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The optimal convergence of the h–p version of the boundary element method with quasiuniform meshes for elliptic problems on polygonal domains

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In the framework of the Jacobi-weighted Besov spaces, we analyze the lower and upper bounds of errors in the h–p version of boundary element solutions on quasiuniform meshes for elliptic problems on polygons. Both lower bound and upper bound are optimal in h and p, and they are of the same order. The optimal convergence of the h–p version of boundary element method with quasiuniform meshes is proved, which includes the optimal rates for h version with quasiuniform meshes and the p version with quasiuniform degrees as two special cases.

Keywords: h–p version with quasiuniform meshes, boundary element method, optimal rate of convergence.

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1. Introduction

In this paper we prove asymptotically exact upper and lower bounds for the approximation error of the h–p version of the boundary element method (BEM) with quasiuniform meshes in two dimensions. More precisely, we analyze elliptic problems on

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polygonal domains whose solutions exhibit typical corner singularities. Our analysis is done within the framework of the Jacobi-weighted Besov spaces which has been already proved the appropriate tool to obtain optimal estimates for the p version of the BEM for this type of problems, see [15]. Here we incorporate the mesh dependence into the analysis and provide optimal estimates for any combination of mesh size and polynomial degree, for the case of quasiuniform meshes and uniform polynomial degrees.

The p version of the Galerkin method (finite elements for differential equations and boundary elements for boundary integral equations) uses a fixed mesh and improves the approximation of the solution by considering piecewise polynomial functions of increasing degrees. The h version is based on mesh refinement and piecewise polynomials of low, fixed degrees. The h–p version combines mesh refinement with increase of degrees. Let us recall the main theoretical achievements for the h–p version since its beginning. For details specific to the p version we refer to [15].

A thorough analysis of the p and h–p versions started with the series of publications by Gui and Babuška [10–12], for the finite element method (FEM) in one dimension. They considered the approximation of typical singularities x^γ, and proved optimal upper and lower bounds of error in the finite element solutions in H^1 and L^2 norms. Problems in two dimensions and their approximations by the h–p version of FEM with quasiuniform meshes are analyzed by Babuška and Suri [6] after improving the approximation results of the p version of FEM [7]. They gave an upper bound of error in FE approximation for elliptic problems with singularities $|x|^\gamma \log^\nu |x|$ ($|x| = r$ is the distance to the origin), which is actually of order $\mathrm{O}(h^{-\gamma} p^{-2\gamma} \log^\nu(p/h))$. This upper bound is optimal for noninteger γ, and it can be sharper for integer γ and $\nu > 0$. The argument was brought to the h–p version of BEM with quasiuniform meshes by Stephan and Suri after introducing the p version of BEM [21], but the estimate on the upper bound of error in the BE solution, measured in $\widetilde{H}^{1/2}$ norm, is not as sharp as in the FE solution, and a rate of $\mathrm{O}(h^{-\gamma} p^{-2\gamma+\epsilon})$, $\epsilon > 0$ arbitrary was claimed for $\nu = 0$ in [22]. Since then, no further improvement on the upper bound for BEM has been seen. Meanwhile, the lower bounds of the error in the h–p FE and BE solutions for elliptic problems with the singularities of $|x|^\gamma \log^\nu |x|$-type has not been addressed up to now. Consequently, the optimal convergence of the h–p version of BEM as well as FEM has not been mathematically established.

The h–p version with quasiuniform meshes is, from methodology and approximation theory, the p version on scaled meshes. The approach of the p version gives the p dependence in the approximation errors, and a proper scaling argument will reveal fully the information of the h dependence. Hence, the analysis for the best approximation of the h–p version with quasiuniform meshes is not feasible unless the optimal convergence of the p version is established. Fortunately, the best a-priori error estimation for the p version has been recently established, we are now ready to pursue the best a-priori error estimation for the h–p version. In the last few years, with a series of papers by Babuška and Guo [2–5], a new analysis of the p version has started in the framework of the Jacobi-weighted Besov spaces. The approximation theory of FEM in two dimensions in this new mathematical framework is systematically developed in these papers, which demonstrates that Jacobi-weighted Besov spaces is the most appropriate tool to obtain optimal upper and lower bounds when dealing with singular solutions on polygons. This framework has been generalized to the p version of FEM in three dimensions [13] and the p and h–p version of BEM. In [15] we showed that the Jacobi-weighted Besov spaces serve equally well for the analysis of polynomial approximations of singular functions in the spaces $\widetilde{H}^{1/2}$ and $\widetilde{H}^{-1/2}$, the energy spaces of hypersingular and weakly singular integral operators, respectively. The Jacobi projection and interpolation have been developed recently in the spectral methods as well, see, e.g., [16–18] and references therein. In this paper we will further generalize the results and methodology to the h–p version with quasiuniform meshes. The generalization for singular problems without logarithmic terms can be easily done by a simple scaling argument, but the generalization for those with logarithmic terms are not trivial, in particular, for the lower bounds of the approximation error of the Jacobi projection.

The h–p version of FEM and BEM with quasiuniform meshes is quite different from the one with geometric meshes, in the methodology and approximation theory. We will not elaborate numerous progresses on the h–p version of FEM and BEM with geometric meshes in the past two decades.

The rest of the paper is organized as follows. In section 2 we shall present the Jacobi-weighted Besov and Sobolev spaces and recall results on the p version of BEM, which we have derived in [15] and will be used later. In section 3, we carry out asymptotic error analysis for the Jacobi projection of singular function $x^\gamma \log^\nu x$ on $P_p(J_h)$, where $P_p(J_h)$ is a set of polynomials of degree p on a scaled interval $J_h = (0, h)$, $\gamma > 0$ and integer $\nu \geqslant 0$. In section 4, the approximation results to these singular functions are applied to the h–p version BE solution on quasiuniform meshes for elliptic problems on polygonal domains, which leads to the optimal lower and upper bounds of approximation error in the h–p version BE solution with quasiuniform meshes, and proves the optimal convergence. In the last section, we will make some concluding remarks.

2. Jacobi-weighted Besov spaces and preliminary results

In the following I denotes the interval $(-1, 1)$. Let $\alpha \geqslant 0$ be an integer and $\beta > -1$ a real number. We introduce a weight function with parameters α and β by

$$W_{\alpha,\beta}(x) = \left(1 - x^2\right)^{\alpha+\beta}$$

and define the spaces $H^{k,\beta}(I)$ for integers $k \geqslant 0$ as the closure of C^∞ functions with respect to the weighted norm

$$\|u\|_{H^{k,\beta}(I)} = \left(\sum_{\alpha=0}^{k} \int_I \left|u^{(\alpha)}\right|^2 W_{\alpha,\beta}(x)\,\mathrm{d}x\right)^{1/2} = \left(\sum_{\alpha=0}^{k} \int_I \left|u^{(\alpha)}\right|^2 \left(1 - x^2\right)^{\alpha+\beta}\,\mathrm{d}x\right)^{1/2}.$$

The semi-norm involving only the highest derivative $u^{(k)}$ is denoted by $|u|_{H^{k,\beta}(I)}$.

For real $s > 0$ the space $H^{s,\beta}(I)$ is defined by interpolation: Let l and k be two integers with $l < k$ and $s = (1-\theta)l + \theta k$ for $\theta \in (0,1)$. Then

$$H^{s,\beta}(I) = \big(H^{l,\beta}(I), H^{k,\beta}(I)\big)_{\theta,2}$$

with norm

$$\|u\|_{H^{s,\beta}(I)} = \left(\int_0^\infty \big(t^{-\theta} K(t,u)\big)^2 \frac{\mathrm{d}t}{t}\right)^{1/2}, \tag{2.1}$$

where

$$K(t,u) = \inf_{u=v+w}\big(\|v\|_{H^{l,\beta}(I)} + t\|w\|_{H^{k,\beta}(I)}\big). \tag{2.2}$$

The spaces $H^{s,\beta}(I)$ are referred to as Jacobi-weighted Sobolev spaces. Interpolating differently we obtain the so-called Jacobi-weighted Besov spaces,

$$B^{s,\beta}(I) = \big(H^{l,\beta}(I), H^{k,\beta}(I)\big)_{\theta,\infty}$$

with norm

$$\|u\|_{B^{s,\beta}(I)} = \sup_{t>0} t^{-\theta} K(t,u),$$

where the functor $K(t,u)$ is defined in (2.2). To analyze the best approximability of the singular function of $x^\gamma \log^\nu x$-type, we need to introduce the modified Jacobi-weighted Besov spaces,

$$B^{s,\beta}_\nu(I) = \big(H^{l,\beta}(I), H^{k,\beta}(I)\big)_{\theta,\infty,\nu}$$

with norm

$$\|u\|_{B^{s,\beta}_\nu(I)} = \sup_{t>0} \frac{t^{-\theta}}{(1+|\log t|)^\nu} K(t,u).$$

For $\nu = 0$ we also write

$$B^{s,\beta}_\nu(I) = B^{s,\beta}(I).$$

The definitions of the above Jacobi-weighted spaces in one dimension are quoted directly from [15]. It is worth indicating that the spaces $H^{s,\beta}(I)$ and $B^{s,\beta}(I)$ are exact interpolation spaces and that the spaces $B^{s,\beta}_\nu(I)$ with $\nu > 0$ are not exact, but only uniform. For the substantial difference between two types of interpolation spaces we refer to [2] and [8].

