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Quadratic spline quasi-interpolants on Powell-Sabin partitions

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Abstract In this paper we address the problem of constructing quasi-interpolants in the space of quadratic Powell-Sabin splines on nonuniform triangulations. Quasi-interpolants of optimal approximation order are proposed and numerical tests are presented.

Keywords quasi-interpolation · quadratic splines · Powell-Sabin refinement · Bézier-Bernstein representation

Mathematics Subject Classifications (2000) 65D05 · 41A05 · 41A25 · 41A50

1 Introduction

It is well known that the term *quasi-interpolation* denotes a general approach to construct, with low computational cost, efficient local approximants to a given set of data or a given function. A quasi-interpolant (q.i.) for a given function f is usually obtained as linear combination of the elements of a suitable set of functions which are required to be positive, to ensure stability, and to have small local support in order

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Dedicated to Prof. Mariano Gasca on the occasion of his 60th birthday.

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to achieve local control. The coefficients of the linear combination are the values of linear functionals depending on f and on its derivatives/integrals.

Quasi-interpolation has received a considerable attention by many authors since the seminal paper [23].

In the univariate case various effective quasi-interpolating schemes are available, based on values of f, and/or its derivatives, and/or its integrals (see for examples [1–3, 14–16, 23] and references quoted therein) and interesting applications have been proposed in different fields.

Similarly, various interesting results have been obtained in the bivariate setting by using quasi-interpolating schemes based on tensor-product polynomial splines (see for example [1, 2, 13, 22] and references quoted therein), on spaces of splines over three directional and four directional partitions both for the uniform and the non-uniform case (see for example [4, 5, 7, 21, 22]) or on other spline spaces as those generated by simplex splines (see [5, 8, 11, 18] and references quoted therein).

In this paper we address the problem of constructing quasi-interpolants in the space of quadratic splines over a Powell-Sabin refinement of a generic triangulation, [19]. The low degree and the simplicity of the Bézier-Bernstein representation coupled together with the possibility of handling arbitrary triangulations of polygonal domains, make this spline space very interesting not only from the theoretical point of view but also for applications. Thus, Powell-Sabin quadratic splines have been widely studied by several authors (see for example [6, 9, 10, 12, 17, 20, 26, 28]).

In the space of Powell-Sabin quadratic splines the Hermite interpolation problem at the vertices of the triangulation has a unique solution and the interpolant can be locally computed in each triangle of the triangulation. So, the Hermite interpolant can be seen as a special quasi-interpolant. However, other interesting quasi-interpolanting schemes can be constructed in this space. In particular, we describe quasi-interpolants, not requiring derivatives of the function f, which reproduce quadratic polynomials and we determine upper bounds for their infinity norm. Therefore, it turns out that the proposed quasi-interpolants provide the full approximation order in the space.

The results we present are based on the properties of the quadratic Powell-Sabin B-splines constructed and analysed by Dierckx and some coauthors, see [9, 24–28].

The remaining of the paper is divided into 5 sections. In the next one we briefly recall from [9] the construction and some salient properties of quadratic B-splines over a Powell-Sabin refinement of a triangulation of a planar domain. In Section 3 we discuss differential quasi-interpolants. In Section 4 we construct some families of discrete quasi-interpolants reproducing bivariate quadratic polynomials and upper bounds of their infinity norms are determined in Section 5. Finally, we end in Section 6 with some numerical examples and some final remarks.

2 Quadratic B-splines over a Powell-Sabin refinement

In this section we briefly summarize from [9] the construction and some properties of quadratic B-splines over a Powell-Sabin refinement of a triangulation of a planar domain (see also [24, 26–28]).

For the sake of simplicity, in the following the Bézier-Bernstein representation will be used to describe polynomials over triangles (see for example [9, 19, 20]).

Let τ be a triangle with vertices $\mathbf{V}_{i_j} := (x_{i_j}, y_{i_j})^T$, $j = 1, 2, 3$, and let (u, v, w) be the barycentric coordinates of a point $(x, y)^T \in \mathbb{R}^2$ with respect to the triangle τ, that is the values determined by the linear system

$$\begin{pmatrix} 1 & 1 & 1 \\ x_{i_1} & x_{i_2} & x_{i_3} \\ y_{i_1} & y_{i_2} & y_{i_3} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}.$$

Let $\mathbb{P}_n$ denote the linear space of algebraic polynomials of degree less than or equal to n. Any element $p \in \mathbb{P}_n$ has a unique representation in barycentric coordinates

$$p(x, y) = \sum_{i+j+k=n} b_{i,j,k} \frac{n!}{i!\,j!k!} u^i v^j w^k.$$

The coefficients $b_{i,j,k}$ are the Bézier ordinates of the polynomial p with respect to the triangle τ. Usually, this representation is called Bézier-Bernstein representation of p and it is schematically represented by associating each coefficient $b_{i,j,k}$ with the domain point

$$(x_{i,j,k}, y_{i,j,k}) \tag{1}$$

having barycentric coordinates $(\frac{i}{n}, \frac{j}{n}, \frac{k}{n})$. The points

$$(x_{i,j,k}, y_{i,j,k}, b_{i,j,k}) \in \mathbb{R}^3,\ i + j + k = n,$$

are the Bézier control points of p.

Let Ω be a polygonal domain in $\mathbb{R}^2$ and let Δ be a regular triangulation of Ω. We denote by

$$\mathbf{V}_l := (x_l, y_l)^T,\ l = 1, \dots, N_V,$$

the vertices of the given triangulation. A Powell-Sabin refinement, Δ_{PS}, of Δ is the refined triangulation, [19], obtained (see Fig. 1) by subdividing each triangle of Δ into six subtriangles as follows. Select a point, say $\mathbf{C}_j$, inside any triangle τ_j of Δ and connect it with the three vertices of τ_j and with the points $\mathbf{C}_{j_1}, \mathbf{C}_{j_2}, \mathbf{C}_{j_3}$ where $\tau_{j_1},\ \tau_{j_2},\ \tau_{j_3}$ are the triangles adjacent to τ_j. If τ_j is a boundary triangle the undefined $\mathbf{C}_{j_i}$ are specified points (usually the midpoints) inside the corresponding boundary edges. We assume that each segment $\mathbf{C}_j\mathbf{C}_{j_i},\ i = 1, 2, 3$, intersects the interior of the common edge of τ_j and τ_{j_i}.

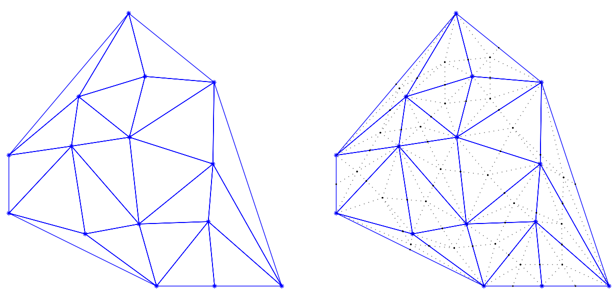


Fig. 1 A triangulation Δ and a Powell-Sabin refinement Δ_{PS} of Δ

We denote by $\mathcal{S}_2^1(\Delta_{PS})$ the space of Powell-Sabin splines, [19], that is the linear space of piecewise quadratic polynomials on Δ_{PS} with C^1 continuity in Ω. It is well known, [9, 19, 20], that:

- $dim(\mathcal{S}_2^1(\Delta_{PS})) = 3N_V$;
- any element of $\mathcal{S}_2^1(\Delta_{PS})$ is determined by its value and its gradient at the vertices of Δ;
- any element of $\mathcal{S}_2^1(\Delta_{PS})$ can be locally computed in each triangle of Δ provided that its values and its gradients at the three vertices of the triangle are given.

In [9] *B-spline bases* of the space $\mathcal{S}_2^1(\Delta_{PS})$ have been proposed as follows. Let us associate three functions to any vertex of Δ

$$\{B_l^{(j)}, \quad j = 1, 2, 3, \quad l = 1, \ldots, N_V\}, \tag{2}$$

such that $s = \sum_{l=1}^{N_V} \sum_{j=1}^{3} c_{l,j} B_l^{(j)}$ for all $s \in \mathcal{S}_2^1(\Delta_{PS})$, and

$$B_l^{(j)}(x, y) \geq 0, \quad \sum_{l=1}^{N_V} \sum_{j=1}^{3} B_l^{(j)}(x, y) = 1. \tag{3}$$

A system satisfying these properties is often called a "blending system". The functions $B_l^{(j)}$ will be referred to as Powell-Sabin B-splines.

Let Ω_l be the subset of Ω consisting of the points belonging to the union of all the triangles of Δ containing the vertex $\mathbf{V}_l$. From the Bézier-Bernstein representation, it is immediate to see that if we ask Powell-Sabin B-splines to have minimal support then the support for any $B_l^{(j)}$ is contained in the cell Ω_l.

