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Dirac delta methods for Helmholtz transmission problems

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Abstract In this paper we use a boundary integral method with single layer potentials to solve a class of Helmholtz transmission problems in the plane. We propose and analyze a novel and very simple quadrature method to solve numerically the equivalent system of integral equations which provides an approximation of the solution of the original problem with linear convergence (quadratic in some special cases). Furthermore, we also investigate a modified quadrature approximation based on the ideas of qualocation methods. This new scheme is again extremely simple to implement and has order three in weak norms.

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1. Introduction

Traditionally, Helmholtz transmission problem (HTP) is the name given to a system of Helmholtz equations with different wave numbers, one on a bounded domain and the other on its complement, coupled through continuity conditions for the unknown and some related fluxes. A relevant field where these problems appear

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is the scattering of acoustic waves in locally homogeneous media in time-harmonic regime. This has led to extensive analytical and numerical studies, aiming at obtaining reliable simulations and at paving the way for two important related problems: electromagnetic waves and associated inverse problems. The books [8, 9] deal with direct and inverse problems for the Helmholtz equation, with an emphasis on exterior boundary value problems and on scattering in non-absorbing media. Although this kind of problems have led research on the Helmholtz equation on unbounded domains, transmission problems have also received attention in the last decades. Different formulations using boundary integral equations can be found for instance in [10, 13, 15, 26, 27].

More recently, HTP have also appeared in the analysis of the scattering of thermal waves [17, 18], based on related work in physical literature [16, 24, 25]. Also, the use of the Laplace transform with numerical quadrature for the inversion formula on special contours [12] allows for the transformation of evolutionary problems into a set of steady-state Helmholtz equations for several wave numbers. In all these cases, transmission problems are more relevant than purely exterior BVP and the media have absorbtion.

In this work we deal with an indirect formulation for HTP based on the use of single layer potentials. This approach can fail when either the interior or the exterior constants for the Helmholtz equation are Dirichlet eigenvalues for the Laplace operator in the interior domain. In these singular cases, our integral system cannot be used. It is however valid in all the situations related to diffusion processes mentioned above and in most purely acoustic situations. The proximity to a Dirichlet eigenvalue for the Laplace operator is in fact detected in the numerical methods as a drastic increase in the condition number. At that point, the formulation should be changed: for instance, a mixed single–double layer potential (Brakhage–Werner or Panich potential, see [8, 9]) can be used. The corresponding system of boundary integral equations has a similar structure but uses hypersingular operators. At the present state of our research we are not able to develop and analyze a quadrature method for these operators. Anyway, the range of applicability of our methods to non-purely acoustic equations or to heat-diffusion problems (and the corresponding inverse problems) supposes a wide set of interesting practical problems.

To solve numerically the integral system, we propose two numerical methods: a family of quadrature methods (in which two particular cases show superconvergence) and an improvement based on ideas of qualocation methods. All these discretizations can be interpreted as non-conforming Petrov–Galerkin methods with discrete sets of Dirac deltas for both trial and test spaces. We give a complete convergence and stability analysis based on classical compact perturbation theory and in the previous results of [4] when dealing with quadrature methods. We carry out the analysis of the modified quadrature method with new techniques based upon some technicalities related to qualocation methods [6, 11].

Quadrature methods for equations of logarithmic type had been previously studied by an equivalent formulation with trigonometric polynomials in [19] or as Dirac delta approximations in [4]. For equations of the second kind, the approach is usually based on the classical analysis of Nyström methods (see, for instance, [2, 14]). Here we give a simple alternative that works for the periodic case with simple rules on equidistant grids based on a non-conforming Dirac delta approximation of the identity operator and a perturbative analysis. Unlike the previous analyses of spline-delta or delta–delta methods in [4], where discrete spaces of Dirac delta masses enter the Sobolev variational setting either in duality products (when Fourier series converge absolutely) or as evaluations of piecewise Hölder functions (when Fourier series are semiconvergent in the point of evaluation), the product of two Dirac delta distributions in the principal part of the operator of the second kind can only be defined at a discrete level. This apparently trivial idea simplifies enormously the way the subsequent analysis is carried out and supposes a novel feature of our variational approach to delta methods that extends the techniques in [4] to new situations that did not follow straightforwardly from the analysis in that work. Also, combinations of two (or more) grids for quadrature methods following the philosophy of qualocation schemes opens new horizons to proposing and analysing extremely simple high order methods. To the best of our knowledge, this is a non-minor novelty of the present work.

We believe that the possibility of obtaining a method of order three for a class of two dimensional HTP requiring no implementation effort at all (setting up the system is trivial, the only problem is solving it) has an interest in itself, since it gives the possibility of obtaining reliable numerical results with very little knowledge on the intricacies of boundary element methods. Both formulation and approximation have, logically, drawbacks: it is not clear which modifications will be needed for non-smooth interfaces (a fully satisfactory analysis of high order convergence collocation on polygons is still lacking); the three dimensional case is even farther away from the theoretical point of view.

We have also limited our analysis to estimates in weak norms. These are used for a posteriori computations, such as the value of the unknown for the transmission problem at some distance of the interface, or far-field computations. Nevertheless, some improvements in this direction are at hand. Following the ideas of [4, 5], we could obtain an asymptotic expansion of the error. From this, it is possible to obtain some pointwise superconvergence results for the unknowns of the integral system (the densities for the single layer potentials) as well as theoretical justification for the use of Richardson extrapolation, either to accelerate convergence or to have a global a posteriori estimate of the error. With the analysis developed in this work and the results in [4], with due adaptations and some new technicalities, these additional results can be easily proven. We do not carry out this analysis here to keep the paper in a reasonable size.

Notation. Throughout the paper C, C', C'' will denote general positive constants independent of the discretization parameter ($h = 1/N$) and of any quantity that is multiplied by them, being possibly different in each occurrence.

2. Statement of the problem

Let $\Omega^- \subset \mathbb{R}^2$ be a simply connected open set and $\Gamma := \partial\Omega^-$ its boundary which is assumed to be a parameterizable regular curve. Our aim is to solve numerically the following transmission problem

$$\Delta u + \lambda^2 u = 0, \qquad \text{in } \Omega^+ := \mathbb{R}^2 \setminus \overline{\Omega}^-, \tag{1}$$

$$\Delta u + \mu^2 u = 0, \qquad \text{in } \Omega^-, \tag{2}$$

$$u|_\Gamma^{\rm int} - u|_\Gamma^{\rm ext} = g_0, \tag{3}$$

$$\alpha\,\partial_n u|_\Gamma^{\rm int} - \beta\,\partial_n u|_\Gamma^{\rm ext} = g_1, \tag{4}$$

$$\lim_{r\to\infty} r^{1/2}(\partial_r u - \imath\lambda u) = 0, \tag{5}$$

where $\alpha \neq -\beta$ are given parameters satisfying $\alpha\beta \neq 0$ and $-\lambda^2,\ -\mu^2$ are not Dirichlet eigenvalues of the Laplace operator in Ω^-. The equality given in (5) is known as the Sommerfeld radiation condition at infinity and has to be satisfied uniformly in all directions.

Conditions on the parameters $\alpha,\ \beta,\ \lambda$ and μ that ensure existence and uniqueness of solution to the problem above can be found for instance in [10, 13, 15, 26, 27]. Throughout this work we will assume that our parameters are such that (1–5) has a unique solution.

Now we give a boundary integral formulation of the problem above. Let $\mathbf{x} : \mathbb{R} \to \Gamma$ be a 1−periodic regular parameterization of Γ. For simplicity, we will assume that $\mathbf{x} \in \mathcal{C}^\infty(\mathbb{R})$, but the results we will see do not need as much regularity.

For a density $\psi : \mathbb{R} \to \mathbb{C}$ we define the single layer potential

$$\mathcal{S}^\rho\psi := \int_0^1 \frac{\imath}{4}\, H_0^{(1)}(\rho|\cdot - \mathbf{x}(t)|)\,\psi(t)\,dt\ :\ \mathbb{R}^2 \longrightarrow \mathbb{C},$$

$H_0^{(1)}$ being the Hankel function of the first kind and order zero. We use an indirect formulation to find the solution to Problem (1–5), that is, we look for a function of the form

$$u := \left|\begin{array}{ll} \mathcal{S}^\lambda\psi^+, & \text{in } \Omega^+, \\ \mathcal{S}^\mu\psi^-, & \text{in } \Omega^-, \end{array}\right.$$

where the densities $\psi^\pm$ have to be determined. With this choice u satisfies Equations (1), (2) and (5). We now consider the following integral operators

$$V^\rho\psi = \int_0^1 V^\rho(\,\cdot\,,t)\,\psi(t)\,dt := \int_0^1 \frac{\imath}{4}\, H_0^{(1)}(\rho|\mathbf{x}(\cdot) - \mathbf{x}(t)|)\,\psi(t)\,dt : \mathbb{R} \longrightarrow \mathbb{C},$$

$$J^\rho\psi = \int_0^1 J^\rho(\,\cdot\,,t)\,\psi(t)\,dt := \int_0^1 \frac{\imath}{4}\,|\mathbf{x}'(\cdot)|\,\partial_{n(\cdot)} H_0^{(1)}(\rho|\mathbf{x}(\cdot) - \mathbf{x}(t)|)\,\psi(t)\,dt : \mathbb{R} \longrightarrow \mathbb{C},$$

where $\partial_{n(s)}$ is the exterior normal derivative at $\mathbf{x}(s)$. The parameterized version of the well-known jump relations of the single layer potential (see [7, Chapter 7], [10]) is

$$\mathcal{S}^\rho\psi|_\Gamma^{\rm int} \circ \mathbf{x} = \mathcal{S}^\rho\psi|_\Gamma^{\rm ext} \circ \mathbf{x} = V^\rho\psi, \tag{6}$$

$$|\mathbf{x}'|\,\partial_n\mathcal{S}^\rho\psi|_\Gamma^{\rm int} \circ \mathbf{x} = \tfrac{1}{2}\,\psi + J^\rho\psi, \qquad |\mathbf{x}'|\,\partial_n\mathcal{S}^\rho\psi|_\Gamma^{\rm ext} \circ \mathbf{x} = -\tfrac{1}{2}\,\psi + J^\rho\psi. \tag{7}$$

