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Stability criteria for exact and discrete solutions of neutral multidelay-integro-differential equations

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Abstract This paper deals with the asymptotic stability of exact and discrete solutions of neutral multidelay-integro-differential equations. Sufficient conditions are derived that guarantee the asymptotic stability of the exact solutions. Adaptations of classical Runge–Kutta and linear multistep methods are suggested for solving such systems with commensurate delays. Stability criteria are constructed for the asymptotic stability of these numerical methods and compared to the stability criteria derived for the continuous problem. It is found that, under suitable conditions, these two classes of numerical methods retain the stability of the continuous systems. Some numerical examples are given that illustrate the theoretical results.

Keywords Asymptotic stability · Neutral multidelay-integro-differential equation · Runge–Kutta method · Linear multistep method

Mathematics Subject Classifications (2000) 65R20 · 65Q05

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1 Introduction

Consider the following complex p-dimensional system of neutral multidelay-integro-differential equations (NMIDEs) with constant delays $\tau_q > 0$,

$$\begin{cases} \dfrac{d}{dt}\left[y(t) - \displaystyle\sum_{q=1}^{d} N_q y(t-\tau_q) \right] = Ly(t) + \displaystyle\sum_{q=1}^{d} M_q y(t-\tau_q) + \sum_{q=1}^{d} Q_q \int_{t-\tau_q}^{t} y(\theta)d\theta, \ t \geq t_0 \\ y(t) = \varphi(t), \quad t \in [t_0 - \hat{\tau}, t_0], \end{cases} \tag{1.1}$$

with matrices $L, M_q, N_q, Q_q \in \mathbb{C}^{p\times p}$ and $\hat{\tau} = \max_{1\leq q\leq d}\{\tau_q\}$. Function $\varphi(t)$ is a given p-dimensional vector-valued function, and $y(t) \in \mathbb{C}^p$ is unknown for $t > t_0$. Such equations, or special cases of these equations, arise in practical applications, e.g., in visco-elasticity, control theory, epidemiology, and population dynamics (cf. [1, 2]).

In the past, many researchers have studied the stability of special cases of (1.1). These studies usually concentrated on non-distributed delay equations, i.e., the case $Q_q = 0$, often with also $N_q = 0$, for $q = 1, 2, \ldots, d$. Their results have been presented, e.g., in the following papers [3–10]. More recently, one has noticed a growing interest in the analysis of delay-integro-differential equations (DIDEs). Baker & Ford [11] studied the asymptotic stability of a class of linear multistep (LM) methods for scalar linear DIDEs; Koto [12] dealt with the linear stability of Runge–Kutta (RK) methods for systems of DIDEs; Huang & Vandewalle [13] gave sufficient and necessary stability conditions for exact and discrete solutions of linear scalar DIDEs, and Luzyanina, Engelborghs & Roose [14] developed computational procedures for determining the stability of DIDEs.

No results have been found in the literature that directly deal with systems as general as (1.1). Under the assumption that $y(t)$ is continuous for $t \geq t_0$, the transformation

$$x_q(t) = \int_{t-\tau_q}^{t} y(\theta)d\theta, \quad q = 1, 2, \ldots, d \tag{1.2}$$

converts (1.1) into a system without distributed delay. With the linear stability theory of [5, 8] some *delay-independent stability* results can be obtained for such a non-distributed-type delay system. This approach is indirect, however, and comes at the price of changing the inherent structure of the system. Here, we prefer to take the more direct route of studying (1.1) immediately. This approach will also enable us to obtain *delay-dependent stability* results.

The paper is structured as follows. In Section 2 we give asymptotic stability criteria for exact solutions of system (1.1). Some examples are given to illustrate the applicability of the criteria. In Section 3 we suggest an adaptation of the classical Linear Multistep and Runge–Kutta methods for solving (1.1). These adaptations are based on the use of a compound quadrature rule to discretize the integrals in the system. In Section 4 and Section 5, we deal with the asymptotic stability of Runge–Kutta and Linear Multistep methods, respectively. The stability criteria presented there can be considered discrete versions of the stability criteria derived in Section 2. The theoretical results are illustrated by some numerical examples.

2 Stability criteria for the exact solution

Before starting the stability analysis, we introduce a complex function

$$\eta(z) = \begin{cases} \frac{1-z}{\ln z}, & z \in \mathbb{C} \setminus \{0, 1\} \\ 0, & z = 0 \\ -1, & z = 1, \end{cases} \tag{2.1}$$

where $\ln z = \ln |z| + i \arg z$ $(z \neq 0, 1; \quad -\pi < \arg z \leq \pi)$, is the principal branch of the multi-valued complex natural logarithm. The function $\eta(z)$ will appear in the characteristic equation of (1.1).

Lemma 2.1 *Function $\eta(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}_0^-$, where $\mathbb{R}_0^- := \{x \in \mathbb{R} : x \leq 0\}$, and satisfies $|\eta(z)| \leq 1$ for $|z| \leq 1$.*

Proof The proof of analyticity in $\mathbb{C} \setminus \{\mathbb{R}_0^- \bigcup \{1\}\}$ is straightforward. To show analyticity at $z = 1$, we only need to verify that $\eta'(1)$ exists. This follows from de l'Hospital's rule:

$$\eta'(1) = \lim_{z \to 1} \frac{\frac{1-z}{\ln z} - (-1)}{z - 1} = -\frac{1}{2}.$$

It remains to prove the bound on $|\eta(z)|$. To that end, we consider the open unit disk with a slit from 0 to -1, i.e., $D := \{z : 0 < |z| < 1, \ |\arg z| < \pi\}$. The function $\eta(z)$ is analytic inside D, and can be continuously extended on the boundary of D. This extension is multi-valued for $z \in (-1, 0)$, because $\lim_{\epsilon \to 0} \eta(z \pm i\epsilon) = (1 - z)/(\ln(-z) \pm i\pi)$. By the maximum modulus principle for analytic functions on Riemann surfaces, the maximum of $|\eta(z)|$ is found on the boundary of D.

