m, x) \;=\; \sum_{s=-\infty}^{\infty} \Lambda^h(\bar{\phi}_m, s) M_{m,s}(x).$$

By replacing ϕ_m by the integral in (45) and f by the series in (40) in the right side of the last equation, we decompose $Q_m(\phi_m)$ into the series:

$$Q_m(\phi_m) \ = \sum_k q_k,$$

where

$$\begin{aligned} q_k(x) \ &:= \sum_{s\in D(\mathbb{I},k)} \sum_{l=-\infty}^{\infty} f_{k,s} \Lambda^h(\bar{\sigma}_{k,s}, l) M_{m,l}(x), \\ \sigma_{k,s}(x) \ &:= \ (-1)^r \int_{-\infty}^{\infty} \Delta_{hu}^r(N_{k,s}, x) N_r(u) du. \end{aligned} \tag{46}$$

We have

$$\|Q_m(f - g_m)\|_p \ = \ \|Q_m(\phi_m)\|_p \ \le \sum_{k=0}^{\infty} \|q_k\|_p. \tag{47}$$

We will estimate the norm $\|q_k\|_p$. From (46) we obtain for each $x \in \mathbb{I}$

$$\begin{aligned} q_k(x) &= \sum_{s\in D(\mathbb{I},k)} \ \sum_{|l-x/h|\le r/2} \ \sum_{|j-l|\le J} f_{k,s} \lambda_{l-j} \bar{\sigma}_{k,s}(hj) M_{m,l}(x) \\ &= \sum_{|l-x/h|\le r/2} M_{m,l}(x) \sum_{|j-l|\le J} \lambda_{l-j} \sum_{s\in D(\mathbb{I},k)} f_{k,s} \bar{\sigma}_{k,s}(hj). \end{aligned} \tag{48}$$

By Hölder's inequality and (41) we have

$$\begin{aligned} \left| \sum_{s\in D(\mathbb{I},k)} f_{k,s} \bar{\sigma}_{k,s}(hj) \right| &\le \left(\sum_{s\in D(\mathbb{I},k)} |f_{k,s}|^p \right)^{1/p} \left(\sum_{s\in D(\mathbb{I},k)} |\bar{\sigma}_{k,s}(hj)|^{p'} \right)^{1/p'} \\ &\ll \ 2^{k/p} \|p_k(f)\|_p \left(\sum_{s\in D(\mathbb{I},k)} |\bar{\sigma}_{k,s}(hj)|^{p'} \right)^{1/p'}. \end{aligned}$$

Therefore, from (48) we receive

$$|q_k(x)| \ll 2^{k/p} \|p_k(f)\|_p \sum_{|l-x/h|\le r/2} M_{m,l}(x) \sum_{|j-l|\le J} |\lambda_{l-j}| \left(\sum_{s\in D(\mathbb{I},k)} |\bar{\sigma}_{k,s}(hj)|^{p'} \right)^{1/p'}. \tag{49}$$

Obviously, the number of $M_{m,l}(x)$ with the restriction $|l - x/h| \le r/2$, does not exceed r. Further, the number of the nonzero $\bar{\sigma}_{k,s}(hj)$, satisfying the condition

$$|l - x/h| \le r/2, \ |j - l| \le J, \ \text{and } s \in D(\mathbb{I}, k), \tag{50}$$

does not exceed

$$A(h, k) := r^2 2^k h + r + 1$$

for each j. Indeed, if either $2^k(hj+r^2h)-s \le 0$ or $2^k hj - s \ge r+1$, then $\Delta_{hu}^r(N_{k,s}, hj) = 0$, and consequently, by (46) $\bar{\sigma}_{k,s}(hj) = 0$. Hence, the estimation (49) can be continued as

$$|q_k(x)| \ \ll 2^{k/p} \|p_k(f)\|_p \, r A^{1/p'}(h, k) \max_{l,j,s} \{M_{m,l}(x) |\bar{\sigma}_{k,s}(hj)|\} \|\Lambda\| \tag{51}$$

where the max is taken over all l, j, s with the restriction (50). From the inequality

$$\left|\Delta_{hu}^r(N_{k,s}, hj)\right| \le 2^r \|N_{k,s}\|_\infty \le 2^r$$

for every h, u, k, s, j, and (46) we obtain

$$\max_{s \in D(\mathbb{I},k)} |\bar{\sigma}_{k,s}(hj)| \le (r+1)2^r. \tag{52}$$

On the other hand, since $N_{k,s}$ are B-splines of order $r' > r$, they are in W^r_∞ and have the rth derivative not exceeding $2^r 2^{rk}$. Hence we have