The three functions $B_l^{(1)}$, $B_l^{(2)}$, $B_l^{(3)}$ can be locally constructed over the cell Ω_l once their values and gradients at any vertex of Δ are given. Due to the structure of the support we have:

$$B_l^{(j)}(\mathbf{V}_k) = 0, \ \frac{\partial}{\partial x} B_l^{(j)}(\mathbf{V}_k) = 0, \ \frac{\partial}{\partial y} B_l^{(j)}(\mathbf{V}_k) = 0, \ \text{ if } k \neq l. \tag{4}$$

Moreover, we denote:

$$B_l^{(j)}(\mathbf{V}_l) =: \alpha_l^{(j)}, \ \frac{\partial}{\partial x} B_l^{(j)}(\mathbf{V}_l) =: \beta_l^{(j)}, \ \frac{\partial}{\partial y} B_l^{(j)}(\mathbf{V}_l) =: \gamma_l^{(j)}. \tag{5}$$

In order to obtain a partition of unity, we have to impose

$$\begin{aligned} \alpha_l^{(1)} + \alpha_l^{(2)} + \alpha_l^{(3)} &= 1, \\ \beta_l^{(1)} + \beta_l^{(2)} + \beta_l^{(3)} &= 0, \\ \gamma_l^{(1)} + \gamma_l^{(2)} + \gamma_l^{(3)} &= 0. \end{aligned} \tag{6}$$

Using the Bézier-Bernstein representation, it is possible to prove, [9], that $B_l^{(j)}$ is non negative if and only if the Bézier ordinates of the Bézier points (1) which are direct neighbours of the vertex $\mathbf{V}_l$ are non negative (see Fig. 2). The positivity of

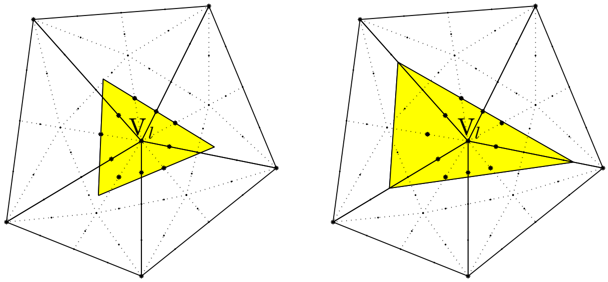


Fig. 2 $*$: location of the Bézier points which are "direct neighbours" of the vertex V_l and (shaded) two possible "valid" triangles for them

the above mentioned coefficients has the following very nice geometric interpretation, [9]. Let us denote $e_{10}(x, y) := x$, $e_{01}(x, y) := y$. For each index l, we want to determine points $\{\mathbf{Q}_l^{(j)},\ j = 1, 2, 3\}$ such that the following expansions hold

$$e_{10} = \sum_{l=1}^{N_V} \sum_{j=1}^{3} e_{10}\left(\mathbf{Q}_l^{(j)}\right) B_l^{(j)}, \quad e_{01} = \sum_{l=1}^{N_V} \sum_{j=1}^{3} e_{01}\left(\mathbf{Q}_l^{(j)}\right) B_l^{(j)}. \tag{7}$$

Since $e_{10}, e_{01} \in \mathcal{S}_2^1(\Delta_{PS})$, then they are uniquely determined by their values and their gradients at the vertices of Δ. So, denoting by $F_{k,x},\ F_{k,y}$ the coordinates of the point $\mathbf{F}_k$, we have

$$M_l \begin{pmatrix} Q_{l,x}^{(1)} \\ Q_{l,x}^{(2)} \\ Q_{l,x}^{(3)} \end{pmatrix} = \begin{pmatrix} x_l \\ 1 \\ 0 \end{pmatrix}, \quad M_l \begin{pmatrix} Q_{l,y}^{(1)} \\ Q_{l,y}^{(2)} \\ Q_{l,y}^{(3)} \end{pmatrix} = \begin{pmatrix} y_l \\ 0 \\ 1 \end{pmatrix}, \quad l = 1, \ldots, N_V, \tag{8}$$

where

$$M_l := \begin{pmatrix} \alpha_l^{(1)} & \alpha_l^{(2)} & \alpha_l^{(3)} \\ \beta_l^{(1)} & \beta_l^{(2)} & \beta_l^{(3)} \\ \gamma_l^{(1)} & \gamma_l^{(2)} & \gamma_l^{(3)} \end{pmatrix}. \tag{9}$$

The Powell-Sabin B-splines are required to be linearly independent, therefore the matrix M_l is non singular. So, for each vertex $\mathbf{V}_l$, systems (8) uniquely determine three points $\mathbf{Q}_l^{(j)},\ j = 1, 2, 3$. From [9], Section 4, we have the following result (see also Fig. 2)

Theorem 1 *The functions $B_l^{(j)},\ j = 1, 2, 3$, are non negative if and only if the triangle with vertices $\mathbf{Q}_l^{(j)},\ j = 1, 2, 3$, contains the Bézier points (1) which are direct neighbours of $\mathbf{V}_l$.*

On the other hand, it is immediate to see that, given three non collinear points $\mathbf{Q}_l^{(j)},\ j = 1, 2, 3$, they uniquely determine, via (8) and (6), the values and gradients of the three functions $B_l^{(j)},\ j = 1, 2, 3$, at the vertex $\mathbf{V}_l$ and so they completely determine the three functions $B_l^{(j)}$. Summarizing, the points $\mathbf{Q}_l^{(j)},\ j = 1, 2, 3$ – and so the triangle they form – are uniquely associated with the triple $B_l^{(j)},\ j = 1, 2, 3$, and can be efficiently used to identify and describe these functions and their properties instead of $\alpha_l^{(j)}$, $\beta_l^{(j)}$, $\gamma_l^{(j)}$. Moreover, [9], triangles with a small area produce B-splines with "better" computational properties. In [9] an optimization strategy has been proposed to select triangles with minimal area ensuring positivity of the corresponding B-splines.

3 Differential quasi-interpolants in $\mathcal{S}_2^1(\Delta_{PS})$

In this section we describe how the B-splines introduced in Section 2 can be used to define q.i.s, based on derivatives of the given function f, which are exact on $\mathbb{P}_2$.

We are interested in q.i.s of the following form

$$\mathcal{Q}f = \sum_{l=1}^{N_V} \sum_{j=1}^{3} \mu_l^{(j)}(f) B_l^{(j)}, \tag{10}$$

where $\mu_l^{(j)}(f)$, $j = 1, 2, 3$, $l = 1, \ldots, N_V$, are suitable linear functionals. First of all we note (see also [27]) that, if

$$f(\mathbf{V}_l), \quad \nabla f(\mathbf{V}_l), \quad l = 1, \ldots, N_V, \tag{11}$$

are given, setting

$$\begin{pmatrix} \mu_l^{(1)}(f) \\ \mu_l^{(2)}(f) \\ \mu_l^{(3)}(f) \end{pmatrix} := M_l^{-1} \begin{pmatrix} f(\mathbf{V}_l) \\ f_x(\mathbf{V}_l) \\ f_y(\mathbf{V}_l) \end{pmatrix}, \tag{12}$$

then, from (5) and (9), expression (10) provides the unique element in $\mathcal{S}_2^1(\Delta_{PS})$ which interpolates the data (11). So, the scheme (10) with coefficients given by (12) is a quasi-interpolating (actually Hermite interpolating) scheme in $\mathcal{S}_2^1(\Delta_{PS})$ which obviously reproduces $\mathbb{P}_2$. Moreover, (12) and (9) show that this is the unique q.i. of the form (10) with

$$\mu_l^{(j)}(f) := a_l^{(j)} f(\mathbf{V}_l) + b_l^{(j)} f_x(\mathbf{V}_l) + c_l^{(j)} f_y(\mathbf{V}_l), \quad a_l^{(j)}, b_l^{(j)}, c_l^{(j)} \in \mathbb{R},$$

which is a projection, [14], in $\mathcal{S}_2^1(\Delta_{PS})$, i.e. satisfies

$$\mathcal{Q}s = s, \quad \forall s \in \mathcal{S}_2^1(\Delta_{PS}).$$

A second q.i. reproducing $\mathbb{P}_2$ can be obtained as a modification of the Schoenberg-Marsden type ([16]) scheme

$$\mathcal{Q}_1 f := \sum_{l=1}^{N_V} \sum_{j=1}^{3} f\left(\mathbf{Q}_l^{(j)}\right) B_l^{(j)}. \tag{13}$$

From (6) and (7) we have

$$(\mathcal{Q}_1 p)(\mathbf{V}_l) = p(\mathbf{V}_l), \quad \nabla(\mathcal{Q}_1 p)(\mathbf{V}_l) = \nabla p(\mathbf{V}_l), \; l = 1, \ldots, N_V, \; \forall p \in \mathbb{P}_1,$$

so that

$$\mathcal{Q}_1 p = p, \quad \forall p \in \mathbb{P}_1. \tag{14}$$

Thus, we modify $\mathcal{Q}_1 f$ in order to obtain reproduction of quadratic polynomials. We have the following result