If we consider the parameterized forms of the data functions, for which we keep the same notation,

$$g_0 := g_0 \circ \mathbf{x}, \qquad g_1 := |\mathbf{x}'|\,g_1 \circ \mathbf{x},$$

then, by the jump relations (6–7), conditions (3–4) are equivalent to the following system of integral equations

$$\mathcal{H}\begin{bmatrix} \psi^- \\ \psi^+ \end{bmatrix} := \begin{bmatrix} V^\mu & -V^\lambda \\ \alpha(\frac{1}{2}I + J^\mu) & \beta(\frac{1}{2}I - J^\lambda) \end{bmatrix} \begin{bmatrix} \psi^- \\ \psi^+ \end{bmatrix} = \begin{bmatrix} g_0 \\ g_1 \end{bmatrix}. \tag{8}$$

In order to study the invertibility and regularity of this operator (and therefore to the solution of the original transmission problem) we deal with the periodic Sobolev spaces (see [14, Chapter 8] or [20, Chapter 5]),

$$H^s := \{\phi \in \mathcal{D}' \,|\, |\widehat{\phi}(0)|^2 + \sum_{0\neq k\in\mathbb{Z}} |k|^{2s} |\widehat{\phi}(k)|^2 < \infty\},$$

where $\mathcal{D}'$ is the space of 1-periodic distributions on the real line and $\widehat{\phi}(k)$ are the Fourier coefficients of ϕ. The $H^0 = L^2(0,1)$ inner product extends to the antiduality bracket between H^s and H^{-s} for all $s \in \mathbb{R}$. Both will be denoted by $(\cdot\,,\,\cdot)$. We will use the notation $\|\cdot\|_s$ for the usual norm in H^s.

It can be shown (see [18, Proposition 3.3]) that

$$\mathcal{H} : H^s \times H^s \longrightarrow H^{s+1} \times H^s$$

is an isomorphism for all $s \in \mathbb{R}$.

3. Quadrature methods

Let $N \in \mathbb{N}$, $h := 1/N$ and

$$t_i := ih, \qquad t_{i+\varepsilon} := (i+\varepsilon)h, \qquad i = 1, \ldots, N,$$

where $0 \neq \varepsilon \in (-1/2, 1/2)$. The method we propose consists of solving the following system of linear equations

$$\left|\begin{array}{l} \boldsymbol{\psi}_h^{\pm} = (\psi_1^{\pm}, \ldots, \psi_N^{\pm})^\top \in \mathbb{C}^N, \\[2mm] \displaystyle\sum_{j=1}^N V^\mu(t_{i+\varepsilon}, t_j)\,\psi_j^- - \sum_{j=1}^N V^\lambda(t_{i+\varepsilon}, t_j)\,\psi_j^+ \;=\; g_0(t_{i+\varepsilon}), \qquad i = 1, \ldots, N, \\[2mm] \displaystyle\frac{\alpha}{2}\,\psi_i^- + \alpha h \sum_{j=1}^N J^\mu(t_i, t_j)\,\psi_j^- + \frac{\beta}{2}\,\psi_i^+ - \beta h \sum_{j=1}^N J^\lambda(t_i, t_j)\,\psi_j^+ = h\, g_1(t_i), \quad i = 1, \ldots, N. \end{array}\right. \tag{9}$$

Note that implementation of this method is trivial. The first group of equations is a quadrature method with displaced nodes whereas the second one corresponds to a classical Nyström method for equations of the second kind. The evaluation of $V^\rho(t_{i+\varepsilon}, t_j)$ is not a problem since $\varepsilon \neq 0$ and the kernel of V^ρ only has a logarithmic singularity on its diagonal. The values $\varepsilon = \pm 1/2$ are not allowed for stability questions (see [4, 20]). In Section 6 we will see that the choices $\varepsilon = \pm 1/6$ provide superconvergent methods.

Once we have the solution $\boldsymbol{\psi}_h^{\pm} \in \mathbb{C}^N$ to (9), we take

$$u_h^\varepsilon(\mathbf{z}) \;:=\; \left|\begin{array}{ll} \displaystyle\frac{\imath}{4} \sum_{j=1}^N H_0^{(1)}(\lambda|\mathbf{z} - \mathbf{x}(t_j)|)\,\psi_j^+, & \text{if } \mathbf{z} \in \Omega^+, \\[2mm] \displaystyle\frac{\imath}{4} \sum_{j=1}^N H_0^{(1)}(\mu|\mathbf{z} - \mathbf{x}(t_j)|)\,\psi_j^-, & \text{if } \mathbf{z} \in \Omega^-, \end{array}\right. \tag{10}$$

as an approximation to the solution to (1–5).

To analyze the discretization above we are going to rewrite it as a generalized Petrov–Galerkin method using the following Dirac delta spaces

$$S_h := \mathbb{C}\langle \delta_i, i = 1, \ldots, N \rangle, \qquad S_h^\varepsilon := \mathbb{C}\langle \delta_{i+\varepsilon}, i = 1, \ldots, N \rangle,$$

δ_i and $\delta_{i+\varepsilon}$ being the 1-periodic Dirac delta distributions on the nodes t_i and $t_{i+\varepsilon}$ respectively.

If f is continuous at x, we will denote

$$\{ f, \delta_x\} = \overline{\{\delta_x, f\}} := f(x).$$

We also introduce the notation

$$\{\delta_i, \delta_j\}_h := \frac{1}{h}\delta_{ij},$$

where δ_{ij} is the Kronecker symbol.

Now the linear system of equations given in (9) can be seen as a generalized Petrov–Galerkin method in the following sense: if $\boldsymbol{\psi}_h^\pm = (\psi_1^\pm, \ldots, \psi_N^\pm)^\top \in \mathbb{C}^N$ is a solution to (9), then $\psi_h^\pm := \sum_{j=1}^N \psi_j^\pm \delta_j \in S_h$ is a solution to

$$\left|\begin{array}{l} \psi_h^\pm \in S_h, \\ \{V^\mu \psi_h^-, \varphi_h^-\} - \{V^\lambda \psi_h^+, \varphi_h^-\} = (g_0, \varphi_h^-), \qquad \forall \varphi_h^- \in S_h^\varepsilon, \\ \{\frac{\alpha}{2}\psi_h^-, \varphi_h^+\}_h + (\alpha J^\mu \psi_h^-, \varphi_h^+) + \{\frac{\beta}{2}\psi_h^+, \varphi_h^+\}_h - (\beta J^\lambda \psi_h^+, \varphi_h^+) = (g_1, \varphi_h^+), \qquad \forall \varphi_h^+ \in S_h, \end{array}\right. \tag{11}$$

and *vice versa*.

We denote by $[\,\cdot\,,\,\cdot\,]$ to the antiduality in the product space $(H^s \times H^r) \times (H^{-s} \times H^{-r})$, that is,

$$[\mathbf{f}, \mathbf{g}] := (f_1, g_1) + (f_2, g_2), \qquad \mathbf{f} = (f_1, f_2)^\top \in H^s \times H^r, \mathbf{g} = (g_1, g_2)^\top \in H^{-s} \times H^{-r}.$$

Given $\boldsymbol{\psi}_h = (\psi_h^-, \psi_h^+)^\top \in S_h \times S_h$ and $\boldsymbol{\varphi}_h = (\varphi_h^-, \varphi_h^+)^\top \in S_h^\varepsilon \times S_h$, we define

$$\begin{aligned}[]
[\mathcal{H}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h := &\{V^\mu \psi_h^-, \varphi_h^-\} - \{V^\lambda \psi_h^+, \varphi_h^-\} + \\
&+ \{\tfrac{\alpha}{2}\psi_h^-, \varphi_h^+\}_h + (\alpha J^\mu \psi_h^-, \varphi_h^+) + \{\tfrac{\beta}{2}\psi_h^+, \varphi_h^+\}_h - (\beta J^\lambda \psi_h^+, \varphi_h^+).
\end{aligned} \tag{12}$$

With these notations and taking $\mathbf{g} := (g_0, g_1)^\top$ we can also write Method (11) in a more compact form:

$$\left|\begin{array}{l} \boldsymbol{\psi}_h \in S_h \times S_h, \\ [\mathcal{H}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h = [\mathbf{g}, \boldsymbol{\varphi}_h], \qquad \forall \boldsymbol{\varphi}_h \in S_h^\varepsilon \times S_h. \end{array}\right. \tag{13}$$

4. Analysis of the principal part

Firstly, we are going to prove some stability and convergence properties of the quadrature method proposed in the previous section applied not to the global operator $\mathcal{H}$ but to its principal part. We begin by introducing the Bessel operator

$$V_0\varphi := -\frac{1}{4\pi}\int_0^1 \log(4e^{-1}\sin^2\pi(\,\cdot\, - t))\,\varphi(t)\,dt, \tag{14}$$

which is a bounded isomorphism from H^s into H^{s+1} for all $s \in \mathbb{R}$ and elliptic from $H^{-1/2}$ into $H^{1/2}$ (see [20, Chapter 5]).

