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Error analysis of spectral method on a triangle

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In this paper, the orthogonal polynomial approximation on triangle, proposed by Dubiner, is studied. Some approximation results are established in certain non-uniformly Jacobi-weighted Sobolev space, which play important role in numerical analysis of spectral and triangle spectral element methods for differential equations on complex geometries. As an example, a model problem is considered.

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1. Introduction

Although spectral methods have gained increasingly popularity in scientific computations during the last 30 years, its applications to problems on complex geometries have been historically limited. Indeed, the standard spectral methods are traditionally confined to problems on regular domains. However, in many areas, the underlying problems are originally set on some complex domains, which usually require the use of numerical methods on irregular meshes. Consequently, the low-order finite element method and finite volume method are preferable in practice, since they allow geometric flexibility. Recently, high-order methods have become popular in computational fluid dynamics, for instance, the viscous flow equations around complex obstacles (cf. [16]). The $h-p$ finite element method and spectral element method are notable among the high-order methods.

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Dubiner [7] considered a system of polynomials derived from Jacobi polynomials:

$$g_{l,m}(x,y) := 2^{l+3/2}(1-y)^l J_l^{(0,0)}(\xi) J_m^{(2l+1,0)}(\eta), \quad \xi = \frac{2x+y-1}{1-y},\ \eta = 2y-1, \quad (1.1)$$

which are $L^2(\mathcal{T})$-orthogonal on the triangle

$$\mathcal{T} := \{(x,y)\colon\ 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1,\ 0 \leqslant x+y \leqslant 1\}. \quad (1.2)$$

Later, Cai [5] extended this basis to the basis functions on curve surface for numerical simulation of electromagnetic scattering. The orthogonal polynomials (1.1) were also used as effective basis functions in $h-p$ finite element method and spectral finite element method in [16,18,19], which exhibit geometric flexibility and spatial accuracy of high order in actual computations. There have been also other families of orthogonal polynomials on triangle. For instance, Appell and de Fériet [1] (also see [8]) proposed some polynomials. Owens [17] constructed the orthogonal polynomials as the eigenfunctions of a singular Sturm–Liouville problem on triangle. However, the explicit expressions of such polynomials are not yet settled. We also refer to the interesting spectral element method developed in [9,14,15].

As we know, the estimate of convergence rate of triangle spectral element method essentially depends on the approximation results on the reference triangle $\mathcal{T}$. Also, the numerical analysis of most pseudospectral methods is closely related to the orthogonal approximation on $\mathcal{T}$. But so far, to our knowledge, there is nearly no result on the convergence analysis of orthogonal polynomial approximation on triangle, especially for the system (1.1). Thereby, we aim in this paper at the approximation properties of this basis, which is the first step for investigating the convergence of triangle spectral element method for partial differential equations on complex geometries.

There are two main difficulties in the error analysis. Firstly, the basis is not formed by the standard tensor product of two one-dimensional basis functions. We see from (1.1) that one of parameters of the second Jacobi polynomial is twice greater than the degree of the first one, and tends to infinity as the mode l goes to infinity. As a consequence, the usual results on one-dimensional Jacobi approximation established in [2,4,10,11] cannot be applied directly, since they are of the form

$$\|v - \pi_N v\|_{S_1} \leqslant c(\alpha,\beta) N^{-r} \|v\|_{S_2}, \quad (1.3)$$

where S_i $(i = 1, 2)$ are two weighted Sobolev spaces, π_N is a certain orthogonal projection upon the set consisting of all polynomials with degree at most N, and α, β are fixed parameters of Jacobi polynomials. In other words, we have to explore the explicit dependence of approximation results on the parameters of Jacobi polynomials. Secondly, the basis (1.1) comes from the Jacobi polynomials on the square $Q := (-1,1)^2$ by using a singular mapping: $(x,y) \in \mathcal{T} \to (\xi,\eta) \in Q$ (cf. (1.1)), which collapses one of edges of the square into a vertex of the triangle. Accordingly, we need some special techniques for dealing with this trouble.

The rest part of this paper is organized as follows. In section 2, we improve some results on the Jacobi approximation. In particular, we present the precise expression of $c(\alpha, \beta)$ in (1.3), so that we can use it for the analysis of orthogonal approximation on triangle. Also, we consider a special one-dimensional orthogonal approximation in this section, which plays an important role in the forthcoming discussions. In section 3, we use the results in section 2 to establish the main results on the orthogonal approximation on triangle by using the basis (1.1). Some numerical results show the efficiency of this approach. As an example of applications to partial differential equations on triangle, we consider a model problem on triangle and prove the convergence of proposed spectral method in section 4. The final section is for some concluding discussions.

2. Jacobi approximations in one dimension

In this section, we shall improve some results on the one-dimensional Jacobi approximation, which describe the explicit dependence of approximation errors on the parameters of Jacobi polynomials, and also consider a special orthogonal projection, which will be used to derive the main results in the next section.

2.1. Improved results on Jacobi approximation

We first introduce some notations. Let $\omega(\xi)$ be a certain weight function defined in $I := (-1, 1)$. Denote by $\mathbb{N}$ the set of all non-negative integers. For any $r \in \mathbb{N}$, we define the weighted Sobolev space $H^r_\omega(I)$ in the usual way, and denote its inner product, semi-norm and norm by $(u, v)_{r,\omega,I}$, $|v|_{r,\omega,I}$ and $\|v\|_{r,\omega,I}$, respectively. In particular, $L^2_\omega(I) = H^0_\omega(I)$, $(u, v)_{\omega,I} = (u, v)_{0,\omega,I}$ and $\|v\|_{\omega,I} = \|v\|_{0,\omega,I}$. For any $r > 0$, $H^r_{0,\omega}(I)$ stands for the closure in $H^r_\omega(I)$ of the set $\mathcal{D}$ consisting of all infinitely differentiable functions with compact support in I. When $\omega(\xi) \equiv 1$, we omit ω in the notations for simplicity.

For any $L \in \mathbb{N}$, let $\mathcal{P}_L(I)$ be the set of all algebraic polynomials of degree at most L, and $\mathcal{P}^0_L(I)$ be the subset of $\mathcal{P}_L(I)$ involving all polynomials vanishing at $\xi = \pm 1$.

Let $\chi^{(\alpha,\beta)}(\xi) = (1-\xi)^\alpha(1+\xi)^\beta$ be the Jacobi weight function. The normalized Jacobi polynomials $J^{(\alpha,\beta)}_l(\xi)$, $l \geqslant 0$, are the eigenfunctions of the Sturm–Liouville problem

$$\partial_\xi\big(\chi^{(\alpha+1,\beta+1)}(\xi)\partial_\xi v(\xi)\big) + \lambda\chi^{(\alpha,\beta)}(\xi)v(\xi) = 0, \quad \xi \in I, \tag{2.1}$$

with corresponding eigenvalue

$$\lambda^{(\alpha,\beta)}_l = l(l+\alpha+\beta+1). \tag{2.2}$$

The normalized Jacobi polynomials fulfill the recurrence relation (cf. [20]):

$$\partial_\xi J^{(\alpha,\beta)}_l(\xi) = \big(l(l+\alpha+\beta+1)\big)^{1/2} J^{(\alpha+1,\beta+1)}_{l-1}(\xi), \quad l \geqslant 1. \tag{2.3}$$

For $\alpha, \beta > -1$, the set $\{J_l^{(\alpha,\beta)}(\xi)\}$ is an $L^2_{\chi^{(\alpha,\beta)}}(I)$-orthonormal system, i.e.,

$$\int_I J_l^{(\alpha,\beta)}(\xi) J_{l'}^{(\alpha,\beta)}(\xi) \chi^{(\alpha,\beta)}(\xi)\, \mathrm{d}\xi = \delta_{l,l'} \tag{2.4}$$

where $\delta_{l,l'}$ is the Kronecker symbol.

For any $v \in L^2_{\chi^{(\alpha,\beta)}}(I)$, we have that

$$v(\xi) = \sum_{l=0}^{\infty} \hat{v}_l^{(\alpha,\beta)} J_l^{(\alpha,\beta)}(\xi), \quad \hat{v}_l^{(\alpha,\beta)} = \left(v, J_l^{(\alpha,\beta)}\right)_{\chi^{(\alpha,\beta)},I}. \tag{2.5}$$

We now consider the orthogonal projection $P_{L,\alpha,\beta} : L^2_{\chi^{(\alpha,\beta)}}(I) \to \mathcal{P}_L(I)$, defined by

$$(P_{L,\alpha,\beta} v - v, \phi)_{\chi^{(\alpha,\beta)},I} = 0, \quad \forall \phi \in \mathcal{P}_L(I). \tag{2.6}$$

To describe approximation results precisely, we introduce the space

$$H^r_{\chi^{(\alpha,\beta)},A}(I) := \left\{v\colon\ v \text{ is measurable and } \left\|\partial_\xi^k v\right\|_{\chi^{(\alpha+k,\beta+k)},I} < \infty, 0 \leqslant k \leqslant r\right\}, \quad r \in \mathbb{N}.$$

Lemma 2.1. For any $v \in H^r_{\chi^{(\alpha,\beta)},A}(I)$, $r, k \in \mathbb{N}$ and $0 \leqslant k \leqslant r$,

$$\left\|\partial_\xi^k (P_{L,\alpha,\beta} v - v)\right\|_{\chi^{(\alpha+k,\beta+k)},I} \leqslant \left(\lambda_{L-r+1}^{(\alpha+r,\beta+r-1)}\right)^{(k-r)/2} \left\|\partial_\xi^r v\right\|_{\chi^{(\alpha+r,\beta+r)},I} \tag{2.7}$$

where $\lambda_{L-r+1}^{(\alpha+r,\beta+r-1)}$ is given by (2.2).