Theorem 2 *Let us define*

$$\tilde{\mathcal{Q}}_2 f := \sum_{l=1}^{N_V} \sum_{j=1}^{3} \tilde{\mu}_l^{(j)}(f) B_l^{(j)}. \tag{15}$$

where, for $j = 1, 2, 3,\ l = 1, \ldots, N_V$

$$\tilde{\mu}_l^{(j)}(f) := f\left(\mathbf{Q}_l^{(j)}\right) - \frac{1}{2}\left(\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right)^T \nabla^2 f(\mathbf{V}_l)\left(\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right). \tag{16}$$

Then,

$$\tilde{\mathcal{Q}}_2 p = p,\ \forall p \in \mathbb{P}_2.$$

Proof Let us consider $p \in \mathbb{P}_2$. From Taylor expansion

$$p\left(\mathbf{Q}_l^{(j)}\right) = p(\mathbf{V}_l) + \nabla^T p(\mathbf{V}_l)\left(\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right) + \frac{1}{2}\left(\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right)^T \nabla^2 p(\mathbf{V}_l)\left(\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right),$$

thus, from (16)

$$\tilde{\mathcal{Q}}_2 p = \sum_{i=1}^{N_V} \sum_{j=1}^{3} \left[p(\mathbf{V}_i) + \nabla^T p(\mathbf{V}_i)\left(\mathbf{Q}_i^{(j)} - \mathbf{V}_i\right)\right] B_i^{(j)}.$$

Hence, from (4)

$$\begin{aligned}(\tilde{\mathcal{Q}}_2 p)(\mathbf{V}_l) &= \sum_{j=1}^{3} \left[p(\mathbf{V}_l) + \nabla^T p(\mathbf{V}_l)\left(\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right)\right] B_l^{(j)}(\mathbf{V}_l) \\ &= p(\mathbf{V}_l)\sum_{j=1}^{3} B_l^{(j)}(\mathbf{V}_l) + \nabla^T p(\mathbf{V}_l)\left(\sum_{j=1}^{3} \mathbf{Q}_l^{(j)} B_l^{(j)}(\mathbf{V}_l)\right) \\ &\quad - \nabla^T p(\mathbf{V}_l)(\mathbf{V}_l)\sum_{j=1}^{3} B_l^{(j)}(\mathbf{V}_l),\end{aligned}$$

so that, from (5), (6) and (8) we obtain

$$(\tilde{\mathcal{Q}}_2 p)(\mathbf{V}_l) = p(\mathbf{V}_l),\ l = 1, \ldots, N_V.$$

Similarly,

$$\nabla(\tilde{\mathcal{Q}}_2 p)(\mathbf{V}_l) = \nabla p(\mathbf{V}_l),\ l = 1, \ldots, N_V.$$

The assertion follows since any element in $\mathcal{S}_2^1(\Delta_{PS})$ is uniquely determined by its values and gradients at the vertices of Δ. □

4 Discrete quasi-interpolants in $\mathcal{S}_2^1(\Delta_{PS})$

The Hermite interpolant and the q.i. (15) require the computation of values of the function f and of some of its derivatives. In this section we describe how it is possible to construct q.i.s of the form (10) which reproduce $\mathbb{P}_2$ and do not require values of derivatives of f. More precisely, we are interested in q.i.s of the form (10) with

$$\mu_l^{(j)}(f) := \sum_{k=1}^{N_l^{(j)}} q_l^{(j,k)} f\left(\mathbf{Z}_l^{(j,k)}\right),$$
$$q_l^{(j,k)} \in \mathbb{R},\ q_l^{(j,k)} \neq 0,\ \mathbf{Z}_l^{(j,k)} \in \mathbb{R}^2,\ 1 \leq N_l^{(j)} \in \mathbb{N}. \tag{17}$$

Since the set of functions (2) forms a basis of $\mathcal{S}_2^1(\Delta_{PS})$, from (13) and (14) we deduce that any q.i. defined via (10) and (17) reproduces $\mathbb{P}_1$ if and only if

$$1 = \sum_{k=1}^{N_l^{(j)}} q_l^{(j,k)}, \quad \mathbf{Q}_l^{(j)} = \sum_{k=1}^{N_l^{(j)}} q_l^{(j,k)} \mathbf{Z}_l^{(j,k)}. \tag{18}$$

The following result shows that, for a q.i. of the form (10) and (17) which reproduces quadratic polynomials, with $N_l^{(j)} \leq 3$, the set of points $\{\mathbf{Z}_l^{(j,k)},\ k = 1, \ldots, N_l^{(j)}\}$ has a specific geometric configuration.

Theorem 3 *Let $\mathcal{Q}$ be any q.i. of the form* (10) *with $\mu_l^{(j)}$ defined according to* (17) *and $N_l^{(j)} \leq 3$. If*

$$\mathcal{Q}p = p, \quad \forall p \in \mathbb{P}_2, \tag{19}$$

then, for any $l \in \{1, \ldots, N_V\}$ and $j \in \{1, 2, 3\}$, the points

$$\mathbf{V}_l, \mathbf{Q}_l^{(j)}, \mathbf{Z}_l^{(j,k)},\ k = 1, \ldots, N_l^{(j)},$$

are collinear.

Proof From Theorem 2 we know that (16) provides the coefficients of the representation of any polynomial of degree less than or equal to 2 with respect to the basis (2). Thus, from (19) we have that

$$\mu_l^{(j)}(p) = \tilde{\mu}_l^{(j)}(p), \quad \forall p \in \mathbb{P}_2,\ l = 1, \ldots, N_V, \quad j = 1, 2, 3. \tag{20}$$

Let us fix $l \in \{1, \ldots, N_V\}$ and $j \in \{1, 2, 3\}$. Without loss of generality we can assume $\mathbf{V}_l = (0, 0)^T$. Specifying equality (20) for the monomial basis of bivariate quadratic polynomials we obtain

$$
\begin{aligned}
\sum_{k=1}^{N_l^{(j)}} q_l^{(j,k)} &= 1 \\
\sum_{k=1}^{N_l^{(j)}} q_l^{(j,k)} Z_{l,x}^{(j,k)} &= \mathrm{Q}_{l,x}^{(j)} \\
\sum_{k=1}^{N_l^{(j)}} q_l^{(j,k)} Z_{l,y}^{(j,k)} &= \mathrm{Q}_{l,y}^{(j)} \\
\sum_{k=1}^{N_l^{(j)}} q_l^{(j,k)} \left(Z_{l,x}^{(j,k)}\right)^2 &= 0 \\
\sum_{k=1}^{N_l^{(j)}} q_l^{(j,k)} \left(Z_{l,y}^{(j,k)}\right)^2 &= 0 \\
\sum_{k=1}^{N_l^{(j)}} q_l^{(j,k)} \left(Z_{l,x}^{(j,k)}\right)\left(Z_{l,y}^{(j,k)}\right) &= 0.
\end{aligned}
\tag{21}
$$

Since $N_l^{(j)} \leq 3$, the system (21) can have solutions only if the rows of the matrix

$$
\begin{pmatrix}
\left(Z_{l,x}^{(j,1)}\right)^2 & \cdots & \left(Z_{l,x}^{(j,N_l^{(j)})}\right)^2 \\
\left(Z_{l,y}^{(j,1)}\right)^2 & \cdots & \left(Z_{l,y}^{(j,N_l^{(j)})}\right)^2 \\
Z_{l,x}^{(j,1)} Z_{l,y}^{(j,1)} & \cdots & Z_{l,x}^{\left(j,N_l^{(j)}\right)} Z_{l,y}^{\left(j,N_l^{(j)}\right)}
\end{pmatrix}
$$

are linearly dependent, that is only if the cartesian coordinates of the points $\mathbf{Z}_l^{(j,k)},\ k = 1, \ldots, N_l^{(j)}$ satisfy the equation

$$
rx^2 + sy^2 + txy = 0, \tag{22}
$$

for some $r, s, t \in \mathbb{R},\ r^2 + s^2 + t^2 > 0$. The locus determined by (22) in the plane is the union of two straight lines through the origin or reduces to the point $(0, 0)^T$. Thus, if system (21) has solutions the points $\mathbf{Z}_l^{(j,k)},\ k = 1, \ldots, N_l^{(j)} \leq 3$, belong to two straight lines through $\mathbf{V}_l$.

If $N_l^{(j)} = 1$, it is clear that system (21) can have solutions only if $\mathbf{Z}_l^{(j,1)} = \mathbf{Q}_l^{(j)} = \mathbf{V}_l$, so that the assertion holds.