For $z \in \{z : |z| = 1, \ |\arg z| < \pi\}$, setting $z = \exp(i\theta)$, we find

$$|\eta(z)| = \left| \frac{1 - \exp(i\theta)}{i\theta} \right| = \frac{\sqrt{(1 - \cos\theta)^2 + \sin^2\theta}}{|\theta|} = \frac{2|\sin\frac{\theta}{2}|}{|\theta|} \leq \frac{2|\frac{\theta}{2}|}{|\theta|} = 1$$

when $\theta \neq 0$, and $|\eta(z)| = 1$ when $\theta = 0$. For $z \in [-1, 0]$, we find for both branches of the multi-valued function that

$$|\eta(z)| = \frac{1 - z}{\sqrt{\ln^2(-z) + \pi^2}} \leq \frac{2}{\pi} < 1$$

when $z \neq 0$, and $|\eta(z)| = 0$ when $z = 0$. This completes the proof. □

Specializing Corollary 3.1 in Hale & Verduyn Lunel [1, Ch.9] to the case of system (1.1) immediately yields the following lemma.

Lemma 2.2 *Assume* $\sup\{\Re(\lambda) : \ P(\lambda) = 0\} < 0$, *where*

$$P(\lambda) := \det \left[\lambda \left(I_p - \sum_{q=1}^{d} e^{-\lambda\tau_q} N_q \right) - L - \sum_{q=1}^{d} e^{-\lambda\tau_q} M_q + \sum_{q=1}^{d} \eta(e^{-\lambda\tau_q}) \tau_q Q_q \right]$$

is the characteristic polynomial of (1.1)*. Then, system* (1.1) *is asymptotically stable.*

Lemma 2.3 *Let $r \in \mathbb{R}$ be such that matrix $\left(I_p - \sum_{q=1}^{d} e^{-\lambda\tau_q} N_q\right)$ is invertible for $\Re\lambda \geq r$. Then the function*

$$\tilde{P}(\lambda) := \det\left[\lambda^2 I_p - \left(I_p - \sum_{q=1}^{d} e^{-\lambda\tau_q} N_q\right)^{-1}\left(\lambda L + \lambda\sum_{q=1}^{d} e^{-\lambda\tau_q} M_q + \sum_{q=1}^{d}(1-e^{-\lambda\tau_q}) Q_q\right)\right]$$

has at most a finite number of zeros when $\Re\lambda \geq r$.

Proof When $\Re\lambda \geq r$, function $\tilde{P}(\lambda)$ can be expanded into the form

$$\tilde{P}(\lambda) = \lambda^{2p} + \psi_{2p-1}\left(e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_d}\right)\lambda^{2p-1} + \ldots + \psi_0\left(e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_d}\right),$$

where $\psi_i\left(e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_d}\right)$ for $i = 0, 1, \ldots, 2p-1$ are rational functions of the expressions $e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \ldots, e^{-\lambda\tau_d}$. Because of the invertibility assumption in the lemma these functions have no poles for $\Re\lambda \geq r$. Since $\tau_i > 0$, we have for $\Re\lambda \geq r$ that

$$\left|e^{-\lambda\tau_i}\right| = e^{-\tau_i\Re\lambda} \leq e^{-\tau_i r},$$

and, hence, there exist constants $K_i > 0$ such that

$$\left|\psi_i\left(e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_d}\right)\right| \leq K_i, \quad i = 0, 1, \ldots, 2p-1.$$

Let M be a positive number large enough so that

$$\frac{K_{2p-1}}{M} + \frac{K_{2p-2}}{M^2} + \ldots + \frac{K_0}{M^{2p}} < 1,$$

which implies, for $\Re\lambda \geq r$ and $|\lambda| \geq M$, that

$$|\tilde{P}(\lambda)| \geq |\lambda|^{2p}\left[1 - \frac{K_{2p-1}}{M} - \frac{K_{2p-2}}{M^2} - \ldots - \frac{K_0}{M^{2p}}\right] > 0.$$

So, $\tilde{P}(\lambda) \neq 0$ in the set $\{\lambda : \Re\lambda \geq r \text{ and } |\lambda| \geq M\}$. Also, by the isolation property of the zeros of analytic functions, $\tilde{P}(\lambda)$ has at most a finite number of zeros in the set $\{\lambda : \Re\lambda \geq r \text{ and } |\lambda| < M\}$. This proves the lemma. □

In the following, we denote the spectrum of a square matrix $\mathcal{A}$ by $\sigma(\mathcal{A})$ and introduce the set $\mathbb{C}^- = \{z \in \mathbb{C} : \Re(z) < 0\}$.

Theorem 2.4 *System (1.1) is asymptotically stable if*

(a) $\det\left[I_p - \sum_{q=1}^{d} \xi_q N_q\right] \neq 0$ *for* $|\xi_q| \leq 1$,

(b) $\sigma(G(\xi)) \subseteq \mathbb{C}^-$ *for* $\xi = (\xi_1, \xi_2, \ldots, \xi_d)^T$ *with* $|\xi_q| \leq 1$, *where* $G(\xi) = \left(I_p - \sum_{q=1}^{d} \xi_q N_q\right)^{-1}\left(L + \sum_{q=1}^{d} \xi_q M_q - \sum_{q=1}^{d} \eta(\xi_q)\tau_q Q_q\right)$.

Proof When $|\xi_q| \leq 1$ for $q = 1, \ldots, d$ condition (a) leads to

$$\hat{P}(\lambda, \xi_1, \xi_2, \ldots, \xi_d) := \det\left[\lambda I_p - \lambda \sum_{q=1}^{d} \xi_q N_q - L - \sum_{q=1}^{d} \xi_q M_q + \sum_{q=1}^{d} \eta(\xi_q)\tau_q Q_q\right]$$
$$= \det\left[I_p - \sum_{q=1}^{d} \xi_q N_q\right] \det[\lambda I_p - G(\xi)].$$

By condition (b) this implies $P(\lambda) = \hat{P}(\lambda, e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \ldots, e^{-\lambda\tau_d}) \neq 0$ for $\Re(\lambda) \geq 0$. Hence,

$$\sup\{\Re(\lambda) : \ P(\lambda) = 0\} \leq 0. \tag{2.2}$$

Next, we show that the strict inequality in (2.2) holds. Consider the function $F(\xi_1, \xi_2, \ldots, \xi_d)$ defined as

$$F(\xi_1, \xi_2, \ldots, \xi_d) := \det\left[I_p - \sum_{q=1}^{d} \xi_q N_q\right].$$

This function is a multivariate polynomial, which by condition (a) is nonzero on the compact domain defined by $|\xi_q| \leq 1, q = 1, \ldots, d$, and equal to 1 at the origin. Hence, its modulus is bounded from below by a value $\epsilon > 0$, i.e.,

$$|F(\xi_1, \xi_2, \ldots, \xi_d)| \geq \epsilon > 0, \ \text{when} \ |\xi_q| \leq 1 \text{ for } q = 1, \ldots, d.$$