The operator $\mathcal{H}$ defined in (8) can be decomposed as $\mathcal{H} = \mathcal{AP} + \mathcal{K}$, where

$$\mathcal{A} := \begin{bmatrix} V_0 & 0 \\ 0 & I \end{bmatrix}, \qquad \mathcal{P} := \begin{bmatrix} I & -I \\ \frac{\alpha}{2} I & \frac{\beta}{2} I \end{bmatrix}, \qquad \mathcal{K} := \begin{bmatrix} V^\mu - V_0 & V_0 - V^\lambda \\ \alpha\, J^\mu & -\beta\, J^\lambda \end{bmatrix}. \tag{15}$$

We also consider

$$\mathcal{H}_0 \;:=\; \mathcal{AP} \;=\; \begin{bmatrix} V_0 & -V_0 \\ \frac{\alpha}{2} I & \frac{\beta}{2} I \end{bmatrix}.$$

For all $s \in \mathbb{R}$, the operators $\mathcal{A}$, $\mathcal{H}_0 : H^s \times H^s \to H^{s+1} \times H^s$ are bounded isomorphisms (see also [18]) and $\mathcal{K} : H^s \times H^s \to H^{s+3} \times H^{s+3}$ is bounded (see [20, Section 7.6.1]). For $r,\ s \in \mathbb{R}$ we denote by $\|\cdot\|_{r,s}$ to the norm in $H^r \times H^s$.

We summarize in the next result some properties related to the approximation properties of the Dirac delta spaces S_h and S_h^ε in a wide range of Sobolev norms. A natural operator onto S_h is $Q_h : H^t \to S_h,\ t > 1/2$, given by

$$Q_h\varphi := h \sum_{i=1}^{N} \varphi(t_i)\,\delta_i.$$

Related to the discrete space S_h^ε, we introduce the Fourier projection (see [1, 4]), $D_h^\varepsilon : \mathcal{D}' \to S_h^\varepsilon$, given by

$$\left| \begin{array}{l} D_h^\varepsilon\varphi \in S_h^\varepsilon, \\ \widehat{D_h^\varepsilon\varphi}(\mu) \;=\; \widehat{\varphi}(\mu), \qquad \forall \mu \in \Lambda_N := \{\, \mu \in \mathbb{Z} \,|\, -N/2 \le \mu < N/2 \,\}. \end{array} \right. \tag{16}$$

Note that this operator is also well-defined for $\varepsilon = 0$.

Lemma 1 (**[4, Lemma 6]**)**.** The following approximation properties hold:

(a) for $t > 1/2$,

$$\|\varphi - Q_h\varphi\|_{-t} \;\le\; C_t\, h^t\, \|\varphi\|_t, \qquad \forall \varphi \in H^t,$$

(b) for $s, t \in \mathbb{R}$ such that $s < -1/2,\ s \le t \le 0$,

$$\|\varphi - D_h^\varepsilon\varphi\|_s \;\le\; C_{s,t}\, h^{t-s}\, \|\varphi\|_t, \qquad \forall \varphi \in H^t.$$

Keeping the notations of Section 3, we define for $\boldsymbol{\psi}_h = (\psi_h^-, \psi_h^+)^\top \in S_h \times S_h$ and $\boldsymbol{\varphi}_h = (\varphi_h^-, \varphi_h^+)^\top \in S_h^\varepsilon \times S_h$, the sesquilinear forms

$$[\mathcal{A}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h := \{V_0\psi_h^-, \varphi_h^-\} \;+\; \{\psi_h^+, \varphi_h^+\}_h, \tag{17}$$

$$[\mathcal{H}_0\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h := \{V_0\psi_h^-, \varphi_h^-\} \;-\; \{V_0\psi_h^+, \varphi_h^-\} \;+\; \{\tfrac{\alpha}{2}\,\psi_h^-, \varphi_h^+\}_h \;+\; \{\tfrac{\beta}{2}\,\psi_h^+, \varphi_h^+\}_h. \tag{18}$$

Theorem 2. There exists $C > 0$, independent of h, such that

$$\sup_{\boldsymbol{\varphi}_h \in S_h^\varepsilon \times S_h} \frac{|\,[\mathcal{H}_0\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h|}{\|\boldsymbol{\varphi}_h\|_{-1,-1}} \ge C\|\boldsymbol{\psi}_h\|_{-1,-1}, \qquad \forall \boldsymbol{\psi}_h \in S_h \times S_h.$$

Proof. Since $\mathcal{H}_0 = \mathcal{A}\mathcal{P}$ and $\mathcal{P}^{-1}|_{(S_h\times S_h)} : S_h \times S_h \to S_h \times S_h$ is uniformly bounded in $H^{-1} \times H^{-1}$, we can equivalently show that

$$\sup_{\boldsymbol{\varphi}_h\in S_h^\varepsilon\times S_h} \frac{|\,[\mathcal{A}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h|}{\|\boldsymbol{\varphi}_h\|_{-1,-1}} \geq C\|\boldsymbol{\psi}_h\|_{-1,-1}, \qquad \forall \boldsymbol{\psi}_h \in S_h \times S_h. \tag{19}$$

On the one hand, by [4, Proposition 8],

$$\sup_{\psi_h\in S_h} \frac{|\{V_0\varphi_h, \psi_h\}|}{\|\psi_h\|_{-1}} \geq \alpha\|\varphi_h\|_{-1}, \qquad \forall \varphi_h \in S_h^\varepsilon, \tag{20}$$

with $\alpha > 0$ independent of h. Since

$$|\{V_0\psi_h, \varphi_h\}| = |\{V_0\varphi_h, \psi_h\}|, \qquad \forall \psi_h \in S_h, \varphi_h \in S_h^\varepsilon,$$

and the spaces S_h and S_h^ε have the same finite dimension, we can reverse the inf-sup condition (20), i.e., with the same constant α, we have

$$\sup_{\varphi_h\in S_h^\varepsilon} \frac{|\{V_0\psi_h, \varphi_h\}|}{\|\varphi_h\|_{-1}} \geq \alpha\|\psi_h\|_{-1}, \qquad \forall \psi_h \in S_h. \tag{21}$$

On the other hand, by [5, Lemma 9] there exists $C > 0$, independent of h, such that

$$\|\psi_h\|_{-1} \leq C\sum_{i=1}^N |\psi_i|, \qquad \forall \psi_h = \sum_{i=1}^N \psi_i\,\delta_i \in S_h.$$

Given $0 \neq \psi_h = \sum_{i=1}^N \psi_i\,\delta_i \in S_h$, we take $\varphi_h := h\sum_{\psi_i\neq 0}(\psi_i/|\psi_i|)\,\delta_i$. Then, $\|\varphi_h\|_{-1} \leq C$ and

$$|\{\psi_h, \varphi_h\}_h| = \sum_{i=1}^N |\psi_i| \geq \frac{1}{C}\,\|\psi_h\|_{-1}.$$

Therefore,

$$\frac{1}{C}\,\|\psi_h\|_{-1} \leq |\{\psi_h, \varphi_h\}_h| \leq C\,\frac{|\{\psi_h, \varphi_h\}_h|}{\|\varphi_h\|_{-1}} \leq C\sup_{\varphi_h\in S_h}\frac{|\{\psi_h, \varphi_h\}_h|}{\|\varphi_h\|_{-1}}. \tag{22}$$

From this and Inequality (21) we trivially obtain (19). □

Theorem 3. Let $\boldsymbol{\psi}_h^0 \in S_h \times S_h$ be the solution to the problem

$$\left|\begin{array}{l} \boldsymbol{\psi}_h^0 \in S_h \times S_h, \\ [\mathcal{H}_0\boldsymbol{\psi}_h^0, \boldsymbol{\varphi}_h]_h = [\mathcal{H}_0\boldsymbol{\psi}, \boldsymbol{\varphi}_h], \qquad \forall \boldsymbol{\varphi}_h \in S_h^\varepsilon \times S_h. \end{array}\right.$$

Then,

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h^0\|_{-1,-1} \leq Ch\|\boldsymbol{\psi}\|_{1,1}, \qquad \forall \boldsymbol{\psi} \in H^1 \times H^1.$$

Proof. Set $\boldsymbol{\xi}_h^0 := \mathcal{P}\boldsymbol{\psi}_h^0$ and $\boldsymbol{\xi} := \mathcal{P}\boldsymbol{\psi}$. Then,

$$\left|\begin{array}{l} \boldsymbol{\xi}_h^0 \in S_h \times S_h, \\ [\mathcal{A}\boldsymbol{\xi}_h^0, \boldsymbol{\varphi}_h]_h = [\mathcal{A}\boldsymbol{\xi}, \boldsymbol{\varphi}_h], \qquad \forall \boldsymbol{\varphi}_h \in S_h^\varepsilon \times S_h. \end{array}\right. \tag{23}$$

Considering the separate components of $\boldsymbol{\xi}_h^0 = (\xi_h^-, \xi_h^+)^\top$ and $\boldsymbol{\xi} = (\xi^-, \xi^+)^\top$, (23) can be written as

$$\xi_h^+ = Q_h \xi^+, \qquad \left| \begin{array}{l} \xi_h^- \in S_h, \\ \{V_0 \xi_h^-, \varphi_h\} \;=\; (V_0 \xi^-, \varphi_h), \qquad \forall \varphi_h \in S_h^\varepsilon. \end{array} \right.$$

By Lemma 1 (a),

$$\|\xi^+ - \xi_h^+\|_{-1} \;\le\; Ch\|\xi^+\|_1. \tag{24}$$

If we show that

$$\|\xi^- - \xi_h^-\|_{-1} \;\le\; Ch\|\xi^-\|_1, \tag{25}$$

then using the relation between $\boldsymbol{\psi}$ and $\boldsymbol{\xi}$ and applying (24–25) we obtain

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h^0\|_{-1,-1} \;=\; \|\mathcal{P}^{-1}(\boldsymbol{\xi} - \boldsymbol{\xi}_h^0)\|_{-1,-1} \;\le\; Ch\|\boldsymbol{\xi}\|_{1,1} \;\le\; C'h\|\boldsymbol{\psi}\|_{1,1},$$

which finishes the result.