Now, let us consider the case $N_l^{(j)} = 3$. If two points, say $\mathbf{Z}_l^{(j,1)},\ \mathbf{Z}_l^{(j,2)}$, and $\mathbf{V}_l$ are collinear, since from (17) $q_l^{(j,3)} \neq 0$, from the last three equations in (21) we have that $\mathbf{Z}_l^{(j,3)}$ is collinear with $\mathbf{Z}_l^{(j,1)},\ \mathbf{Z}_l^{(j,2)}$ and $\mathbf{V}_l$ too. Thus, $\mathbf{Z}_l^{(j,1)},\ \mathbf{Z}_l^{(j,2)},\ \mathbf{Z}_l^{(j,3)}$ and $\mathbf{V}_l$ are collinear and $\mathbf{Q}_l^{(j)}$ has to belong to the same line otherwise the first three equations of (21) cannot hold.

In a similar way we can conclude that the same result is valid in the case $N_l^{(j)} = 2$. □

As noticed in the proof of the previous theorem, q.i.s of the form (10), (17) with $N_l^{(j)} = 1$, can reproduce quadratic polynomials only if $\mathbf{Q}_l^{(j)} = \mathbf{V}_l$. From Theorem 1, this condition can be satisfied only by boundary vertices. Thus, we will concentrate our interest in the cases $N_l^{(j)} = 2, 3$.

First, considering the results of Theorem 3 we completely describe the q.i.s of the form (10) with coefficients given by (17), reproducing quadratic polynomials, in the case $N_l^{(j)} = 2$.

Theorem 4 *Let $\mathcal{Q}$ be any q.i. of the form (10) with $\mu_l^{(j)}$ defined according to (17). Then*

$$\mathcal{Q}p = p, \ \forall p \in \mathbb{P}_2$$

if and only if, for any $l = 1, \dots, N_V, \ j = 1, 2, 3$, with $N_l^{(j)} = 2$ and $\mathbf{Q}_l^{(j)} \neq \mathbf{V}_l$ we have

$$\mathbf{Z}_l^{(j,1)} := \left(1 - \frac{1}{\zeta_l^{(j)}}\right)\mathbf{V}_l + \frac{1}{\zeta_l^{(j)}}\mathbf{Q}_l^{(j)},$$

$$\mathbf{Z}_l^{(j,2)} := -\frac{\zeta_l^{(j)}}{\left(1-\zeta_l^{(j)}\right)}\mathbf{V}_l + \frac{1}{\left(1-\zeta_l^{(j)}\right)}\mathbf{Q}_l^{(j)},$$

$$q_l^{(j,1)} := \frac{\left(\zeta_l^{(j)}\right)^2}{2\zeta_l^{(j)}-1}, q_l^{(j,2)} := -\frac{\left(1-\zeta_l^{(j)}\right)^2}{2\zeta_l^{(j)}-1}, \quad \zeta_l^{(j)} \in \mathbb{R}, \ \zeta_l^{(j)} \neq 0, \frac{1}{2}, 1. \tag{23}$$

Proof Let us put $\mathbf{Q}_l^{(j)} = (1-\zeta_l^{(j,k)})\mathbf{V}_l + \zeta_l^{(j,k)}\mathbf{Z}_l^{(j,k)}, \ k=1,2$. Since $\mathbf{Q}_l^{(j)} \neq \mathbf{V}_l$, imposing

$$\mu_l^{(j)}(p) = \tilde{\mu}_l^{(j)}(p), \quad p \in \{1, x, y, x^2, y^2, xy\},$$

results (see also the proof of Theorem 3) in the nonlinear system

$$q_l^{(j,1)} + q_l^{(j,2)} = 1$$

$$\frac{q_l^{(j,1)}}{\zeta_l^{(j,1)}} + \frac{q_l^{(j,2)}}{\zeta_l^{(j,2)}} = 1$$

$$\frac{q_l^{(j,1)}}{\left(\zeta_l^{(j,1)}\right)^2} + \frac{q_l^{(j,2)}}{\left(\zeta_l^{(j,2)}\right)^2} = 0$$

whose solutions are the values of $q_l^{(j,k)}, \zeta_l^{(j,k)}, k=1,2$, given in (23) setting $\zeta_l^{(j)} = \zeta_l^{(j,1)}$. □

For the case $N_l^{(j)} = 3$, the following Theorem provides a family of q.i.s reproducing quadratic polynomials.

Theorem 5 *For $l = 1, \dots, N_V, \ j = 1, 2, 3$ let $\mathbf{W}_{l_j} \in \mathbb{R}^2, \ \mathbf{W}_{l_j} \neq \mathbf{V}_l$, be so that*

$$\mathbf{Q}_l^{(j)} = \left(1 - \lambda_l^{(j)}\right)\mathbf{V}_l + \lambda_l^{(j)}\mathbf{W}_{l_j}, \ \ \lambda_l^{(j)} \in \mathbb{R}.$$

Let $\mathcal{Q}$ *be any q.i. of the form* (10) *with* $\mu_l^{(j)}$ *defined according to* (17) *where, for* $l = 1, \ldots, N_V,\ j = 1, 2, 3,\ N_l^{(j)} = 3$,

$$\begin{aligned}
\mathbf{Z}_l^{(j,1)} &:= \mathbf{V}_l, \\
\mathbf{Z}_l^{(j,2)} &:= \left(1 - \zeta_l^{(j)}\right)\mathbf{V}_l + \zeta_l^{(j)}\mathbf{W}_{l_j},\ \ \zeta_l^{(j)} \in \mathbb{R},\ \zeta_l^{(j)} \neq 0, 1, \\
\mathbf{Z}_l^{(j,3)} &:= \mathbf{W}_{l_j},
\end{aligned} \tag{24}$$

and

$$q_l^{(j,1)} := 1 - \lambda_l^{(j)} - \frac{\lambda_l^{(j)}}{\zeta_l^{(j)}},\ q_l^{(j,2)} := \frac{\lambda_l^{(j)}}{\zeta_l^{(j)}\left(1 - \zeta_l^{(j)}\right)},\ q_l^{(j,3)} := -\frac{\lambda_l^{(j)}\zeta_l^{(j)}}{1 - \zeta_l^{(j)}}, \tag{25}$$

then

$$\mathcal{Q}p = p, \forall p \in \mathbb{P}_2.$$

Proof For any $l = 1, \ldots, N_V$, and $j = 1, 2, 3$, we consider the triple of points $\mathbf{Z}_l^{(j,k)},\ k = 1, 2, 3$, given in (24) and we impose

$$\mu_l^{(j)}(p) = \tilde{\mu}_l^{(j)}(p), \quad p \in \left\{1, x, y, x^2, y^2, xy\right\}.$$

This results (see also the proof of Theorem 3) in the system

$$\begin{aligned}
q_l^{(j,1)} + q_l^{(j,2)} + q_l^{(j,3)} &= 1 \\
q_l^{(j,1)} + \left(1 - \zeta_l^{(j)}\right) q_l^{(j,2)} &= 1 - \lambda_l^{(j)} \\
q_l^{(j,2)}\zeta_l^{(j)} + q_l^{(j,3)} &= \lambda_l^{(j)} \\
q_l^{(j,2)}\left(\zeta_l^{(j)}\right)^2 + q_l^{(j,3)} &= 0.
\end{aligned} \tag{26}$$

It is immediate to verify that the values (25) uniquely solve (26) so that the corresponding $\mu_l^{(j)}(p),\ l = 1, \ldots, N_V,\ j = 1, 2, 3$, exactly provide the coefficients of $p \in \mathbb{P}_2$ with respect to the basis (2). □

The q.i.s constructed in Theorem 5 require evaluating f at points collinear with the vertices of Δ and with the points $\mathbf{Q}_l^{(j)}$ defining the used family of B-splines. So, it seems natural to investigate the case where the above mentioned points lie over the edges of Δ. In order to do that we have to assume that the partition Δ and the family of B-splines we are dealing with satisfy the following hypothesis (see Figs. 2 and 3).

Hypothesis 6 *For any pair* $(\mathbf{V}_l,\ \mathbf{Q}_l^{(j)})$ *there exists a vertex* $\mathbf{V}_{l_j}$ *of* Δ *such that* $\mathbf{V}_l$, $\mathbf{Q}_l^{(j)}$, $\mathbf{V}_{l_j}$ *are collinear.*

If Hypothesis 6 holds, we can easily estimate

$$\frac{1}{2}\left(\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right)^T \nabla^2 f(\mathbf{V}_l)\left(\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right)$$