$$\left|\Delta_{hu}^r(N_{k,s}, hj)\right| \le 2^r (h2^k)^r,$$

and consequently,

$$\max_{s \in D(\mathbb{I},k)} |\bar{\sigma}_{k,s}(hj)| \le (r+1)2^r (h2^k)^r.$$

Combining the last inequality and (52) gives for each j

$$\max_{s \in D(\mathbb{I},k)} |\bar{\sigma}_{k,s}(hj)| \le (r+1)2^r \min\{1, (h2^k)^r\},$$

and therefore, the max in (51) can be estimated by

$$\max_{l,j,s} \{\{J_{m,l}(x)| f_{k,s}\bar{\sigma}_{k,s}(hj)|\} \ll \min\{1, (h2^k)^r\} \max_l M_{m,l}(x), \tag{53}$$

where the max in the right side is taken over all l such that $|l - x/h| \le r/2$. Notice that the norm $\|\Lambda\|$ is an absolute constant and the quantity $A(h,k)$ does not exceed $h2^k$ multiplied by an absolute constant. Hence, by (51) and (53) we have

$$|q_k(x)| \ll 2^{k/p}(h2^k)^{1/p'} \min\{1, (h2^k)^r\} \|p_k(f)\|_p \max_l \{M_r(h^{-1}x - l).$$

From this inequality we derive

$$\begin{aligned}
\|q_k\|_p^p &= \int_0^1 |q_k(x)|^p dx \\
&\ll 2^k (h2^k)^{p/p'} \min\{1, (h2^k)^{rp}\} \|p_k(f)\|_p^p \int_0^1 \left(\max_l \ M_r(h^{-1}x - l)\right)^p dx \\
&\ll 2^k (h2^k)^{p/p'} \min\{1, (h2^k)^{rp}\} \|p_k(f)\|_p^p h \int_0^{1/h} \left(\max_l \ M_r(y - l)\right)^p dy \\
&\ll (h2^k)^p \min\{1, (h2^k)^{rp}\} \|p_k(f)\|_p^p.
\end{aligned}$$

Thus, we have obtained the following estimate for the norms $\|q_k\|_p$:

$$\|q_k\|_p \ll (h2^k) \min\{1, (h2^k)^r\} \|p_k(f)\|_p,$$

which together with (47) implies

$$\|Q_m(f - g_m)\|_p \ll \sum_{k=0}^{\infty} (h2^k) \min\{1, (h2^k)^r\} \|p_k(f)\|_p.$$

The last inequality yields

$$\|2^m Q_m(f-g_m)\|_p \ll \left(\sum_{k=0}^{m} 2^{r(k-m)}\|2^k p_k(f)\|_p + \sum_{k=m+1}^{\infty}\|2^k p_k(f)\|_p\right).$$

Since $\alpha > 1$, applying the discrete Hardy inequality (35)–(36) gives

$$\|\{2^k Q_k(f-g_k)\}\|_{l_\theta^{\alpha-1}(L_p)} \ll \|\{2^k p_k(f)\}\|_{l_\theta^{\alpha-1}(L_p)},$$

or equivalently,

$$\|\{Q_k(f-g_k)\}\|_{l_\theta^{\alpha}(L_p)} \ll \|\{p_k(f)\}\|_{l_\theta^{\alpha}(L_p)} = B_3(f). \tag{54}$$

In a way similar to the proof of (54), one can derive that

$$B_1(F,\mathbb{G}) \ll B_3(F,\mathbb{G}) \ll B_3(f).$$

Consequently, from the inequalities (43) and (44) we can see that

$$\begin{aligned}\|\{f-g_k\}\|_{l_\theta^{\alpha}(L_p)} &\ll \|\{\omega(2^{-k})\}\|_{l_\theta^{\alpha}}\\ &= B_1(F,\mathbb{G}) \ll B_3(f),\end{aligned} \tag{55}$$

and

$$\begin{aligned}\|\{g_k - Q_k(g_k)\}\|_{l_\theta^{\alpha}(L_p)} &\ll \|\{\omega(2^{-k})\}\|_{l_\theta^{\alpha}}\\ &= B_1(F,\mathbb{G}) \ll B_3(f).\end{aligned} \tag{56}$$