In the boundary element analysis, the approximation errors are measured in the norms of $\widetilde{H}^{1/2}(I)$ and $H^{1/2}(I)$,

$$H^{1/2}(I) = \big(L^2(I), H^1(I)\big)_{1/2,2}, \qquad \widetilde{H}^{1/2}(I) = \big(L^2(I), H^1_0(I)\big)_{1/2,2}.$$

Here $\widetilde{H}^{1/2}(I)$ is the energy space for hypersingular operators, $\widetilde{H}^{-1/2}(I)$ and $H^{-1/2}(I)$ are the dual spaces of $H^{1/2}(I)$ and $\widetilde{H}^{1/2}(I)$, in which one analyzes the error for weakly singular operators. Hence, it is essential to explore the relation between these spaces and the corresponding Chebyshev-weighted spaces, which are Jacob-weighted Besov spaces with $\beta = -1/2$. To this end, we introduce the space

$$H_0^{1,-1/2}(I) = \{u \in H^{1,-1/2}(I) \mid u(\pm 1) = 0\},$$

and the interpolation space

$$\widetilde{H}^{1/2,-1/2}(I) = \left(H^{0,-1/2}(I), H_0^{1,-1/2}(I)\right)_{1/2,2}$$

with norm analogously to (2.1) for

$$K(t,u) = \inf_{u=v+w, v\in H^{0,-1/2}(I), w\in H_0^{1,-1/2}(I)} \left(\|v\|_{H^{0,-1/2}(I)} + t\|w\|_{H^{1,-1/2}(I)}\right).$$

The following two propositions indicate the equivalence between the usual Sobolev spaces and the Chebyshev-weighted spaces.

Proposition 2.1 ([15, Theorem 2.2]).

$$H^{1/2,-1/2}(I) \cong H^{1/2}(I), \qquad \widetilde{H}^{1/2,-1/2}(I) \cong \widetilde{H}^{1/2}(I),$$

i.e. there exist constants $c_1, c_2 > 0$ such that

$$c_1\|u\|_{H^{1/2}(I)} \leqslant \|u\|_{H^{1/2,-1/2}(I)} \leqslant c_2\|u\|_{H^{1/2}(I)} \tag{2.3}$$

and

$$c_1\|u\|_{\widetilde{H}^{1/2}(I)} \leqslant \|u\|_{\widetilde{H}^{1/2,-1/2}(I)} \leqslant c_2\|u\|_{\widetilde{H}^{1/2}(I)}. \tag{2.4}$$

Let us consider a function $\hat{u}(\xi) = u(\cos\xi)$ with cosine expansion

$$\hat{u}(\xi) = \sum_{k=0}^{\infty} a_k \cos(k\xi). \tag{2.5}$$

This leads to the Chebyshev expansion of $u \in H^{0,-1/2}(I)$,

$$u(x) = \sum_{k=0}^{\infty} a_k T_k(x) \tag{2.6}$$

with $T_k(x) = \cos(k \arccos x)$. By Corollary 2.1 in [15] there holds

$$\|u\|^2_{H^{1/2}(I)} \cong \|u\|^2_{H^{1/2,-1/2}(I)} \cong \sum_{k=0}^{\infty} a_k^2\left(1+k^2\right)^{1/2}. \tag{2.7}$$

Finally let us recall the technical and approximation results for singular functions of the type

$$u(x) = x^\gamma \log^\nu(x)\chi(x), \quad x \in J := (0,1), \tag{2.8}$$

with $\gamma > 0$ and integer $\nu \geqslant 0$. Here, $\chi \in C^\infty(J)$ with $\chi(x) = 1$ for $0 < x < \delta_0/2$ and $\chi(x) = 0$ for $\delta_0 < x < 1$ where $\delta_0 < 1$ is a positive constant.

In the case $\nu = 0$ we have the following Chebyshev expansion of u.

Proposition 2.2 ([15, Lemma 5.1]). For the function $u(x) = (1+x)^\gamma$ with $\gamma > 0$, let $\sum_{k=0}^{\infty} a_k(\gamma) T_k$ be its Chebyshev expansion. Then there holds for $k > 0$

$$a_k(\gamma) = \frac{2^{\gamma+1}}{\pi} \Gamma(1/2)\Gamma(\gamma+1/2) \frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{\Gamma(\gamma+k+1)}, \tag{2.9}$$

and for noninteger γ

$$\left|a_k(\gamma)\right| \sim k^{-2\gamma-1}\left(1 + \mathrm{O}\left(\frac{1}{k}\right)\right) \quad (k \to \infty).$$

In the case $\nu > 0$ we have the following Chebyshev expansion.

Proposition 2.3 ([15, Lemma 5.1]). For the function $u(x) = (1+x)^\gamma \log^\nu(1+x)$ with $\gamma > 0$ and $\nu > 0$, let $\sum_{k=0}^{\infty} b_k(\gamma) T_k$ be its Chebyshev expansion. Then there holds for $k > 0$

$$b_k = a_k^{(\nu)}(\gamma) T_k,$$

where $a_k^{(\nu)}(\gamma)$ denotes the νth derivative of $a_k(\gamma)$ with respect to γ. For noninteger γ,

$$\left|b_k(\gamma)\right| \sim k^{-2\gamma-1} \log^\nu k\left(1 + \mathrm{O}\left(\log^{-1} k\right)\right),$$

and for integer $\gamma > 0$ and $\nu \geqslant 1$

$$\left|b_k(\gamma)\right| \sim k^{-2\gamma-1} \log^{\nu-1} k\left(1 + \mathrm{O}\left(\log^{-1} k\right)\right).$$

One of the main results of [15] are the following optimal upper and lower bounds for polynomial approximation of singular functions.

Proposition 2.4. (i) Let u be given by (2.8). Then, for $p > 0$, there exists a polynomial ψ of degree p such that

$$\|u - \psi\|_{\tilde{H}^{1/2}(J)} \leqslant C p^{-2\gamma}.$$

(ii) Let $v = (1+x)^\gamma \log^\nu(1+x)$ with $\gamma > 0$, $\nu > 0$. Then, for any polynomial ϕ of degree p there holds

$$\|v - \phi\|_{H^{1/2}(J)} \geqslant c p^{-2\gamma} (1 + \log p)^{\nu^*}.$$

Here, the positive constants C and c are independent of p, and $\nu^* = \nu$ if γ is noninteger or $\nu = 0$ and $\nu^* = \nu - 1$ if γ is integer and $\nu \geqslant 1$.

Proof. For the existence of a polynomial ψ satisfying the upper bound see theorem 4.4 in [15]. The lower bound for the approximation of v is given by [15, Theorem 5.2]. $\square$

3. Asymptotic error analysis for Jacobi projection of singular functions of $x^\gamma \log^\nu x$ on a scaled interval

Let

$$u = x^\gamma \log^\nu x, \quad x \in J_h = (0, h),$$

with real $\gamma > 0$ and integer $\nu \geqslant 0$, and let $P_p(J_h)$ be a set of polynomials of degree $\leqslant p$ on J_h, let $u_{h,p}$ denote its Jacobi projection on $P_p(J_h)$ with the weight $\beta = -1/2$, which is called the Chebyshev projection as well. Then we have the following asymptotics of the approximation error of Jacobi projections.

Theorem 3.1. Let $u = x^\gamma$, and let $u_{h,p}$ be its Chebyshev projection on $P_p(J_h)$. Then for noninteger $\gamma > 0$

$$\|u - u_{h,p}\|_{H^{1/2}(J_h)} \cong (h/2)^\gamma p^{-2\gamma}. \tag{3.1}$$

Hereafter $\cong$ means equivalence with constant independent of h and p.