Proof. Let

$$J_{l,k}^{(\alpha,\beta)}(\xi) := \partial_\xi^k J_l^{(\alpha,\beta)}(\xi). \tag{2.8}$$

Then by (2.5),

$$\partial_\xi^k v(\xi) = \sum_{l=0}^{\infty} \hat{v}_l^{(\alpha,\beta)} J_{l,k}^{(\alpha,\beta)}(\xi). \tag{2.9}$$

Moreover, by virtue of (2.3),

$$J_{l,k}^{(\alpha,\beta)}(\xi) = \left(\frac{\Gamma(l+\alpha+\beta+k+1)\Gamma(l+1)}{\Gamma(l+\alpha+\beta+1)\Gamma(l-k+1)}\right)^{1/2} J_{l-k}^{(\alpha+k,\beta+k)}(\xi).$$

Hence, $J_{l,k}^{(\alpha,\beta)}(\xi)$ is the same as $J_{l-k}^{(\alpha+k,\beta+k)}(\xi)$, apart from a constant. Therefore, (2.1) implies that

$$\partial_\xi\left(\chi^{(\alpha+k+1,\beta+k+1)}(\xi)\partial_\xi J_{l,k}^{(\alpha,\beta)}(\xi)\right) + \lambda_{l-k}^{(\alpha+k,\beta+k)} \chi^{(\alpha+k,\beta+k)}(\xi) J_{l,k}^{(\alpha,\beta)}(\xi) = 0, \quad l \geqslant k.$$

Multiplying the above equality by $J_{l,k}^{(\alpha,\beta)}(\xi)$ and integrating the result by parts, we find that

$$\left\|\partial_\xi J_{l,k}^{(\alpha,\beta)}\right\|^2_{\chi^{(\alpha+k+1,\beta+k+1)},I} = \lambda_{l-k}^{(\alpha+k,\beta+k)} \left\|J_{l,k}^{(\alpha,\beta)}\right\|^2_{\chi^{(\alpha+k,\beta+k)},I}. \tag{2.10}$$

Due to (2.8),

$$\partial_\xi J_{l,k}^{(\alpha,\beta)}(\xi) = \partial_\xi^{k+1} J_l^{(\alpha,\beta)}(\xi) = J_{l,k+1}^{(\alpha,\beta)}(\xi).$$

This with (2.10) leads to

$$\left\| J_{l,k}^{(\alpha,\beta)} \right\|_{\chi^{(\alpha+k,\beta+k)},I}^2 = \lambda_{l-k+1}^{(\alpha+k-1,\beta+k-1)} \left\| J_{l,k-1}^{(\alpha,\beta)} \right\|_{\chi^{(\alpha+k-1,\beta+k-1)},I}^2 = \cdots = c_{l,k}^{(\alpha,\beta)}, \tag{2.11}$$

where

$$c_{l,k}^{(\alpha,\beta)} = \prod_{j=0}^{k-1} \lambda_{l-j}^{(\alpha+j,\beta+j)} = \prod_{j=0}^{k-1} (l-j)(l+j+\alpha+\beta+1). \tag{2.12}$$

So we obtain from (2.9) and (2.11) that, for any $k, r \in \mathbb{N}$ and $k \leqslant r$,

$$\begin{aligned}
\left\| \partial_\xi^k (P_{L,\alpha,\beta} v - v) \right\|_{\chi^{(\alpha+k,\beta+k)},I}^2 &= \sum_{l=L+1}^{\infty} \left(\hat{v}_l^{(\alpha,\beta)}\right)^2 \left\| J_{l,k}^{(\alpha,\beta)} \right\|_{\chi^{(\alpha+k,\beta+k)},I}^2 \\
&= \sum_{l=L+1}^{\infty} \left(\hat{v}_l^{(\alpha,\beta)}\right)^2 c_{l,k}^{(\alpha,\beta)} = \sum_{l=L+1}^{\infty} \frac{c_{l,k}^{(\alpha,\beta)}}{c_{l,r}^{(\alpha,\beta)}} \left(\hat{v}_l^{(\alpha,\beta)}\right)^2 c_{l,r}^{(\alpha,\beta)} \\
&= \sum_{l=L+1}^{\infty} \frac{c_{l,k}^{(\alpha,\beta)}}{c_{l,r}^{(\alpha,\beta)}} \left(\hat{v}_l^{(\alpha,\beta)}\right)^2 \left\| J_{l,r}^{(\alpha,\beta)} \right\|_{\chi^{(\alpha+r,\beta+r)},I}^2.
\end{aligned} \tag{2.13}$$

Further, for $l \geqslant L+1$, we have from (2.2) and (2.12) that

$$\begin{aligned}
\frac{c_{l,k}^{(\alpha,\beta)}}{c_{l,r}^{(\alpha,\beta)}} &= \prod_{j=k}^{r-1} \left(\lambda_{l-j}^{(\alpha+j,\beta+j)}\right)^{-1} = \prod_{j=k}^{r-1} \frac{1}{(l-j)(l+j+\alpha+\beta+1)} \\
&\leqslant \prod_{j=k}^{r-1} \frac{1}{(L-j+1)(L+j+\alpha+\beta+2)} \\
&\leqslant \prod_{j=k}^{r-1} \frac{1}{(L-j)(L+j+\alpha+\beta+2)} = \prod_{j=k}^{r-1} \frac{1}{\lambda_{L-j}^{(\alpha+j+1,\beta+j)}}.
\end{aligned} \tag{2.14}$$

Next, set

$$f(z) = (L-z)(L+\alpha+\beta+2+z).$$

Since $\alpha, \beta > -1$, we have

$$f'(z) = -(\alpha+\beta+2) - 2z < 0, \quad \forall z \geqslant 0.$$

This implies that for $\alpha, \beta > -1$ and $k \leqslant j \leqslant r-1$,

$$\lambda_{L-j}^{(\alpha+j+1,\beta+j)} = f(j) \geqslant f(r-1) = \lambda_{L-r+1}^{(\alpha+r,\beta+r-1)}. \tag{2.15}$$

A combination of (2.9) and (2.13)–(2.15) leads to the desired result. □

We can use lemma 2.1 to improve the corresponding result on the $H^1(I)$-projection, which will be used in section 4. Basically, we consider the orthogonal projection $P_L^{1,0}: H_0^1(I) \to \mathcal{P}_L^0(I)$, such that for any $v \in H_0^1(I)$,

$$\left(\partial_\xi (P_L^{1,0} v - v), \partial_\xi \phi\right)_I = 0, \quad \forall \phi \in \mathcal{P}_L^0(I). \tag{2.16}$$

Hereafter, we denote by c a generic positive constant independent of L, α, β and any function.

Lemma 2.2. If $v \in H_0^1(I)$, $\partial_\xi v \in H^{r-1}_{\chi^{(0,0)},A}(I)$ and $r \in \mathbb{N}$, then for $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$\left\| P_L^{1,0} v - v \right\|_{\mu,I} \leqslant c L^{\mu - r} \left\| \partial_\xi^r v \right\|_{\chi^{(r-1,r-1)},I}. \tag{2.17}$$

Proof. We can follow the standard procedure (cf. [6]) to obtain the desired result by using lemma 2.1. To do this, let

$$\phi^*(\xi) = \int_{-1}^{\xi} P_{L-1,0,0} \partial_\zeta v(\zeta)\, \mathrm{d}\zeta, \qquad \phi(\xi) = \phi^*(\xi) - \frac{1}{2}\phi^*(1)(1+\xi).$$

Clearly $\phi \in \mathcal{P}_L^0(I)$. Moreover

$$\left|\phi^*(1)\right| = \left| \int_I \left(P_{L-1,0,0}\partial_\zeta v(\zeta) - \partial_\zeta v(\zeta)\right) \mathrm{d}\zeta \right| \leqslant \sqrt{2} \| P_{L-1,0,0} \partial_\xi v - \partial_\xi v \|_I.$$

By projection theorem,

$$\begin{aligned}
\left| P_L^{1,0} v - v \right|_{1,I} &\leqslant |\phi - v|_{1,I} \leqslant \| P_{L-1,0,0}\partial_\xi v - \partial_\xi v \|_I + \left|\phi^*(1)\right| \\
&\leqslant \left(\sqrt{2} + 1\right) \| P_{L-1,0,0}\partial_\xi v - \partial_\xi v \|_I \\
&\leqslant \left(\sqrt{2} + 1\right) \left(\lambda_{L-r}^{(r,r-1)}\right)^{(1-r)/2} \left\| \partial_\xi^r v \right\|_{\chi^{(r-1,r-1)},I} \\
&\leqslant c L^{1-r} \left\| \partial_\xi^r v \right\|_{\chi^{(r-1,r-1)},I}.
\end{aligned} \tag{2.18}$$

By this fact and the Poincaré inequality, we obtain (2.17) with $\mu = 1$. We next consider the case $\mu = 0$. Let $g \in L^2(I)$, and consider the auxiliary problem

$$(\partial_\xi w, \partial_\xi z)_I = (g, z)_I, \quad \forall z \in H_0^1(I).$$

It has a unique solution and $\|w\|_{2,I} \leqslant c\|g\|_I$. In view of this fact, we have from (2.18) that

$$\left| P_L^{1,0} w - w \right|_{1,I} \leqslant c L^{-1} \left\| \partial_\xi^2 w \right\|_{\chi^{(1,1)},I} \leqslant c L^{-1} \|g\|_I.$$

Finally, we can derive the desired result by a standard duality argument and space interpolation. □

2.2. Spectral approximation by using orthogonal system $\{K_l^{(\sigma)}\}$

In this subsection, we consider a special orthogonal approximation by using the family of polynomials:

$$K_l^{(\sigma)}(\eta) := (1-\eta)^\sigma J_l^{(2\sigma+1,0)}(\eta), \quad \sigma \in \mathbb{N},\ \eta \in I. \tag{2.19}$$

This system possesses the following properties:

- According to (2.1),

  $$(1-\eta)^{-\sigma}\partial_\eta\big((1-\eta)^{2\sigma+2}(1+\eta)\partial_\eta\big((1-\eta)^{-\sigma}K_l^{(\sigma)}(\eta)\big)\big) + \gamma_l^{(\sigma)}(1-\eta)K_l^{(\sigma)}(\eta) = 0,$$

  where

  $$\gamma_l^{(\sigma)} = l(l+2\sigma+2).$$

- Thanks to (2.3),

  $$\partial_\eta\big((1-\eta)^{-\sigma}K_l^{(\sigma)}(\eta)\big) = \big(l(l+2\sigma+2)\big)^{1/2}J_{l-1}^{(2\sigma+2,1)}(\eta), \quad l \geqslant 1.$$