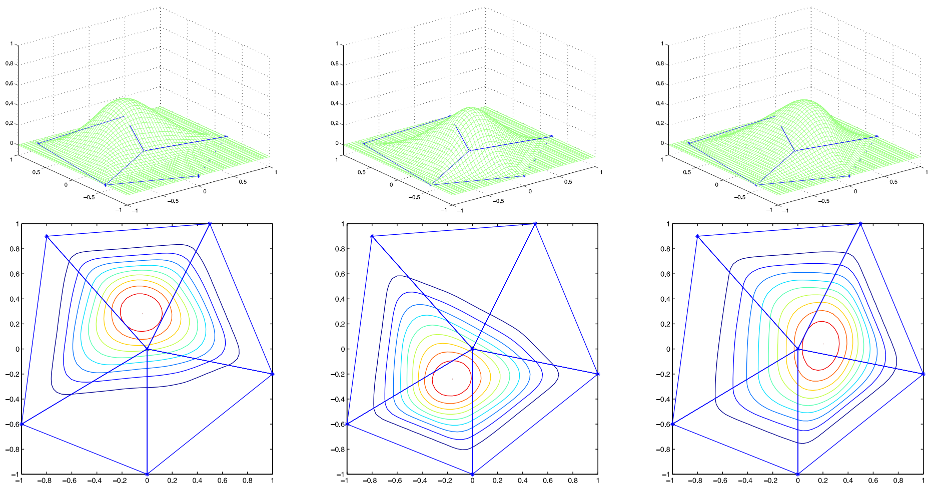


Fig. 3 The three B-splines, $B_l^{(j)}$, $j=1,2,3$ associated with the vertex V_l corresponding to the triangle in Fig. 2 right. Top: the functions $B_l^{(j)}$, Bottom: contour lines for $B_l^{(j)}$

by the second (univariate) divided difference of f along $\mathbf{V}_l\mathbf{Q}_l^{(j)}$,

$$\left\|\mathbf{V}_l\mathbf{Q}_l^{(j)}\right\|^2\left[\mathbf{V}_l,\ \mathbf{Q}_l^{(j)},\mathbf{V}_{l_j}\right]f.$$

This estimate is exact for any element of $\mathbb{P}_2$, and involves values of f at $\mathbf{V}_l,\ \mathbf{Q}_l^{(j)},\mathbf{V}_{l_j}$ only. Thus, from Theorem 2 we immediately have the following result

Theorem 7 *If Hypothesis 6 holds, define*

$$\mathcal{Q}_2 f := \sum_{l=1}^{N_V}\sum_{j=1}^{3}\mu_l^{(j)}(f)B_l^{(j)}$$

$$\mu_l^{(j)}(f) := f\left(\mathbf{Q}_l^{(j)}\right)-\left\|\mathbf{V}_l\mathbf{Q}_l^{(j)}\right\|^2\left[\mathbf{V}_l,\ \mathbf{Q}_l^{(j)},\mathbf{V}_{l_j}\right]f. \qquad (27)$$

then

$$\mathcal{Q}_2 p = p,\ \forall p\in\mathbb{P}_2.$$

Remark 8 When Hypothesis 6 holds, if we choose $\mathbf{W}_{l_j}:=\mathbf{V}_{l_j},\ l=1,\ldots,N_V,\ j=1,2,3$ in (24) and $\zeta_l^{(j)}=\lambda_l^{(j)}$ in (24)-(25) (if $\lambda_l^{(j)}\neq 0,1$) then the q.i. defined by (10), (17), (24) and (25) coincides with (27).

Remark 9 The q.i. (27) reproduces quadratic polynomials and only uses linear combination of the values of the given function f at the vertices of Δ or at points belonging to edges of the triangulation (or to their prolongation). However, though it seems reasonable to assume that the function f can be evaluated at some (given) points belonging to the edges of Δ, on the other hand it seems too restrictive to ask for values of f at any point of the edges of Δ. So, it can be useful to deal with q.i.s where the evaluation points can be selected according to the available data. In this respect, if Hypothesis 6 holds, we can apply Theorem 5 with the choice

$$\mathbf{W}_{l_j} := \mathbf{V}_{l_j},\ l = 1, \ldots, N_V,\ \ j = 1, 2, 3,$$

and selecting $\zeta_l^{(j)}$ so that $\mathbf{Z}_l^{(j,2)}$ in (24) does not necessarily coincides with $\mathbf{Q}_l^{(j)}$ but is a point where the value of f is available.

We end this Section by noting that, starting from the Hermite interpolant, (see (10), (12) and (9)), and using the same approach as in Theorem 7, it is not difficult to construct a q.i. which reproduces $\mathbb{P}_2$ and only uses values of the given function f. It suffices to approximate $\nabla f(\mathbf{V}_l)$ by means of a linear combination of values of f which provides an exact estimate for quadratic polynomials. As an example, if $\mathbf{S}_l^{(1)},\ \mathbf{S}_l^{(2)}$ are two points not collinear with $\mathbf{V}_l$ we can estimate the directional derivative

$$\nabla^T f(\mathbf{V}_l)\left(\mathbf{S}_l^{(k)} - \mathbf{V}_l\right),\ k = 1, 2,$$

by

$$d_l^{(k)} := \frac{f\left(\mathbf{D}_l^{(k)}\right) + \nu_{l,k}(\nu_{l,k} - 2) f(\mathbf{V}_l) - (1 - \nu_{l,k})^2 f\left(\mathbf{S}_l^{(k)}\right)}{\nu_{k,l}(1 - \nu_{k,l})}, \tag{28}$$

where

$$\mathbf{D}_l^{(k)} := \nu_{k,l}\mathbf{V}_l + (1 - \nu_{k,l})\mathbf{S}_l^{(k)},\ \nu_{k,l} \neq 0, 1,\ k = 1, 2.$$

From (28) we immediately obtain an estimate for the gradient of f which can be substituted to $\nabla f(\mathbf{V}_l)$ in (12) providing the following new expression for the coefficients of the q.i. (10)

$$\begin{pmatrix} \mu_l^{(1)} \\ \mu_l^{(2)} \\ \mu_l^{(3)} \end{pmatrix} := M_l^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & S_{l,x}^{(1)} - V_{l,x} & S_{l,y}^{(1)} - V_{l,y} \\ 0 & S_{l,x}^{(2)} - V_{l,x} & S_{l,y}^{(2)} - V_{l,y} \end{pmatrix}^{-1} \begin{pmatrix} f(\mathbf{V}_l) \\ d_l^{(1)} \\ d_l^{(2)} \end{pmatrix}. \tag{29}$$

From (8) we have

$$M_l^{-1} = \begin{pmatrix} 1 & Q_{l,x}^{(1)} - \mathbf{V}_{l,x} & Q_{l,y}^{(1)} - \mathbf{V}_{l,y} \\ 1 & Q_{l,x}^{(2)} - \mathbf{V}_{l,x} & Q_{l,y}^{(2)} - \mathbf{V}_{l,y} \\ 1 & Q_{l,x}^{(3)} - \mathbf{V}_{l,x} & Q_{l,y}^{(3)} - \mathbf{V}_{l,y} \end{pmatrix}.$$

Thus, setting

$$\mathbf{Q}_l^{(j)} = \left(1 - \zeta_l^{(j,1)} - \zeta_l^{(j,2)}\right)\mathbf{V}_l + \zeta_l^{(j,1)}\mathbf{S}_l^{(1)} + \zeta_l^{(j,2)}\mathbf{S}_l^{(2)},\quad \zeta_l^{(j,k)} \in \mathbb{R},\ j = 1, 2, 3,$$

after some manipulations we obtain a new and more elegant form of (29)

$$\mu_l^{(j)}(f) = f(\mathbf{V}_l) + \zeta_l^{(j,1)} d_l^{(1)} + \zeta_l^{(j,2)} d_l^{(2)}, \;\; j = 1, 2, 3. \tag{30}$$

As estimates (28) of the directional derivatives are exact for quadratic polynomials, the q.i. (10) with coefficients given by (29), (or 30) reproduces $\mathbb{P}_2$. Since each coefficient $\mu_l^{(j)}$ consists in a linear combination of five values of f, the obtained q.i. is of the form (17) with $N_l^{(j)} = 5$.

In order to minimize the number of needed values of f, it is convenient to select the points $\mathbf{S}_l^{(k)}$ as vertices of Δ. On the other hand, since any element of $\mathcal{S}_2^1(\Delta_{PS})$ is a quadratic polynomial in each subtriangle of the Powell-Sabin refinement of Δ, if the points $\mathbf{V}_l, \mathbf{S}_l^{(i)}, \mathbf{D}_l^{(i)}, \; i = 1, 2$ belong to the same subtriangle of Δ_{PS} then

$$\begin{pmatrix} S_{l,x}^{(1)} - \mathbf{V}_{l,x} & S_{l,y}^{(1)} - \mathbf{V}_{l,y} \\ S_{l,x}^{(2)} - \mathbf{V}_{l,x} & S_{l,y}^{(2)} - \mathbf{V}_{l,y} \end{pmatrix}^{-1} \begin{pmatrix} d_l^{(1)} \\ d_l^{(2)} \end{pmatrix} = \nabla^T f(\mathbf{V_l}), \forall f \in S_2^1(\Delta_{PS}).$$

Thus we can state the following:

Theorem 10 *Let $\mathcal{Q}$ be any q.i. of the form* (10) *with $\mu_l^{(j)}$ defined according to* (29). *If the points of each triple $\mathbf{V}_l, \mathbf{S}_l^{(i)}, \mathbf{D}_l^{(i)}, \; i = 1, 2$ belong to the same subtriangle of the Powell-Sabin refinement of Δ, then*

$$\mathcal{Q}s = s, \quad \forall s \in \mathcal{S}_2^1(\Delta_{PS}).$$

5 Bounding the norm of discrete quasi-interpolants

In this section we will provide upper bounds of the infinity norms of the discrete q.i.s in the space $\mathcal{S}_2^1(\Delta_{PS})$ described in the previous section.