Now we prove (25). By Lemma 1 (b),

$$\|\xi^- - \xi_h^-\|_{-1} \;\le\; Ch\|\xi^-\|_0 + \|D_h^0 \xi^- - \xi_h^-\|_{-1}.$$

Applying now (21),

$$\|D_h^0 \xi^- - \xi_h^-\|_{-1} \le C \sup_{\varphi_h \in S_h^\varepsilon} \frac{|\{V_0(D_h^0 \xi^- - \xi_h^-), \varphi_h\}|}{\|\varphi_h\|_{-1}} = C \sup_{\varphi_h \in S_h^\varepsilon} \frac{|\{V_0 D_h^0 \xi^-, \varphi_h\} - (V_0 \xi^-, \varphi_h)|}{\|\varphi_h\|_{-1}}.$$

Inequality (25) is then proven once we have

$$|\{V_0 D_h^0 \psi, \varphi_h\} - (V_0 \psi, \varphi_h)| \;\le\; Ch\|\psi\|_1 \|\varphi_h\|_{-1}, \qquad \forall \psi \in H^1, \;\; \varphi_h \in S_h^\varepsilon. \tag{26}$$

We consider the traslation operator $t_h^\varepsilon \phi := \phi(\,\cdot\, - \varepsilon h)$, defined by transposition ($t_h^{-\varepsilon}$ is the inverse traslation),

$$\langle t_h^\varepsilon \varphi, \phi \rangle_{\mathcal{D}' \times \mathcal{D}} \;=\; \langle \varphi, t_h^{-\varepsilon} \phi \rangle_{\mathcal{D}' \times \mathcal{D}}.$$

Obviously it is an isometric isomorphism in H^s for all $s \in \mathbb{R}$. It also satisfies the following straightforward properties:

$$D_h^0 = t_h^\varepsilon D_h^{-\varepsilon} t_h^{-\varepsilon}, \tag{27}$$

$$(V_0 \phi, \varphi) = (V_0 t_h^\varepsilon \phi, t_h^\varepsilon \varphi), \quad \forall \phi \in H^0, \;\; \varphi \in H^{-1}, \tag{28}$$

$$\{V_0 \delta_x, \delta_y\} = \{V_0 t_h^\varepsilon \delta_x, t_h^\varepsilon \delta_y\}, \quad \text{if } x - y \notin \mathbb{Z}. \tag{29}$$

Since by [4, Theorem 7]

$$|\{V_0 D_h^\varepsilon \varphi, \psi_h\} - (V_0 \varphi, \psi_h)| \;\le Ch\|\varphi\|_1 \|\psi_h\|_{-1}, \qquad \forall \varphi \in H^1, \;\; \psi_h \in S_h, \tag{30}$$

trivial computations using (27–29) show now that

$$\begin{aligned} |\{V_0 D_h^0 \psi, \varphi_h\} - (V_0 \psi, \varphi_h)| &= |\{V_0\, t_h^\varepsilon D_h^{-\varepsilon} t_h^{-\varepsilon} \psi, t_h^\varepsilon\, t_h^{-\varepsilon} \varphi_h\} - (V_0\, t_h^{-\varepsilon} \psi, t_h^{-\varepsilon} \varphi_h)| \\ &= |\{V_0 D_h^{-\varepsilon} t_h^{-\varepsilon} \psi, t_h^{-\varepsilon} \varphi_h\} - (V_0\, t_h^{-\varepsilon} \psi, t_h^{-\varepsilon} \varphi_h)| \\ &\le Ch\|t_h^{-\varepsilon} \psi\|_1 \, \|t_h^{-\varepsilon} \varphi_h\|_{-1} \;=\; Ch\|\psi\|_1 \|\varphi_h\|_{-1}, \end{aligned}$$

i.e., (26) holds. □

The theorems above imply that the operator $G_h^0 : H^1 \times H^1 \to S_h \times S_h$ given by

$$[\mathcal{H}_0 G_h^0 \boldsymbol{\psi}, \boldsymbol{\varphi}_h]_h = [\mathcal{H}_0 \boldsymbol{\psi}, \boldsymbol{\varphi}_h], \qquad \forall \boldsymbol{\varphi}_h \in S_h^\varepsilon \times S_h, \tag{31}$$

is well-defined and furthermore,

$$\|(\mathcal{I} - G_h^0)\boldsymbol{\psi}\|_{-1,-1} \le Ch\|\boldsymbol{\psi}\|_{1,1}, \qquad \forall \boldsymbol{\psi} \in H^1 \times H^1. \tag{32}$$

5. Convergence analysis

In this section we prove a uniform inf-sup condition for the global operator $\mathcal{H}$. From it, existence and uniqueness of solution to (13), and therefore to (9), follow readily.

Proposition 4. There exists $C > 0$, independent of h, such that for all h small enough

$$\sup_{\boldsymbol{\varphi}_h \in S_h^\varepsilon \times S_h} \frac{|\,[\mathcal{H}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h|}{\|\boldsymbol{\varphi}_h\|_{-1,-1}} \ge C\|\boldsymbol{\psi}_h\|_{-1,-1}, \qquad \forall \boldsymbol{\psi}_h \in S_h \times S_h.$$

Proof. Notice that by the mapping properties of the integral operators given at the beginning of Section 4, the operator $\mathcal{H}_0^{-1}\mathcal{K} : H^{-1} \times H^{-1} \to H^1 \times H^1$ is bounded. Thus, from (32),

$$\|(\mathcal{I} - G_h^0)\mathcal{H}_0^{-1}\mathcal{K}\boldsymbol{\varphi}\|_{-1,-1} \le Ch\|\boldsymbol{\varphi}\|_{-1,-1}, \quad \forall \boldsymbol{\varphi} \in H^{-1} \times H^{-1},$$

and hence

$$\|(\mathcal{I} + \mathcal{H}_0^{-1}\mathcal{K}) - (\mathcal{I} + G_h^0\mathcal{H}_0^{-1}\mathcal{K})\|_{\mathcal{L}(H^{-1}\times H^{-1})} \to 0.$$

As $\mathcal{I} + \mathcal{H}_0^{-1}\mathcal{K} = \mathcal{H}_0^{-1}\mathcal{H}$ is invertible, by a classical operator approximation result (see for instance [3, Theorem 11.1.2]), for h small enough $\mathcal{I} + G_h^0\mathcal{H}_0^{-1}\mathcal{K}$ is invertible, with uniformly bounded inverse. Besides,

$$\mathcal{I} + G_h^0\mathcal{H}_0^{-1}\mathcal{K}|_{(S_h\times S_h)} : S_h \times S_h \to S_h \times S_h.$$

Therefore, applying now Theorem 2, the definition of the operator G_h^0 given in (31) and the decomposition $\mathcal{H} = \mathcal{H}_0 + \mathcal{K}$, it follows that

$$\begin{aligned}\|\boldsymbol{\psi}_h\|_{-1,-1} &\le C\|(\mathcal{I} + G_h^0\mathcal{H}_0^{-1}\mathcal{K})\boldsymbol{\psi}_h\|_{-1,-1} \\ &\le C' \sup_{\boldsymbol{\varphi}_h \in S_h^\varepsilon \times S_h} \frac{|\,[\mathcal{H}_0(\mathcal{I} + G_h^0\mathcal{H}_0^{-1}\mathcal{K})\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h|}{\|\boldsymbol{\varphi}_h\|_{-1,-1}} = C' \sup_{\boldsymbol{\varphi}_h \in S_h^\varepsilon \times S_h} \frac{|\,[\mathcal{H}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h|}{\|\boldsymbol{\varphi}_h\|_{-1,-1}},\end{aligned}$$

and the result is proven. □

Theorem 5. The problem

$$\left|\begin{aligned} &\boldsymbol{\psi}_h \in S_h \times S_h, \\ &[\mathcal{H}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h = [\mathcal{H}\boldsymbol{\psi}, \boldsymbol{\varphi}_h], \qquad \forall \boldsymbol{\varphi}_h \in S_h^\varepsilon \times S_h, \end{aligned}\right. \tag{33}$$

is uniquely solvable for all h small enough. Moreover,

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{-1,-1} \le Ch\|\boldsymbol{\psi}\|_{1,1}, \qquad \forall \boldsymbol{\psi} \in H^1 \times H^1.$$