- Let $\chi(\eta) = 1-\eta$. By (2.4), the set $\{K_l^{(\sigma)}(\eta)\}$ is a normalized $L^2_\chi(I)$-orthogonal system, i.e.,

  $$\int_I K_l^{(\sigma)}(\eta)K_{l'}^{(\sigma)}(\eta)\chi(\eta)\,\mathrm{d}\eta = \int_I J_l^{(2\sigma+1,0)}(\eta)J_{l'}^{(2\sigma+1,0)}(\eta)(1-\eta)^{2\sigma+1}\,\mathrm{d}\eta = \delta_{l,l'}. \tag{2.20}$$

Now, let

$$\widetilde{L}^2_\chi(I) := \big\{v\colon\ v = (1-\eta)^\sigma u,\ u \in L^2_{\chi^{(2\sigma+1,0)}}(I)\big\},$$

and

$$\widetilde{\mathcal{P}}_{M,\sigma}(I) := \operatorname{span}\big\{K_0^{(\sigma)}, K_1^{(\sigma)}, \ldots, K_M^{(\sigma)}\big\}.$$

Further, let ${}^0\widetilde{\mathcal{P}}_{M,\sigma}(I)$ and $\widetilde{\mathcal{P}}^0_{M,\sigma}(I)$ be the subsets of $\widetilde{\mathcal{P}}_{M,\sigma}(I)$ consisting of all polynomials vanishing at $\eta = 1$ and $\eta = \pm 1$, respectively.

The orthogonal projection $\widetilde{P}_{M,\sigma} : \widetilde{L}^2_\chi(I) \to \widetilde{\mathcal{P}}_{M,\sigma}(I)$ is defined by

$$\big(\widetilde{P}_{M,\sigma}v - v, \phi\big)_{\chi,I} = 0, \quad \forall \phi \in \widetilde{\mathcal{P}}_{M,\sigma}(I). \tag{2.21}$$

To describe approximation results more precisely, we introduce the weighted space

$$A^r_\sigma(I) = \big\{v \mid v \text{ is measurable on } I \text{ and } \|v\|_{A^r_\sigma(I)} < \infty\big\}, \quad r \in \mathbb{N},$$

where

$$\|v\|_{A^r_\sigma(I)} = \left(\sum_{k=0}^{r}(\sigma+r-k-1)^{2r-2k}\big\|(1-\eta)^{k-r/2}(1+\eta)^{r/2}\partial_\eta^k v\big\|^2_{\chi,I}\right)^{1/2}. \tag{2.22}$$

Lemma 2.3. For any $v \in \widetilde{L}^2_\chi(I) \cap A^r_\sigma(I)$, $r \in \mathbb{N}$ and $r \geqslant 0$,

$$\big\| \widetilde{P}_{M,\sigma} v - v \big\|_{\chi,I} \leqslant c\big(\lambda^{(2\sigma+r+1,r-1)}_{M-r+1}\big)^{-r/2} \|v\|_{A^r_\sigma(I)}. \tag{2.23}$$

Proof. Let $P_{M,2\sigma+1,0}$ be the $L^2_{\chi^{(2\sigma+1,0)}}(I)$-orthogonal projection as in (2.6), and

$$u(\eta) = v(\eta)\chi^{(-\sigma,0)}(\eta), \qquad \phi(\eta) = \chi^{(\sigma,0)}(\eta)(P_{M,2\sigma+1,0}u)(\eta).$$

Clearly, $u \in L^2_{\chi^{(2\sigma+1,0)}}(I)$ and $\phi \in \widetilde{\mathcal{P}}_{M,\sigma}(I)$. By projection theorem and lemma 2.1,

$$\begin{aligned}\big\| \widetilde{P}_{M,\sigma} v - v \big\|_{\chi,I} &\leqslant \|\phi - v\|_{\chi,I} = \|P_{M,2\sigma+1,0}u - u\|_{\chi^{(2\sigma+1,0)},I} \\ &\leqslant c\big(\lambda^{(2\sigma+r+1,r-1)}_{M-r+1}\big)^{-r/2} \|\partial^r_\eta u\|_{\chi^{(r+2\sigma+1,r)},I}.\end{aligned}$$

A direct calculation gives

$$\big\|\partial^r_\eta u\big\|_{\chi^{(r+2\sigma+1,r)},I} \leqslant c\|v\|_{A^r_\sigma(I)}.$$

Then the conclusion follows immediately. □

In lemma 2.3, we presented a basic approximation result. But for approximating partial differential equations on triangle, we need to consider another orthogonal projection in the following non-uniformly weighted space:

$$\widetilde{H}^1_\chi(I) = \big\{v\colon\ v \in L^2_{\chi^{-1}}(I) \cap \widetilde{L}^2_\chi(I) \text{ and } \partial_\eta v \in L^2_\chi(I)\big\}$$

equipped with the norm

$$\|v\|_{1,\chi,\sim,I} = \big(\|\partial_\eta v\|^2_{\chi,I} + \|v\|^2_{\chi^{-1},I}\big)^{1/2}.$$

The spaces ${}^0\widetilde{H}^1_\chi(I)$ and $\widetilde{H}^1_{0,\chi}(I)$ are the subspaces of $\widetilde{H}^1_\chi(I)$ consisting of all functions vanishing at $\eta = 1$ and $\eta = \pm 1$, respectively.

As a preparation, we first consider the orthogonal projection ${}^0\widetilde{P}^1_{M,\sigma} : {}^0\widetilde{H}^1_\chi(I) \to {}^0\widetilde{\mathcal{P}}_{M,\sigma}(I)$, defined by

$$\big(\partial_\eta\big({}^0\widetilde{P}^1_{M,\sigma} v - v\big), \partial_\eta\phi\big)_{\chi,I} + \big({}^0\widetilde{P}^1_{M,\sigma} v - v, \phi\big)_{\chi^{-1},I} = 0, \quad \forall \phi \in {}^0\widetilde{\mathcal{P}}_{M,\sigma}(I). \tag{2.24}$$

For better description of approximation errors, we introduce the space

$$B^r_\sigma(I) = \big\{v\colon\ v \text{ is measurable on } I \text{ and } \|v\|_{B^r_\sigma(I)} < \infty\big\}, \quad r \in \mathbb{N},$$

where

$$\|v\|_{B^r_\sigma(I)} = \left(\sum_{k=0}^{r} (\sigma + r - k - 1)^{2r-2k} \big\|(1-\eta)^{k-r/2-1/2}(1+\eta)^{r/2}\partial^k_\eta v\big\|^2_{\chi,I}\right)^{1/2}.$$

Lemma 2.4. If $v \in {}^0\widetilde{H}^1_\chi(I)$, $\partial_\eta v \in B^{r-1}_{\sigma-1}(I)$, $r, \sigma \in \mathbb{N}$ and $r, \sigma \geqslant 1$, then

$$\big\|{}^0\widetilde{P}^1_{M,\sigma} v - v\big\|_{1,\chi,\sim,I} \leqslant c\big(\lambda^{(2\sigma+r-3,r-2)}_{M-r+2}\big)^{(1-r)/2} \|\partial_\eta v\|_{B^{r-1}_{\sigma-1}(I)}. \tag{2.25}$$

Proof. By the Hardy inequality (cf. [13]), we know that for any measurable function ψ and real number $d < 1$,

$$\int_I \left(\frac{1}{1-\eta} \int_\eta^1 \psi(\zeta)\,\mathrm{d}\zeta \right)^2 (1-\eta)^d\,\mathrm{d}\eta \leqslant \frac{4}{1-d} \int_I \psi^2(\eta)(1-\eta)^d\,\mathrm{d}\eta.$$

Due to $v(1) = 0$, we can take $\psi = \partial_\eta v$ in the above inequality, and so for $\alpha < 1$,

$$\|v\|_{\chi^{(\alpha-2,0)},I} \leqslant c\|\partial_\eta v\|_{\chi^{(\alpha,0)},I}. \tag{2.26}$$

Hence, by projection theorem and (2.26), we have that for any $\phi \in {}^0\widetilde{\mathcal{P}}_{M,\sigma}(I)$,

$$\left\| {}^0\widetilde{P}^1_{M,\sigma} v - v \right\|_{1,\chi,\sim,I} \leqslant \|\phi - v\|_{1,\chi,\sim,I} \leqslant c\left\| \partial_\eta(\phi - v) \right\|_I. \tag{2.27}$$

Let $P_{M,2\sigma-2,0}$ be the projector as in (2.6), and take

$$\phi(\eta) = \int_\eta^1 \chi^{(\sigma-1,0)}(\zeta)\big(P_{M,2\sigma-2,0}\big(\chi^{(1-\sigma,0)}\partial_\zeta v\big)\big)(\zeta)\,\mathrm{d}\zeta.$$