This is motivated by the fact that, denoting by $\|g\|_{\infty,\Upsilon} := \sup_{\mathbf{x}\in\Upsilon} |g(\mathbf{x})|$, and by $\|\mathcal{Q}\|_{\infty,\Upsilon}$ the corresponding induced norm, if $\mathcal{Q}$ reproduces quadratic polynomials we have

$$\|\mathcal{Q}f - f\|_{\infty,\Omega} \leq \max_{l=1,\ldots,N_V} (1 + \|\mathcal{Q}\|_{\infty,\Omega_l}) \inf_{p\in\mathbb{P}_2} \|f - p\|_{\infty,\Omega_l}. \tag{31}$$

Thus, bounding $\|\mathcal{Q}\|_{\infty,\Omega_l}$ allows us to affirm that the q.i. is third order accurate, that is it provides the optimal approximation order in $\mathcal{S}_2^1(\Delta_{PS})$.

From the blending properties of the B-spline basis (see (3)), for any q.i. of the form (10) with coefficients $\mu_l^{(j)}$ given by (17), if $\mathbf{Z}_l^{(j,k)} \in \Omega_l$, we immediately have

$$\|\mathcal{Q}\|_{\infty,\Omega_l} \leq \max_{l=1,\ldots,N_V} \max_{j=1,2,3} \sum_{k=1}^{N_l^{(j)}} \left| q_l^{(j,k)} \right| =: \|\mathcal{Q}\|. \tag{32}$$

With some elementary manipulations we have the following results

Theorem 11 *Let $\mathcal{Q}$ be any q.i. of the form* (10) *with $\mu_l^{(j)}$ defined according to* (17) *and* (23) *then*

$$\left|\zeta_l^{(j)}\right| = \frac{\left\|\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right\|}{\left\|\mathbf{Z}_l^{(j,1)} - \mathbf{V}_l\right\|},$$

$$\left|1 - \zeta_l^{(j)}\right| = \frac{\left\|\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right\|}{\left\|\mathbf{Z}_l^{(j,2)} - \mathbf{V}_l\right\|},$$

$$\frac{\left|\left(1 - \zeta_l^{(j)}\right)\zeta_l^{(j)}\right|}{\left|2\zeta_l^{(j)} - 1\right|} = \frac{\left\|\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right\|}{\left\|\mathbf{Z}_l^{(j,2)} - \mathbf{Z}_l^{(j,1)}\right\|},$$

so that

$$\|\mathcal{Q}\| = \max_{l=1,\ldots,N_V} \max_{j=1,2,3} \left\{ \frac{\left\|\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right\|}{\left\|\mathbf{Z}_l^{(j,1)} - \mathbf{Z}_l^{(j,2)}\right\|} \frac{\left\|\mathbf{Z}_l^{(j,1)} - \mathbf{V}_l\right\|^2 + \left\|\mathbf{Z}_l^{(j,2)} - \mathbf{V}_l\right\|^2}{\left\|\mathbf{Z}_l^{(j,1)} - \mathbf{V}_l\right\| \left\|\mathbf{Z}_l^{(j,2)} - \mathbf{V}_l\right\|} \right\}. \tag{33}$$

As far as q.i.s based on three points are concerned (see Theorem 5 and Remarks 8 and 9) we limit ourselves to present an upper bound for a particularly significant q.i. of the family. However, completely similar results can be obtained for other choices of the values $\zeta_l^{(j)}$.

Theorem 12 *Let $\mathcal{Q}$ be any q.i. of the form* (10) *with $\mu_l^{(j)}$ defined according to* (17) *and* (24)*, with $\zeta_l^{(j)} = \lambda_l^{(j)}$, then*

$$|\zeta_l^{(j)}| = \frac{\left\|\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right\|}{\left\|\mathbf{Z}_l^{(j,3)} - \mathbf{V}_l\right\|} = \frac{\left\|\mathbf{Z}_l^{(j,2)} - \mathbf{V}_l\right\|}{\left\|\mathbf{Z}_l^{(j,3)} - \mathbf{V}_l\right\|}, \quad \left|1 - \zeta_l^{(j)}\right| = \frac{\left\|\mathbf{Z}_l^{(j,2)} - \mathbf{Z}_l^{(j,3)}\right\|}{\left\|\mathbf{Z}_l^{(j,3)} - \mathbf{V}_l\right\|},$$

so that

$$\|\mathcal{Q}\| = \max_{l=1,\ldots,N_V} \max_{j=1,2,3} \left\{ \frac{\left\|\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right\|^2}{\left\|\mathbf{Z}_l^{(j,3)} - \mathbf{V}_l\right\| \left\|\mathbf{Z}_l^{(j,3)} - \mathbf{Z}_l^{(j,2)}\right\|} + \frac{\left\|\mathbf{Z}_l^{(j,3)} - \mathbf{V}_l\right\|}{\left\|\mathbf{Z}_l^{(j,3)} - \mathbf{Z}_l^{(j,2)}\right\|} + \frac{\left\|\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right\|}{\left\|\mathbf{Z}_l^{(j,3)} - \mathbf{V}_l\right\|} \right\}. \tag{34}$$

It is worthwhile to notice that, in both cases, the upper bounds for $\|\mathcal{Q}\|_{\infty,\Omega_l}$ depend on the position of the points $\mathbf{Z}_l^{(j,k)}$ with respect to the points $\mathbf{Q}_l^{(j)}$ and $\mathbf{V}_l$.

Roughly speaking, once the family of B-splines, that is the points $\mathbf{Q}_l^{(j)}$, has been selected, the upper bound of the norm of the q.i. decreases as the two distances of $\mathbf{Z}_l^{(j,k)}$ from $\mathbf{V}_l$ and of $\mathbf{Z}_l^{(j,k)}$ from $\mathbf{Z}_l^{(j,r)}$ increase. Thus, " less localized" q.i.s have a lower upper bound for the infinity norm.

On the other hand, if we fix the points of $\mathbf{Z}_l^{(j,k)}$, and we select a family of B-splines, we have that the upper bound of the infinity norm of the corresponding q.i. decreases as the distance of the points $\mathbf{Q}_l^{(j)}$ from the vertex $\mathbf{V}_l$ decreases. Thus, *B-splines determined by points* $\mathbf{Q}_l^{(j)}$ *having minimal distance from* $\mathbf{V}_l$ (taking into account positivity of the functions $B_l^{(j)}$, see Theorem 1) *are preferable because they produce q.i.s with a lower bound for the infinity norm.*

In both cases it is of interest to provide upper bounds for $\|\mathbf{Q}_l^{(j)} - \mathbf{V}_l\|$. Let us denote by $\mathcal{I}_l$ the set of indices of vertices of Δ, different from $\mathbf{V}_l$, belonging to Ω_l (see Section 2). Let us put

$$H_l := \max_{j\in\mathcal{I}_l} \|\mathbf{V}_j - \mathbf{V}_l\|, \quad h_l := \min_{j\in\mathcal{I}_l} \|\mathbf{V}_j - \mathbf{V}_l\|.$$

With some elementary geometry we have that any equilateral triangle circumscribed to the circle centered in $\mathbf{V}_l$ and having radius $\frac{H_l}{2}$ contains all the Bézier points (1) which are direct neighbours of $\mathbf{V}_l$. Thus, choosing the points $\mathbf{Q}_l^{(j)}$, $j = 1, 2, 3$, as the vertices of such a triangle, the corresponding $B_l^{(j)}$ are non negative (see Theorem 1). Hence, if no specific configuration of the points $\mathbf{Q}_l^{(j)}$, $j = 1, 2, 3$, is required we can limit ourselves to work with families of B-splines such that

$$\left\|\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right\| \leq H_l. \tag{35}$$

On the other hand, if we deal with points $\mathbf{Q}_l^{(j)}$ belonging to the edges of Δ (or to their prolongation), see Hypothesis 6 and Fig. 4, we can observe (see Fig. 4) that the distance of $\mathbf{V}_l$ from any point of the edge $\mathbf{Q}_l^{(j)}\mathbf{Q}_l^{(j+1)}$ (superscripts modulo 3) is greater than

$$\min\left\{\left\|\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right\|, \left\|\mathbf{Q}_l^{(j+1)} - \mathbf{V}_l\right\|\right\} \cos\left(\frac{\theta_l^{(j)}}{2}\right),$$

Thus, assuming as it is reasonable, that

$$\frac{\left\|\mathbf{Q}_l^{(i)} - \mathbf{V}_l\right\|}{\left\|\mathbf{Q}_l^{(j)} - \mathbf{V}_l\right\|} \leq \frac{H_l}{h_l}, \; i, j = 1, 2, 3, \tag{36}$$