Proof. Proposition 4 implies unique solvability of (33) and by (32) it is enough to show that

$$\|\boldsymbol{\psi}_h - G_h^0\boldsymbol{\psi}\|_{-1,-1} \le Ch\|\boldsymbol{\psi}\|_{1,1}, \qquad \forall \boldsymbol{\psi} \in H^1 \times H^1.$$

From the definitions of $G_h^0\boldsymbol{\psi}$ and $\boldsymbol{\psi}_h$, we have for all $\boldsymbol{\varphi}_h \in S_h^\varepsilon \times S_h$,

$$[\mathcal{H}(\boldsymbol{\psi}_h - G_h^0\boldsymbol{\psi}), \boldsymbol{\varphi}_h]_h = [\mathcal{H}\boldsymbol{\psi}, \boldsymbol{\varphi}_h] - [\mathcal{H}G_h^0\boldsymbol{\psi}, \boldsymbol{\varphi}_h]_h = [\mathcal{K}(\boldsymbol{\psi} - G_h^0\boldsymbol{\psi}), \boldsymbol{\varphi}_h].$$

We can make use now of Proposition 4 to obtain that

$$\|\boldsymbol{\psi}_h - G_h^0\boldsymbol{\psi}\|_{-1,-1} \le C \sup_{\boldsymbol{\varphi}_h \in S_h^\varepsilon \times S_h} \frac{|\,[\mathcal{K}(\boldsymbol{\psi} - G_h^0\boldsymbol{\psi}), \boldsymbol{\varphi}_h]\,|}{\|\boldsymbol{\varphi}_h\|_{-1,-1}}$$

$$\le C\|\mathcal{K}(\boldsymbol{\psi} - G_h^0\boldsymbol{\psi})\|_{1,1} \le C'\|\boldsymbol{\psi} - G_h^0\boldsymbol{\psi}\|_{-1,-1} \le C''h\|\boldsymbol{\psi}\|_{1,1},$$

where we have applied once again (32) for the last inequality. □

The previous bounds also lead us to an estimation of the error in the approximation of u by the function u_h^ε defined in (10).

Theorem 6. Let $\boldsymbol{\psi}$ be the solution to (8). If $\boldsymbol{\psi} \in H^1 \times H^1$, then,

$$|u(\mathbf{z}) - u_h^\varepsilon(\mathbf{z})| \le Ch\|\boldsymbol{\psi}\|_{1,1}, \qquad \mathbf{z} \in \mathbb{R}^2 \setminus \Gamma,$$

where $C > 0$ only depends on $\mathbf{z}$.

Proof. Assume that $\mathbf{z} \in \Omega^-$. In this case,

$$u_h^\varepsilon(\mathbf{z}) = \frac{\imath}{4}\sum_{j=1}^N H_0^{(1)}(\mu|\mathbf{z} - \mathbf{x}(t_j)|)\,\psi_j^- = \Big(\frac{\imath}{4}H_0^{(1)}(\mu|\mathbf{z} - \mathbf{x}(\,\cdot\,)|), \sum_{j=1}^N \overline{\psi_j^-}\,\delta_j\Big) = \mathcal{S}^\mu\psi_h^-(\mathbf{z}).$$

Therefore, by Theorem 5,

$$|u(\mathbf{z}) - u_h^\varepsilon(\mathbf{z})| = |\mathcal{S}^\mu\psi^-(\mathbf{z}) - \mathcal{S}^\mu\psi_h^-(\mathbf{z})|$$

$$\le C\|H_0^{(1)}(\mu|\mathbf{z} - \mathbf{x}(\,\cdot\,)|)\|_1\|\psi^- - \psi_h^-\|_{-1} \le C_{\mathbf{z}}h\|\boldsymbol{\psi}\|_{1,1}.$$

Obviously, when $\mathbf{z} \in \Omega^+$, the proof is exactly the same since $u_h^\varepsilon(\mathbf{z}) = \mathcal{S}^\lambda\psi_h^+(\mathbf{z})$. □

Although the constant in the theorem above depends on the point, it only blows up when we are very close to Γ. Moreover, when $\lambda \notin \mathbb{R}$ it is uniformly bounded in the exterior of any ball containing Γ, whereas for $\lambda \in \mathbb{R}$ we can only assure uniform boundness in compact sets.

6. Superconvergent methods

There are two special methods with better convergence properties belonging to the family analyzed in the previous sections: those associated to the parameters $\varepsilon = \pm 1/6$. Notice that this special choice for logarithmic integral equations had already been observed in [4, 19].

Theorem 7. Let $\boldsymbol{\psi}_h \in S_h \times S_h$ be the solution to any of the problems

$$\left|\begin{array}{l} \boldsymbol{\psi}_h \in S_h \times S_h, \\ [\mathcal{H}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h \;=\; [\mathcal{H}\boldsymbol{\psi}, \boldsymbol{\varphi}_h], \qquad \forall \boldsymbol{\varphi}_h \in S_h^{\pm 1/6} \times S_h. \end{array}\right.$$

Then,

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{-2,-2} \leq Ch^2 \|\boldsymbol{\psi}\|_{2,2}, \qquad \forall \boldsymbol{\psi} \in H^2 \times H^2.$$

Consequently, if $\boldsymbol{\psi} \in H^2 \times H^2$, then,

$$|u(\mathbf{z}) - u_h^{\pm 1/6}(\mathbf{z})| \;\leq\; C_{\mathbf{z}} h^2 \|\boldsymbol{\psi}\|_{2,2}, \qquad \mathbf{z} \in \mathbb{R}^2 \setminus \Gamma.$$

Proof. As a direct consequence of [4, Theorem 7],

$$|\{V_0 D_h^{\pm 1/6}\varphi, \psi_h\} - (V_0\varphi, \psi_h)| \leq Ch^2 \|\varphi\|_2 \|\psi_h\|_{-1}, \qquad \forall \varphi \in H^2,\;\; \psi_h \in S_h, \tag{34}$$

and following step by step the proof of Theorem 3 we prove the existence of $C > 0$, independent of h, such that

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h^0\|_{-2,-2} \;\leq\; Ch^2 \|\boldsymbol{\psi}\|_{2,2}, \qquad \forall \boldsymbol{\psi} \in H^2 \times H^2,$$

where $\boldsymbol{\psi}_h^0 \in S_h \times S_h$ is the solution to the problem

$$\left|\begin{array}{l} \boldsymbol{\psi}_h^0 \in S_h \times S_h, \\ [\mathcal{H}_0 \boldsymbol{\psi}_h^0, \boldsymbol{\varphi}_h]_h \;=\; [\mathcal{H}_0\boldsymbol{\psi}, \boldsymbol{\varphi}_h], \qquad \forall \boldsymbol{\varphi}_h \in S_h^{\pm 1/6} \times S_h. \end{array}\right. \tag{35}$$

Thus,

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{-2,-2} \;\leq\; Ch^2 \|\boldsymbol{\psi}\|_{2,2} + \|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^0\|_{-2,-2},$$

and following the proof of Theorem 5 (recall that $\mathcal{K} : H^s \times H^s \to H^{s+3} \times H^{s+3}$ is bounded),

$$\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^0\|_{-1,-1} \;\leq\; C\|\mathcal{K}(\boldsymbol{\psi} - \boldsymbol{\psi}_h^0)\|_{1,1} \;\leq\; C'\|\boldsymbol{\psi} - \boldsymbol{\psi}_h^0\|_{-2,-2} \;\leq\; C"h^2 \|\boldsymbol{\psi}\|_{2,2}.$$

The last assertion can be shown as in Theorem 6. □

Remark. To prove the theorem above it would be enough to have continuity of $\mathcal{K} : H^s \times H^s \to H^{s+3} \times H^{s+2}$. We can also obtain the result by analyzing the transposed method of Scheme (35) and using some techniques that will be used in Section 9. By this way the proof is more involved.

7. Modified quadrature method

In the detailed convergence analysis of the quadrature methods, even in the superconvergent cases with $\varepsilon = \pm 1/6$, we observe that the order of convergence in the second group of equations in (9) can be increased by considering weaker norms when we deal with regular solutions. This motivates the search of an improvement in the test for the first N equations. We will replace the 1-periodic displaced Dirac deltas by linear combinations of some of them. However, we will keep untouched the remaining N equations in (9), which correspond to a Nyström method for equations of the second kind.

We introduce the weighted averages

$$V_{ij}^{\rho} := \frac{5}{6}\left(V^{\rho}(t_{i-1/6}, t_j) + V^{\rho}(t_{i+1/6}, t_j)\right) + \frac{1}{6}\left(V^{\rho}(t_{i-5/6}, t_j) + V^{\rho}(t_{i+5/6}, t_j)\right),$$

following the ideas of qualocation methods (see [6, 21, 22, 23]). Again we are avoiding the logarithmic singularity of the kernel of V^{ρ}. For the right hand side we likewise define

$$\widehat{g}_0^{\,i} := \frac{5}{6}\left(g_0(t_{i-1/6}) + g_0(t_{i+1/6})\right) + \frac{1}{6}\left(g_0(t_{i-5/6}) + g_0(t_{i+5/6})\right).$$

The new method consists of solving the linear system of equations

$$\left|\begin{array}{l} \boldsymbol{\psi}_h^{\pm} = (\psi_1^{\pm}, \ldots, \psi_N^{\pm})^{\top} \in \mathbb{C}^N, \\[2mm] \displaystyle\sum_{j=1}^{N} V_{ij}^{\mu}\, \psi_j^{-} - \sum_{j=1}^{N} V_{ij}^{\lambda}\, \psi_j^{+} \;=\; \widehat{g}_0^{\,i}, \quad i = 1, \ldots, N, \\[2mm] \displaystyle\frac{\alpha}{2}\, \psi_i^{-} + \alpha h \sum_{j=1}^{N} J^{\mu}(t_i, t_j)\, \psi_j^{-} + \frac{\beta}{2}\, \psi_i^{+} - \beta h \sum_{j=1}^{N} J^{\lambda}(t_i, t_j)\, \psi_j^{+} \;=\; h\, g_1(t_i), \quad i = 1, \ldots, N. \end{array}\right. \tag{36}$$

From the solution to this problem we construct an approximation u_h to the solution of the original transmission problem as in (10).