Proof. Introducing a linear mapping

$$x = \frac{(1+\xi)h}{2}, \quad \xi \in (-1, 1), \tag{3.2}$$

we define a function $\tilde{u}(\xi) = u(\frac{h(1+\xi)}{2}) = (h/2)^\gamma (1+\xi)^\gamma$. Due to proposition 2.2, $\tilde{u}(\xi)$ has a Chebyshev expansion

$$\tilde{u}(\xi) = \sum_{k=0}^{\infty} c_k(\gamma) T_k(\xi)$$

with $c_k(\gamma) = (h/2)^\gamma a_k(\gamma)$, where the coefficients $a_k(\gamma)$ are given in (2.9), and

$$\left|c_k(\gamma)\right| \sim (h/2)^\gamma k^{-2\gamma-1}\left(1 + \mathrm{O}\left(\frac{1}{k}\right)\right) \quad (k \to \infty).$$

Therefore,

$$u = x^\gamma = \sum_{k=0}^{\infty} c_k(\gamma) T_k\left(\frac{2}{h}x - 1\right)$$

is the Jacobi expansion of $u(x)$ on J_h, and

$$u_{h,p} = \sum_{k=0}^{p} c_k(\gamma) T_k\left(\frac{2}{h}x - 1\right)$$

is the Chebyshev projection of $u(x)$ on $P_p(J_h)$. Note that

$$\begin{aligned}\|u - u_{h,p}\|^2_{H^{1/2}(J_h)} &\cong \frac{h}{2}\|\tilde{u} - \tilde{u}_{h,p}\|^2_{L^2(-1,1)} + |\tilde{u} - \tilde{u}_{h,p}|^2_{H^{1/2}(-1,1)} \\ &\cong \frac{h}{2}\sum_{k=p+1}^{\infty}|c_k(\gamma)|^2 + \sum_{k=p+1}^{\infty}|c_k(\gamma)|^2 k \cong \sum_{k=p+1}^{\infty}|c_k(\gamma)|^2 k, \quad (3.3)\end{aligned}$$

and (3.1) follows immediately. □

Theorem 3.2. Let $u = x^\gamma \log^\nu x$ with integer $\nu \geqslant 1$, and let $u_{h,p}$ be its Chebyshev projection on $P_p(J_h)$. Then for noninteger $\gamma > 0$

$$\|u - u_{h,p}\|_{H^{1/2}(J_h)} \cong (h/2)^\gamma p^{-2\gamma} \log^\nu \frac{2p^2}{h}, \tag{3.4}$$

and for integer $\gamma > 0$ and $\nu > 0$

$$\|u - u_{h,p}\|_{H^{1/2}(J_h)} \cong (h/2)^\gamma p^{-2\gamma} \log^{\nu-1} \frac{2p^2}{h}. \tag{3.5}$$

Proof. By the mapping (3.2), we have a function on $I = (-1, 1)$

$$\begin{aligned}\tilde{u}(\xi) &= u\left(\frac{h(1+\xi)}{2}\right) = h^\gamma\left(\frac{(1+\xi)}{2}\right)^\gamma \log^\nu \frac{h(1+\xi)}{2} \\ &= h^\gamma\left(\frac{(1+\xi)}{2}\right)^\gamma \sum_{m=0}^{\nu}\binom{\nu}{m}\log^{\nu-m}(h/2)\log^m(1+\xi) = \frac{\mathrm{d}^\nu}{\mathrm{d}\gamma^\nu}\left(\frac{h(1+\xi)}{2}\right)^\gamma.\end{aligned}$$

By Proposition 2.2

$$\tilde{u}(\xi) = \sum_{k=0}^{\infty}\frac{\mathrm{d}^\nu}{\mathrm{d}\gamma^\nu}\big(c_k(\gamma)\big)T_k(\xi) = \sum_{k=0}^{\infty}\frac{\mathrm{d}^\nu}{\mathrm{d}\gamma^\nu}\big((h/2)^\gamma a_k(\gamma)\big)T_k(\xi),$$

where $c_k = (h/2)\gamma a_k(\gamma)$ is the coefficient of the Chebyshev expansion of the function $u_0^h(\xi) = (\frac{h(1+\xi)}{2})^\gamma$, and a_k is given in (2.9). For $0 \leqslant \ell \leqslant \nu$, let

$$\tilde{u}_\ell^h(\xi) = \frac{\mathrm{d}^\nu}{\mathrm{d}^\nu\gamma}\left(\frac{h(1+\xi)}{2}\right)^\gamma.$$

First we consider the case $\nu = 1$. There holds

$$\tilde{u}_1^h(\xi) = \sum_{k=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}\gamma}\big((h/2)^\gamma a_k(\gamma)\big)T_k(\xi).$$

By (2.9) the coefficients a_k can be written like

$$a_k(\gamma) = (-1)^{k-k^*}\widetilde{C}_0(\gamma)\frac{\Gamma(k-\gamma)}{\Gamma(\gamma+k+1)} \quad (k > \gamma)$$

with

$$\widetilde{C}_0(\gamma) = \frac{2^{\gamma+1}}{\pi}\Gamma(1/2)\Gamma(\gamma+1/2)\gamma(\gamma-1)\cdots\bigl(\gamma-k^*\bigr).$$

Here, k^* is the maximum integer less than or equal to γ. For $k > \gamma$, $\widetilde{C}_0$ does not depend on k, and $\widetilde{C}_0 \neq 0$ for noninteger γ.

We obtain

$$\begin{aligned}(-1)^{k-k^*}\frac{\mathrm{d}}{\mathrm{d}\gamma}\bigl((h/2)^\gamma a_k(\gamma)\bigr) = (h/2)^\gamma\Biggl(&\log(h/2)\widetilde{C}_0(\gamma)\frac{\Gamma(k-\gamma)}{\Gamma(\gamma+k+1)}\\ &+\widetilde{C}_0'(\gamma)\frac{\Gamma(k-\gamma)}{\Gamma(\gamma+k+1)} - \widetilde{C}_0(\gamma)\frac{\Gamma'(k-\gamma)}{\Gamma(\gamma+k+1)}\\ &-\widetilde{C}_0(\gamma)\frac{\Gamma(k-\gamma)\Gamma'(\gamma+k+1)}{\Gamma(\gamma+k+1)^2}\Biggr).\end{aligned}$$

Using

$$\frac{\Gamma(k+\alpha)}{\Gamma(k+\beta)} = k^{\alpha-\beta}\bigl(1+\mathrm{O}(1/k)\bigr) \quad (k\to\infty)$$

(see [1, formula 6.1.46]), and the relation $\Gamma'(z) = \Gamma(z)(\log z + \mathrm{O}(z^{-1}))$ for $z > 1$ (see [1, Formula in 6.3.1 and 6.3.18], we obtain

$$\begin{aligned}&(-1)^{k-k^*+1}\frac{\mathrm{d}}{\mathrm{d}\gamma}\bigl((h/2)^\gamma a_k(\gamma)\bigr)\\ &\quad= (h/2)^\gamma\bigl(\log(2/h)\widetilde{C}_0(\gamma)k^{-2\gamma-1}\bigl(1+\mathrm{O}\bigl(k^{-1}\bigr)\bigr)\bigr)\\ &\qquad-\widetilde{C}_0'(\gamma)k^{-2\gamma-1}\bigl(1+\mathrm{O}\bigl(k^{-1}\bigr)\bigr) + 2\widetilde{C}_0(\gamma)k^{-2\gamma-1}\log(k)\bigl(1+\mathrm{O}\bigl(k^{-1}\bigr)\bigr)\\ &\quad= (h/2)^\gamma k^{-2\gamma-1}\bigl(\widetilde{C}_0(\gamma)\log(2/h) - \widetilde{C}_0'(\gamma) + 2\widetilde{C}_0(\gamma)\log(k)\bigr)\bigl(1+\mathrm{O}\bigl(k^{-1}\bigr)\bigr)\\ &\quad= (h/2)^\gamma k^{-2\gamma-1}\left(\widetilde{C}_0(\gamma)\log\left(\frac{2k^2}{h}\right) - \widetilde{C}_0'(\gamma)\right)\bigl(1+\mathrm{O}\bigl(k^{-1}\bigr)\bigr).\end{aligned} \tag{3.6}$$

For noninteger γ, there holds

$$\widetilde{C}_0(\gamma)\log\left(\frac{2k^2}{h}\right) - \widetilde{C}_0'(\gamma) = \widetilde{C}_0(\gamma)\log\left(\frac{2k^2}{h}\right)\left(1+\mathrm{O}\left(\log^{-1}\frac{2k^2}{h}\right)\right) \quad \left(\frac{2k^2}{h}\to\infty\right)$$

which together with (3.6) yields

$$\begin{aligned}&(-1)^{k-k^*}\frac{\mathrm{d}}{\mathrm{d}\gamma}\bigl(h^\gamma a_k(\gamma)\bigr)\\ &\quad= -(h/2)^\gamma k^{-2\gamma-1}\widetilde{C}_0(\gamma)\log\left(\frac{2k^2}{h}\right)\left(1+\mathrm{O}\left(\log^{-1}\frac{2k^2}{h}\right)\right).\end{aligned} \tag{3.7}$$

The Chebyshev projection of $\tilde{u}_1^h(\xi)$ on $P_p(I)$,

$$\psi_p(\xi) = \sum_{k=0}^{p} \frac{\mathrm{d}}{\mathrm{d}\gamma}\big((h/2)^\gamma a_k(\gamma)\big)T_k(\xi)$$

satisfies

$$\begin{aligned}\left|\tilde{u}_1^h - \psi_p\right|^2_{H^{1/2}(I)} &\cong \sum_{k=p+1}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}\gamma}\big((h/2)^\gamma a_k\big)\right)^2 k \\ &= \widetilde{C}_0^2(\gamma)(h/2)^{2\gamma} \sum_{k=p+1}^{\infty} k^{-4\gamma-1} \log^{2\nu}\left(\frac{2k^2}{h}\right)\left(1 + \mathrm{O}\left(\log^{-1}\frac{2k^2}{h}\right)\right)^2\end{aligned}$$

which leads to the assertion (3.4) for $\nu = 1$.