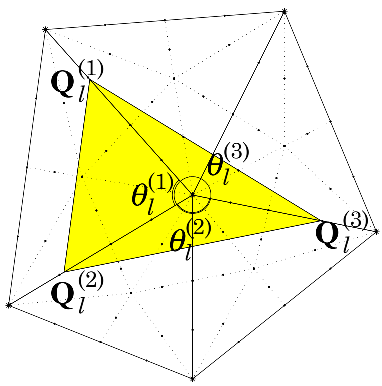


Fig. 4 Points $\mathrm{Q}_l^{(j)}$ satisfying Hypothesis 6 and corresponding angles $\theta_l^{(j)}$

we can have positive B-splines $B_l^{(j)}$ considering

$$\left\| \mathbf{Q}_l^{(j)} - \mathbf{V}_l \right\| \leq \frac{H_l}{h_l} \frac{H_l}{2} \max_{i=1,2,3} \left[\cos\left(\frac{\theta_l^{(i)}}{2} \right) \right]^{-1}. \tag{37}$$

Finally, bounds similar to (33) and (34) can be obtained for the family of q.i.s of the form (10) with $\mu_l^{(j)}$ defined according to (29). As an example we have

Theorem 13 *Let $\mathcal{Q}$ be any q.i. of the form* (10) *with $\mu_l^{(j)}$ defined according to* (28), (30) *with $\nu_{l,k} = \frac{1}{2}$. If $\mathbf{S}_l^{(i)}$ is collinear with $\mathbf{V}_l$ and $\mathbf{Q}_l^{(j_i)}$, $j_1 \neq j_2$, $j_1, j_2 \in \{1, 2, 3\}$, then*

$$\|\mathcal{Q}\| \leq 1 + 8 \max_{l=1,\ldots,N_V} \max \left\{ \frac{\left\| \mathbf{Q}_l^{(j_1)} - \mathbf{V}_l \right\|}{\left\| \mathbf{S}_l^{(1)} - \mathbf{V}_l \right\|}, \frac{\left\| \mathbf{Q}_l^{(j_2)} - \mathbf{V}_l \right\|}{\left\| \mathbf{S}_l^{(2)} - \mathbf{V}_l \right\|}, A_1 \frac{\left\| \mathbf{Q}_l^{(j_1)} - \mathbf{V}_l \right\|}{\left\| \mathbf{S}_l^{(1)} - \mathbf{V}_l \right\|} + A_2 \frac{\left\| \mathbf{Q}_l^{(j_2)} - \mathbf{V}_l \right\|}{\left\| \mathbf{S}_l^{(2)} - \mathbf{V}_l \right\|} \right\} \tag{38}$$

where A_i denotes the ratio among the area of the triangle $\mathbf{V}_l\mathbf{Q}_l^{(j_i+1)}\mathbf{Q}_l^{(j_i+2)}$ and that one of the triangle $\mathbf{V}_l\mathbf{Q}_l^{(j_1)}\mathbf{Q}_l^{(j_2)}$ (superscripts modulo 3).

In particular, the points $\mathbf{S}_l^{(i)}$ can be selected so that the triangle $\mathbf{V}_l\mathbf{Q}_l^{(j_1)}\mathbf{Q}_l^{(j_2)}$ has area of maximum value among the three triangles obtained connecting $\mathbf{V}_l$ with $\mathbf{Q}_l^{(j)}$, $j = 1, 2, 3$, so that

$$A_1, A_2 \leq 1. \tag{39}$$

Moreover, if Hypothesis 6 and (36) hold, choosing $\mathbf{S}_l^{(i)}, i = 1, 2,$ as vertices of Δ, different from $\mathbf{V}_l$, belonging to Ω_l and such that (39) holds, from (37) and (38) we obtain

$$\|\mathcal{Q}\| \leq 1 + 8 \max_{l=1,\ldots,N_V} \left(\frac{H_l}{h_l} \right)^2 \max_{i=1,2,3} \left[\cos\left(\frac{\theta_l^{(i)}}{2} \right) \right]^{-1}.$$

Remark 14 For a q.i. as in Theorem 10, estimates for $\|\mathcal{Q}f - f\|_{\infty,\Omega}$ can be also be obtained by using the error bounds for Hermite interpolation presented in [20] and considering bounds on differences between exact and approximate gradients.

6 Numerical examples

In this section we illustrate the numerical performances of some quasi-interpolating schemes presented above.

Before presenting the numerical results we note that, in order to construct in practice the q.i.s analysed in Theorem 7 and in Remark 9, we have to construct a family of (positive) B-splines whose associated points $\mathbf{Q}_l^{(j)}$ satisfy Hypothesis 6. Since, in order to ensure positivity of B-splines

$$B_l^{(1)},\ B_l^{(2)},\ B_l^{(3)}, \tag{40}$$

the triangle $\mathbf{Q}_l^{(1)}\mathbf{Q}_l^{(2)}\mathbf{Q}_l^{(3)}$ has to contain the Bézier points which are "direct neighbours" of $\mathbf{V}_l$ (see Theorem 1 and Fig. 2), then, in order to satisfy Hypothesis 6, it is necessary that the edges emanating from any interior vertex $\mathbf{V}_l$ belong at least to three different directions (special conditions can also be necessary for boundary vertices).

In case Hypothesis 6 can be satisfied, we try to obtain a B-spline basis of $\mathcal{S}_2^1(\Delta_{PS})$ with good performances from the computational point of view (see Section 2 and [9]) and producing q.i.s with a small upper bound (32) for the infinity norm (see Section 5). For any vertex $\mathbf{V}_l$ of Δ we will determine the triangle $\mathbf{Q}_l^{(1)}\mathbf{Q}_l^{(2)}\mathbf{Q}_l^{(3)}$ by imposing that its vertices belong to some edges of Δ emanating from $\mathbf{V}_l$ (or to their prolongation) and it has a small – possibly minimal – area, see Figs. 2 and 3.

The minimal area procedure proposed in [9] does not produce in general triangles with vertices belonging to the edges of Δ (see Fig. 2 and [9]). Algorithm 15 summarizes the procedure we have used, for the interior vertices $\{\mathbf{V}_l,\ l = 1, \ldots, N_I\}$, in order to satisfy the above properties. Of course, the outlined procedure has to be suitably adapted for the boundary vertices. Moreover, we mention that, from the numerous numerical tests we performed, it turns out that other "initial" strategies can be sometimes efficiently used instead of that one described in step **2.1**, which is suggested by (37). For the sake of brevity we omit the related details.

Algorithm 15.