We identify now (36) with a generalized Petrov–Galerkin method. With this purpose, we introduce the 1-periodic distributions

$$\delta_i^{*} := \frac{5}{6}\,(\delta_{i-1/6} + \delta_{i+1/6}) + \frac{1}{6}\,(\delta_{i-5/6} + \delta_{i+5/6}),$$

and the space $S_h^{*} := \mathbb{C}\langle\, \delta_i^{*},\ i = 1, \ldots, N\,\rangle$. It is simple to see that S_h^{*} is an N-dimensional subspace of H^s for all $s < -1/2$. Moreover, $S_h^{*} \subset S_h^{1/6} + S_h^{-1/6}$. Therefore, we can define for $\boldsymbol{\psi}_h \in S_h \times S_h$ and $\boldsymbol{\varphi}_h \in S_h^{*} \times S_h$ the quantities

$$[\mathcal{H}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h, \qquad [\mathcal{A}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h, \qquad [\mathcal{H}_0\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h,$$

as in (12), (17), and (18). With these notations, the modified quadrature method (36) can be equivalently written as

$$\left|\begin{array}{l} \boldsymbol{\psi}_h \in S_h \times S_h, \\[1mm] [\mathcal{H}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h \;=\; [\mathbf{g}, \boldsymbol{\varphi}_h], \quad \forall \boldsymbol{\varphi}_h \in S_h^{*} \times S_h, \end{array}\right. \tag{37}$$

in the sense that if we define $\psi_h^{\pm} := \sum_{j=1}^{N} \psi_j^{\pm} \delta_j$ from a solution to (36), then $\boldsymbol{\psi}_h := (\psi_h^{-}, \psi_h^{+})$ is a solution to (37) and *vice versa*.

8. Technical results

The starting point for the study of the quadrature methods was the independent analysis of the corresponding numerical methods associated to the identity operator and to V_0. The only difference between the quadrature methods and the modified quadrature method lies in the test for the first group of equations, related to the logarithmic operator, where we have replaced the space S_h^{ε} by S_h^{*}. Thus, the aim of this section is to analyze the numerical scheme

$$\left|\begin{array}{l}\psi_h \in S_h,\\ \{V_0\psi_h, \varphi_h^*\} = (f, \varphi_h^*), \qquad \forall \varphi_h^* \in S_h^*,\end{array}\right. \tag{38}$$

for solving the logarithmic equation $V_0\psi = f$. We will prove a uniform inf-sup condition analogous to (21) and some convergence results. To do this we will deal with qualocation methods (see [11, 21, 22, 23] and references therein). We introduce the spaces of periodic smoothest splines of degrees zero and one,

$$S_h^0 := \{\, v_h \in H^0 | v_h|_{[t_i,t_{i+1}]} \in \mathbb{P}_0, \forall i\,\}, \qquad S_h^1 := \{\, \mu_h \in \mathcal{C}^0 | \mu_h|_{[t_i,t_{i+1}]} \in \mathbb{P}_1, \forall i\,\},$$

and consider the usual basis $\{\eta_i\}_{i=1}^N$ of S_h^1 such that $\eta_i(t_j) = \delta_{ij}$. We also define the discrete sesquilinear form

$$\langle f, g\rangle_h := \frac{h}{2}\sum_{i=1}^N \Big(f(t_{i-1/6})\,\overline{g(t_{i-1/6})} + f(t_{i+1/6})\,\overline{g(t_{i+1/6})}\Big) \approx (f, g) = \int_0^1 f(t)\,\overline{g(t)}\,dt.$$

Notice that the operator $T_h : S_h^1 \to S_h^*$ given by $T_h(\sum_{i=1}^N \mu_i\,\eta_i) := \frac{h}{2}\sum_{i=1}^N \mu_i\,\delta_i^*$, satisfies

$$\langle u, \mu_h\rangle_h = (u, T_h\mu_h), \qquad \forall u \in H^1,\ \mu_h \in S_h^1.$$

By the Riesz-Fréchet theorem, $T_h\mu_h$ is the unique element in H^{-1} satisfying the identity above. Moreover, by [11, Propositions 1 and 3], there exist $C_1, C_2 > 0$, independent of h, such that

$$C_1\|\mu_h\|_{-1} \le \|T_h\mu_h\|_{-1} \le C_2\|\mu_h\|_{-1}, \qquad \forall \mu_h \in S_h^1. \tag{39}$$

These notations allow us to write (38) in the equivalent form

$$\left|\begin{array}{l}\psi_h \in S_h,\\ \langle V_0\psi_h, \mu_h\rangle_h = \langle f, \mu_h\rangle_h, \qquad \forall \mu_h \in S_h^1.\end{array}\right.$$

From this point of view, (38) can be seen as a non-conforming qualocation method with a discrete set of Dirac deltas as trial space, instead of the commonly used periodic splines. The study of the properties of (38) will be carried out by analyzing a standard qualocation method for a singular integral equation. With this purpose, we consider the isomorphism $D + J : H^s \to H^{s-1}$, where D is the differential operator and $Jv := \widehat{v}(0)$, and define $A_0 := V_0(D + J)$. It can be easily verified that

$$A_0 v = \frac{1}{2}\,\text{p.v.}\int_0^1 \cot\pi(\,\cdot\, - t)\,v(t)\,dt + \frac{1}{4\pi}\int_0^1 v(t)\,dt,$$

(p.v. stands for the Cauchy principal value). A_0 is a periodic pseudodifferential operator and is therefore pseudolocal. Hence, if $v_h \in S_h^0$, $A_0 v_h$ is indefinitely differentiable in the intervals (t_i, t_{i+1}).

The solution to the following qualocation method with S_h^0 and S_h^1 as trial and test spaces

$$\left|\begin{array}{l}v_h \in S_h^0,\\ \langle A_0 v_h, \mu_h\rangle_h = \langle A_0 v, \mu_h\rangle_h, \qquad \forall \mu_h \in S_h^1,\end{array}\right. \tag{40}$$

satisfies (see [6, Theorems 2 and 5]) for $k \in \{1, 2, 3\}$,

$$\|v - v_h\|_{-k+1} \leq Ch^k \|v\|_k, \qquad \forall v \in H^k. \tag{41}$$

Since

$$\{A_0 v_h, T_h \mu_h\} = \langle A_0 v_h, \mu_h \rangle_h, \qquad \forall \mu_h \in S_h^1, \tag{42}$$

then (40) is equivalent to

$$\left| \begin{array}{l} v_h \in S_h^0, \\ \{A_0 v_h, \varphi_h^*\} = \{A_0 v, \varphi_h^*\}, \qquad \forall \varphi_h^* \in S_h^*. \end{array} \right.$$

Proposition 8. There exists $C > 0$, independent of h, such that

$$\sup_{\varphi_h^* \in S_h^*} \frac{|\{A_0 v_h, \varphi_h^*\}|}{\|\varphi_h^*\|_{-1}} \geq C \|v_h\|_0, \qquad \forall v_h \in S_h^0.$$

Proof. The solution to (40) satisfies the inequality $\|v_h\|_0 \leq C\|v\|_1$ (take $k = 1$ in (41)). Using then the same techniques as in [11], it can be proven that there exists $C > 0$, independent of h, such that

$$\sup_{\mu_h \in S_h^1} \frac{|\langle A_0 v_h, \mu_h \rangle_h|}{\|\mu_h\|_{-1}} \geq C \|v_h\|_0, \qquad \forall v_h \in S_h^0. \tag{43}$$

Applying now (39) and (42),

$$\sup_{\varphi_h^* \in S_h^*} \frac{|\{A_0 v_h, \varphi_h^*\}|}{\|\varphi_h^*\|_{-1}} = \sup_{\mu_h \in S_h^1} \frac{|\{A_0 v_h, T_h \mu_h\}|}{\|T_h \mu_h\|_{-1}} \geq C \sup_{\mu_h \in S_h^1} \frac{|\langle A_0 v_h, \mu_h \rangle_h|}{\|\mu_h\|_{-1}}, \qquad \forall v_h \in S_h^0,$$

and the result follows from (43). □

We introduce the element $\xi_h := h \sum_{i=1}^N \delta_i \in S_h$. It is very easy to prove that

$$\|1 - \xi_h\|_{-s} \leq C_s h^s, \qquad s > 1/2. \tag{44}$$

Lemma 9. For all $\varphi_h^* \in S_h^*$, $\{V_0(\xi_h - 1), \varphi_h^*\} = 0$.