In general, for integer $\nu \geqslant 1$ and noninteger $\gamma > 0$, we have

$$\begin{aligned}&(-1)^{k-k^*}\frac{\mathrm{d}^\nu}{\mathrm{d}\gamma^\nu}\big((h/2)^\gamma a_k(\gamma)\big) \\ &\quad= (-1)^\nu \widetilde{C}_0(\gamma)(h/2)^\gamma k^{-2\gamma-1}\log^\nu\left(\frac{2k^2}{h}\right)\left(1 + \mathrm{O}\left(\log^{-1}\frac{2k^2}{h}\right)\right). \qquad (3.8)\end{aligned}$$

For the details of the induction, we refer to [14]. Therefore, the Chebyshev projection of $\tilde{u}_\nu^h(\xi)$ on $P_p(I)$, denoted again by $\psi_p(\xi)$,

$$\psi_p = \sum_{k=0}^{p} \frac{\mathrm{d}^\nu}{\mathrm{d}\gamma^\nu}\big((h/2)^\gamma a_k(\gamma)\big)T_k(\xi)$$

has the asymptotic error estimation

$$\begin{aligned}\left|\tilde{u}_\nu^h - \psi_p\right|^2_{H^{1/2}(-1,1)} &\cong \sum_{k=p+1}^{\infty} \left(\frac{\mathrm{d}^\nu}{\mathrm{d}\gamma^\nu}((h/2)^\gamma a_k)\right)^2 k \\ &= \widetilde{C}_0^2(\gamma)(h/2)^{2\gamma} \sum_{k=p+1}^{\infty} k^{-4\gamma-1} \log^{2\nu}\left(\frac{2k^2}{h}\right)\left(1 + \mathrm{O}\left(\log^{-1}\frac{2k^2}{h}\right)\right)^2\end{aligned}$$

which tends to zero uniformly with respect to γ for $\gamma \geqslant \gamma_0 > 0$. Therefore,

$$u(x) = \tilde{u}_\nu^h\left(\frac{2}{h}x - 1\right) = \sum_{k=0}^{\infty} \frac{\mathrm{d}^\nu}{\mathrm{d}\gamma^\nu}\big((h/2)^\gamma a_k(\gamma)\big)T_k\left(\frac{2}{h}x - 1\right),$$

and its Jacobi projection on $P_p(J_h)$

$$u_{hp} = \sum_{k=0}^{p} \frac{\mathrm{d}^\nu}{\mathrm{d}\gamma^\nu}\big((h/2)^\gamma a_k(\gamma)\big)T_k\left(\frac{2}{h}x - 1\right).$$

Then, due to the relation (3.3), the assertion (3.4) in general follows easily.

Note that $\widetilde{C}(\gamma) = 0$ and $\widetilde{C}'(\gamma) \neq 0$ for integer $\gamma > 0$. We have instead of (3.7)

$$\begin{aligned}(-1)^{k-k^*} \frac{\mathrm{d}}{\mathrm{d}\gamma}\big((h/2)^\gamma a_k(\gamma)\big) &= (h/2)^\gamma \widetilde{C}_0'(\gamma) \frac{\Gamma(k-\gamma)}{\Gamma(\gamma+k+1)} \\ &= \widetilde{C}_0'(\gamma)(h/2)^\gamma k^{-2\gamma-1}\big(1+\mathrm{O}(k^{-1})\big) \qquad (3.9)\end{aligned}$$

which leads to the assertion (3.5) for $\nu = 1$. In general for integer $\gamma > 0$ and $\nu \geqslant 1$, we have, instead of (3.8)

$$\begin{aligned}&(-1)^{k-k^*} \frac{\mathrm{d}^\nu}{\mathrm{d}\gamma^\nu}\big((h/2)^\gamma a_k(\gamma)\big) \\ &\quad = (-1)^\nu \widetilde{C}_0'(\gamma)(h/2)^\gamma k^{-2\gamma-1} \log^{\nu-1}\left(\frac{2k^2}{h}\right)\left(1+\mathrm{O}\left(\log^{-1}\frac{2k^2}{h}\right)\right) \qquad (3.10)\end{aligned}$$

which gives the assertion (3.5) in general for integer γ and $\nu \geqslant 1$. □

4. Optimal rate of convergence of the h–p version with quasiuniform meshes

In this section we demonstrate how the optimal approximation results obtained in the previous section lead to optimal a priori upper and lower error estimates for the h–p version of BEM with quasiuniform meshes. We follow the presentation in [15, section 6] where we analyzed the p version.

For a polygonal domain Ω with boundary Γ we study the h–p Galerkin approximation of the integral equations

$$V\psi = \left(\frac{1}{2}I + K\right)f \quad \text{on } \Gamma, \qquad (4.1)$$

$$Wv = \left(\frac{1}{2}I - K'\right)g \quad \text{on } \Gamma. \qquad (4.2)$$

The operators V, K are the single layer and double layer potential operators and K', W are obtained by taking the normal derivatives of V and $-K$, respectively. The equation (4.1) models the Dirichlet problem for the Laplacian with Dirichlet datum f on Γ and unknown function ψ, the normal derivative of the solution of the Dirichlet problem. The integral equation (4.2) with hypersingular operator W is the corresponding equation for the Neumann problem, with unknown function v being the trace on Γ of the solution.

It is also well-known that there exist a unique solution $\psi \in H^{-1/2}(\Gamma)$ of (4.1) if the conformal radius of Γ is less than one (which can be obtained by a scaling). The operator W has a kernel which consists of constant functions. In the space

$$H_0^{1/2}(\Gamma) = \left\{ w \in H^{1/2}(\Gamma);\ \int_\Gamma w \,\mathrm{d}s = 0 \right\}$$

(4.2) is uniquely solvable.

In order to study the convergence of the h–p version of the boundary element method for solving (4.1) and (4.2) we recall some regularity results. In the following we consider for simplicity piecewise analytic given data (f for (4.1) and g for (4.2)).

Let us denote the vertices of Ω by t_j ($1 \leqslant j \leqslant J$, $t_{J+1} = t_1$) and let Γ^j be the open edge connecting t_j and t_{j+1}. The internal angle at t_j is ω_j. We consider a partition of unity $(\chi_1, \dots, \chi_J)$ where χ_j is the restriction of a $C_0^\infty(\mathbb{R}^2)$ function to Γ such that $\chi_j = 1$ in a neighborhood of the vertex t_j and $\operatorname{supp}(\chi_j) \subset \Gamma^{j-1} \cup \{t_j\} \cup \Gamma^j$ ($\Gamma^0 = \Gamma^J$). In this way we may write any function φ on Γ like

$$\varphi = \sum_{j=1}^{J} (\varphi_-, \varphi_+)\chi_j,$$

where a pair (φ_-, φ_+) corresponds to φ on $\Gamma^{j-1} \cup \{t_j\} \cup \Gamma^j$ with

$$\varphi_- = \varphi|_{\Gamma^{j-1}} \quad \text{and} \quad \varphi_+ = \varphi|_{\Gamma^j}.$$

Then we have the following regularity result.

Proposition 4.1 ([20]). Let $\alpha_{jk} := k\frac{\pi}{\omega_j}$ (integer $k \geqslant 1$, $j = 1, \dots, J$) and, for $t \geqslant 1/2$, let n be an integer with $n + 1 > \frac{\omega_j}{\pi}(t - 1/2) \geqslant n$.

(i) If f is a piecewise analytic function, then there exists a function ψ_0 with $\psi_0|_{\Gamma^j} \in H^{t-1}(\Gamma^j)$ such that, for the solution ψ of (4.1), there holds

$$\psi = \sum_{j=1}^{J} \sum_{k=1}^{n} \bigl((\psi_{jk})_-, (\psi_{jk})_+\bigr)\chi_j + \psi_0.$$

Here

$$(\psi_{jk})_\pm(x) = c|x - t_j|^{\alpha_{jk}-1} \qquad (\alpha_{jk} \text{ is not an integer}),$$
$$(\psi_{jk})_\pm(x) = c_1|x - t_j|^{\alpha_{jk}-1} + c_2|x - t_j|^{\alpha_{jk}-1} \log|x - t_j| \qquad (\alpha_{jk} \text{ is an integer}).$$

(ii) If g is a piecewise analytic function, then there exists a function v_0 with $v_0|_{\Gamma^j} \in H^t(\Gamma^j)$ such that, for the solution v of (4.2), there holds

$$v = \sum_{j=1}^{J} \sum_{k=1}^{n} \bigl((v_{jk})_-, (v_{jk})_+\bigr)\chi_j + v_0.$$

Here

$$(v_{jk})_\pm(x) = c|x - t_j|^{\alpha_{jk}} \qquad (\alpha_{jk} \text{ is not an integer}),$$
$$(v_{jk})_\pm(x) = c_1|x - t_j|^{\alpha_{jk}} + c_2|x - t_j|^{\alpha_{jk}} \log|x - t_j| \qquad (\alpha_{jk} \text{ is an integer}).$$

The constants c, c_1 and c_2 above are generic.