By the continuity of F, there exists a $\delta > 0$ such that

$$|F(\xi_1, \xi_2, \ldots, \xi_d)| > 0, \ \text{when} \ |\xi_q| \leq 1 + \delta \text{ for } q = 1, \ldots, d.$$

From this, it follows that

$$\det\left[I_p - \sum_{q=1}^{d} e^{-\lambda\tau_q} N_q\right] \neq 0, \ \text{when} \ |e^{-\lambda\tau_q}| \leq 1 + \delta \text{ for } q = 1, \ldots, d.$$

Let r be the strictly positive number $r = \ln(1 + \delta)/\max\limits_{q=1,\ldots,d} \tau_q$ then

$$\det\left[I_p - \sum_{q=1}^{d} e^{-\lambda\tau_q} N_q\right] \neq 0, \quad \text{for } \Re\lambda \geq -r.$$

Thus, we can apply Lemma 2.3 to show that equation $\tilde{P}(\lambda) = 0$ has only a finite number of roots when $\Re\lambda \geq -r$. By condition (a), the same holds true for the equation $P(\lambda) = 0$. Combined with (2.2), this shows that the characteristic equation has at most a finite number of roots in the region $\{\lambda : -r \leq \Re\lambda < 0\}$. Set $-\gamma$ equal to the maximum real part of any of those roots, or equal to $-r$, if there are no such roots. Hence, $\gamma > 0$. Then, when $\Re\lambda > -\gamma$, the characteristic equation $P(\lambda) = 0$ has no root. Hence, a strict inequality holds in (2.2). Application of Lemma 2.2 completes the proof. □

In order to be able to derive our next stability criteria, we need to introduce some more notations. Throughout this paper, $\rho(\cdot)$ will denote the spectral radius of a matrix; $\lambda_l(\cdot)$ will denote the lth eigenvalue, and $\mu(\cdot)$ is the logarithmic norm subject to a given matrix norm $\|\cdot\|$; matrix $|M|$ satisfies $|M| = (|m_{ij}|)$ when $M = (m_{ij})$, and the inequality $M \leq \tilde{M}$ means $m_{ij} \leq \tilde{m}_{ij}$ for $M = (m_{ij})$ and $\tilde{M} = (\tilde{m}_{ij}) \in \mathbb{R}^{n\times n}$. By Theorem 1 in [15, Ch.15] and by classical properties of spectral radius and norm, we can prove the following inequalities for any matrix norm $\|\cdot\|$:

$$\rho\left(\sum_{q=1}^{d} \xi_q N_q\right) \leq \rho\left(\sum_{q=1}^{d} |N_q|\right) \quad \text{and} \quad \rho\left(\sum_{q=1}^{d} \xi_q N_q\right) \leq \sum_{q=1}^{d} \|N_q\| \;\; \text{for} \;\; |\xi_q| \leq 1.$$

With this information we can now reformulate Theorem 2.4.

Theorem 2.5 *System* (1.1) *is asymptotically stable when condition* (b) *from Theorem* 2.4 *holds and when there exists a matrix norm* $\|\cdot\|$ *such that*

$$\min\left\{\rho\left(\sum_{q=1}^{d} |N_q|\right), \sum_{q=1}^{d} \|N_q\|\right\} < 1. \tag{2.3}$$

Remark 2.6 Theorem 1.1 in [8] and Lemma 2.1 in [5] can be derived immediately from Theorem 2.5 when one sets $Q_q = 0$ $(q = 1, 2, \ldots, d)$.

Remark 2.7 For the p-dimensional one-delay system

$$\begin{cases} \dfrac{d}{dt}[y(t)-Ny(t-\tau)] = Ly(t)+My(t-\tau)+Q\displaystyle\int_{t-\tau}^{t} y(\theta)d\theta, \;\; t \geq t_0, \\ y(t) = \varphi(t), \quad t \in [t_0-\tau, t_0], \end{cases} \tag{2.4}$$

condition (a) in Theorem 2.4 can be guaranteed by $\rho(N) < 1$ since the following inequalities hold for $|\xi| \leq 1$:

$$\rho(\xi N) \leq |\xi|\rho(N) \leq \rho(N).$$

Hence (2.3) can be replaced by $\rho(N) < 1$ in the case of one-delay systems. This also implies that Theorem 2.4 covers Theorem 2.1 in [4], where the case $Q = 0$ was considered.

Example 2.1 Consider system (2.4) with $\tau = 1/10$, and with

$$L = \begin{pmatrix} -10 & 2 & 2 \\ 2 & -8 & -3 \\ 1 & 5 & -9 \end{pmatrix}, \qquad M = \begin{pmatrix} 2 & -4 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & -4 \end{pmatrix},$$

$$N = \frac{1}{10}\begin{pmatrix} -5 & 1 & 2 \\ 6 & 0 & -3 \\ 0 & 3 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix},$$

and any initial function $\varphi(t)$. A direct calculation yields $\rho(N) \cong 0.4767 < 1$. Also, using a MATLAB code, we numerically computed that

$$\max_{|\xi|\le 1}\{\Re(\lambda) : \lambda \in \sigma[G(\xi)]\} \cong -4.7104 < 0.$$

This implies that $\sigma(G(\xi)) \subseteq \mathbb{C}^-$ for $|\xi| \le 1$. Thus, by Theorem 2.5 and Remark 2.7 we may conclude that this system is asymptotically stable.

Obviously, it is quite difficult in the multidelay case to check the conditions of Theorem 2.4. To deal with this, we derive some sufficient conditions that are more easily verified.

Theorem 2.8 *System* (1.1) *is asymptotically stable if there exists a matrix norm* $\|\cdot\|$ *such that*

(â) $\sum_{q=1}^{d} \|N_q\| < 1$,

(b̂) $$\mu(L)+\sum_{q=1}^{d}\|M_q\|+\sum_{q=1}^{d}\tau_q\|Q_q\|+\frac{\sum_{q=1}^{d}\left[\|N_qL\|+\sum_{\hat q=1}^{d}(\|N_qM_{\hat q}\|+\tau_{\hat q}\|N_qQ_{\hat q}\|)\right]}{1-\sum_{q=1}^{d}\|N_q\|}<0$$