Clearly, $\phi(1) = 0$, $\partial_\eta\phi \in \widetilde{\mathcal{P}}_{M,\sigma-1}(I)$ and $\phi \in {}^0\widetilde{\mathcal{P}}_{M,\sigma}(I)$. Therefore by lemma 2.1 and (2.27),

$$\begin{aligned}
\left\| {}^0\widetilde{P}^1_{M,\sigma} v - v \right\|_{1,\chi,\sim,I} &\leqslant c\left\| \chi^{(\sigma-1,0)}\big(P_{M,2\sigma-2,0}\big(\chi^{(1-\sigma,0)}\partial_\eta v\big) - \chi^{(1-\sigma,0)}\partial_\eta v\big) \right\|_I \\
&\leqslant c\left\| P_{M,2\sigma-2,0}\big(\chi^{(1-\sigma,0)}\partial_\eta v\big) - \chi^{(1-\sigma,0)}\partial_\eta v \right\|_{\chi^{(2\sigma-2,0)},I} \\
&\leqslant c\big(\lambda^{(2\sigma+r-3,r-2)}_{M-r+2}\big)^{(1-r)/2} \left\| \partial_\eta^{r-1}\big(\chi^{(1-\sigma,0)}\partial_\eta v\big) \right\|_{\chi^{(2\sigma+r-3,r-1)},I}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left\| \partial_\eta^{r-1}\big(\chi^{(1-\sigma,0)}\partial_\eta v\big) \right\|^2_{\chi^{(2\sigma+r-3,r-1)},I} \\
&\quad = \int_I \left(\sum_{k=0}^{r-1} (-1)^{r-k-1} C^k_{r-1} \partial_\eta^{k+1} v (1-\eta)^{-\sigma-r+k+2} \prod_{j=-1}^{r-k-3} (\sigma + j) \right)^2 \chi^{(2\sigma+r-3,r-1)}(\eta)\,\mathrm{d}\eta \\
&\quad \leqslant c \sum_{k=0}^{r-1} (\sigma + r - k - 3)^{2(r-k-1)} \left\| (1-\eta)^{k-r/2}(1+\eta)^{(r-1)/2} \partial_\eta^{k+1} v \right\|^2_{\chi,I} \\
&\quad \leqslant c\|\partial_\eta v\|_{B^{r-1}_{\sigma-1}(I)}.
\end{aligned}$$

This ends the proof. □

In the end of this section, we introduce the orthogonal projection $\widetilde{P}^{1,0}_{M,\sigma} : \widetilde{H}^1_{0,\chi}(I) \to \widetilde{\mathcal{P}}^0_{M,\sigma}(I)$, which will be used in section 4. It is defined by

$$\big(\partial_\eta\big(\widetilde{P}^{1,0}_{M,\sigma} v - v\big), \partial_\eta\phi\big)_{\chi,I} + \big(\widetilde{P}^{1,0}_{M,\sigma} v - v, \phi\big)_{\chi^{-1},I} = 0, \quad \forall\phi \in \widetilde{\mathcal{P}}^0_{M,\sigma}(I). \tag{2.28}$$

Lemma 2.5. If $v \in \widetilde{H}^1_{0,\chi}(I)$, $\partial_\eta((1+\eta)^{-1}v) \in B^{r-1}_{\sigma-1}(I)$, $r, s \in \mathbb{N}$ and $r, \sigma \geqslant 1$, then

$$\big\| \widetilde{P}^{1,0}_{M,\sigma} v - v \big\|_{1,\chi,\sim,I} \leqslant c\big(\lambda^{(2\sigma+r-3,r-2)}_{M-r+1}\big)^{(1-r)/2} \big\| \partial_\eta\big((1+\eta)^{-1}v\big)\big\|_{B^{r-1}_{\sigma-1}(I)}. \tag{2.29}$$

Proof. Take

$$\phi(\eta) = (1+\eta)^0 \widetilde{P}^1_{M-1,\sigma}\big((1+\eta)^{-1}v\big) \in \widetilde{\mathcal{P}}^0_{M,\sigma}(I).$$

We have from projection theorem, (2.26) and lemma 2.4 that

$$\begin{aligned}\big\| \widetilde{P}^{1,0}_{M,\sigma} v - v \big\|_{1,\chi,\sim,I} &\leqslant \|\phi - v\|_{1,\chi,\sim,I} \\ &= \big\|(1+\eta)\big({}^0\widetilde{P}^1_{M-1,\sigma}\big((1+\eta)^{-1}v\big) - (1+\eta)^{-1}v\big)\big\|_{1,\chi,\sim,I} \\ &\leqslant c\big\|{}^0\widetilde{P}^1_{M-1,\sigma}\big((1+\eta)^{-1}v\big) - (1+\eta)^{-1}v\big\|_{1,\chi,\sim,I} \\ &\leqslant c\big(\lambda^{(2\sigma+r-3,r-2)}_{M-r+1}\big)^{(1-r)/2}\big\|\partial_\eta\big((1+\eta)^{-1}v\big)\big\|_{B^{r-1}_{\sigma-1}(I)}.\end{aligned}$$

This completes the proof. □

Remark 2.1. Lemmas 2.3–2.5 will play important role in derivation of the main results of this paper. But the technique in this section is also useful for other problems. For instance, if the derivatives of orders up to integer $\sigma - 1 \geqslant 0$ of solutions of differential equations vanish at $\eta = 1$, then it seems reasonable to approximate it by using the basis $\{K^{(\sigma)}_l(\eta)\}$. On the other hand, if the solution belongs to $L^2_{\chi^{(\lambda,0)}}(I)$ and has the corner singularity like $(1-\eta)^\sigma, \sigma < 0$, then we may use the basis functions $\widetilde{K}^{(\sigma,\lambda)}_l(\eta) := (1-\eta)^\sigma J^{(2\sigma+\lambda,0)}_l(\eta)$.

3. Orthogonal approximation on a triangle

In this section, we establish the main results on the orthogonal approximation on the reference triangle $\mathcal{T}$, given in (1.2). We shall use the notations $L^2(\mathcal{T})$, $H^r(\mathcal{T})$, $H^r_0(\mathcal{T})$, $(u,v)_\mathcal{T}$, $\|v\|_\mathcal{T}$, $|v|_{r,\mathcal{T}}$ and $\|v\|_{r,\mathcal{T}}$, etc.

Let $I_\xi = \{\xi \mid |\xi| < 1\}$, $I_\eta = \{\eta \mid |\eta| < 1\}$ and $Q = I_\xi \times I_\eta$ be the reference square. By the variable transformation

$$\xi = \frac{2x+y-1}{1-y}, \qquad \eta = 2y - 1, \tag{3.1}$$

the triangle $\mathcal{T}$ becomes the square Q. The mapping (3.1) collapses the top edge of Q into the vertex (0, 1) of $\mathcal{T}$. The Jacobian of this mapping is

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \frac{4}{1-y} = \frac{8}{1-\eta}. \tag{3.2}$$

Its inverse

$$J^{-1} = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{1}{8}(1-\eta) = \frac{1}{4}(1-y). \tag{3.3}$$

The orthogonal polynomials on the triangle $\mathcal{T}$ (cf. [7,16,18,19]) are of the form

$$g_{l,m}(x,y) = 2^{l+3/2}(1-y)^l J_l^{(0,0)}\left(\frac{2x+y-1}{1-y}\right) J_m^{(2l+1,0)}(2y-1),$$
$$0 \leqslant l \leqslant L, \quad 0 \leqslant m \leqslant M, \quad 0 \leqslant l+m \leqslant M, \quad L \leqslant M. \tag{3.4}$$

By (3.1) and (3.4),

$$g_{l,m}(x,y) = \tilde{g}_{l,m}(\xi,\eta) = 2\sqrt{2}(1-\eta)^l J_l^{(0,0)}(\xi) J_m^{(2l+1,0)}(\eta). \tag{3.5}$$

Therefore, by (3.3) and (3.5),

$$\begin{aligned}
&\iint_{\mathcal{T}} g_{l,m}(x,y) g_{l',m'}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \frac{1}{8}\int_Q \tilde{g}_{l,m}(\xi,\eta)\tilde{g}_{l',m'}(\xi,\eta)(1-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad = \int_{I_\xi} J_l^{(0,0)}(\xi) J_{l'}^{(0,0)}(\xi)\,\mathrm{d}\xi \int_{I_\eta} J_m^{(2l+1,0)}(\eta) J_{m'}^{(2l'+1,0)}(\eta)(1-\eta)^{l+l'+1}\,\mathrm{d}\eta = \delta_{l,l'}\delta_{m,m'}.
\end{aligned}$$

Moreover, it can be checked by an argument as in the proof of lemma 2 of [5] that the set $\{g_{l,m}\}$ is complete in $L^2(\mathcal{T})$. Thus, for any $v \in L^2(\mathcal{T})$, we can write

$$v(x,y) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \hat{v}_{l,m} g_{l,m}(x,y), \tag{3.6}$$

where

$$\hat{v}_{l,m} = \iint_{\mathcal{T}} v(x,y) g_{l,m}(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.7}$$

3.1. $L^2(\mathcal{T})$-orthogonal projection

Now, let

$$\mathcal{P}_{L,M}(\mathcal{T}) = \operatorname{span}\bigl\{g_{l,m}(x,y) \mid 0 \leqslant l \leqslant L,\ 0 \leqslant m \leqslant M\bigr\},$$
$$\mathcal{P}_{L,M}^0(\mathcal{T}) = \bigl\{v \mid v \in \mathcal{P}_{L,M}(\mathcal{T}) \text{ and } v|_{\partial\mathcal{T}} = 0\bigr\}.$$

We consider the most important orthogonal projection $P_{L,M} : L^2(\mathcal{T}) \to \mathcal{P}_{L,M}(\mathcal{T})$, defined by

$$(P_{L,M}v - v, \phi)_{\mathcal{T}} = 0, \quad \forall \phi \in \mathcal{P}_{L,M}(\mathcal{T}). \tag{3.8}$$

For simplicity and clarity of description of the main approximation result, we introduce the following non-isotropic weighted space:

$$H^{r,s}(\mathcal{T}) = \bigl\{v \mid v \text{ is measurable on } \mathcal{T} \text{ and } \|v\|_{H^{r,s}(\mathcal{T})} < \infty\bigr\}, \quad r,s \in \mathbb{N},$$

with the norm

$$\|v\|_{H^{r,s}(\mathcal{T})}=\left(\sum_{k=0}^{r}\sum_{j=0}^{k}\left\|x^{j}y^{r/2}(1-y)^{k-j-r/2}\partial_x^j\partial_y^{k-j}v\right\|^2_{L^2(\mathcal{T})}\right.$$
$$\left.+\left\|x^{s/2}(1-x-y)^{s/2}\partial_x^s v\right\|^2_{L^2(\mathcal{T})}\right)^{1/2}. \qquad (3.9)$$

Theorem 3.1. For any $v\in H^{r,s}(\mathcal{T}),\ r,s\in\mathbb{N}$ and $r,s\geqslant 0$,

$$\|P_{L,M}v-v\|_{\mathcal{T}}\leqslant c\left(\left(\frac{M(M+L)}{L^2}\right)^{-r}+L^{-s}\right)\|v\|_{H^{r,s}(\mathcal{T})}. \qquad (3.10)$$