1. Let the triangulation Δ and the Powell-Sabin refinement Δ_{PS} be given;

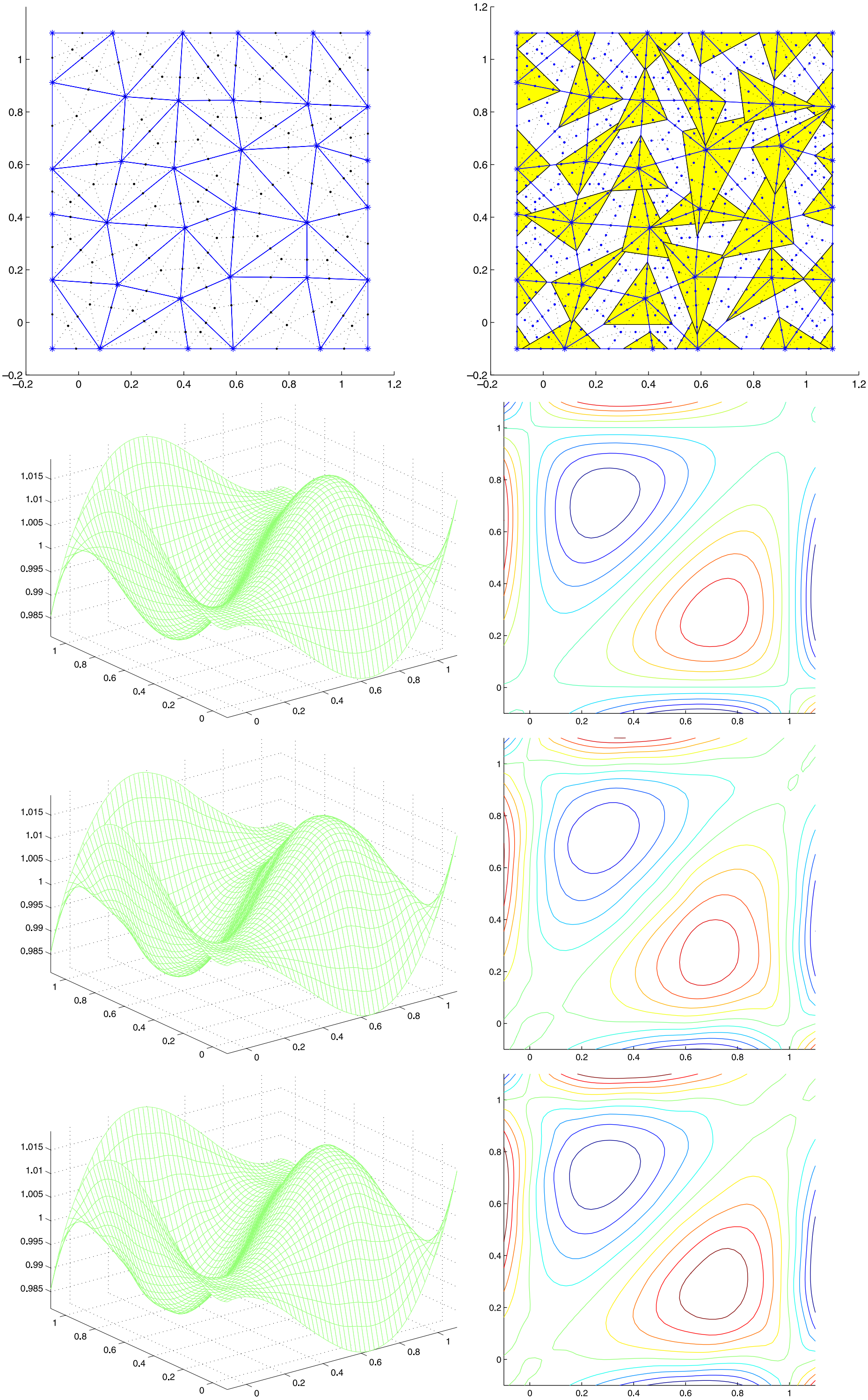
2. for $l = 1, \ldots, N_I$

- **2.1.** determine (if it is possible) a triangle $\hat{\mathbf{Q}}_l^{(1)}\hat{\mathbf{Q}}_l^{(2)}\hat{\mathbf{Q}}_l^{(3)}$ containing $\mathbf{V}_l$ in its interior, having vertices belonging to edges of Δ emanating from $\mathbf{V}_l$ (or onto their prolongation), which minimizes

$$\max_{i=1,2,3} \theta_l^{(i)}.$$

- **2.2.** select a point $\mathbf{Q}_l^{(j)}$ over the line $\hat{\mathbf{Q}}_l^{(j)}\mathbf{V}_l,\ j = 1, 2, 3,$ so that the triangle $\mathbf{Q}_l^{(1)}\mathbf{Q}_l^{(1)}\mathbf{Q}_l^{(3)}$ has minimal area and contains all the Bézier points which are direct neighbours of the vertex $\mathbf{V}_l$;
- **2.3.** compute the three B-splines $B_l^{(j)},\ j = 1, 2, 3,$ corresponding to the three points $\mathbf{Q}_l^{(j)},\ j = 1, 2, 3.$

Fig. 5 Example. Top to Bottom: triangulation and triangles determining the set of Powell-Sabin B-splines; f and its contour lines; $\mathcal{Q}_2 f$ and its contour lines; the Hermite interpolant (10)-(12) and its contour lines ▶



−0.2 0 0.2 0.4 0.6 0.8 1 1.2
−0.2 0 0.2 0.4 0.6 0.8 1 1.2
0.985 0.99 0.995 1 1.005 1.01 1.015
0 0.2 0.4 0.6 0.8 1
0 0.2 0.4 0.6 0.8 1

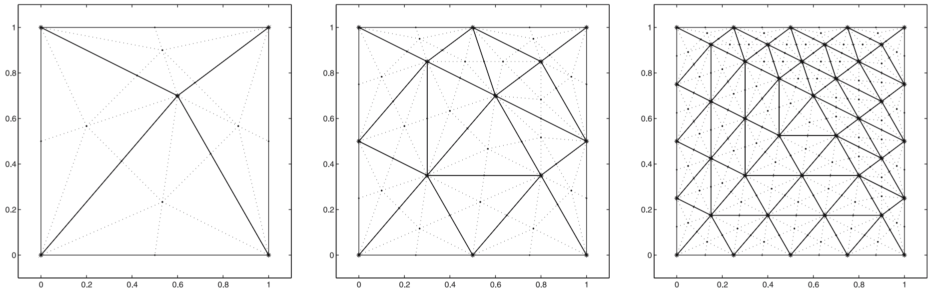


Fig. 6 Left to right: triangulations $\Delta^{(k)}$, $k = 0, 1, 2$, and their Powell-Sabin refinements

We remark that, due to the completely local construction of the q.i.s, different definitions for the coefficients $\mu_l^{(j)}$ can be used for different vertices. Moreover, for the practical construction of the q.i.s, a special attention must be payed for the boundary vertices. For such vertices it may happen that $\mathbf{V}_l = \mathbf{Q}_l^{(j)}$ for some j or that two distinct points $\mathbf{Q}_l^{(j)}$ are collinear with $\mathbf{V}_l$, so that some of the quasi-interpolating schemes presented in Section 4 cannot be applied around this vertices. However, as an example, the schemes presented in Theorem 5 (and/or Remark 9) are well defined even for boundary vertices.

For a practical example we have considered the function

$$f(x, y) = \exp(x(1-x)(1-y)y(x-y)). \tag{41}$$

Figure 5 top shows the considered triangulation with the Powell-Sabin refinement and the triangles determining the used B-splines. All the vertices of the triangles lie on the edges of Δ (or on their prolongations). The second row of Fig. 5 shows the graph of f and its contour lines while in the third row we have the graph of $\mathcal{Q}_2 f$ (see (27)) and its contour lines. For comparison we depict in the last row of Fig. 5 the graph and the contour lines of the the classical Powell-Sabin Hermite interpolant (see (10)-(12)). The two approximants are completely comparable and the contour lines of $\mathcal{Q}_2 f$ present a little bit smoother behaviour.

Finally, to numerically confirm the approximation power of the proposed q.i.s, we have considered the triangulation $\Delta^{(0)}$ depicted in Fig. 6 (left) and the refined triangulations $\Delta^{(k)}$ (see Fig. 6, for $k = 0, 1, 2$) obtained considering the midpoint of any edge of $\Delta^{(k-1)}$ and taking the Delaunay triangulation of this new set of vertices. For all the triangulations we have considered the B-splines obtained according to Algorithm 15 so that Hypothesis 6 holds.

Table 1 Error behaviour of different q.i.s for triangulations $\Delta^{(k)}$

k	N_V	interp.	q.i. (23)	q.i. (24)	q.i. Th. 10
0	5	0.016997	0.018356	0.016837	0.013519
1	13	0.003767	0.003681	0.006568	0.002856
2	41	0.000578	0.000573	0.001051	0.000448
3	145	0.000078	0.000143	0.000143	0.000093

We have applied different q.i.s to the function (41) over the partitions $\Delta^{(k)}$, $k = 0, 1, 2, 3,$ and we have computed in each case

$$\max_{r,s=1,\dots,50} |f(x_r, y_s) - \mathcal{Q}f(x_r, y_s)| \tag{42}$$

where x_r, y_s are equally spaced points in [0, 1].

The results are depicted in Table 1. Any row of the table refers to a triangulation $\Delta^{(k)}$ for fixed k. The first column indicates the refinement level while the second one shows the number of vertices of the triangulation. In the remaining columns we have the values of the tabulated absolute error (42) for different q.i.s. In column 3 we have considered the Hermite interpolant see (10), (12). Column 4 shows the results for the q.i. presented in Theorem 4 (see (23)) with $\zeta_l^{(j)} = .7$ (setting $\mathbf{Z}_l^{(j,k)} = \mathbf{V}_l, k = 1, 2$ for boundary vertices where $\mathbf{V}_l = \mathbf{Q}_l^{(j)}$). In column 5 we present the behaviour of the q.i. of Remark 9 with $\zeta_l^{(j)} = .5$. Finally, column 6 refers to the q.i. (10) with $\mu_l^{(j)}$ defined by (29) taking $\mathbf{S}_l^{(i)}$ along two edges emanating from $\mathbf{V}_l$ as points of split of the Powell-Sabin refinement of $\Delta^{(k)}$, and $\nu_{k,l} = 0.5$.

Since the maximum length of an edge of the triangulation halves at each level of refinement, the numerical results in Table 1 are in agreement with the expected cubic reduction of the error (see (31)) .

Summarizing, we have presented and analysed several families of discrete q.i.s in the space $\mathcal{S}_2^1(\Delta_{PS})$ reproducing quadratic polynomials. For any vertex of Δ, the q.i.s are obtained via linear combinations of values of the given function f at (few) points around the vertex. The fact that the evaluation points can be chosen (possibly under some special assumptions on the used family of B-splines) as vertices of the triangulation Δ and/or belonging to its edges, makes the schemes attractive in practical applications.

Moreover, upper bounds for the infinity norm of the presented q.i.s have been provided. Thanks to quadratic polynomial reproduction, these upper bounds completely describe the optimal approximation order of the presented schemes.

The structure of the mentioned upper bounds indicates that, even in the context of quasi-interpolation, families of B-splines defined in terms of triangles (see Section 2) with small (minimal) area, are preferable.

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