Proof By Theorem 2.5 and condition (â), it suffices to show that condition (b) from Theorem 2.4 holds. In fact, with Theorem 1 in [15, Ch.11], properties of the logarithmic norm (cf. [16, Ch.7]), Lemma 2.1 and condition (â), one can infer the bound given below for all l and $|\xi_q| \le 1$. Using the notation $H := \sum_{q=1}^{d} \xi_q N_q$, we have

$$\begin{aligned}
\Re[\lambda_l(G(\xi))] &\le \mu[G(\xi)]\\
&= \mu\left[\left(I_p+\sum_{i=1}^{\infty}H^i\right)\left(L+\sum_{q=1}^{d}\xi_qM_q-\sum_{q=1}^{d}\eta(\xi_q)\tau_qQ_q\right)\right]\\
&= \mu\Bigg[L+\sum_{q=1}^{d}\xi_qM_q-\sum_{q=1}^{d}\eta(\xi_q)\tau_qQ_q+\left(I_p+\sum_{i=1}^{\infty}H^i\right)\\
&\qquad\cdot\left(HL+H\sum_{q=1}^{d}\xi_qM_q-H\sum_{q=1}^{d}\eta(\xi_q)\tau_qQ_q\right)\Bigg]\\
&\le \mu(L)+\sum_{q=1}^{d}\|M_q\|+\sum_{q=1}^{d}\tau_q\|Q_q\|+\left[1+\sum_{i=1}^{\infty}\left(\sum_{q=1}^{d}\|N_q\|\right)^i\right]\\
&\qquad\cdot\left[\sum_{q=1}^{d}\|N_qL\|+\sum_{q=1}^{d}\sum_{\hat q=1}^{d}\|N_qM_{\hat q}\|+\sum_{q=1}^{d}\sum_{\hat q=1}^{d}\tau_{\hat q}\|N_qQ_{\hat q}\|\right]
\end{aligned} \tag{2.5}$$

Hence condition ($\hat{\text{b}}$) guarantees $\Re[\lambda_l(G(\xi))] < 0$, and (b) in Th. 2.4 holds. □

Remark 2.9 Theorem 2.8 can be simplified when $N_q = 0$ for all q, which leads to the first conclusion of Theorem 4 in [6]. In addition, when $N_q = Q_q = 0$ for all q, Theorem 2.8 becomes Theorem 2.2 in [10].

Example 2.2 Consider a system of the form (1.1) with

$$L = \begin{pmatrix} -68 & 0 & 2 \\ 1 & -79 & -3 \\ 1 & -4 & -82 \end{pmatrix}, \; M_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -2 & 2 \\ 3 & -2 & 5 \end{pmatrix}, \; M_2 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -1 & 3 \\ 0 & 2 & 1 \end{pmatrix},$$

$$N_1 = \frac{1}{25}\begin{pmatrix} 1 & -3 & 0 \\ 2 & 0 & 2 \\ -1 & 3 & 2 \end{pmatrix}, N_2 = \frac{1}{20}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 0 & 0 \end{pmatrix}, Q_1 = \begin{pmatrix} -1 & 2 & 4 \\ 2 & -1 & 3 \\ 1 & 0 & -2 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 2 & 0 & 1 \\ -1 & -3 & 1 \\ -3 & 2 & 1 \end{pmatrix}, \quad \tau_1 = \frac{1}{20} \quad \tau_2 = \frac{1}{10}.$$

With the 2-norm $\|\mathcal{A}\|_2 = \sqrt{\rho(\mathcal{A}\mathcal{A}^T)}$ and its induced logarithmic norm $\mu_2(\mathcal{A}) = \max\{\sigma(\frac{\mathcal{A}+\mathcal{A}^T}{2})\}$, a simple computation yields $\sum_{q=1}^{2} \|N_q\|_2 \cong 0.3769 < 1$, $\Omega \cong -5.6623 < 0$, with Ω denoting the left-hand side of the inequality in condition ($\hat{\text{b}}$). Hence, this system is asymptotically stable.

3 Runge–Kutta and linear multistep methods

In this section, we will confine our discussion to systems of NMIDEs with commensurate delays, i.e., systems of the form (1.1) with $\tau_q = q\tau$:

$$\begin{cases} \dfrac{d}{dt}\left[y(t) - \sum\limits_{q=1}^{d} N_q y(t-q\tau)\right] = Ly(t) + \sum\limits_{q=1}^{d} M_q y(t-q\tau) + \sum\limits_{q=1}^{d} Q_q \displaystyle\int_{t-q\tau}^{t} y(\theta)d\theta, \; t \geq t_0, \\ y(t) = \varphi(t), \quad t \in [t_0 - d\tau, t_0], \end{cases} \tag{3.1}$$

where $\tau > 0$ is a constant and $L, M_q, N_q, Q_q \in \mathbb{C}^{p\times p}$. Before constructing discrete schemes for this system, we review classical Runge–Kutta (RK) methods, Linear Multistep (LM) methods and related concepts. This is done mainly for setting some notation and for future reference. For solving ODE systems of the form $y'(t) = f(t, y(t))$, $t \geq t_0$, with $y(t_0) = y_0$, there are two classes of classical methods. RK methods are of the form

$$\begin{cases} y_i^{(n)} = y_n + h \sum\limits_{j=1}^{s} a_{ij} f(t_n + c_j h, y_j^{(n)}), \quad i = 1, 2, ..., s, \\ y_{n+1} = y_n + h \sum\limits_{j=1}^{s} b_j f(t_n + c_j h, y_j^{(n)}), \quad n \geq 0, \end{cases} \tag{3.2}$$

where the abscissae c_j, the weights b_j and the coefficients a_{ij} are characteristics of the method; h denotes the stepsize; $t_n = t_0 + nh$, and $y_i^{(n)}$ and y_n are approximations to $y(t_n + c_i h)$ and $y(t_n)$, respectively. The other class of methods are LM methods, compactly denoted as

$$\mathcal{P}(E)y_n = h\mathcal{Q}(E)f_n, \tag{3.3}$$

where E is the shift operator, $\mathcal{P}(\xi)$ and $\mathcal{Q}(\xi)$ are irreducible polynomials,

$$\mathcal{P}(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j \quad \textit{and} \quad \mathcal{Q}(\xi) = \sum_{j=0}^{k} \beta_j \xi^j,$$

and y_n and f_n are approximations to $y(t_n)$ and $f(t_n, y(t_n))$, respectively. The stability regions (cf. [18]) of these methods are given by the sets:

$$\mathbb{S}_{\mathbb{RK}} := \{\zeta \in \mathbb{C} : \ (I_s - \zeta A) \text{ is invertible and } |1 + \zeta b^T (I_s - \zeta A)^{-1} e| < 1\},$$
$$\mathbb{S}_{\mathbb{LM}} := \{\zeta \in \mathbb{C} : \ \mathcal{P}(z) = \zeta \mathcal{Q}(z) \Rightarrow |z| < 1\},$$

where $b = (b_1, b_2, \ldots, b_s)^T$, $A = (a_{ij})$ and $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^s$. If there exists an $\alpha \in (0, \frac{\pi}{2}]$ such that the stability region contains the sector $\mathbb{S}_\alpha := \{\zeta \in \mathbb{C} : |\arg(-\zeta)| < \alpha\}$, the method is called $A(\alpha)$-stable. In particular, an $A(\frac{\pi}{2})$-stable method is called A-stable.