Now we define and analyze the h–p Galerkin method for the approximate solution of (4.1) and (4.2). To this end we introduce piecewise polynomial spaces. Let Γ be decomposed into straight line pieces Γ_j, $j = 1, \dots, n$, such that the corners of the polygon Γ coincide with endpoints of some elements. We assume that the length h of the longest element is bounded by a constant times the length of the smallest element. This is the so-called quasiuniformity of the mesh $\Gamma_h := \{\Gamma_1, \dots, \Gamma_n\}$ (the number of elements n is proportional to h^{-1}). For a given integer $p > 0$ we define

$$S^1_{h,p}(\Gamma) := \{v \in C^0(\Gamma);\ v|_{\Gamma_j} \in P_p(\Gamma_j),\ j = 1, \dots, n\}$$

and

$$S^0_{h,p-1}(\Gamma) := \{v;\ v|_{\Gamma_j} \in P_{p-1}(\Gamma_j),\ j = 1, \dots, n\}.$$

Here, $P_p(\Gamma_j)$ denotes the set of polynomials on Γ_j (with respect to the arc length) up to degree p. There holds $S^1_{h,p}(\Gamma) \subset H^{1/2}(\Gamma)$ and $S^0_{h,p-1}(\Gamma) \subset H^{-1/2}(\Gamma)$. However, the condition of integral mean zero is not satisfied by the functions in $S^1_{h,p}(\Gamma)$. This condition can be incorporated by a Lagrangian multiplier, see, e.g., [9]. The h–p Galerkin schemes then are as follows.

For a given mesh Γ_h and $p \geqslant 0$, find $\psi_{h,p} \in S^0_{h,p}(\Gamma)$ satisfying

$$\langle V\psi_{h,p}, \phi\rangle_{L^2(\Gamma)} = \left\langle \left(\frac{1}{2}I + K\right) f, \phi \right\rangle_{L^2(\Gamma)} \quad \forall \phi \in S^0_{h,p}(\Gamma), \tag{4.3}$$

and, for $p \geqslant 1$, find $v_{h,p} \in S^1_{h,p}(\Gamma)$ and a real number a satisfying

$$\begin{aligned} \langle Wv_{h,p}, w\rangle_{L^2(\Gamma)} + \langle w, a\rangle_{L^2(\Gamma)} &= \left\langle \left(\frac{1}{2}I - K'\right) g, w \right\rangle_{L^2(\Gamma)} \quad \forall w \in S^1_{h,p}(\Gamma), \\ \langle v_{h,p}, 1\rangle_{L^2(\Gamma)} &= 0. \end{aligned} \tag{4.4}$$

Before analyzing the convergence of the Galerkin schemes let us present sharp approximations results in $\widetilde{H}^{1/2}(J)$ for smooth and singular functions over one edge $J = \Gamma^j$ by piecewise polynomials of $S_{h,p}(J) = S_{h,p}(\Gamma)|_J$.

Proposition 4.2 ([22, theorem 3.1]). Let $r > 1/2$ and $p \geqslant 1$. Then for $v \in H^r(J)$ there exists $v_{hp} \in S_{h,p}(J)$ such that

$$\|v - v_{hp}\|_{\widetilde{H}^{1/2}(J)} \leqslant ch^{\mu-1/2}p^{-(r-1/2)}(1 + \log p)^{1/2}\|v\|_{H^r(J)}.$$

Here, $\mu = \min\{r, p + 1\}$ and the constant $c > 0$ is independent of h, p and v, but depends on r.

Now let us consider the approximation of a singular function of the type

$$u(x) = x^\gamma \log^\nu(x)\chi(x), \quad x \in J := (0, 1) \tag{4.5}$$

($\gamma > 0$ and integer $\nu \geqslant 0$). Here, $\chi \in C^\infty(J)$ with $\chi(x) = 1$ for $0 < x < \delta_0/2$ and $\chi(x) = 0$ for $\delta_0 < x < 1$ where $\delta_0 < 1$ is a positive constant.

Theorem 4.1. Let u be given by (4.5) with noninteger γ. Then, there exists $u_{hp} \in S_{h,p}(J)$ with $p > 2\gamma - 1/2$ such that

$$\|u - u_{hp}\|_{\widetilde{H}^{1/2}(J)} \leqslant Ch^{\gamma} p^{-2\gamma}\left(1 + \log^{\nu}\left(\frac{p}{h}\right)\right), \tag{4.6}$$

where the constant C is independent of h and p. If γ is an integer, there holds

$$\|u - u_{hp}\|_{\widetilde{H}^{1/2}(J)} \leqslant Ch^{\gamma} p^{-2\gamma}\left(1 + \log^{\nu-1}\left(\frac{p}{h}\right)\right). \tag{4.7}$$

Proof. We adapt the proof of the optimal estimates for the p version from [15] by incorporating a proper scaling argument. Let us assume without loss of generality that all the elements are of the same size h and $h < \delta_0/2$. We represent the singular function u like $u = u_1 + u_2$ with

$$u_1(x) = u(x)\chi(x/h), \qquad u_2(x) = u(x)\bigl(1 - \chi(x/h)\bigr).$$

For simplicity we have taken the same cut-off function χ as in the representation of u in (4.5). There holds $\operatorname{supp}(u_1) \subset \bar{I}_1 = [0, h]$. Defining $v_1(\xi) := u_1(h\xi)$ we obtain

$$v_1(\xi) = u_1(h\xi)\chi(\xi) = h^{\gamma}\xi^{\gamma}\log^{\nu}(h\xi)\chi(\xi) = h^{\gamma}\xi^{\gamma}\sum_{k=0}^{\nu}\binom{\nu}{k}\log^{k}(h)\log^{\nu-k}(\xi)\tilde{\chi}(\xi)$$

for $\xi \in J = (0,1)$, $\chi(\xi) = 1$ for $\xi \in (0, 1/2)$, and $\chi(\xi) = 0$ for $\xi > 1$. By proposition 2.4 there exists, for $l = 0, \ldots, \nu$, a polynomial $\psi^{l}_{1,p}(\xi) \in P_p(J)$ satisfying for noninteger $\gamma > 0$

$$\bigl\|\xi^{\gamma}\log^{l}(\xi)\tilde{\chi}(\xi) - \psi^{l}_{1,p}(\xi)\bigr\|_{\widetilde{H}^{1/2}(J)} \leqslant Cp^{-2\gamma}(1 + \log p)^{l}. \tag{4.8}$$

Letting

$$\psi(\xi) := h^{\gamma}\sum_{k=0}^{\nu}\binom{\nu}{k}\log^{k}(h)\psi^{\nu-k}_{1,p}(\xi)$$

we have the estimate

$$\begin{aligned}
\|v_1 - \psi\|_{\widetilde{H}^{1/2}(J)} &= \left\| v_1 - h^{\gamma}\sum_{k=0}^{\nu}\binom{\nu}{k}\log^{k}(h)\psi^{\nu-k}_{1,p}\right\|_{\widetilde{H}^{1/2}(0,1)} \\
&\leqslant h^{\gamma}\sum_{k=0}^{\nu}\binom{\nu}{k}\log^{k}\left(\frac{1}{h}\right)\bigl\|\xi^{\gamma}\log^{\nu-k}(\xi)\tilde{\chi}(\xi) - \psi^{\nu-k}_{1,p}(\xi)\bigr\|_{\widetilde{H}^{1/2}(J)} \\
&\leqslant h^{\gamma}\sum_{k=0}^{\nu}\binom{\nu}{k}C(k)\log^{k}\left(\frac{1}{h}\right)p^{-2\gamma}(1 + \log p)^{(\nu-k)} \\
&\leqslant C(\nu)h^{\gamma}p^{-2\gamma}\left(1 + \log\left(\frac{p}{h}\right)\right)^{\nu}.
\end{aligned}$$

Let $\phi_1(x) = \psi(x/h)$. Due to the scalability of the $\widetilde{H}^{1/2}$-norm (see, e.g., [22, lemma 3.1]), there holds

$$\begin{aligned}\big\|u_1(x) - \phi_1(x)\big\|_{\widetilde{H}^{1/2}(I_1)} &\cong \big\|v_1(\xi) - \psi(\xi)\big\|_{\widetilde{H}^{1/2}(J)} \\ &\leqslant C(\nu)h^{\gamma} p^{-2\gamma}\left(1 + \log\left(\frac{p}{h}\right)\right)^{\nu}. \end{aligned} \tag{4.9}$$