Proof. Straightforward calculations show that for all $\mu \in \Lambda_N$ and $m \in \mathbb{Z}$, the Fourier coefficients of ξ_h satisfy

$$\widehat{\xi}_h(\mu + mN) = \begin{cases} 1, \text{ if } \mu = 0, \\ 0, \text{ otherwise.} \end{cases}$$

Then, by using the Fourier expansion of the Bessel operator (see for instance [20, Section 5.6])

$$4\pi\, V_0 u = \widehat{u}(0) + \sum_{m \neq 0} \frac{1}{|m|} \widehat{u}(m)\, e_m,$$

we obtain

$$4\pi\, V_0(\xi_h - 1) = \sum_{m \neq 0} \frac{1}{|m|N} e_{mN} = -\frac{1}{N} \log(4 \sin^2(\pi N \cdot)).$$

Thus,

$$\{V_0(\xi_h - 1), \delta_{i\pm 1/6}\} = -(1/N)\,\log(4\sin^2(\pi/6)) = 0, \qquad i = 1, \ldots, N,$$

which completes the proof. □

We consider now the discrete operator $J_h u := \widehat{u}(0)\,\xi_h$ which can be understood as an approximation of J. Furthermore, for all $s < -1/2$, the operator $D + J_h : S_h^0 \to S_h$ satisfies (see the proof of [4, Proposition 16])

$$C_s \|(D + J_h)v_h\|_s \;\le\; \|v_h\|_{s+1} \;\le\; C_s' \|(D + J_h)v_h\|_s, \qquad \forall v_h \in S_h^0, \tag{45}$$

where $C_s,\ C_s' > 0$ are independent of h. Note that Lemma 9 implies that

$$\{V_0(J - J_h)v, \varphi_h^*\} = 0, \qquad \forall v \in \mathcal{D}',\ \ \varphi_h^* \in S_h^*. \tag{46}$$

Proposition 10. There exists $C > 0$, independent of h, such that

$$\sup_{\varphi_h^* \in S_h^*} \frac{|\{V_0\psi_h, \varphi_h^*\}|}{\|\varphi_h^*\|_{-1}} \;\ge\; C\|\psi_h\|_{-1}, \qquad \forall \psi_h \in S_h.$$

Proof. Let $\psi_h \in S_h$ and take $v_h := (D + J_h)^{-1}\psi_h \in S_h^0$. Notice that $\widehat{\psi}_h(0) = \widehat{v}_h(0)$. Then,

$$V_0\psi_h \;=\; V_0(D + J)v_h + V_0(J_h - J)v_h.$$

We apply now Proposition 8 and (46) to deduce that

$$\sup_{\varphi_h^* \in S_h^*} \frac{|\{V_0\psi_h, \varphi_h^*\}|}{\|\varphi_h^*\|_{-1}} \;\ge\; C\|v_h\|_0 \;\ge\; C'\|\psi_h\|_{-1},$$

where we have used (45) for the last inequality. □

Proposition 11. Let $\psi_h^0 \in S_h$ be the solution to the problem

$$\left|\begin{array}{l} \psi_h^0 \in S_h, \\ \{V_0\psi_h^0, \varphi_h^*\} \;=\; (V_0\psi, \varphi_h^*), \qquad \forall \varphi_h \in S_h^*. \end{array}\right. \tag{47}$$

Then, for $k \in \{1, 2, 3\}$,

$$\|\psi - \psi_h^0\|_{-k} \;\le\; Ch^k\|\psi\|_{k-1}, \qquad \forall \psi \in H^{k-1}. \tag{48}$$

Proof. Given $\psi \in H^{k-1}$ we define $v := (D + J)^{-1}\psi \in H^k$ and take the solution $v_h \in S_h^0$ to (40). Then, by the definition of A_0 and (46),

$$(V_0\psi, \varphi_h^*) \;=\; (A_0 v, \varphi_h^*) \;=\; \{A_0 v_h, \varphi_h^*\} \;=\; \{V_0(D + J_h)v_h, \varphi_h^*\}, \qquad \forall \varphi_h^* \in S_h^*.$$

Since $(D + J_h)v_h \in S_h$, then $\psi_h^0 = (D + J_h)v_h$. Finally, as

$$\psi - \psi_h^0 = (D + J_h)(v - v_h) + (J - J_h)v,$$

applying (44) and (41) we obtain that

$$\|\psi - \psi_h^0\|_{-k} \;\le\; C\|v - v_h\|_{-k+1} + \|1 - \xi_h\|_{-k}|\widehat{v}(0)| \;\le\; C'h^k\|v\|_k \;\le\; C''h^k\|\psi\|_{k-1},$$

that is, (48) holds. □

We finally need a result concerning the transposed method to (47).

Proposition 12. Let $\varphi_h^* \in S_h^*$ be the solution to the problem

$$\left|\begin{array}{ll} \varphi_h^* \in S_h^*, & \\ \{V_0\varphi_h^*, \psi_h\} = (V_0\varphi, \psi_h), & \forall \psi_h \in S_h. \end{array}\right. \tag{49}$$

Then, for $k \in \{1, 2, 3\}$,

$$\|\varphi - \varphi_h^*\|_{-k} \le Ch^k \|\varphi\|_{k-1}, \qquad \forall \varphi \in H^{k-1}.$$

Proof. It is a simple transposition argument. Taking into account that $V_0^{-1} : H^k \to H^{k-1}$ is bounded,

$$\|\varphi - \varphi_h^*\|_{-k} = \sup_{\phi \in H^k} \frac{|(\phi, \varphi - \varphi_h^*)|}{\|\phi\|_k} \le C \sup_{\psi \in H^{k-1}} \frac{|(V_0\psi, \varphi - \varphi_h^*)|}{\|\psi\|_{k-1}}. \tag{50}$$

Taking now for $\psi \in H^{k-1}$ the solution $\psi_h \in S_h$ to (47), we have that

$$(V_0\psi, \varphi_h^*) = \{V_0\psi_h, \varphi_h^*\} = (V_0\psi_h, \varphi),$$

and by Proposition 11,

$$\begin{aligned} |(V_0\psi, \varphi - \varphi_h^*)| = |(V_0(\psi - \psi_h), \varphi)| &\le \|V_0(\psi - \psi_h)\|_{-k+1} \|\varphi\|_{k-1} \\ &\le C\|\psi - \psi_h\|_{-k}\|\varphi\|_{k-1} \le C'h^k \|\psi\|_{k-1}\|\varphi\|_{k-1}. \end{aligned}$$

From (50) we deduce the result. □

9. Analysis of the modified quadrature method

In this section we proceed as in the study of the quadrature methods, beginning by analyzing the numerical method applied to $\mathcal{H}_0$. From its properties we derive easily the desired results for the modified quadrature method applied to the global operator $\mathcal{H}$.

Proposition 13. There exists $C > 0$, independent of h, such that

$$\sup_{\boldsymbol{\varphi}_h \in S_h^* \times S_h} \frac{|\,[\mathcal{H}_0\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h|}{\|\boldsymbol{\varphi}_h\|_{-1,-1}} \ge C\|\boldsymbol{\psi}_h\|_{-1,-1}, \qquad \forall \boldsymbol{\psi}_h \in S_h \times S_h. \tag{51}$$

Moreover, if $\boldsymbol{\psi}_h^0 \in S_h \times S_h$ is the solution to

$$\left|\begin{array}{ll} \boldsymbol{\psi}_h^0 \in S_h \times S_h, & \\ [\mathcal{H}_0\boldsymbol{\psi}_h^0, \boldsymbol{\varphi}_h]_h = [\mathcal{H}_0\boldsymbol{\psi}, \boldsymbol{\varphi}_h], & \forall \boldsymbol{\varphi}_h \in S_h^* \times S_h, \end{array}\right. \tag{52}$$

then, for $k \in \{1, 2, 3\}$,

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h^0\|_{-k,-k} \le C\,h^k \|\boldsymbol{\psi}\|_{k,k}, \qquad \forall \boldsymbol{\psi} \in H^k \times H^k, \tag{53}$$

whereas if $\boldsymbol{\varphi}_h^0 \in S_h^* \times S_h$ is the solution to

$$\left|\begin{array}{ll} \boldsymbol{\varphi}_h^0 \in S_h^* \times S_h, & \\ [\mathcal{H}_0\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h^0]_h = [\mathcal{H}_0\boldsymbol{\psi}_h, \boldsymbol{\varphi}], & \forall \boldsymbol{\psi}_h \in S_h \times S_h, \end{array}\right. \tag{54}$$

then, for $k \in \{1,2,3\}$,

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h^0\|_{-k,-k} \le C h^k \|\boldsymbol{\varphi}\|_{k-1,k}, \qquad \forall \boldsymbol{\varphi} \in H^{k-1} \times H^k. \tag{55}$$

Proof. From Proposition 10 and (22) we obtain (51). Notice that in particular this implies that (52) and (54) are uniquely solvable. To show Estimate (53) we can proceed as at the beginning of Theorem 3 and apply Lemma 1 (a), Proposition 11 and the fact that $\mathcal{P} : S_h \times S_h \to S_h \times S_h$ is bounded in $H^k \times H^k$ as well as $\mathcal{P}^{-1}$ in $H^{-k} \times H^{-k}$.