Adapting (3.2) to (3.1), for $h = \frac{\tau}{m}$, with m a positive integer, yields, for $n \geq 0$ and $j = 1, 2, \ldots, s$,

$$\begin{cases} f_j^{(n)} = Ly_j^{(n)} + \sum\limits_{q=1}^{d} M_q y_j^{(n-qm)} + \sum\limits_{q=1}^{d} N_q f_j^{(n-qm)} + h \sum\limits_{q=1}^{d} \sum\limits_{r=0}^{qm} v_r Q_q y_j^{(n-r)}, \\ y_i^{(n)} = y_n + h \sum\limits_{j=1}^{s} a_{ij} f_j^{(n)}, \\ y_{n+1} = y_n + h \sum\limits_{j=1}^{s} b_j f_j^{(n)}, \end{cases} \tag{3.4}$$

where $y_n, y_i^{(n)}$ and $f_j^{(n)}$ are approximations to $y(t_n), y(t_n + c_i h)$ and $y'(t_n + c_j h)$, respectively. For the discretization of the integral item $\int_{t_n + c_j h - q\tau}^{t_n + c_j h} y(\theta) d\theta$, a convergent compound quadrature formula with weights v_r has been used. In (3.4), when mesh points t_n or off-step points $t_n + c_j h$ belong to the initial set $[t_0 - \hat{\tau}, \ t_0]$, we will set

$$y_n = \varphi(t_n), \quad y_j^{(n)} = \varphi(t_n + c_j h) \quad \text{and} \quad f_j^{(n)} = \varphi'(t_n + c_j h).$$

With the similar argument of Theorem 4.2.5 (compare also Theorem 4.1.6) in [16], we can deduce the following convergence result.

Theorem 3.1 *Suppose the underlying RK method* (3.2) *is of order p and the employed quadrature rule is of order q. Then the induced method* (3.4) *is convergent of order* $\min\{p, \ q\}$.

Applying the LM method (3.3), together with a convergent compound quadrature formula with weights υ_r, gives, for $n \geq -(k-1)$ and $j = 1, 2, \ldots, k$,

$$\begin{cases} f_{n+j} = L y_{n+j} + \sum_{q=1}^{d} M_q y_{n+j-qm} + \sum_{q=1}^{d} N_q f_{n+j-qm} + h \sum_{q=1}^{d} \sum_{r=0}^{qm} \upsilon_r Q_q y_{n+j-r}, \\ \sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \end{cases} \tag{3.5}$$

where y_n and f_n are approximations to $y(t_n)$ and $y'(t_n)$. Moreover, when mesh points t_n belong to the initial set $[t_0 - \hat{\tau},\ t_0]$, we will set

$$y_n = \varphi(t_n) \quad \text{and} \quad f_n = \varphi'(t_n).$$

The convergence order of method (3.5) follows from a small modification of the proof of Theorem 2.1 in [17].

Theorem 3.2 *Suppose the underlying LM method* (3.3) *is of order p and the employed quadrature rule is of order q. Then the induced method* (3.5) *is convergent of order* $\min\{p,\ q\}$.

Remark 3.3 If interpolatory-type quadrature formulae are used in (3.4) and (3.5) with non-negative weights, one has the following property for all t, which will be used further on to simplify certain stability conditions,

$$\tau_q = \int_{t-q\tau}^{t} d\upsilon = h \sum_{r=0}^{qm} \upsilon_r = h \sum_{r=0}^{qm} |\upsilon_r|, \qquad q = 1, 2, \ldots, d. \tag{3.6}$$

Next, we will study the asymptotic stability of methods (3.4) and (3.5).

Definition 3.4 A numerical method for (1.1) is called asymptotically stable if the numerical solution y_n generated by the method satisfies $\lim_{n \to \infty} y_n = 0$.

4 Stability criteria for the adapted RK methods

With the Kronecker product $\otimes$ and the notations:

$$F^{(n)} = \left(f_1^{(n)T}, f_2^{(n)T}, \ldots, f_s^{(n)T}\right)^T \quad \text{and} \quad Y^{(n)} = \left(y_1^{(n)T}, y_2^{(n)T}, \ldots, y_s^{(n)T}\right)^T,$$

the method (3.4) can be written more compactly as

$$\begin{cases} F^{(n)} = (e \otimes L) y_n + h(A \otimes L) F^{(n)} \\ \qquad + \sum_{q=1}^{d} \left[(e \otimes M_q) y_{n-qm} + h(A \otimes M_q) F^{(n-qm)} + (I_s \otimes N_q) F^{(n-qm)} \right] \\ \qquad + h \sum_{q=1}^{d} \sum_{r=0}^{qm} \upsilon_r \left[(e \otimes Q_q) y_{n-r} + h(A \otimes Q_q) F^{(n-r)} \right] \\ y_{n+1} = y_n + h(b^T \otimes I_p) F^{(n)}, \end{cases} \tag{4.1}$$

Define

$$Y_{n+1} = \left(F^{(n)\,T}, y_{n+1}^T\right)^T, \; \bar{L} = hL, \; \bar{M}_q = hM_q, \; \bar{Q}_q = hQ_q \quad \text{and} \quad \bar{\bar{Q}}_q = h^2 Q_q.$$

Then, (4.1) can be transformed into the following difference equation:

$$\begin{aligned}\begin{pmatrix} I_s \otimes I_p - A \otimes \bar{L} & 0 \\ -h(b^T \otimes I_p) & I_p \end{pmatrix} Y_{n+1} &= \begin{pmatrix} 0 & e \otimes L \\ 0 & I_p \end{pmatrix} Y_n + \sum_{q=1}^{d} \begin{pmatrix} 0 & e \otimes M_q \\ 0 & 0 \end{pmatrix} Y_{n-qm} \\ &+ \sum_{q=1}^{d} \begin{pmatrix} A \otimes \bar{M}_q + I_s \otimes N_q & 0 \\ 0 & 0 \end{pmatrix} Y_{n-qm+1} \\ &+ \sum_{q=1}^{d} \sum_{r=0}^{qm} v_r \begin{pmatrix} 0 & e \otimes \bar{Q}_q \\ 0 & 0 \end{pmatrix} Y_{n-r} \\ &+ \sum_{q=1}^{d} \sum_{r=0}^{qm} v_r \begin{pmatrix} A \otimes \bar{\bar{Q}}_q & 0 \\ 0 & 0 \end{pmatrix} Y_{n-r+1}.\end{aligned}$$