We now are in position to prove the inequality (39). From the identity (42) and the inequality

$$\|\{u_k+v_k\}\|_{l_\theta^{\alpha}(L_p)} \le \|\{u_k\}\|_{l_\theta^{\alpha}(L_p)} + \|\{v_k\}\|_{l_\theta^{\alpha}(L_p)}$$

for the case $1 \le \theta \le \infty$, and the inequality

$$\|\{u_k+v_k\}\|_{l_\theta^{\alpha}(L_p)} \le 2^{1/\theta}\big(\|\{u_k\}\|_{l_\theta^{\alpha}(L_p)} + \|\{v_k\}\|_{l_\theta^{\alpha}(L_p)}\big)$$

for the case $0 < \theta < 1$, we obtain

$$\begin{aligned}\|\{f-Q_k(f)\}\|_{l_\theta^{\alpha}(L_p)} &\ll \|\{f-g_k\}\|_{l_\theta^{\alpha}(L_p)} + \|\{g_k-Q_k(g_k)\}\|_{l_\theta^{\alpha}(L_p)}\\ &\quad + \|\{Q_k(f-g_k)\}\|_{l_\theta^{\alpha}(L_p)}.\end{aligned} \tag{57}$$

Now we can see that (39) is true by (54)–(57).

Further, since $\|q_k(f)\|_p \le \|f-Q_k(f)\|_p + \|f-Q_{k-1}(f)\|_p$, we have

$$B_6(f) \le 2B_5(f). \tag{58}$$

On the other hand, due to the inequality

$$\|f-Q_m(f)\|_p \le \sum_{k=m+1}^{\infty}\|q_k(f)\|_p,$$

we receive by the discrete Hardy inequality (35)–(36)

$$B_5(f) \le B_6(f). \tag{59}$$

Finally, by definition

$$B_4(f) \leq B_5(f). \tag{60}$$

Combining (39) and (58)–(60) completes the proof of Theorem 5. □

We will deduce from Theorem 5 a quasi-interpolant wavelet representation of a function in $B^{\alpha}_{p,\theta}$ in terms of the B-splines $M_{k,s} \in \mathbf{M}$, and a associated discrete equivalent quasi-norm for the functional coefficients. We assume that the order of the B-splines $M_{k,s}$ is $r = 2\rho$ an even natural number.

Let

$$J(k) := \{s \in \mathbb{Z} : -\rho < s < 2^k + \rho\}$$

be the set of s for which $M_{k,s}$ do not vanish identically on $\mathbb{I}$. We have by (19)

$$\begin{aligned} q_k(f,x) &= Q_k(f,x) - Q_{k-1}(f,x) \\ &= \sum_{s \in J(k)} a_{k,s}(f) M_{k,s}(x) - \sum_{s \in J(k-1)} a_{k-1,s}(f) M_{k-1,s}(x), \end{aligned} \tag{61}$$

From the equation (10) it follows that

$$M_{k-1,s}(x) = 2^{-r+1} \sum_{s'=0}^{r} \binom{r}{s'} M_{k,s+s'-\rho}(x).$$

Hence, we get for each $x \in \mathbb{I}$

$$\begin{aligned} \sum_{s \in J(k-1)} a_{k-1,s}(f) M_{k-1,s}(x) &= 2^{-r+1} \sum_{s \in J(k-1)} a_{k-1,s}(f) \sum_{s'=0}^{r} \binom{r}{s'} M_{k,s+s'-\rho}(x) \\ &= \sum_{s \in J(k)} a'_{k,s}(f) M_{k,s}(x), \end{aligned}$$

where

$$a'_{k,s}(f) := 2^{-r+1} \sum_{j=s+\rho}^{s+3\rho} \binom{r}{j-s-\rho} a_{k-1,j}(f).$$

The last equation and (61) give

$$q_k(f,x) = \sum_{s \in J(k)} c_{k,s}(f) M_{k,s}(x), \tag{62}$$

where for $s \in J(k)$

$$c_{k,s}(f) := \begin{cases} a_{k,s}(f), & \text{if } 2^{k-1} + \rho \leq s < 2^k + \rho, \\ a_{k,s}(f) - a'_{k,s}(f), & \text{if } -\rho < s < 2^{k-1} + \rho. \end{cases} \tag{63}$$

Let $\mathbf{M}_k := \{ M_{k,s} \}_{s \in J(k)}$ and Σ^*_k be the space spanned by $\mathbf{M}_k$. If $1 \leq q \leq \infty$, for all non-negative integers k and all functions

$$g = \sum_{s \in J(k)} a_s M_{k,s}$$

from Σ_k^*, there is the norm equivalence

$$\|g\|_q \asymp 2^{-k/q}\|\{a_s\}\|_q, \tag{64}$$

where

$$\|\{a_s\}\|_p := \left(\sum_{s\in J(k)} |a_s|^p\right)^{1/p}.$$

From the last relation, (62) and Theorem 5 we obtain

Corollary 1 *Under the assumptions of Theorem 5 let $r = 2\rho$ be an even natural number. A function f on $\mathbb{I}$ belongs to the Besov space $B^\alpha_{p,\theta}$ if and only if f has a quasi-interpolant wavelet representation*