Finally, by using again the invertibility of $\mathcal{P}$ we deduce that (54) is equivalent to

$$\left|\begin{array}{l} \boldsymbol{\varphi}_h^0 \in S_h^* \times S_h, \\ [\mathcal{A}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h^0]_h = [\mathcal{A}\boldsymbol{\psi}_h, \boldsymbol{\varphi}], \qquad \forall \boldsymbol{\psi}_h \in S_h \times S_h. \end{array}\right.$$

Therefore, setting $\boldsymbol{\varphi}_h^0 = (\varphi_h^-, \varphi_h^+)^\top$ and $\boldsymbol{\varphi} = (\varphi^-, \varphi^+)^\top$, these equations are also equivalent to

$$\varphi_h^+ = Q_h \varphi^+, \qquad \left|\begin{array}{l} \varphi_h^- \in S_h^*, \\ \{V_0 \psi_h, \varphi_h^-\} = (V_0 \psi_h, \varphi^-), \qquad \forall \psi_h \in S_h. \end{array}\right.$$

Now (55) is a consequence of Lemma 1 (a) and Proposition 12. □

Theorem 14. There exists $C > 0$, independent of h, such that for all h small enough

$$\sup_{\boldsymbol{\varphi}_h \in S_h^* \times S_h} \frac{|\,[\mathcal{H}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h|}{\|\boldsymbol{\varphi}_h\|_{-1,-1}} \ge C \|\boldsymbol{\psi}_h\|_{-1,-1}, \qquad \forall \boldsymbol{\psi}_h \in S_h \times S_h.$$

Proof. Since (51) holds, we can follow step by step the proof of Proposition 4. □

Theorem 15. The problem

$$\left|\begin{array}{l} \boldsymbol{\psi}_h \in S_h \times S_h, \\ [\mathcal{H}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h]_h = [\mathcal{H}\boldsymbol{\psi}, \boldsymbol{\varphi}_h], \qquad \forall \boldsymbol{\varphi}_h \in S_h^* \times S_h. \end{array}\right.$$

is uniquely solvable for all h small enough. Furthermore, for $k \in \{1,2,3\}$,

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{-k,-k} \le C h^k \|\boldsymbol{\psi}\|_{k,k}, \qquad \forall \boldsymbol{\psi} \in H^k \times H^k. \tag{56}$$

Proof. The estimates for $k = 1$ and $k = 2$ given in (56) can be shown as in Theorems 5 and 7 applying (53). To prove the result for $k = 3$, we start by noticing that

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{-3,-3} \le C \sup_{\boldsymbol{\varphi} \in H^2 \times H^3} \frac{|\,[\mathcal{H}(\boldsymbol{\psi} - \boldsymbol{\psi}_h), \boldsymbol{\varphi}]\,|}{\|\boldsymbol{\varphi}\|_{2,3}}, \tag{57}$$

since $(\mathcal{H}^*)^{-1} : H^3 \times H^3 \to H^2 \times H^3$ is bounded. Given $\boldsymbol{\varphi} \in H^2 \times H^3$, we take the solution $\boldsymbol{\varphi}_h^0 \in S_h^* \times S_h$ to (54). Then,

$$[\mathcal{H}\boldsymbol{\psi}, \boldsymbol{\varphi}_h^0] - [\mathcal{H}_0 \boldsymbol{\psi}_h, \boldsymbol{\varphi}] = [\mathcal{H}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h^0]_h - [\mathcal{H}_0 \boldsymbol{\psi}_h, \boldsymbol{\varphi}_h^0]_h = [\mathcal{K}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h^0].$$

Therefore, by easy manipulations we obtain the following equalities

$$\begin{aligned} [\mathcal{H}(\boldsymbol{\psi} - \boldsymbol{\psi}_h), \boldsymbol{\varphi}] &= [\mathcal{H}_0 \boldsymbol{\psi}, \boldsymbol{\varphi}] + [\mathcal{K}(\boldsymbol{\psi} - \boldsymbol{\psi}_h), \boldsymbol{\varphi}] + [\mathcal{K}\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h^0] - [\mathcal{H}\boldsymbol{\psi}, \boldsymbol{\varphi}_h^0] \\ &= [\mathcal{H}_0 \boldsymbol{\psi}, \boldsymbol{\varphi} - \boldsymbol{\varphi}_h^0] + [\mathcal{K}(\boldsymbol{\psi} - \boldsymbol{\psi}_h), \boldsymbol{\varphi} - \boldsymbol{\varphi}_h^0]. \end{aligned}$$

From this, by (55) and the result for $k = 1$, we deduce that (recall the continuity of $\mathcal{H}_0$ and $\mathcal{K}$),

$$\begin{aligned} |\,[\mathcal{H}(\boldsymbol{\psi} - \boldsymbol{\psi}_h), \boldsymbol{\varphi}]\,| &\le \|\mathcal{H}_0\boldsymbol{\psi}\|_{4,3}\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h^0\|_{-4,-3} + \|\mathcal{K}(\boldsymbol{\psi} - \boldsymbol{\psi}_h)\|_{2,2}\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h^0\|_{-2,-2} \\ &\le Ch^3\|\boldsymbol{\psi}\|_{3,3}\|\boldsymbol{\varphi}\|_{2,3} + C'\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{-1,-1}\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h^0\|_{-2,-2} \\ &\le Ch^3\|\boldsymbol{\psi}\|_{3,3}\|\boldsymbol{\varphi}\|_{2,3} + C''h^3\|\boldsymbol{\psi}\|_{1,1}\|\boldsymbol{\varphi}\|_{1,2}. \end{aligned}$$

By the inequality given in (57) we prove (56). □

Pointwise error estimates of the form

$$|u(\mathbf{z}) - u_h(\mathbf{z})| \le C_{\mathbf{z}}h^k\|\boldsymbol{\psi}\|_{k,k}, \qquad \mathbf{z} \in \mathbb{R}^2 \setminus \Gamma,$$

follow readily from (56).

10. Numerical results

We test our numerical methods on a problem (1–5) whose exact solution is known explicitly. Let Γ be given by the 1-periodic regular and smooth parameterization

$$\mathbf{x}(t) := (r(t)\cos(2\pi t), r(t)\sin(2\pi t))^\top, \qquad r(t) := 7 + 4\cos(2\pi t) + 2\sin(4\pi t).$$

In this case, Ω^- (the bounded domain defined by Γ) is non-convex.

We take the parameters

$$\mu = 1, \qquad \lambda = (1+\imath)/100, \qquad \alpha = 2, \qquad \beta = 1,$$

and choose the functions g_0 and g_1 for the right hand side such that the solution to Problem (1–5) is

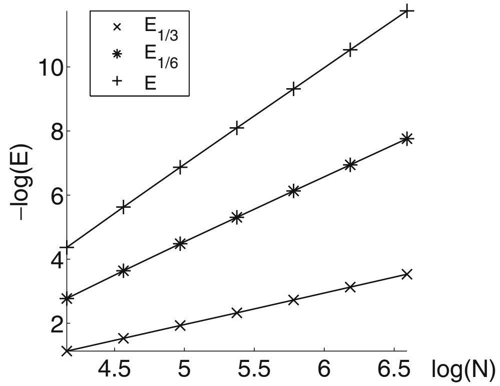
$$u(\mathbf{z}) = \left| \begin{array}{ll} e^{\imath\mu\,\mathbf{d}\cdot\mathbf{z}}, & \text{if } \mathbf{z} \in \Omega^-, \\ \dfrac{\imath}{4}\,H_0^{(1)}(\lambda|\mathbf{z}_0 - \mathbf{z}|), & \text{if } \mathbf{z} \in \Omega^+, \end{array} \right.$$

where $\mathbf{z}_0 := (5,5)^\top$ and $\mathbf{d} := (\sqrt{2}/2, -\sqrt{2}/2)^\top$.

Table 1 Pointwise errors and estimated convergence rates.

N	$E_{1/3}$	e.c.r.	$E_{1/6}$	e.c.r.	E	e.c.r.
64	3.22(-1)		6.26(-2)		1.27(-2)	
96	2.16(-1)	0.986	2.63(-2)	2.140	3.60(-3)	3.122
144	1.45(-1)	0.975	1.13(-2)	2.073	1.04(-3)	3.059
216	9.78(-2)	0.981	4.96(-3)	2.041	3.05(-4)	3.024
324	6.55(-2)	0.986	2.18(-3)	2.023	9.02(-5)	3.010
486	4.38(-2)	0.990	9.65(-4)	2.014	2.66(-5)	3.004
729	2.93(-2)	0.993	4.27(-4)	2.008	7.90(-6)	3.002

Figure 1 Pointwise errors.



We consider the points $\mathbf{z}_j := (-6 + 3.6(j-1), 1)^\top$, $j = 1, \ldots, 6$, (three of them are in Ω^-) and compute for $N = 64, 96, 144, 216, 324, 486$ and 729 the pointwise error

$$\sum_{j=1}^{6} |u(\mathbf{z}_j) - u_h(\mathbf{z}_j)|, \tag{58}$$

where u_h is defined in (10). Notice that the ratio between two consecutive grids is 3/2. Table 1 shows the error defined by the expression (58) when using the quadrature method with $\varepsilon = 1/3$ ($E_{1/3}$) and $\varepsilon = 1/6$ ($E_{1/6}$) and with the modified quadrature method (E). We also compute the estimated convergence rates (e.c.r.) by comparing these errors on consecutive grids in the usual way.

In figure 1 we represent the errors in logarithmic scale, obtaining three lines whose slopes give us the estimated convergence rates.

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