Its characteristic equation is given by

$$\det \begin{bmatrix} T_1(z) & T_2(z) \\ T_3(z) & T_4(z) \end{bmatrix} = 0, \quad z \in \mathbb{C} \tag{4.2}$$

where, with $m_0 = md$, we have that

$$T_1(z) = z^{m_0+1} \left[I_s \otimes \left(I_p - \sum_{q=1}^{d} z^{-qm} N_q \right) - A \otimes \left(\bar{L} + \sum_{q=1}^{d} z^{-qm} \bar{M}_q + \sum_{q=1}^{d} \sum_{r=0}^{qm} v_r z^{-r} \bar{\bar{Q}}_q \right) \right],$$

$$T_2(z) = -z^{m_0} \left[e \otimes \left(L + \sum_{q=1}^{d} z^{-qm} M_q + \sum_{q=1}^{d} \sum_{r=0}^{qm} v_r z^{-r} \bar{Q}_q \right) \right],$$

$$T_3(z) = -z^{m_0+1} h \left(b^T \otimes I_p \right), \quad T_4(z) = z^{m_0} (z-1) I_p.$$

It follows from the theory on difference equations (cf. [15]) that $\lim_{n\to\infty} Y_n = 0$ if all the zeros of (4.2) satisfy $|z| < 1$. Hence, we can formulate the following lemma.

Lemma 4.1 *Numerical method* (4.1) *satisfies* $\lim_{n\to\infty} y_n = 0$ *if all the zeros of* (4.2) *satisfy* $|z| < 1$.

Lemma 4.2 *Assume that condition* (a) *from Theorem* 2.4 *holds and assume that the matrices* $I_s - \lambda_l(r(z))A$ $(1 \le l \le p)$ *are invertible for* $|z| \ge 1$, *where*

$$r(z) = \left(I_p - \sum_{q=1}^{d} z^{-qm} N_q \right)^{-1} \left(\bar{L} + \sum_{q=1}^{d} z^{-qm} \bar{M}_q + \sum_{q=1}^{d} \sum_{r=0}^{qm} v_r z^{-r} \bar{\bar{Q}}_q \right). \tag{4.3}$$

Then, $\det[T_1(z)] \ne 0$ *for* $|z| \ge 1$.

Proof Condition (a) in Theorem 2.4 implies that matrix $I_p - \sum_{q=1}^{d} z^{-qm} N_q$ is invertible for $|z| \geq 1$. A simple computation using properties of the Kronecker product (cf. [19, Ch.4]) lead to

$$T_1(z) = z^{m_0+1}\left[I_s \otimes \left(I_p - \sum_{q=1}^{d} z^{-qm} N_q\right)\right]\left[I_s \otimes I_p - A \otimes r(z)\right] \quad \text{for } |z| \geq 1.$$

With this, we have for $|z| \geq 1$ that

$$\begin{aligned}\det[T_1(z)] &= (z^{m_0+1})^{sp}\left[\det\left(I_p - \sum_{q=1}^{d} z^{-qm} N_q\right)\right]^s \det[I_s \otimes I_p - A \otimes r(z)] \\ &= (z^{m_0+1})^{sp}\left[\det\left(I_p - \sum_{q=1}^{d} z^{-qm} N_q\right)\right]^s \prod_{l=1}^{p}\prod_{j=1}^{s}[1 - \lambda_l(r(z))\lambda_j(A)].\end{aligned} \tag{4.4}$$

The invertibility of the matrices $I_s - \lambda_l(r(z))A$ means that $\lambda_l(r(z))\lambda_j(A) \neq 1$ for all l, j. Hence, $\det[T_1(z)] \neq 0$ for $|z| \geq 1$. □

Theorem 4.3 *Method* (3.4) *is asymptotically stable if condition* (a) *in Theorem* 2.4 *holds and* $\sigma[r(z)] \subseteq \mathbb{S}_{\mathbb{RK}}$ *for* $|z| \geq 1$.

Proof By Lemma 4.1, we need to prove that all the zeros of (4.2) satisfy $|z| < 1$. If this were not true, there would exist a $z_0 \in \mathbb{C} : |z_0| \geq 1$ such that

$$\det\begin{bmatrix} T_1(z_0) & T_2(z_0) \\ T_3(z_0) & T_4(z_0)\end{bmatrix} = 0. \tag{4.5}$$

By Lemma 4.2 one has that $\det[T_1(z_0)] \neq 0$; hence, (4.5) is equivalent to

$$\det[T_4(z_0) - T_3(z_0)T_1^{-1}(z_0)T_2(z_0)] = 0. \tag{4.6}$$

Using properties of the Kronecker product (cf. [19, Ch.4]) and the Jordan canonical form $J(z_0)$ of $r(z_0)$, we find

$$\begin{aligned}&\det[T_4(z_0) - T_3(z_0)T_1^{-1}(z_0)T_2(z_0)] \\ &= z_0^{m_0 p}\det\{z_0 I_p - [I_p + (b^T \otimes I_p)(I_s \otimes I_p - A \otimes r(z_0))^{-1}(e \otimes r(z_0))]\} \\ &= z_0^{m_0 p}\det\{z_0 I_p - [I_p + (I_p \otimes b^T)(I_p \otimes I_s - r(z_0) \otimes A)^{-1}(r(z_0) \otimes e)]\} \\ &= z_0^{m_0 p}\det\{z_0 I_p - [I_p + (I_p \otimes b^T)(I_p \otimes I_s - J(z_0) \otimes A)^{-1}(J(z_0) \otimes e)]\} \\ &= z_0^{m_0 p}\prod_{l=1}^{p}\{z_0 - [1 + \lambda_l(r(z_0))b^T(I_s - \lambda_l(r(z_0))A)^{-1}e]\}.\end{aligned} \tag{4.7}$$

Combining (4.7) with (4.6) shows that there is an l such that

$$|1 + \lambda_l(r(z_0))b^T(I_s - \lambda_l(r(z_0))A)^{-1}e| = |z_0| \geq 1.$$

This implies $\lambda_l(r(z_0)) \notin \mathbb{S}_{\mathbb{RK}}$, which contradicts the earlier assumption $\sigma[r(z)] \subseteq \mathbb{S}_{\mathbb{RK}}$ for $|z| \geq 1$. Hence, the theorem is proven. □

A combination of Theorem 4.3 and the definitions of $A(\alpha)$- and A-stability leads to the following results.