Proof. The set $\{\tilde{g}_{l,m}(\xi,\eta)\}$ is mutually orthogonal on the square Q, associated with the weight $\chi(\eta)=1-\eta$. Moreover $\|\tilde{g}_{l,m}\|_{\mathcal{T}}=2\sqrt{2}$. Let

$$u(\xi,\eta)=v\left(\frac{1}{4}(1+\xi)(1-\eta),\frac{1}{2}(1+\eta)\right). \qquad (3.11)$$

Then

$$v(x,y)=u(\xi,\eta)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\hat{u}_{l,m}\tilde{g}_{l,m}(\xi,\eta) \qquad (3.12)$$

where

$$\hat{u}_{l,m}=\frac{1}{8}\iint_Q u(\xi,\eta)\tilde{g}_{l,m}(\xi,\eta)(1-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \qquad (3.13)$$

Define the $L^2_\chi(Q)$-orthogonal projection as

$$P^*_{L,M}u(\xi,\eta)=\sum_{l=0}^{L}\sum_{m=0}^{M}\hat{u}_{l,m}\tilde{g}_{l,m}(\xi,\eta).$$

Thanks to (3.1), (3.3), (3.5) and (3.7),

$$P_{L,M}v(x,y)=P^*_{L,M}u(\xi,\eta),$$

whence

$$v(x,y)-P_{L,M}v(x,y)=u(\xi,\eta)-P^*_{L,M}u(\xi,\eta). \qquad (3.14)$$

Thus by (3.3),

$$\|P_{L,M}v-v\|^2_{\mathcal{T}}=\left\|u-P^*_{L,M}u\right\|^2_{L^2(I_\xi;L^2_\chi(I_\eta))}. \qquad (3.15)$$

We now estimate the right side of (3.15). To do this, let $\widetilde{P}_{M,l}$ be the $\widetilde{L}^2_\chi(I_\eta)$-orthogonal projection as in (2.21), and let

$$u_l(\eta) = \int_{I_\xi} u(\xi,\eta) J_l^{(0,0)}(\xi)\,\mathrm{d}\xi, \quad l \geqslant 0.$$

Then we use (3.5), (3.12), (3.13) and the definition of $\widetilde{P}_{M,l}$ to reach that

$$\begin{aligned} P^*_{L,M}u(\xi,\eta) &= \sum_{l=0}^{L}\sum_{m=0}^{M}\left(\int_{I_\eta}\left(\int_{I_\xi} u(\xi,\eta)(1-\eta)^{l+1} J_l^{(0,0)}(\xi) J_m^{(2l+1,0)}(\eta)\,\mathrm{d}\xi\right)\mathrm{d}\eta\right)\\ &\quad \times (1-\eta)^l J_l^{(0,0)}(\xi) J_m^{(2l+1,0)}(\eta)\\ &= \sum_{l=0}^{L}\left(\sum_{m=0}^{M}\left(\int_{I_\eta} u_l(\eta) K_m^{(l)}(\eta)(1-\eta)\,\mathrm{d}\eta\right) K_m^{(l)}(\eta)\right) J_l^{(0,0)}(\xi)\\ &= \sum_{l=0}^{L} \widetilde{P}_{M,l} u_l(\eta) J_l^{(0,0)}(\xi). \end{aligned}$$

On the other hand,

$$u(\xi,\eta) = \sum_{l=0}^{\infty} u_l(\eta) J_l^{(0,0)}(\xi).$$

Let $P_{L,0,0}$ be the $L^2(I_\xi)$-orthogonal projection as in (2.6), and $\widetilde{P}_{M,l}$ be the orthogonal projection as before. Then by the above equality, (2.4), and lemmas 2.1 and 2.3,

$$\begin{aligned} \left\|u - P^*_{L,M}u\right\|^2_{L^2(I_\xi;L^2_\chi(I_\eta))} &\leqslant \sum_{l=0}^{L}\left\|u_l - \widetilde{P}_{M,l}u_l\right\|^2_{L^2_\chi(I_\eta)} + \sum_{l=L+1}^{\infty}\|u_l\|^2_{L^2_\chi(I_\eta)}\\ &\leqslant c\sum_{l=0}^{L}\left(\lambda_{M-r+1}^{(2l+r+1,r-1)}\right)^{-r}\|u_l\|^2_{A_l^r(I_\eta)} + \|P_{L,0,0}u - u\|^2_{L^2(I_\xi;L^2_\chi(I_\eta))}. \end{aligned} \tag{3.16}$$

So it remains to estimate the terms at the right side of (3.16).

We have from (3.1) that

$$\frac{\partial x}{\partial \xi} = \frac{1}{2}(1-y), \qquad \frac{\partial x}{\partial \eta} = \frac{x}{2(y-1)}, \qquad \frac{\partial y}{\partial \xi} = 0, \qquad \frac{\partial y}{\partial \eta} = \frac{1}{2} \tag{3.17}$$

from which and (3.11),

$$\partial_\xi^s u = 2^{-s}(1-y)^s\partial_x^s v, \qquad \partial_\eta^r u = \sum_{j=0}^{r}(-1)^j 2^{-r} C_r^j x^j (1-y)^{-j}\partial_x^j\partial_y^{r-j} v. \tag{3.18}$$

Also by (3.1),

$$1-\xi^2 = 4x(1-x-y)(1-y)^{-2}. \tag{3.19}$$

Using lemma 2.1, (3.18) and (3.19), we obtain that

$$\|P_{L,0,0}u - u\|^2_{L^2(I_\xi;L^2_\chi(I_\eta))} \leqslant cL^{-2s}\left\|\partial_\xi^s u\right\|^2_{L^2_{\chi^{(s,s)}}(I_\xi;L^2_\chi(I_\eta))} \leqslant cL^{-2s}\left\|x^{s/2}(1-x-y)^{s/2}\partial_x^s v\right\|^2_{L^2(\mathcal{T})}. \tag{3.20}$$

Next, for $l \leqslant L$,

$$l^2\big(\lambda_{M-r+1}^{(2l+r+1,r-1)}\big)^{-1} = l^2(M-r+1)^{-1}(M+r+2l+1)^{-1} \leqslant \frac{cL^2}{M(M+L)}. \tag{3.21}$$

By the above and the definition of the norm of space $A_l^r(I_\eta)$,

$$\begin{aligned}
&\big(\lambda_{M-r+1}^{(2l+r+1,r-1)}\big)^{-r}\|u_l\|^2_{A_l^r(I_\eta)} \\
&= \big(\lambda_{M-r+1}^{(2l+r+1,r-1)}\big)^{-r}\sum_{k=0}^{r}\int_{I_\eta}(l+r-k-1)^{2r-2k}(1-\eta)^{2k-r+1}(1+\eta)^r\big(\partial_\eta^k u_l\big)^2\,\mathrm{d}\eta \\
&\leqslant c\Big(l^2\big(\lambda_{M-r+1}^{(2l+r+1,r-1)}\big)^{-1}\Big)^r\sum_{k=0}^{r}\int_{I_\eta}(1-\eta)^{2k-r+1}(1+\eta)^r\big(\partial_\eta^k u_l\big)^2\,\mathrm{d}\eta \\
&\leqslant c\left(\frac{M(M+L)}{L^2}\right)^{-r}\sum_{k=0}^{r}\int_{I_\eta}(1-\eta)^{2k-r+1}(1+\eta)^r\big(\partial_\eta^k u_l\big)^2\,\mathrm{d}\eta.
\end{aligned}$$

Therefore, we use (3.1), (3.2), (3.18) and lemma 2.1 to obtain that

$$\begin{aligned}
&\sum_{l=0}^{L}\big(\lambda_{M-r+1}^{(2l+r+1,r-1)}\big)^{-r}\|u_l\|^2_{A_l^r(I_\eta)} \\
&\quad\leqslant c\left(\frac{M(M+L)}{L^2}\right)^{-r}\sum_{k=0}^{r}\int_{I_\eta}(1-\eta)^{2k-r+1}(1+\eta)^r\sum_{l=0}^{L}\big(\partial_\eta^k u_l\big)^2\,\mathrm{d}\eta \\
&\quad= c\left(\frac{M(M+L)}{L^2}\right)^{-r}\sum_{k=0}^{r}\int_{I_\eta}(1-\eta)^{2k-r+1}(1+\eta)^r\left\|P_{L,0,0}\partial_\eta^k u\right\|^2_{L^2(I_\xi)}\,\mathrm{d}\eta \\
&\quad\leqslant c\left(\frac{M(M+L)}{L^2}\right)^{-r}\sum_{k=0}^{r}\int_{I_\eta}(1-\eta)^{2k-r+1}(1+\eta)^r\left\|\partial_\eta^k u\right\|^2_{L^2(I_\xi)}\,\mathrm{d}\eta \\
&\quad\leqslant c\left(\frac{M(M+L)}{L^2}\right)^{-r}\sum_{k=0}^{r}\sum_{j=0}^{k}\iint_{\mathcal{T}}x^{2j}y^r(1-y)^{2k-r-2j}\big(\partial_x^j\partial_y^{k-j}v\big)^2\,\mathrm{d}x\,\mathrm{d}y \\
&\quad= c\left(\frac{M(M+L)}{L^2}\right)^{-r}\sum_{k=0}^{r}\sum_{j=0}^{k}\left\|x^j y^{r/2}(1-y)^{k-j-r/2}\partial_x^j\partial_y^{k-j}v\right\|^2_{L^2(\mathcal{T})}.
\end{aligned} \tag{3.22}$$

Finally, a combination of (3.15), (3.16), (3.20) and (3.22) leads to the desired result. □

Remark 3.1. We can see from theorem 3.1 that the best approximation result is obtained as long as $M = \mathrm{O}(L^{1+s/(2r)})$, and in this case,

$$\|P_{L,M}v - v\|_{\mathcal{T}} \leqslant cL^{-s}\|v\|_{H^{r,s}(\mathcal{T})},$$

or equivalently,

$$\|P_{L,M}v - v\|_{\mathcal{T}} \leqslant cM^{-2rs/(2r+s)}\|v\|_{H^{r,s}(\mathcal{T})}.$$