Due to proposition 4.2, the function $u_2 \in C^{\infty}(J)$ with support in $[h/2, 1]$ can be approximated by a piecewise polynomial $\phi_2 \in S_{hp}(J)$ satisfying

$$\|u_2 - \phi_2\|_{\widetilde{H}^{1/2}(J)} \leqslant C(r)h^{\mu-1/2} p^{-(r-1/2)}(1 + \log p)^{1/2}\|u_2\|_{H^r(J)}, \quad r > 1/2,$$

with $\mu = \min\{r, p + 1\}$. It is trivial that

$$\|u_2\|_{H^r(J)} \leqslant \begin{cases} C(r)h^{\gamma+1/2-r}\log^{\nu}\left(\dfrac{1}{h}\right), & r > \gamma + 1/2, \\ C\log^{\nu+1/2}\left(\dfrac{1}{h}\right), & r = \gamma + 1/2, \\ C, & r < \gamma + 1/2 \end{cases}$$

which implies for $r > \gamma + 1/2$

$$\|u_2 - \phi_2\|_{\widetilde{H}^{1/2}(J)} \leqslant C(r)h^{\mu-r+\gamma}\log^{\nu}\left(\frac{1}{h}\right)p^{-(r-1/2)}(1 + \log p)^{1/2}. \tag{4.10}$$

Extending ϕ_1 onto J by a zero extension outside of I_1 and defining $u_{hp} := \phi_1 + \phi_2$, combination of (4.9) and (4.10) gives for any $r > \gamma + 1/2$

$$\begin{aligned}&\|u - u_{hp}\|_{\widetilde{H}^{1/2}(J)} \\ &\quad\leqslant C\big(\|u_1 - \phi_1\|_{\widetilde{H}^{1/2}(I_1)} + \|u_2 - \phi_2\|_{\widetilde{H}^{1/2}(J)}\big) \\ &\quad\leqslant C\max\big\{h^{\gamma}p^{-2\gamma}, C(r)\,h^{\mu-r+\gamma}p^{-(r-1/2)}(1 + \log p)^{1/2}\big\}\left(1 + \log\left(\frac{p}{h}\right)\right)^{\nu} \\ &\quad\leqslant C\max\big\{h^{\gamma}p^{-2\gamma}, p^{-(r-1/2)}(1 + \log p)^{1/2}\max\big\{C(r)\,h^{\gamma}, C(r)\,h^{p+\gamma+1-r}\big\}\big\} \\ &\qquad\times\left(1 + \log\left(\frac{p}{h}\right)\right)^{\nu}.\end{aligned}$$

Since $p > 2\gamma - 1/2$, selecting an integer $r \in (2\gamma + 1/2, p + 1]$, we have

$$\max\big\{h^{\gamma}p^{-2\gamma}, p^{-(r-1/2)}(1 + \log p)^{1/2}\max\big\{C(r)\,h^{\gamma}, C(r)\,h^{p+\gamma+1-r}\big\}\big\} \leqslant C(\gamma)\,h^{\gamma}p^{-2\gamma}$$

which leads immediately to (4.6).

If γ is an integer, we introduce a linear mapping $x = h\frac{1+\xi}{2}$ before separating the function into smooth and singular parts. Let

$$\tilde{u}(\xi) = u(h\xi) = h^{\gamma}\xi^{\gamma}\log^{\nu}(h\xi) = h^{\gamma}\xi^{\gamma}\left(\log^{\nu} h + \sum_{k=0}^{\nu-1}\binom{\nu}{k}\log^k h\,\log^{\nu-k}\xi\right).$$

Note that the first term $h^\gamma \xi^\gamma \log^\nu h$ is a polynomial of degree γ in $P_p(J)$ with $p > 2\gamma - 1/2$, for which there is no approximation error. Then we separate the function $\tilde{u}(\xi) - h^\gamma \xi^\gamma \log^\nu h$ into smooth and singular parts by a cut-off function $\tilde{\chi}(\xi)$, and apply the approximation result of proposition 2.4 with integer γ for each term $h^\gamma \xi^\gamma \log^k h \log^{\nu-k} \xi \chi(\xi), 0 \leqslant k \leqslant \nu - 1$, as before. We have (4.7) instead of (4.6). $\square$

Now we are ready to give a sharp upper bound for the approximation error concerning the hypersingular integral equation on polygonal domains.

Theorem 4.2. Let v be the exact solution of (4.2) with piecewise analytic g, and let $v_{h,p}$ be the BE approximation defined by (4.4) with $p > 2\pi/\omega^* - 1/2$. There holds

$$\|v - v_{h,p}\|_{\widetilde{H}^{1/2}(\Gamma)} \leqslant C h^{\pi/\omega^*} p^{-2\pi/\omega^*}. \tag{4.11}$$

Here, $\omega^* = \max\{\omega_j;\ j = 1, \ldots, J\}$ and the constant C does not depend on h and p.

Proof. The Galerkin scheme (4.4) converges quasioptimally in $\widetilde{H}^{1/2}(\Gamma) \times \mathbb{R}$ [9]. Therefore, we only need to find an element $w \in S^1_{h,p}(\Gamma)$ such that $\|v - w\|_{\widetilde{H}^{1/2}(\Gamma)}$ satisfies the bound stated by the theorem. We will define this approximation piecewise on the edges Γ^j by functions $w^j \in S_{h,p}(\Gamma^j) := S_{h,p}(\Gamma)|_{\Gamma^j}$.

By proposition 4.1(ii), on Γ^j we can represent the solution v of (4.2) like

$$v = v_1 + v_2, \tag{4.12}$$

where v_1 contains the singularities at the corner t_j and v_2 contains the singularities at t_{j+1}. More precisely, we can find a representation

$$v_1 = v_{11} + v_{12} + v_{10} \tag{4.13}$$

with $v_{10} \in H^{s_1}(\Gamma^j)$, $s_1 = 3\pi/\omega_j + 1/2 - \epsilon$ ($\epsilon > 0$) and singularities

$$\begin{aligned} v_{11}(x) &:= c_{11}|x - t_j|^{\pi/\omega_j} \chi_j + c_{12}|x - t_j|^{\pi/\omega_j} \log|x - t_j| \chi_j, \\ v_{12}(x) &:= c_{13}|x - t_j|^{2\pi/\omega_j} \chi_j + c_{14}|x - t_j|^{2\pi/\omega_j} \log|x - t_j| \chi_j. \end{aligned}$$

The constant c_{12} vanishes for noninteger $\alpha_{j1} = \pi/\omega_j$, and c_{14} vanishes for noninteger $\alpha_{j2} = 2\pi/\omega_j$. χ_j is the C^∞ cut-off function. Accordingly, for v_2 in (4.12) we find a representation

$$v_2 = v_{21} + v_{22} + v_{20}$$

with $v_{20} \in H^{s_2}(\Gamma^j)$, $s_2 = 3\pi/\omega_{j+1} + 1/2 - \epsilon$ ($\epsilon > 0$) and singularities

$$\begin{aligned} v_{21}(x) &:= c_{21}|x - t_{j+1}|^{\pi/\omega_{j+1}} \chi_{j+1} + c_{22}|x - t_{j+1}|^{\pi/\omega_{j+1}} \log|x - t_{j+1}| \chi_{j+1}, \\ v_{22}(x) &:= c_{23}|x - t_{j+1}|^{2\pi/\omega_{j+1}} \chi_{j+1} + c_{24}|x - t_{j+1}|^{2\pi/\omega_{j+1}} \log|x - t_{j+1}| \chi_{j+1}. \end{aligned}$$

By theorem 4.1 there exist $w_{11}, w_{12} \in S_{h,p}(\Gamma^j)$ with $p > 2\pi/\omega_j - 1/2$ such that

$$\|v_{11} - w_{11}\|_{\widetilde{H}^{1/2}(\Gamma^j)} \leqslant c h^{\pi/\omega_j} p^{-2\pi/\omega_j}, \tag{4.14}$$

$$\|v_{12} - w_{12}\|_{\widetilde{H}^{1/2}(\Gamma^j)} \leqslant c h^{2\pi/\omega_j} p^{-4\pi/\omega_j}. \tag{4.15}$$

Note that c_{12} in the representation of v_{11} is nonzero only for integer π/ω_j, and Theorem 4.1 tells us that the logarithmic term does not appear in the estimates (4.14) and (4.15).