$$f = \sum_{k=0}^\infty q_k(f) = \sum_{k=0}^\infty \sum_{s\in J(k)} c_{k,s}(f) M_{k,s} \tag{65}$$

with the convergence in the space $B^\alpha_{p,\theta}$, and in addition the quasi-norm of the Besov space $B(f)$ is equivalent to the discrete quasi-norm

$$B_7(f) := \left(\sum_{k=0}^\infty \left(2^{(\alpha-1/p)k}\|\{c_{k,s}(f)\}\|_p\right)^\theta\right)^{1/\theta}.$$

Remark From (20) and (63) we can see that for each pair k, s the coefficient $c_{k,s}(f)$ in the decomposition (65) is a function of the values $f(2^{-k}(s-j))$, and $f(2^{-k+1}(s'-j))$, $|j| \le J$, $s' = s-\rho, s-\rho+1, ..., s+\rho$. The number of these values does not exceed $2J + r$.

3 Non-linear sampling recovery of functions

3.1

Before constructing the non-linear sampling recovery methods based on quasi-interpolant wavelet representations, we will briefly consider a linear sampling recovery method using a quasi-interpolant Q_k for $C(\mathbb{I})$, given in (19)–(20). We will show that it is a linear sampling method of recovery with nice approximation and local properties. In this section, we assume that the order of the B-splines $M_{k,s}$ is $r = 2\rho$ an even natural number.

We have

$$Q_k(f, x) := \sum_{s\in J(k)} a_{k,s}(f) M_{k,s}(x) \tag{66}$$

where we recall that $J(k) := \{s \in \mathbb{Z} : -\rho < s < \rho 2^k + \rho\}$ is the set of s for which $M_{k,s}$ do not vanish identically on $\mathbb{I}$, and the coefficients $a_{k,s}(f)$ are given in (20).

The formula (66) defines a linear sampling method of recovery of a function f from its sampled values $f(2^{-k}j)$, $j \in Z(k)$, where

$$Z(k) := \{s \in \mathbb{Z} : -J + \rho < s < 2^k + J + \rho\}.$$

The number of sampled values is $|Z(k)|$ which does not exceed $2^k + 2J + r + 1$.

As mentioned above, for each pair k, s the coefficient $a_{k,s}(f)$ is a linear combination of the values $f(2^{-k}(s-j))$, $|j| \le J$, and maybe, $f(2^{-k}j)$ for $j = 0, 1, ..., r-1$ or $j = 2^k - r + 2, 2^k - r + 3, ..., 2^k$, if the point $2^{-k}s$ is near to the ends 0 or 1 of the interval $\mathbb{I}$, respectively. Moreover, the number of these values does not exceed the $2J + r$ and not depend on neither functions f and nor k.

It is easy to see that for a given point $x \in \mathbb{I}$, we have

$$Q_k(f, x) = \sum_{|2^{-k}s - x| < 2^{-k}\rho} a_{k,s}(f) M_{k,s}(x).$$

Hence, the points $2^{-k}j$, $j \in Z(k)$, at which the sampled values are taken in the linear sampling method (19) for recovering $f(x)$, are in the neighborhood of x

$$U(x) := \left\{y \in \mathbb{I} : |y - x| < 2^{-k}(J + \rho)\right\}$$

whose size does not depend on x, and is 2^{-k} multiplied by an absolute constant. The number of these points does not exceed $2J + r$. This shows that the linear sampling recovery method (19) possesses a good local property.

We recall that $U^\alpha_{p,\theta}$ denote the unit ball in $B^\alpha_{p,\theta}$. For $1 \le p, q \le \infty$ and $\alpha > (1/p - 1/q)_+$, the space $B^\alpha_{p,\theta}$ is embedded into the space $B^{\alpha - (1/p - 1/q)_+}_{q,\theta}$, that is,

$$U^\alpha_{p,\theta} \subset \mu U^{\alpha - (1/p - 1/q)_+}_{q,\theta}$$

with a multiplier μ. Hence, by Theorem 5 we obtain the following estimates of the error for the linear sampling recovery method $Q_k(f)$ in (19):

Corollary 2 *Under the assumptions of Theorem* 5 *let $r = 2\rho$ be an even natural number. Then there is the inequality*

$$\sup_{f \in U^\alpha_{p,\theta}} \|f - Q_k(f)\|_q \le C 2^{-(\alpha - (1/p - 1/q)_+)k}, \tag{67}$$

and the number of sampled values of a function in $Q_k(f)$ does not exceed $\lambda 2^k$ with some absolute constants C and λ. Moreover, the linear sampling method $Q_k(f)$ with $n \asymp \lambda 2^k \le n$, is asymptotically optimal for $\gamma_n(U^\alpha_{p,\theta})_q$ and $\varrho_n(U^\alpha_{p,\theta})_q$.