In other words, for $M = \mathrm{O}(L^{1+\alpha})$, $\alpha > 0$, we have

$$\|P_{L,M}v - v\|_{\mathcal{T}} \leqslant cM^{-2\alpha r/(\alpha+1)}\|v\|_{H^{r,2\alpha r}(\mathcal{T})}. \tag{3.23}$$

Remark 3.2. The previous statements give some general approximation results. In practice, the convergence rate depends on the asymptotic behavior at the collapsed vertex (0, 1), of the approximated function v. To show this, let

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta + 1.$$

We consider ρ^σ-type function as in [3], i.e., $v \sim \rho^\sigma$ as $\rho \to 0$. By the definition (3.9), the norm $\|v\|_{H^{r,2r}(\mathcal{T})}$ is composed of two parts. A calculation shows that

$$x^j y^{r/2}(1-y)^{k-j-r/2}\partial_x^j\partial_y^{k-j}v \sim \rho^{\sigma-r/2}, \qquad x^r(1-x-y)^r\partial_x^{2r}v \sim \rho^\sigma.$$

If $\|v\|_{H^{r,2r}(\mathcal{T})}$ is finite, then $\rho^{2\sigma-r+1} = \mathrm{o}(1/\rho)$. In other words, $r \leqslant 2\sigma + 2 - \varepsilon$, ε being any arbitrary positive constant. Therefore by (3.23), for ρ^σ-type function,

$$\|P_{L,M}v - v\|_{\mathcal{T}} \leqslant c^*M^{-r}, \quad r \leqslant 2\sigma + 2 - \varepsilon.$$

Obviously, this convergence rate is at least the same as the recent approximation result on a square given in [3,12], for which the convergence rate is $2\sigma+1$. This fact shows the efficiency of the orthogonal approximation by using the basis functions defined by (3.4).

3.2. *Numerical results*

In the end of this section, we present some numerical results. We take the test function

$$v(x, y) = (1-y)^\beta \sin(a\pi x)\cos(b\pi y), \quad (x, y) \in \mathcal{T}, \tag{3.24}$$

where a, b and β are some constants specified below.

In order to compute $P_{L,M}v$, we need a quadrature formula on $\mathcal{T}$. As suggested in [16], in the square Q, we can use the Gauss–Legendre–Lobatto interpolation nodes $\{\xi_l\}_{l=0}^L$ in the ξ-direction, and the Gauss–Jacobi–Radau interpolation nodes $\{\eta_m\}_{m=0}^M$ ($\eta_0 = -1$) in the η-direction, respectively, with respect to the Jacobi weight function $\chi^{(1,0)}$. Then we use (3.1) to transform them into the nodes in the triangle $\mathcal{T}$, i.e.,

$$x_{lm} = \frac{1}{4}(1+\xi_l)(1-\eta_m), \qquad y_m = \frac{1}{2}(1+\eta_m), \quad 0 \leqslant l \leqslant L,\ 0 \leqslant m \leqslant M,\ L \leqslant M. \tag{3.25}$$

Accordingly, we can define the corresponding discrete L^2-norm on $\mathcal{T}$.

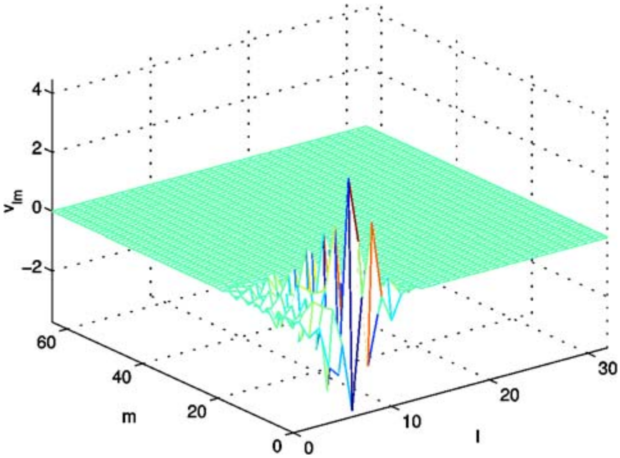

Figure 1. The coefficients in terms of the basis (3.4).

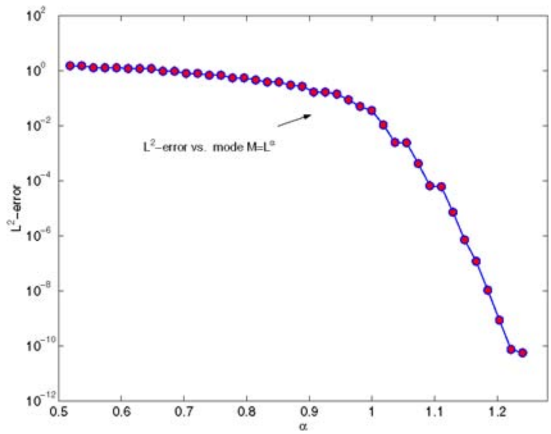

Figure 2. Discrete L^2-error vs. α.

As we know, the errors between $P_{L,M}v$ and v are usually dominated by the leading truncated coefficient $\hat{v}_{L+1,M+1}$ (see (3.7)). We now take $\beta = 2$, $a = 5$ and $b = 6$ in (3.24). We plot in figure 1 the coefficients $\hat{v}_{lm}$, $0 \leqslant l \leqslant 32$, $0 \leqslant m \leqslant 64$, in terms of the basis (3.4). We see that $\hat{v}_{lm}$ decay rapidly as l and m increase, which demonstrate that very accurate approximation might be achieved by only using suitably small modes L and M.

We now examine the convergence rate predicted in theorem 3.1. To do this, let $L = 24$ and $M = L^\alpha$. We take $a = b = \beta = 2$ in (3.24), and plot in figure 2 the discrete L^2-errors between $P_{L,M}v$ and v vs. the power α. It indicates that the better approximations can be obtained with $M = L^\alpha$, $\alpha > 1$, which coincide well with the theoretical result in theorem 3.1. In particular, $\beta = 2$ implies $r = s = \infty$ in remark 3.1. Thus the best approximation is valid for $\alpha \sim 1.5$. The numerical results confirm this prediction.

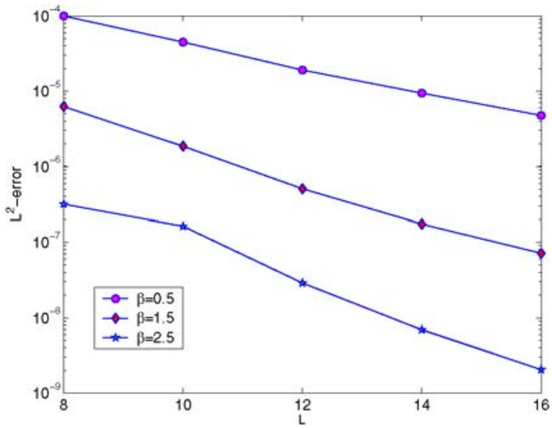

Figure 3. Discrete L^2-error vs. L.

Finally, we use the basis (3.4) to approximate functions with corner singularities on the triangle $\mathcal{T}$. We take $a = b = 1$ and $\beta = 0.5, 1.5, 2.5$ in (3.24). In these cases, the derivatives of $v(x, y)$ have singularities when $y \to 1$. According to remark 3.2, the expected convergence rate is $L^{\varepsilon-2\beta-2}$, which is somehow similar to the numerical results (with $M = L^{1.2}$) illustrated in figure 3.

4. Application to a model equation

As an example of applications, we now consider a model problem, and show how to apply the Dubiner-type orthogonal approximation to partial differential equations on a triangle. Indeed, it is of great interests to consider nonlinear problems on complex geometries, which will be one of the main subjects in our forthcoming paper. Here we focus on the Poisson equation on the reference triangle:

$$\begin{cases} -\Delta U(x, y) = f(x, y), & (x, y) \in \mathcal{T}, \\ U(x, y) = g(x, y), & (x, y) \in \partial\mathcal{T}, \end{cases} \tag{4.1}$$

where f and g are given functions. For simplicity of statements, we assume that $g \equiv 0$. Otherwise, we make the variable transformation

$$\begin{aligned} U(x, y) = V(x, y) &+ (1 - x - y)g(x, 0) + (1 - x - y)(1 - y)^{-1}g(0, y) \\ &+ x(1 - y)^{-1}g(x, y) - (1 - x - y)g(0, 0). \end{aligned}$$

Then $V(x, y) = 0$ on $\partial\mathcal{T}$. Indeed, the convergence analysis is the same no matter $U(x, y)$ vanishes on the boundary or not. A weak formulation of (4.1) with $g \equiv 0$ is to find $U(x, y) \in H_0^1(\mathcal{T})$ such that

$$(\nabla U, \nabla v)_{\mathcal{T}} = (f, v)_{\mathcal{T}}, \quad \forall v \in H_0^1(\mathcal{T}). \tag{4.2}$$

If $f \in H^{-1}(\mathcal{T})$, then we know from the Lax–Milgram lemma that (4.2) has a unique solution $U \in H_0^1(\mathcal{T})$.

The spectral scheme for (4.2) is to find $u_{L,M}(x,y) \in \mathcal{P}^0_{L,M}(\mathcal{T})$ such that

$$(\nabla u_{L,M}, \nabla\phi)_{\mathcal{T}} = (f,\phi)_{\mathcal{T}}, \quad \forall \phi \in \mathcal{P}^0_{L,M}(\mathcal{T}). \tag{4.3}$$

This approximate problem is also unisolvent, and we have the following result on its convergence:

Theorem 4.1. If $U \in H^1_0(\mathcal{T}) \cap H^{r,s}_*(\mathcal{T})$, $r, s \in \mathbb{N}$ and $r, s \geqslant 1$, then

$$\|u_{L,M} - U\|_{1,\mathcal{T}} \leqslant c\left(\left(\frac{M(M+L)}{L^2}\right)^{1-r} + L^{1-s}\right)\|v\|_{H^{r,s}_*(\mathcal{T})}. \tag{4.4}$$

In particular, if $M = \mathrm{O}(L^{1+(s-1)/(2r-2)})$, then

$$\|u_{L,M} - U\|_{1,\mathcal{T}} \leqslant cL^{1-s}\|U\|_{H^{r,s}_*(\mathcal{T})}. \tag{4.5}$$

Here, the space $H^{r,s}_*(\mathcal{T})$ and its norm will be specified below.