For the smooth remainder v_{10} of v_1 we find by Proposition 4.2 a piecewise polynomial $w_{10} \in S_{h,p}(\Gamma^j)$ which satisfies

$$\|v_{10} - w_{10}\|_{\widetilde{H}^{1/2}(\Gamma^j)} \leqslant ch^{\mu-1/2}p^{-s_1+1/2}(1+\log p)^{1/2},$$

where $\mu = \min\{s_1, p+1\}$. Noting that $s_1 = 3\pi/\omega_j + 1/2 - \epsilon > 2\pi/\omega_j + 1/2$ for $\epsilon < \pi/\omega_j$, e.g., $\epsilon = \frac{1}{2}\pi/\omega_j$, and $p+1 > 2\pi/\omega_j + 1/2$, we have

$$\|v_{10} - w_{10}\|_{\widetilde{H}^{1/2}(\Gamma^j)} \leqslant ch^{2\pi/\omega_j}p^{-\frac{5}{2}\pi/\omega_j}(1+\log p)^{1/2} \leqslant ch^{2\pi/\omega_j}p^{-2\pi/\omega_j}. \tag{4.16}$$

Analogously for v_2 we find piecewise polynomials $w_{21}, w_{22} \in S_{h,p}(\Gamma^j)$ with

$$\|v_{21} - w_{21}\|_{\widetilde{H}^{1/2}(\Gamma^j)} \leqslant ch^{\pi/\omega_{j+1}}p^{-2\pi/\omega_{j+1}} \tag{4.17}$$

and

$$\|v_{22} - w_{22}\|_{\widetilde{H}^{1/2}(\Gamma^j)} \leqslant ch^{2\pi/\omega_{j+1}}p^{-4\pi/\omega_{j+1}}. \tag{4.18}$$

Also, there exists $w_{20} \in S_{h,p}(\Gamma^j)$ such that

$$\|v_{20} - w_{20}\|_{\widetilde{H}^{1/2}(\Gamma^j)} \leqslant ch^{2\pi/\omega_{j+1}}p^{-2\pi/\omega_{j+1}}. \tag{4.19}$$

The approximation $w^j \in S^1_{h,p}(\Gamma^j)$ of v on Γ^j is constructed by the pieces defined above:

$$w^j := w_{11} + w_{12} + w_{10} + w_{21} + w_{22} + w_{20}.$$

Combining (4.14)–(4.19) we obtain

$$\left\|v - w^j\right\|_{\widetilde{H}^{1/2}(\Gamma^j)} \leqslant \sum_{i=1}^{2}\sum_{j=0}^{2}\|v_{ij} - w_{ij}\|_{\widetilde{H}^{1/2}(\Gamma^j)} \leqslant \max\left\{h^{\pi/\omega_i}p^{-2\pi/\omega_i};\ i = j, j+1\right\}.$$

Now, proceeding in the same way an all the edges Γ^j of Γ, we define $w := w^j$ on Γ^j and conclude that there holds

$$\|v - w\|_{\widetilde{H}^{1/2}(\Gamma)} \leqslant c\left(\sum_{j=1}^{J}\left\|v - w^j\right\|^2_{\widetilde{H}^{1/2}(\Gamma^j)}\right)^{1/2} \leqslant c\max\left\{h^{\pi/\omega_i}p^{-2\pi/\omega_i};\ i = 1, \ldots, J\right\}$$

which proves the theorem. □

Now we analyze a lower bound for the error of the Galerkin approximation of the hypersingular integral equation (4.2).

Theorem 4.3. Let v be the exact solution of (4.2) with piecewise analytic g, and let $v_{h,p}$ be the BE approximation defined by (4.4) with $p > 2\pi/\omega^* - 1/2$. Suppose that the strongest singularity $|x-t_{j_0}|^{\pi/\omega_{j_0}}$ or $|x-t_{j_0}|^{\pi/\omega_{j_0}}\log|x-t_{j_0}|$ occurs at some vertex t_{j_0} of Ω with $\omega_{j_0}=\omega^*=\max\{\omega_j;\ j=1,\dots,J\}$. Then there holds

$$\|v-v_{h,p}\|_{\widetilde{H}^{1/2}(\Gamma)}\geqslant ch^{\pi/\omega^*}p^{-2\pi/\omega^*} \tag{4.20}$$

with $c>0$ independent of h and p.

Proof. We may assume that $j_0=1, \omega_1=\omega^*$ and that $\Gamma_1^1=J_h=(0,h)$ is the corner element connected to the vertex t_1 located at $x=0$. There holds

$$\|v-v_{h,p}\|_{\widetilde{H}^{1/2}(\Gamma)}\cong\|v-v_{h,p}\|_{H^{1/2}(\Gamma)}\geqslant\|v-v_{h,p}\|_{H^{1/2}(\Gamma_1^1)}.$$

By proposition 4.1, $v=v_1+v_2+v_0$ on Γ_1^1 with

$$v_1(x)=c_1x^{\pi/\omega_1}+c_2x^{\pi/\omega_1}\log x,$$
$$v_2(x)=c_3x^{2\pi/\omega_1}+c_4x^{2\pi/\omega_1}\log x,$$

and

$$v_0(r)\in H^t\left(\Gamma_1^1\right)\quad\forall t=3\pi/\omega_1+1/2-\epsilon.$$

Here, $\epsilon>0$ is arbitrary, c_2 and c_4 are not zero if π/ω_1 and $2\pi/\omega_1$ are integers, respectively. Note that in Proposition 4.1 the singular functions v_{11} and v_{12} are associated with cut-off functions, but v_1 and v_2 are plain singular functions. However, both representations differ only by C^∞-perturbations which become a part of v_0. Now we assume that the assertion of the theorem does not hold. Therefore, there exists a piecewise polynomial $w\in S_p^1(\Gamma_h)$ and a function $\delta(p,h)$ such that

$$\begin{aligned}&\|v-v_{h,p}\|_{H^{1/2}(\Gamma_1^1)}\leqslant\|v-v_{h,p}\|_{H^{1/2}(\Gamma)}\leqslant ch^{\pi/\omega_1}p^{-2\pi/\omega_1}\delta(p,h),\\&\delta(p,h)\to0\ (p\to\infty,\ \text{or } h\to0).\end{aligned} \tag{4.21}$$

By Theorem 4.1, there exists a polynomial w_2 of degree p such that

$$\|v_2-w_2\|_{H^{1/2}(\Gamma_1^1)}\leqslant ch^{2\pi/\omega_1}p^{-4\pi/\omega_1}.$$

Moreover, by a standard approximation argument, there exists a polynomial w_0 of degree p such that

$$\|v_0-w_0\|_{H^{1/2}(\Gamma_1^1)}\leqslant ch^{\mu-1/2}p^{-(t-1/2)},$$

where $\mu=\min\{t,p+1\}$. Noting that $p+1>2\pi/\omega_1+1/2$ and $t=\frac{5}{2}\pi/\omega_1+1/2$ for $\epsilon=\frac{1}{2}\pi/\omega_1$, there holds

$$\|v_0-w_0\|_{H^{1/2}(\Gamma_1^1)}\leqslant ch^{2\pi/\omega_1}p^{-\frac{5}{2}\pi/\omega_1}.$$

Therefore, combining the last two estimates and the assumption (4.21) we obtain

$$\begin{aligned}
&\big\|v_1 - (v_{h,p} - w_2 - w_0)\big\|_{H^{1/2}(\Gamma_1^1)} \\
&\quad \leqslant \|v - v_{h,p}\|_{H^{1/2}(\Gamma_1^1)} + \|v_2 - w_2\|_{H^{1/2}(\Gamma_1^1)} + \|v_0 - w_0\|_{H^{1/2}(\Gamma_1^1)} \\
&\quad \leqslant c h^{\pi/\omega_1} p^{-2\pi/\omega_1}\big(\delta(p,h) + h^{\pi/\omega_1} p^{-2\pi/\omega_1} + h^{\pi/\omega_1} p^{-\frac{1}{2}\pi/\omega_1}\big).
\end{aligned}$$

Note that in either case π/ω_1 is an integer or not, by theorems 3.2

$$\big\|v_1 - (v_{h,p} - w_2 - w_0)\big\|_{H^{1/2}(\Gamma_1^1)} \geqslant \|v_1 - \Pi v_1\|_{H^{1/2}(J_h)} \cong h^{\pi/\omega_1} p^{-2\pi/\omega_1},$$

where Πv_1 is the Chebyshev projection of v_1 on $P_p(J_h)$, which leads to a contradiction. Thus we complete the proof. □

Corollary 4.1. Let v be the exact solution of (4.2) with piecewise analytic g, and let v_p be the BE approximation defined by (4.4) with $p > 2\pi/\omega^* - 1/2$. Suppose that the most severe singularity $|x - t_{j_0}|^{\pi/\omega_{j_0}}$ or $|x - t_{j_0}|^{\pi/\omega_{j_0}} \log|x - t_{j_0}|$ occurs at some vertex t_{j_0} of Ω with $\omega_{j_0} = \omega^* = \max\{\omega_j;\ j = 1, \ldots, J\}$. Then there holds the optimal error estimate

$$c h^{\pi/\omega^*} p^{-2\pi/\omega^*} \leqslant \|v - v_{h,p}\|_{\widetilde{H}^{1/2}(\Gamma)} \leqslant C h^{\pi/\omega^*} p^{-2\pi/\omega^*}, \tag{4.22}$$

with $C > 0$ and $c > 0$ independent of h and p.