3.2

Let us first establish the upper bound of $\nu_n(U^\alpha_{p,\theta}, \mathbf{M})_q$ in (6) of Theorem 2 for the most difficult and interesting case where $1 \le p < q \le \infty$. In this case a linear sampling recovery method does not work and therefore, we should construct a non-linear one.

Recall that

$$\mathbf{M} := \bigcup_{k=0}^{\infty} \mathbf{M}_k = \{M_{k,s} : (k,s) \in \mathbf{K}^*\},$$

is the family of B-spline wavelets $M_{k,s}$ which do not vanish identically on $\mathbb{I}$, where

$$\mathbf{K}^* := \{(k,s) : s \in J(k), \quad k = 0, 1, 2, ...\}.$$

Let

$$\mathbf{D}^* := \left\{\xi_{k,s} = 2^{-k}s : (k,s) \in \mathbf{K}^*\right\}$$

be the set of dyadic points indexed by $\mathbf{K}^*$.

For each function $f \in U^\alpha_{p,\theta}$, we will choose a triple of a sequence $\xi = \{\xi^j\}_{j=1}^n$ of n points in $\mathbf{D}^*$, a sequence $a = \{a_j\}_{j=1}^n$ of n functions defined on $\mathbb{R}^n$ and a sequence $\{M_{k_j,s_j}\}_{j=1}^n$ of n B-spline wavelets from $\mathbf{M}$. This choice will define a non-linear sampling method of recovery of f from its values $f(\xi^s)$, $s = 1, 2, ..., n$ by

$$S_n(f,x) := \sum_{j=1}^n a_j(f(\xi^1), ..., f(\xi^n) M_{k_j,s_j}(x).$$

To establish the upper bound of (6) for the case where $1 \le p < q \le \infty$, we will show that such a S_n can be explicitly constructed so that there hold the inequalities

$$\nu_n\left(U^\alpha_{p,\theta}, \mathbf{M}\right)_q \le \sup_{f \in U^\alpha_{p,\theta}} \|f - S_n(f)\|_q \ll n^{-\alpha}.$$

From embedding theorems (see [1]) it follows that the space $B^\alpha_{p,\theta}$ can be considered as a subspace of the largest space $B^\alpha_{p,\infty}$. Hence, it is sufficient to construct S_n for $U := U^\alpha_{p,\infty}$. It will be constructed on the basic of the following representation of functions from U.

Corollary 1 says that for arbitrary positive integer m, a function $f \in U$ can be represented by a series

$$f = Q_m(f) + \sum_{k>m} q_k(f) \tag{68}$$

with the functions

$$q_k(f) := \sum_{s \in J(k)} c_{k,s}(f) M_{k,s}(x) \tag{69}$$

from the subspace Σ^*_k and $c_{k,s}(f)$ given in (63). Moreover, q_k satisfy the condition

$$\|q_k(f)\|_p \asymp 2^{-k/p}\|\{c_{k,s}(f)\}\|_p, \ll 2^{-\alpha k}, \quad k = m+1, m+2, ... \tag{70}$$

Our strategy of using the representation (68)–(69) for construction of a recovery approximant $S_n(f)$ is as follows. We will choose two appropriate integers $\bar{k}$ and k^*. Then we take the quasi-interpolant $Q_{\bar{k}}(f)$ as the main linear part of $S_n(f)$. The non-linear part is constructed as a sum of greedy algorithms G_k with regard to the representations (69) for non-linear approximation of each component function $q_k(f)$ in the subspaces $\Sigma_k^*,\ k = 0, 1, \ldots$ for $\bar{k} < k \le k^*$.