To prove the above convergence result, we need some approximation results in $H^1(\mathcal{T})$, which can be applied to numerical analysis of various problems on triangles or on complex domains.

Firstly, we have the following Poincaré inequality on $\mathcal{T}$:

Lemma 4.1. If $v \in H^1(\mathcal{T})$ and $v(0,y) = 0$ for all $y \in (-1,1)$, then

$$\|v\|_{L^2(\mathcal{T})} \leqslant 2\|\nabla v\|_{L^2(\mathcal{T})}. \tag{4.6}$$

Proof. Let $u(\xi,\eta)$ be the same as in (3.11). Clearly, $u(-1,\eta) = 0$ for all $\eta \in I_\eta$, and

$$\begin{aligned} u^2(\xi,\eta)(1-\xi) &= \int_{-1}^{\xi} \partial_\zeta\big(u^2(\zeta,\eta)(1-\zeta)\big)\,\mathrm{d}\zeta \\ &= -\int_{-1}^{\xi} u^2(\zeta,\eta)\,\mathrm{d}\zeta + 2\int_{-1}^{\xi} u(\zeta,\eta)\partial_\zeta u(\zeta,\eta)(1-\zeta)\,\mathrm{d}\zeta. \end{aligned}$$

By the above equality and the Cauchy inequality,

$$\int_{-1}^{\xi} u^2(\zeta,\eta)\,\mathrm{d}\zeta \leqslant 2\left(\int_{-1}^{\xi} u^2(\zeta,\eta)\,\mathrm{d}\zeta\right)^{1/2}\left(\int_{-1}^{\xi}\big(\partial_\zeta u(\zeta,\eta)\big)^2(1-\zeta)^2\,\mathrm{d}\zeta\right)^{1/2}.$$

This implies that

$$\int_{I_\xi} u^2(\xi,\eta)\,\mathrm{d}\xi \leqslant 16\int_{I_\xi}\big(\partial_\xi u(\xi,\eta)\big)^2\,\mathrm{d}\xi. \tag{4.7}$$

On the other hand, we use (3.1), (3.3) and (3.17) to verify that

$$\|\partial_x v\|^2_{L^2(\mathcal{T})} = \iint_Q \left(\partial_\xi u(\xi,\eta)\frac{4}{1-\eta}\right)^2 \frac{1-\eta}{8}\,\mathrm{d}\xi\,\mathrm{d}\eta = \iint_Q \big(\partial_\xi u(\xi,\eta)\big)^2\frac{2}{1-\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{4.8}$$

Since

$$\frac{1-\eta}{2} \leqslant \frac{2}{1-\eta}, \quad \forall \eta \in I_\eta,$$

we use (3.2), (3.3), (4.7) and (4.8) to deduce that

$$\begin{aligned}\|v\|^2_{L^2(\mathcal{T})} &= \frac{1}{8}\iint_Q u^2(\xi,\eta)(1-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \leqslant 2\iint_Q \big(\partial_\xi u(\xi,\eta)\big)^2(1-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\leqslant \iint_Q \big(\partial_\xi u(\xi,\eta)\big)^2 \frac{8}{1-\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta = 4\|\partial_x v\|^2_{L^2(\mathcal{T})} \leqslant 4\|\nabla v\|^2_{L^2(\mathcal{T})}.\end{aligned}$$

This ends the proof. □

Remark 4.1. The same result as in lemma 4.1 holds, provided that $v = 0$ at least on one of edges of the triangle $\mathcal{T}$.

Next, we consider the orthogonal projection $P^{1,0}_{L,M} : H^1_0(\mathcal{T}) \to \mathcal{P}^0_{L,M}(\mathcal{T})$, such that for any $v \in H^1_0(\mathcal{T})$,

$$\big(\nabla\big(P^{1,0}_{L,M}v - v\big), \nabla\phi\big)_{\mathcal{T}} = 0, \quad \forall \phi \in \mathcal{P}^0_{L,M}(\mathcal{T}). \tag{4.9}$$

We specify the space used in theorem 4.1:

$$H^{r,s}_*(\mathcal{T}) = \big\{v \mid v \text{ is measurable on } \mathcal{T} \text{ and } \|v\|_{H^{r,s}_*(\mathcal{T})} < \infty\big\}, \quad r, s \in \mathbb{N},$$

equipped with the norm

$$\begin{aligned}\|v\|_{H^{r,s}_*(\mathcal{T})} = \Bigg(&\sum_{k=1}^{r}\sum_{j=0}^{k}\sum_{l=0}^{j}\Big(\big\|x^l y^{j-k+r/2-3/2}(1-y)^{k-l-r/2-1}\partial^l_x\partial^{j-l}_y v\big\|^2_{L^2(\mathcal{T})} \\ &+ \big\|x^{l-1}y^{j-k+r/2-3/2}(1-y)^{k-r-l}(x\partial_x+1)\partial^l_x\partial^{j-l}_y v\big\|^2_{L^2(\mathcal{T})}\Big) \\ &+ \Big\|x^{(s-1)/2}(1-x-y)^{(s-1)/2}(1-y)^{1-s}\left(\frac{x}{y-1}\partial_x + \partial_y\right) \\ &\times \big((1-y)^{s-1}\partial^{s-1}_x v\big)\Big\|^2_{L^2(\mathcal{T})} \\ &+ \big\|x^{(s-1)/2}(1-x-y)^{(s-1)/2}\partial^s_x v\big\|^2_{L^2(\mathcal{T})}\Bigg)^{1/2}.\end{aligned} \tag{4.10}$$

Lemma 4.2. For any $v \in H^1_0(\mathcal{T}) \cap H^{r,s}_*(\mathcal{T})$ $r, s \in \mathbb{N}$ and $r, s \geqslant 1$,

$$\big\|P^{1,0}_{L,M}v - v\big\|_{1,\mathcal{T}} \leqslant c\left(\left(\frac{M(M+L)}{L^2}\right)^{1-r} + L^{1-s}\right)\|v\|_{H^{r,s}_*(\mathcal{T})}. \tag{4.11}$$

Proof. By lemma 4.1 and projection theorem, for any $\phi \in \mathcal{P}^0_{L,M}(\mathcal{T})$,

$$\|P^{1,0}_{L,M}v - v\|_{1,\mathcal{T}} \leqslant \sqrt{5}\|\nabla(P^{1,0}_{L,M}v - v)\|_{\mathcal{T}} \leqslant \sqrt{5}\|\nabla(\phi - v)\|_{\mathcal{T}}. \tag{4.12}$$

Let $u(\xi, \eta)$ be the same as in (3.11), and

$$\psi(\xi, \eta) = \phi\left(\frac{1}{4}(1+\xi)(1-\eta), \frac{1}{2}(1+\eta)\right).$$

By virtue of (3.1) and (3.17),

$$\partial_x(\phi - v) = \frac{4}{1-\eta}\partial_\xi(\psi - u).$$

Using (3.1) again yields

$$\partial_y(\phi - v) = \frac{2(1+\xi)}{1-\eta}\partial_\xi(\psi - u) + 2\partial_\eta(\psi - u).$$

So a direct calculation with (3.3) gives

$$\begin{aligned}
\|\nabla(\phi - v)\|^2_{\mathcal{T}} &= \frac{1}{2}\iint_Q \big(\partial_\xi\big(\psi(\xi,\eta) - u(\xi,\eta)\big)\big)^2 \frac{4 + (1+\xi)^2}{1-\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad + \frac{1}{2}\iint_Q \big(\partial_\eta\big(\psi(\xi,\eta) - u(\xi,\eta)\big)\big)^2 (1-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad + \iint_Q \partial_\xi\big(\psi(\xi,\eta) - u(\xi,\eta)\big)\partial_\eta\big(\psi(\xi,\eta) - u(\xi,\eta)\big)(1+\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\leqslant c\Big(\|\partial_\xi(\psi - u)\|^2_{L^2(I_\xi; L^2_{\chi^{-1}}(I_\eta))} + \|\partial_\eta(\psi - u)\|^2_{L^2(I_\xi; L^2_\chi(I_\eta))}\Big).
\end{aligned} \tag{4.13}$$

We now choose ψ. Let $P^{1,0}_L$ be the $H^1_0(I_\xi)$-orthogonal projection as in (2.16), and $\widetilde{P}^{1,0}_{M,L}$ be the orthogonal projection as in (2.28). Take

$$\psi(\xi, \eta) = \big(P^{1,0}_L \circ \widetilde{P}^{1,0}_{M,L} u\big)(\xi, \eta).$$

We obtain from lemmas 2.2 and 2.5 that

$$\begin{aligned}
&\|\partial_\xi(\psi - u)\|^2_{L^2(I_\xi; L^2_{\chi^{-1}}(I_\eta))} \\
&\quad = \|\partial_\xi\big(P^{1,0}_L \circ \widetilde{P}^{1,0}_{M,L}u - u\big)\|_{L^2(I_\xi; L^2_{\chi^{-1}}(I_\eta))} \\
&\quad \leqslant \|\partial_\xi\big(P^{1,0}_L u - u\big)\|_{L^2(I_\xi; L^2_{\chi^{-1}}(I_\eta))} + \|\partial_\xi P^{1,0}_L\big(\widetilde{P}^{1,0}_{M,L}u - u\big)\|_{L^2(I_\xi; L^2_{\chi^{-1}}(I_\eta))} \\
&\quad \leqslant c\Big(L^{1-s}\|\partial^s_\xi u\|_{L^2_{\chi^{(s-1,s-1)}}(I_\xi; L^2_{\chi^{-1}}(I_\eta))} + \|\partial_\xi\big(\widetilde{P}^{1,0}_{M,L}u - u\big)\|_{L^2(I_\xi; L^2_{\chi^{-1}}(I_\eta))}\Big) \\
&\quad \leqslant c\Big(L^{1-s}\|\partial^s_\xi u\|_{L^2_{\chi^{(s-1,s-1)}}(I_\xi; L^2_{\chi^{-1}}(I_\eta))} \\
&\qquad + \big(\lambda^{(2L+r-3,r-2)}_{M-r+1}\big)^{(1-r)/2}\|\partial_\xi\partial_\eta\big((1+\eta)^{-1}u\big)\|_{L^2(I_\xi; B^{r-1}_{L-1}(I_\eta))}\Big).
\end{aligned} \tag{4.14}$$