Theorem 4.4 *Assume that the classical RK method* (3.2) *is* $A(\alpha)$*-stable* (*resp.* A*-stable*) *and system* (3.1) *satisfies condition* (a) *in Theorem* 2.4*, and assume that* $\sigma[r(z)] \subseteq \mathbb{S}_\alpha$ (*resp.* $\sigma[r(z)] \subseteq \mathbb{C}^-$) *for* $|z| \geq 1$*. Then, the method* (3.4) *is asymptotically stable.*

Theorem 4.5 *Assume that the classical RK method* (3.2) *is* $A(\alpha)$*-stable* (*resp.* A*-stable*) *and system* (3.1) *satisfies* (2.3)*, and assume that* $\sigma[r(z)] \subseteq \mathbb{S}_\alpha$ (*resp.* $\sigma[r(z)] \subseteq \mathbb{C}^-$) *for* $|z| \geq 1$*. Then, the method* (3.4) *is asymptotically stable.*

Moreover, with Theorem 4.5 and a similar derivation as the one in the proof of Theorem 2.8, we can show the correctness of the following theorem.

Theorem 4.6 *Assume that the classical RK method* (3.2) *is* A*-stable and that system* (3.1) *satisfies conditions* ($\hat{a}$) *and* ($\hat{b}$) *from Theorem* 2.8*. Then, the method* (3.4) *is asymptotically stable when stepsize* h *satisfies*

$$h \sum_{r=0}^{qm} |v_r| \leq \tau_q, \qquad q = 1, 2, \ldots, d. \tag{4.8}$$

Remark 4.7 When the employed quadrature rule is of interpolatory type and the coefficients v_r are all nonnegative, the condition (4.8) can be dropped, because of (3.6).

Theorems 4.4, 4.5 and 4.6 can be viewed as the discrete counterparts of Theorems 2.4, 2.5 and 2.8, respectively. Furthermore, It is well-known that the classical Radau IA, Radau IIA, Gauss, Lobatto IIIA, Lobatto IIIB and Lobatto IIIC methods are all A-stable (cf. [18]). Combining this with Theorems 4.5 and 4.6 yields the next two corollaries.

Corollary 4.8 *Assume that system* (3.1) *satisfies* (2.3) *and that* $\sigma[r(z)] \subseteq \mathbb{C}^-$ *for* $|z| \geq 1$*. Then, method* (3.4) *based on a Radau IA, Radau IIA, Gauss, Lobatto IIIA, Lobatto IIIB or Lobatto IIIC method is asymptotically stable.*

Corollary 4.9 *Assume that system* (3.1) *satisfies conditions* ($\hat{a}$)–($\hat{b}$) *from Theorem* 2.8*. Then, method* (3.4) *based on a Radau IA, Radau IIA, Gauss, Lobatto IIIA, Lobatto IIIB or Lobatto IIIC method is asymptotically stable whenever* (4.8) *holds.*

As an illustration for Corollaries 4.8 and 4.9, we give two examples.

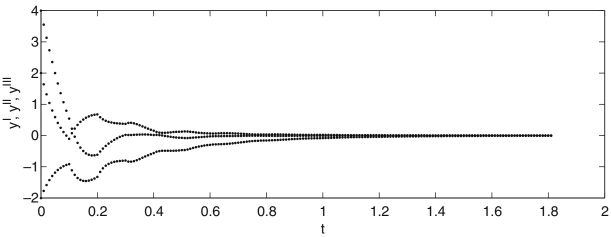

Fig. 1 Numerical solution $y = (y^I, y^{II}, y^{III})^T$ obtained with the adapted two-stage Gauss method and the compound Simpson rule with $m = 10$ for the system in Example 2.1

Example 4.1 We consider method (3.4) based on the 2-stage classical Gauss method and the compound Simpson rule

$$y_{i,\,1}^{(n)} = \frac{h}{3}\left(y_i^{(n)} + 4\sum_{r=1}^{m/2} y_i^{(n-2r+1)} + 2\sum_{r=1}^{(m-2)/2} y_i^{(n-2r)} + y_i^{(n-m)}\right) \text{ where } m = 10.$$

We apply this method to the system of Example 2.1 with initial condition $\varphi(t) = (-2, 2, 4)^T$ for $t \in [-1/10, 0]$). Numerically one can verify that

$$\max_{|z|\geq 1}\{\Re(\lambda) : \lambda \in \sigma[r(z)]\} \cong -0.0471 < 0,$$

which shows that $\sigma[r(z)] \subseteq \mathbb{C}^-$ for $|z| \geq 1$. Moreover, we have that $\rho(N) \cong 0.4767 < 1$. Hence, by Corollary 4.8 and Remark 2.7 this will produce an asymptotically stable numerical solution, as shown in Fig. 1.

Example 4.2 Consider method (3.4) induced by the 2-stage classical Radau IIA method and the compound Gregory rule

$$y_{i,\,q}^{(n)} = \tfrac{h}{12}\left(5y_i^{(n)} + 13y_i^{(n-1)} + 12\sum_{r=2}^{qm-2} y_i^{(n-r)} + 13y_i^{(n-qm+1)} + 5y_i^{(n-qm)}\right),$$

where $q = 1, 2$ and $m = 50$. We apply this method to the system in Example 2.2 with initial condition $\varphi(t) = (-1, 0.5, 1)^T$ for $t \in [-1/10, 0]$. We have shown earlier that this system satisfies conditions $(\hat{\text{a}})$–$(\hat{\text{b}})$ of Theorem 2.8 with $\tau_q = q/20$ $(q = 1, 2)$. By Remark 4.7, condition (4.8) is satisfied since the quadrature rule coefficients are positive. Thus, it follows from Corollary 4.9 that the numerical solution is asymptotically stable, see Fig. 2.