Making use of the approximation results for the hypersingular integral equation we prove analogous results for the weakly singular integral equation (4.1).

Theorem 4.4. Let ψ be the exact solution of (4.1) with piecewise analytic f, and let $\psi_{h,p}$ be the BE approximation defined by (4.3) with $p > 2\pi/\omega^* - 1/2$. Suppose that the strongest singularity $|x - t_{j_0}|^{\pi/\omega_{j_0}-1}$ or $|x - t_{j_0}|^{\pi/\omega_{j_0}-1} \log|x - t_{j_0}|$ occurs at some vertex t_{j_0} of Ω with $\omega_{j_0} = \omega^* = \max\{\omega_j;\ j = 1, \ldots, J\}$. Then there holds

$$c h^{\pi/\omega^*} p^{-2\pi/\omega^*} \leqslant \|\psi - \psi_{h,p}\|_{\widetilde{H}^{-1/2}(\Gamma)} \leqslant C h^{\pi/\omega^*} p^{-2\pi/\omega^*} \tag{4.23}$$

with $C > 0$ and $c > 0$ independent of h and p.

Proof. We apply the results of Theorems 4.2 and 4.3 by considering antiderivatives of ψ, which have singularities like the solution v of (4.2). This technique has been used previously by Stephan and Suri [21] and in our paper on the p version [15].

First we prove the upper bound for $\|\psi - \psi_{h,p}\|_{\widetilde{H}^{-1/2}(\Gamma)}$. By the quasi-optimality of the Galerkin scheme (4.3) we only need to define an element $\phi \in S^0_{h,p}(\Gamma)$ such that $\|\psi - \phi\|_{\widetilde{H}^{-1/2}(\Gamma)}$ satisfies the upper bound stated by the theorem.

We consider an edge Γ^j which we identify with interval $J = (0, a)$, $a = |\Gamma^j|$. We define

$$u(x) = \int_0^x \big(\psi - \bar{\psi}\big)(t)\,\mathrm{d}t \quad (x \in J), \tag{4.24}$$

where $\bar{\psi} = \frac{1}{a}\int_0^a \psi(t)\,\mathrm{d}t$. Then u vanishes at the endpoints of J and $u \in \widetilde{H}^{1/2}(J)$. By Proposition 4.1(i) and standard calculation

$$\begin{aligned}u(x) &= \int_0^x \big(\psi - \bar{\psi}\big)(t)\,\mathrm{d}t\\ &= c_{11}x^{\alpha_{j1}} + c_{12}x^{\alpha_{j1}}\log(x) + c_{13}x^{\alpha_{j2}} + c_{14}x^{\alpha_{j2}}\log(x)\\ &\quad + c_{21}|x-a|^{\alpha_{j+1,1}} + c_{22}|x-a|^{\alpha_{j+1,1}}\log|x-a|\\ &\quad + c_{23}|x-a|^{\alpha_{j+1,2}} + c_{24}|x-a|^{\alpha_{j+1,2}}\log|x-a| + u_0(x)\end{aligned}$$

with $\alpha_{ji} = i\pi/\omega_j$ and a function $u_0 \in H^s(J)$, $s = \min\{3\pi/\omega_j + 1/2, 3\pi/\omega_{j+1} + 1/2\} - \epsilon$, $\epsilon > 0$. As in the proof of Theorem 4.2, e.g., by using Theorem 4.1, there exists a piecewise polynomial $w^j \in S_{h,p+1}(J)$ such that (now writing Γ^j instead of J)

$$\big\|u - w^j\big\|_{\widetilde{H}^{1/2}(\Gamma^j)} \leqslant ch^{\pi/\omega_j^*}p^{-2\pi/\omega_j^*} \tag{4.25}$$

with $\omega_j^* = \max\{\omega_j, \omega_{j+1}\}$. By [22, lemma 3.4] differentiation is a mapping

$$\widetilde{H}^{1/2}(J) \to \widetilde{H}^{-1/2}(J).$$

Therefore, defining $\phi^j = (w^j)' + \bar{\psi} \in S_{h,p}(\Gamma^j)$ (differentiation with respect to the arc length) we obtain

$$\big\|\psi - \phi^j\big\|_{\widetilde{H}^{-1/2}(\Gamma^j)} \leqslant c\big\|u - w^j\big\|_{\widetilde{H}^{1/2}(\Gamma^j)}.$$

Estimate (4.25) gives the needed approximation result on an edge Γ^j. We define a piecewise polynomial $\phi \in S_{h,p}(\Gamma)$ by piecing together the local constructions ϕ^j and the stated upper bound for $\|\psi - \phi\|_{\widetilde{H}^{-1/2}(\Gamma)}$ follows.

We prove the lower bound in (4.23) indirectly. To this end assume that there exists a piecewise polynomial $\phi \in S_p^0(\Gamma_h)$ and a function $\delta(h,p)$ such that

$$\|\psi - \phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \leqslant ch^{\pi/\omega^*}p^{-2\pi/\omega^*}\delta(h,p), \quad \delta(h,p) \to 0 \text{ as } p \to \infty, \text{ or } h \to 0. \tag{4.26}$$

We define with $\bar{\psi} = \frac{1}{|\Gamma|}\int_\Gamma \psi\,\mathrm{d}s$ the function

$$u(s) = \int_0^s (\psi - \bar{\psi})(t)\,\mathrm{d}t,$$

s being the arc length of Γ, e.g. starting at the vertex t_1. By [19, Lemma 3] the anti-derivative operator is continuous as a mapping from $H_0^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ ($H_0^{-1/2}(\Gamma)$ is the space of $H^{-1/2}(\Gamma)$-functions with integral mean zero). Therefore, defining the piecewise polynomial $w(s) = \int_0^s(\phi(t) - \bar{\phi})\,\mathrm{d}t \in S_{p+1}^1(\Gamma_h)$ (with $\bar{\phi} = \frac{1}{|\Gamma|}\int_\Gamma \phi\,\mathrm{d}s$), we obtain

$$\begin{aligned}\|u-w\|_{\tilde H^{1/2}(\Gamma)} &\leqslant c\big\|\psi-\bar\psi-\big(\phi-\bar\phi\big)\big\|_{\tilde H^{-1/2}(\Gamma)}\\ &\leqslant c\|\psi-\phi\|_{\tilde H^{-1/2}(\Gamma)}+c\frac{1}{|\Gamma|}\left|\int_\Gamma(\psi-\phi)\,\mathrm{d}s\right|\|1\|_{\tilde H^{-1/2}(\Gamma)}\\ &\leqslant c\left(1+\|1\|_{\tilde H^{-1/2}(\Gamma)}\frac{\|1\|_{\tilde H^{1/2}(\Gamma)}}{|\Gamma|}\right)\|\psi-\phi\|_{\tilde H^{-1/2}(\Gamma)}.\end{aligned}$$

Since u is an antiderivative of ψ it possesses corner singularities which are exactly of the type given in Proposition 4.1(ii). Without loss of generality we may assume that the most severe singularity $|x-t_1|^{\pi/\omega^*}$ or $|x-t_1|^{\pi/\omega^*}\log|x-t_1|$ occurs at the vertex t_1 of Ω with $\omega_1=\omega^*=\max\{\omega_j;\ j=1,\dots,J\}$. Further we assume that Γ_1^1 is the corner element at this vertex. Repeating the arguments of Theorem 4.3 we have

$$\|u-w\|_{H^{1/2}(\Gamma_1^1)}\leqslant ch^{\pi/\omega_1}p^{-2\pi/\omega_1}\delta(h,p)$$

which contradicts Theorem 3.2 (if the singularity contains a logarithmic term) or Theorem 3.1 (otherwise). Thus, the assumption (4.26) does not hold and the proof of the theorem is finished. □

5. Concluding remarks

Based on the analysis of the optimal convergence for the p version of BEM in the framework of the Jacobi-weighted Besov and Sobolev spaces, we prove, by incorporating a properly designed scaling argument, the optimal rate of convergence of the h–p version with quasiuniform meshes for the hypersingular and weakly singular integral equations on polygonal domains where singularity of $|x|^\gamma$-type and $|x|^\gamma\log|x|$-type occur. The results include the h and p versions of BEM as two special cases. For fixed h, it coincides with the optimal convergence of the p version of BEM [15], and for fixed p, it gives the optimal convergence of the h version. Also it is parallel to the results of the h–p version of FEM with quasiuniform meshes [14].

The concepts, methods and techniques in analysis can be generalized to three dimensional problems, but such a generalization will be substantial, and will be feasible only when the analysis for optimal convergence of the p version of BEM in three dimensions is available, which, unfortunately, is an open problem now.

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