Let $m_k := |J(k)| = 2^k + 2\rho - 1$ be the number of elements of $\mathbf{M}_k$. We define a integer $\bar{k}$ from the condition

$$(2J + r)m_{\bar{k}+2} \le n < (2J + r)m_{\bar{k}+3}. \tag{71}$$

Next, we will select an integer k^* and a sequence of non-negative integers $\{n_k\}_{k=\bar{k}+1}^{k^*}$ such that

$$(2J + r)m_{\bar{k}} + (2J + r)\sum_{k=\bar{k}+1}^{k^*} n_k \ \le n. \tag{72}$$

To do this we fix a number ε satisfying the inequalities

$$0 < \varepsilon < (\alpha - \delta)/\delta, \tag{73}$$

where $0 < \delta := 1/p - 1/q < \alpha$. Then an appropriate selection of k^* and $\{n_k\}_{k=\bar{k}+1}^{k^*}$ is

$$k^* := \left[\varepsilon^{-1}\log(\lambda n)\right] + \bar{k} + 1. \tag{74}$$

and

$$n_k \ = \left[\lambda n 2^{-\varepsilon(k-\bar{k})}\right], \quad k = \bar{k} + 1, \bar{k} + 2, \ldots, k^*, \tag{75}$$

with a positive constant λ chosen such that there holds the inequalities (72) and $n_k < m_k$. Here $[t]$ denotes the integer part of $t \in \mathbb{R}$.

Thus, the integers $\bar{k}$ and k^* as well the sequence $\{n_k\}_{k=\bar{k}+1}^{k^*}$ have selected. We are now in position to construct a non-linear sampling recovery method which will give the upper bound of (6) for the case where $1 \le p < q \le \infty$.

For a non-linear approximation of $q_k(f)$ we define the greedy algorithms G_k with regard to the decomposition (69) in the subspace Σ_k^* as follows. We reorder the indexes $s \in J(k)$ as $\{s_j\}_{j=1}^{m_k}$ so that

$$|c_{k,s_1}(f)| \ge |c_{k,s_2}(f)| \ge \cdots |c_{k,s_n}(f)| \ge \cdots |c_{k,m_k}(f)|,$$

and then take the first largest n term for a non-linear approximation of $q_k(f)$ by forming the linear combination

$$G_k(q_k(f)) := \sum_{j=1}^{n_k} c_{k,s_j}(f) M_{k,s_j}. \tag{76}$$

The worst case error of this approximation for all $f \in U$ is

$$\sup_{f \in U} \|q_k(f) - G_k(q_k(f))\|_q \ll 2^{-\alpha k} 2^{\delta k} n_k^{-\delta}. \tag{77}$$

Indeed, we have (see [15])

$$\left(\sum_{j=n_k+1}^{m_k} |c_{k,s_j}(f)|^q\right)^{1/q} \le n_k^{-\delta} \|\{c_{k,s}(f)\}\|_p. \tag{78}$$

By the norm equivalence (64) and (70), we derive

$$\|q_k(f) - G_k(q_k)\|_q = \|\sum_{j=n_k+1}^{m_k} c_{k,s_j}(f) M_{k,s_j}\|_q$$

$$\asymp 2^{-k/q} \left(\sum_{j=n_k+1}^{m_k} |c_{k,s_j}(f)|^q\right)^{1/q} \ll 2^{-k/q} n_k^{-\delta} \|\{c_{k,s}(f)\}\|_p$$

$$\ll 2^{-\alpha k} 2^{\delta k} n_k^{-\delta}.$$

Thus, (77) has been verified.

We define the non-linear operator S_n^* by

$$S_n^*(f,x) := Q_{\bar{k}}(f,x) + G_n^*(f,x)$$

where

$$G_n^*(f,x) := \sum_{k=\bar{k}+1}^{k^*} G_k(q_k(f),x).$$

By (66) and (76) we have

$$S_n^*(f,x) = \sum_{s \in J(\bar{k})} a_{k,s}(f) M_{k,s}(x) + \sum_{k=\bar{k}+1}^{k^*} \sum_{j=1}^{n_k} c_{k,s_j}(f) M_{k,s_j}(x). \tag{79}$$

Thus, S_n^* is a sum of the linear quasi-interpolant $Q_{\bar{k}}$ and non-linear operator G_n^*. The last one is the sum of the greedy algorithms G_k in the subspaces Σ_k^*. Since the number of the sampled values determining each coefficient $a_{k,s}(f)$ or $c_{k,s}(f)$ in (79) does not exceed $2J+r$, by (72) the total number of sampled values determining all the coefficients $a_{k,s}(f)$ and $c_{k,s}(f)$ in (79) does not exceed n and consequently, we can consider $c_{k,s}(f)$ and $a_{k,s}(f)$ as a function of values of f at certain n points. Further, also by (72) the number of B-spline wavelets $M_{k,s} \in \mathbf{M}$ in (79) does not exceed n. This means that S_n^* is a non-linear sampling recovery method of the form (5) with regard to the family of B-spline wavelets $\mathbf{M}$ and with the sampled values at points in $\mathbf{D}^*$.