Similarly, by lemmas 2.2 and 2.5,

$$\begin{aligned}
&\left\|\partial_\eta(\psi-u)\right\|^2_{L^2(I_\xi;L^2_\chi(I_\eta))}\\
&\quad=\left\|\partial_\eta\left(P_L^{1,0}\circ\widetilde{P}^{1,0}_{M,L}u-u\right)\right\|_{L^2(I_\xi;L^2_\chi(I_\eta))}\\
&\quad\leqslant\left\|\partial_\eta\left(\widetilde{P}^{1,0}_{M,L}u-u\right)\right\|_{L^2(I_\xi;L^2_\chi(I_\eta))}+\left\|\partial_\eta\widetilde{P}^{1,0}_{M,L}\left(P_L^{1,0}u-u\right)\right\|_{L^2(I_\xi;L^2_\chi(I_\eta))}\\
&\quad\leqslant c\Big(\big(\lambda^{(2L+r-3,r-2)}_{M-r+1}\big)^{(1-r)/2}\left\|\partial_\eta\left((1+\eta)^{-1}u\right)\right\|_{L^2(I_\xi;B^{r-1}_{L-1}(I_\eta))}\\
&\qquad+\left\|\partial_\eta\left(P^{1,0}_{L,0,0}u-u\right)\right\|_{L^2(I_\xi;L^2_\chi(I_\eta))}\Big)\\
&\quad\leqslant c\Big(\big(\lambda^{(2L+r-3,r-2)}_{M-r+1}\big)^{(1-r)/2}\left\|\partial_\eta\left((1+\eta)^{-1}u\right)\right\|_{L^2(I_\xi;B^{r-1}_{L-1}(I_\eta))}\\
&\qquad+L^{1-s}\left\|\partial_\xi^{s-1}\partial_\eta u\right\|_{L^2_{\chi^{(s-1,s-1)}}(I_\xi;L^2_\chi(I_\eta))}\Big).
\end{aligned}\tag{4.15}$$

Thus it remains to estimate the upper-bounds of the terms at the right sides of (4.14) and (4.15). To this end, we first use (3.2), (3.18) and (3.19) to obtain that

$$\left\|\partial_\xi^s u\right\|^2_{L^2_{\chi^{(s-1,s-1)}}(I_\xi;L^2_{\chi^{-1}}(I_\eta))}=\frac{1}{2}\iint_{\mathcal{T}}x^{s-1}(1-x-y)^{s-1}\left(\partial_x^s v\right)^2\,\mathrm{d}x\,\mathrm{d}y.\tag{4.16}$$

Next, (3.17) and (3.18) imply that

$$\partial_\xi^{s-1}\partial_\eta u=\frac{1}{2}\left(\frac{x}{y-1}\partial_x+\partial_y\right)\left(\partial_\xi^{s-1}u\right)=\frac{1}{2^s}\left(\frac{x}{y-1}\partial_x+\partial_y\right)\left((1-y)^{s-1}\partial_x^{s-1}v\right).$$

The above with (3.2) and (3.18) leads to

$$\begin{aligned}
&\left\|\partial_\xi^{s-1}\partial_\eta u\right\|^2_{L^2_{\chi^{(s-1,s-1)}}(I_\xi;L^2_\chi(I_\eta))}\\
&\quad=2\iint_{\mathcal{T}}x^{s-1}(1-x-y)^{s-1}(1-y)^{2-2s}\left(\left(\frac{x}{y-1}\partial_x+\partial_y\right)\right.\\
&\qquad\left.\times\left((1-y)^{s-1}\partial_x^{s-1}v\right)\right)^2\mathrm{d}x\,\mathrm{d}y.
\end{aligned}\tag{4.17}$$

Further, by the definition of $B^{r-1}_{L-1}(I_\eta)$,

$$\begin{aligned}
&\left\|\partial_\eta\left((1+\eta)^{-1}u\right)\right\|^2_{L^2(I_\xi;B^{r-1}_{L-1}(I_\eta))}\\
&\quad=\sum_{k=1}^{r}\iint_Q(L+r-k-2)^{2r-2k}(1-\eta)^{2k-r-1}(1+\eta)^{r-1}\\
&\qquad\times\left(\partial_\eta^k\left((1+\eta)^{-1}u\right)\right)^2\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned}\tag{4.18}$$

We can use (3.18) to check that

$$\partial_\eta^k\big((1+\eta)^{-1}u\big) = \sum_{j=0}^{k}(-1)^{k-j}C_k^j(1+\eta)^{-k+j-1}\partial_\eta^j u$$
$$= \sum_{j=0}^{k}\sum_{l=0}^{j}(-1)^{k-j+l}2^{-k-1}C_k^l C_j^l y^{j-k-1}(1-y)^{-l}x^l\partial_x^l\partial_y^{j-l}v. \quad (4.19)$$

On use of (3.2), (4.18) and (4.19), we assert that

$$\big\|\partial_\eta^k\big((1+\eta)^{-1}u\big)\big\|^2_{L^2(I_\xi;B_{L-1}^{r-1}(I_\eta))}$$
$$\leqslant cL^{2r-2}\sum_{k=1}^{r}\sum_{j=0}^{k}\sum_{l=0}^{j}\iint_{\mathcal{T}} x^{2l}y^{2j-2k+r-3}(1-y)^{2k-r-2l-2}\big(\partial_x^l\partial_y^{j-l}v\big)^2\,\mathrm{d}x\,\mathrm{d}y. \quad (4.20)$$

Thanks to $\partial_\xi\partial_\eta^k((1+\eta)^{-1}u) = \frac{1}{2}(1-y)\partial_x\partial_\eta^k((1+\eta)^{-1}u)$, we follow the same line as in the previous paragraph to reach that

$$\big\|\partial_\xi\partial_\eta^k\big((1+\eta)^{-1}u\big)\big\|^2_{L^2(I_\xi;B_{L-1}^{r-1}(I_\eta))}$$
$$\leqslant cL^{2r-2}\sum_{k=1}^{r}\sum_{j=0}^{k}\sum_{l=0}^{j}\iint_{\mathcal{T}} x^{2l-2}y^{2j-2k+r-3}(1-y)^{2k-r-l}\big((x\partial_x+1)\partial_x^l\partial_y^{j-l}v\big)^2\,\mathrm{d}x\,\mathrm{d}y. \quad (4.21)$$

In addition,

$$L^2\big(\lambda_{M-r+1}^{(2L+r-3,r-2)}\big)^{-1} \leqslant \frac{cL^2}{M(M+L)}. \quad (4.22)$$

Finally, substituting (4.16), (4.17) and (4.20)–(4.22) into (4.14) and (4.15), we use (4.12) and (4.13) to reach the desired result. □

Proof of theorem 4.1. We have from (4.2), (4.3) and (4.9) that

$$\big(\nabla(u_{L,M}-U),\nabla\phi\big)_{\mathcal{T}} = \big(\nabla\big(u_{L,M}-P_{L,M}^{1,0}U\big),\nabla\phi\big)_{\mathcal{T}} = 0, \quad \forall\phi\in\mathcal{P}_{L,M}^0,$$

which implies $u_{L,M} = P_{L,M}^{1,0}U$. Then the conclusion of theorem 4.1 follows from lemma 4.2 immediately. □

Numerical results

We next present some numerical results for (4.1), by using the scheme (4.3). We take the test solution

$$U(x,y) = xy\big(\mathrm{e}^{x+y}-\mathrm{e}\big), \quad (x,y)\in\mathcal{T}.$$

In figure 4, we plot the maximum errors and L^2-errors (in log scale) against various L (with $M = L^{1.2}$), which indicates an exponential decay of the errors, as predicted by our theoretical analysis.

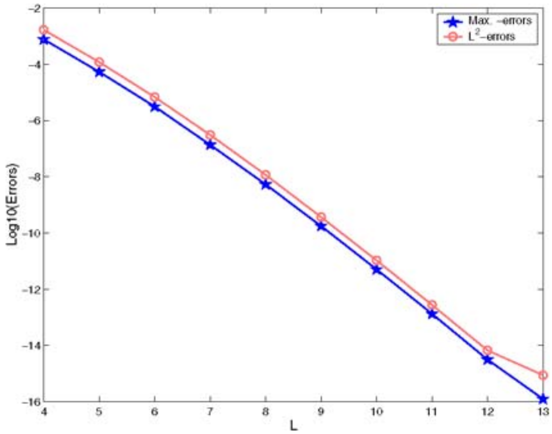

Figure 4. Maximum and L^2-errors vs. L.

5. Concluding discussions

In this work, we considered the orthogonal approximation on a triangle by using the base functions (3.4). We derived some basic results on the convergence rate, which showed the efficiency of this approximation, and played an important role in the numerical analysis of spectral methods on a triangle. A more interesting problem is how to analyze the corresponding interpolation on triangles, which are related to the analysis of p-version finite element method and triangle spectral element method (see [16,18,19]). The main difficulty of that work is how to build up some approximation results on the interpolation on a triangle.

Another important problem is the spectral element method using orthogonal approximations on triangles. The key point is how to match boundary conditions between elements. It is possible to modify the construction, by building a new basis, including vertex functions, edge functions and internal functions (bubbles), see [5]. But the convergence analysis of such approach is still open.

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