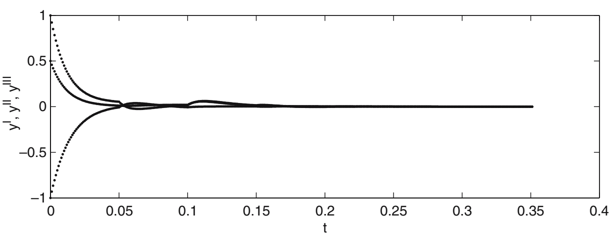

Fig. 2 Numerical solution $y = (y^I, y^{II}, y^{III})^T$ obtained with the adapted two-stage Radau IIA method and the compound Gregory rule with $m = 50$ for the system in Example 2.2

5 Stability criteria for the adapted LM methods

In our study of the asymptotic stability of the methods (3.5), we will again use the symbols $\bar{L}$, $\bar{M}$, $\bar{Q}$ and $\bar{\bar{Q}}$, which were introduced in Section 4. Substituting the first equation of (3.5) into the second one and using the equality

$$\sum_{j=0}^{k} \alpha_j y_{n+j-qm} = h \sum_{j=0}^{k} \beta_j f_{n+j-qm}$$

lead to

$$\sum_{j=0}^{k} \alpha_j \left(y_{n+j} - \sum_{q=1}^{d} N_q y_{n+j-qm} \right) = \sum_{j=0}^{k} \beta_j \left(\bar{L} y_{n+j} + \sum_{q=1}^{d} \bar{M}_q y_{n+j-qm} + \sum_{q=1}^{d} \bar{\bar{Q}}_q \sum_{r=0}^{qm} v_r y_{n+j-r} \right). \tag{5.1}$$

The characteristic equation of this difference equation for $z \in \mathbb{C}$ is given by

$$\det \left[\mathcal{P}(z) \left(I_p - \sum_{q=1}^{d} z^{-qm} N_q \right) - \mathcal{Q}(z) \left(\bar{L} + \sum_{q=1}^{d} z^{-qm} \bar{M} + \sum_{q=1}^{d} \sum_{r=0}^{qm} v_r z^{-r} \bar{\bar{Q}}_q \right) \right] = 0. \tag{5.2}$$

In the theory on difference equations one finds the next result.

Lemma 5.1 *Numerical method* (5.1) *is asymptotically stable if all the zeros of* (5.2) *satisfy* $|z| < 1$.

Theorem 5.2 *Method* (3.5) *is asymptotically stable if condition* (a) *in Theorem* 2.4 *holds and* $\sigma[r(z)] \subseteq \mathbb{S}_{\mathbb{LM}}$ *for* $|z| \geq 1$, *where* $r(z)$ *is given by* (4.3).

Proof By Lemma 5.1, we need to prove that all the zeros of (5.2) satisfy $|z| < 1$. Assume that there exists a z_0 with $|z_0| \geq 1$ such that (5.2) holds with $z = z_0$. Since matrix $I_p - \sum_{q=1}^{d} z_0^{-qm} N_q$ is assumed to be invertible, (5.2) is equivalent to $\det[\mathcal{P}(z_0) I_p - \mathcal{Q}(z_0) r(z_0)] = 0$, which implies that there exists an l such that $\mathcal{P}(z_0) = \lambda_l(r(z_0)) \mathcal{Q}(z_0)$. Since $|z_0| \geq 1$ one has $\lambda_l(r(z_0)) \notin \mathbb{S}_{\mathbb{LM}}$. This contradicts the earlier assumption $\sigma[r(z)] \subseteq \mathbb{S}_{\mathbb{LM}}$ for $|z| \geq 1$. Hence, the theorem holds. □

Combining Theorem 5.2 with the definitions of $A(\alpha)$- and A-stability, we obtain the following results.

Theorem 5.3 *Assume that the LM method* (3.3) *is* $A(\alpha)$*-stable* (*resp.* A*-stable*) *and system* (3.1) *satisfies condition* (a) *in Theorem* 2.4, *and assume that* $\sigma[r(z)] \subseteq \mathbb{S}_\alpha$ (*resp.* $\sigma[r(z)] \subseteq \mathbb{C}^-$) *for* $|z| \geq 1$. *Then, the method* (3.5) *is asymptotically stable.*

Theorem 5.4 *Assume that the LM method (3.3) is $A(\alpha)$-stable (resp. A-stable) and system (3.1) satisfies (2.3), and assume that $\sigma[r(z)] \subseteq \mathbb{S}_\alpha$ (resp. $\sigma[r(z)] \subseteq \mathbb{C}^-$) for $|z| \geq 1$. Then, the method (3.5) is asymptotically stable.*

Theorem 5.5 *Assume that the LM method (3.3) is A-stable and system (3.1) satisfies conditions* (â)–($\hat{\text{b}}$) *in Theorem* 2.8. *Then, the method (3.5) is asymptotically stable whenever (4.8) holds.*

Remark 5.6 With (3.6), condition (4.8) can be dropped in Theorem 5.5 if the employed quadrature rule is of interpolatory type and the coefficients v_r are all nonnegative.

We may regard Theorems 5.3, 5.4 and 5.5 as discrete versions of Theorems 2.4, 2.5 and 2.8, respectively. The following examples illustrate Theorems 5.4 and 5.5.

Example 5.1 The classical BDF method of order 3 is $A(86.03°)$-stable. Adapting this method towards (3.5) with the compound Gregory rule

$$y_{n+j,\,1} = \frac{h}{12}\left(5y_{n+j} + 13y_{n+j-1} + 12\sum_{r=2}^{m-2} y_{n+j-r} + 13y_{n+j-m+1} + 5y_{n+j-m}\right),$$

where $m = 10$, we obtain a numerical method for the system in Example 2.1. Earlier, we showed that $\rho(N) \cong 0.4767 < 1$; in addition it can be numerically verified that the system satisfies

$$\max_{|z|\geq 1}\{|\arg(-\lambda)| :\ \lambda \in \sigma[r(z)]\} \cong 54.90°,$$

which shows that $\sigma[r(z)] \subseteq \mathbb{S}_{86.03°}$ for $|z| \geq 1$. Hence, it follows from Theorem 5.4 and Remark 2.7 that the method will produce an asymptotically stable numerical solution.

Example 5.2 The classical second order BDF method is A-stable. Based on this method and the compound trapezoidal rule

$$y_{n+j,\,q} = \frac{h}{2}\left(y_{n+j} + 2\sum_{r=1}^{qm-1} y_{n+j-r} + y_{n+j-qm}\right), \quad q = 1, 2; \quad m = 50,$$

we obtain a numerical scheme of the form (3.5). If the method is applied to the system in Example 2.2, which has been proven to satisfy conditions (â)–($\hat{\text{b}}$) for $\tau_q = q/20$ $(q = 1, 2)$, an asymptotically stable numerical solution will be generated. This follows from Theorem 5.5 together with Remark 5.6.

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