Since

$$S_n^*(f) = Q_{\bar{k}}(f) + \sum_{k=\bar{k}+1}^{k^*} G_k(q_k(f)),$$

we obtain by (68)

$$f - S_n^*(f) = \sum_{k=\bar{k}+1}^{k^*} \{q_k - G_k(q_k(f))\} + \sum_{k>k^*} q_k(f).$$

From (70), (64) and (77), one can derive that for each function $f \in U$

$$
\begin{aligned}
\|f - S_n^*(f)\|_q &\le \sum_{k=\bar{k}+1}^{k^*} \|q_k(f) - G_k(q_k(f))\|_q + \sum_{k>k^*} \|q_k(f)\|_q \\
&\ll \sum_{k=\bar{k}+1}^{k^*} 2^{-\alpha k} 2^{\delta k} n_k^{-\delta} + \sum_{k>k^*} 2^{-\alpha k} 2^{\delta k}.
\end{aligned}
$$

By using (71), (73)–(74) and the inequalities $\alpha > 1 \ge \delta$, we can continue the last inequality as follows

$$
\begin{aligned}
&\ll n^{-\delta} 2^{-(\alpha-\delta)\bar{k}} \sum_{k=\bar{k}+1}^{k^*} 2^{-(\alpha-\delta+\delta\varepsilon)(k-\bar{k})} + 2^{-(\alpha-\delta)k^*} \sum_{k>k^*} 2^{-(\alpha-\delta)(k-k^*)} \\
&\ll n^{-\delta} 2^{-(\alpha-\delta)\bar{k}} + 2^{-(\alpha-\delta)k^*} \\
&\ll n^{-\alpha}.
\end{aligned}
$$

Thus, we have proven the following theorem.

Theorem 6 *The non-linear sampling recovery method S_n^* given in (79) is of the form (5) for the family $\Phi = \mathbf{M}$. Moreover, it gives the upper bound of (6) in Theorem 2 for the case where $1 \le p < q \le \infty$. Namely, there are the following upper estimates*

$$
\nu_n\left(U^\alpha_{p,\theta}, \mathbf{M}\right)_q \le \sup_{f \in U^\alpha_{p,\theta}} \|f - S_n^*(f)\|_q \ll n^{-\alpha}.
$$

3.3

Proof of Theorem 2 The upper bound of (6) for the case where $1 \le p < q \le \infty$ is in Theorem 6 and can be obtained from (67) in Corollary 2 by applying the linear sampling recovery method (19) for the case where $1 \le q \le p \le \infty$. Let us prove the lower bound.

We define the quantity of n-term approximation $\sigma_n\left(U^\alpha_{p,\theta}, \mathbf{M}\right)_q$ of $U^\alpha_{p,\theta}$ in the L_q-norm with regard to $\mathbf{M}$ by

$$
\sigma_n\left(U^\alpha_{p,\theta}, \mathbf{M}\right)_q := \sup_{f \in U^\alpha_{p,\theta}} \inf_{\varphi \in \Sigma_n(\mathbf{M})} \|f - \varphi\|_q
$$

as the worst case error of the approximation of $f \in U^\alpha_{p,\theta}$ in the L_q-norm by elements from the set

$$
\Sigma_n(\mathbf{M}) := \left\{ \varphi = \sum_{j=1}^{n} a_j M_{k_j,s_j} : (k_j, s_j) \in \mathbf{K}^* \right\}.
$$

Obviously,

$$
\nu_n\left(U^\alpha_{p,\theta}, \mathbf{M}\right)_q \ge \sigma_n\left(U^\alpha_{p,\theta}, \mathbf{M}\right)_q.
$$

Then the lower bound of (6) in Theorem 2 immediately follows from the last inequality and the lower bound for $\sigma_n(U^\alpha_{p,\theta}, \mathbf{M})_q$ established in [14, 15]:

$$\sigma_n \left(U^\alpha_{p,\theta}, \mathbf{M}\right)_q \gg n^{-\alpha}.$$

□

3.4

Some remarks As mentioned in Introduction the investigation of computation complexity and cost of asymptotically optimal non-linear sampling recovery methods (algorithms) for ν_n in comparing with asymptotically optimal linear methods and non-linear methods for λ_n, γ_n and ρ_n, is of great interest.

Theorem 2 is proven for univariate functions with the restrictions $1 \le p, q \le \infty$ and $\alpha > 1$. It is natural to extend it to the case $0 < p, q \le \infty$ and $\alpha \ge 1/p$, and generalize it for multivariate functions on the cube $[0, 1]^d$ or more general, on a Lipschitz domain as in [20] for the quantities $\lambda_n(W)_q$ and $\rho_n(W)_q$.

Unlike $\lambda_n(W)_q$, $\rho_n(W)_q$ and $\gamma_n(W)_q$ and depending on the family Φ, the quantity $\nu_n(W, \Phi)_q$ is not absolute in the sense of n-widths or optimal methods. Similarly an approach to the quantity of n-term approximation $\sigma_n(W, \Phi)_q$ (see [24] for details), one can consider the quantity

$$\nu_n(W, \mathcal{B})_q := \inf_{\Phi \in \mathcal{B}} \nu_n(W, \Phi)_q,$$

for a collection $\mathcal{B}$ of families Φ with a given property, and find reasonable restrictions to have nontrivial lower bounds of $\nu_n(W, \mathcal{B})_q$.

Some of these problems will be discussed in a forthcoming paper.

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References

1. Besov, O.V., Il'in, V.P., Nikol'skii, S.M.: Integral Representations of Functions and Embedding Theorems. V.H. Winston, Washington, D.C. (1978)
2. Butzer, P.L., Schmidt, M.: Central factorial numbers and their role in finite difference calculus and approximation. In: Szabados, J., Tandory, K. (eds.) Approximation Theory. CMSB, vol. 58. NH, pp. 127–150 (1990)
3. Butzer, P.L., Schmidt, M., Stark, E.L., Vogt, L.: Central factorial numbers, their main properties and some applications. Numer. Funct. Anal. Optim. **10**, 419–488 (1989)
4. Chui, C.K.: An Introduction to Wavelets. Academic, New York (1992)
5. Chui, C.K., Diamond, H.: A natural formulation of quasi-interpolation by multivariate splines. Proc. Amer. Math. Soc. **99**, 643–646 (1987)
6. de Boor, C., Fix, G.J.: Spline approximation by quasiinterpolants. J. Approx. Theory **8**, 19–45 (1973)
7. Dahlke, S., Novak, E., Sickel, W.: Optimal approximation of elliptic problems by linear and nonlinear mappings I. Complexity **22**, 29–49 (2006)
8. Dahlke, S., Novak, E., Sickel, W.: Optimal approximation of elliptic problems by linear and nonlinear mappings II. Complexity **22**, 549–603 (2006)
9. de Boor, C., Höllig, K., Riemenschneider, S.: Box Spline. Springer, Berlin (1993)
10. DeVore, R.A.: Nonlinear approximation. Acta Numer. **7**, 51–150 (1998)
11. DeVore, R.A., Lorentz, G.G.: Constructive Approximation. Springer, New York (1993)

12. DeVore, R.A., Popov, V.A.: Interpolation of Besov spaces. Trans. Amer. Math. Soc. **305**, 397–413 (1988)
13. Dung, D.: On interpolation recovery for periodic functions. In: Koshi, S. (ed.) Functional Analysis and Related Topics, pp. 224–233. World Scientific, Singapore (1991)
14. Dung, D.: On nonlinear n-widths and n-term approximation. Vietnam Math. J. **26**, 165–176 (1998)
15. Dung, D.: Continuous algorithms in n-term approximation and non-linear n-widths. J. Approx. Theory **102**, 217–242 (2000)
16. Dung, D.: Using quasi-interpolant wavelet representations for non-linear sampling recovery. Vietnam Math. J. **35**, 331–338 (2007)
17. Kydryatsev, S.N.: The best accuracy of reconstruction of finitely smooth functions from their values at a given number of points. Izv. Math. **62**, 19–53 (1998)
18. Nikol'skii, S.: Approximation of Functions of Several Variables and Embedding Theorems. Springer, Berlin (1975)
19. Novak, E.: Deterministic and Stochastic Error Bounds in Numerical Analysis. Lecture Notes in Mathematics, vol. 1349. Springer, Berlin (1988)
20. Novak, E., Triebel, H.: Function spaces in Lipschitz domains and optimal rates of convergence for sampling. Constr. Approx. **23**, 325–350 (2006)
21. Sablonnière, P.: Recent progress on univariate and multivariate polynomial and spline quasi-interpolants. In: de Bruin, M.G., Mache, D.H., Szabados, J. (eds.) Trends and applications in constructive approximation. Int. Ser. Numer. Math. **151**, 229–245 (2005)
22. Schoenberg, I.J.: Cardinal spline interpolation. CBMS-NSF Regional Conferences Series in Applied Mathematics, vol. 12. SIAM, Philadelphia (1973)
23. Temlyakov, V.: Approximation of Periodic Functions. Nova Science, New York (1993)
24. Temlyakov, V.: Nonlinear methods of approximation. Fund. Comput. Math. **3**, 33–107 (2003)