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A new method of fundamental solutions applied to nonhomogeneous elliptic problems

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The classical method of fundamental solutions (MFS) has only been used to approximate the solution of homogeneous PDE problems. Coupled with other numerical schemes such as domain integration, dual reciprocity method (with polynomial or radial basis functions interpolation), the MFS can be extended to solve the nonhomogeneous problems. This paper presents an extension of the MFS for the direct approximation of Poisson and nonhomogeneous Helmholtz problems. This can be done by using the fundamental solutions of the associated eigenvalue equations as a basis to approximate the nonhomogeneous term. The particular solution of the PDE can then be evaluated. An advantage of this mesh-free method is that the resolution of both homogeneous and nonhomogeneous equations can be combined in a unified way and it can be used for multiscale problems. Numerical simulations are presented and show the quality of the approximations for several test examples.

Keywords: mesh-free method, fundamental solutions, Poisson equation, nonhomogeneous helmholtz equation, density theorems

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1. Introduction

In this paper we extend a technique introduced by Kupradze and Aleksidze [13] in 1964, called the method of fundamental solutions (MFS), to nonhomogeneous equations. The MFS has been continuously developed by numerous mathematicians and scientists over the past 40 years [9,12], and apparently the MFS was rediscovered independently by different scientists in various fields. As a result, the MFS is also known as the desingularized method, the charge simulation method or the superposition method in the mathematical and engineering literature. During the first three decades after 1964, the MFS was essentially restricted to solving homogeneous elliptic equations, such as the Laplace and the Biharmonic equations [9]. In the early 90's, Golberg and Chen [12] began to explore the possibility of coupling radial basis functions (RBFs) with the MFS for solving various types of linear, nonlinear, nonhomogeneous, and time dependent problems. The concept of such extensions was based on the evaluation of a particular solution using the dual reciprocity method (DRM) [16] and RBFs [12]. The MFS-DRM approach has lead to a mesh-free scheme that has become popular in recent years. However, the main difficulty of such a scheme is the need of deriving an approximate analytical particular solutions which requires a certain mathematical skill [14].

This paper builds on two previous reports from the authors [1,2]. In those reports an extended MFS based on the fundamental solutions of the eigenvalue equation was used for the first time. In those reports it was mainly considered the approximation of functions in a domain and the numerical resolution of a Poisson problem. Here we will present theoretical results and numerical simulations for Poisson and nonhomogeneous Helmholtz problems, but the same approach can be used for other elliptic problems with appropriate modifications.

We develop an approximation scheme in which a set of frequencies and point-sources leads to an extended MFS, that we will call MFS-D, used to approximate a function in a bounded domain and derive a straightforward approximation for a particular solution of the PDE. The resolution of a nonhomogeneous PDE is then simple, combining the MFS-D with the classical boundary MFS. In the case of nonhomogeneous Helmholtz equations, we can obtain approximations for particular solutions with different wave numbers by a simple coefficient multiplication from the same MFS-D expansion of the nonhomogeneous term. Another advantage is that MFS and MFS-D can be integrated in a unified and computationally simple approach and it can be useful for multiscale problems [5]. The approximation of the nonhomogeneous term uses a certain set of frequencies and each coefficient in the expansion is independently rescaled allowing a separate approximation. If a small set of test frequencies provides an initial rough approximation, we can get a sharper approximation just by adding the independent contribution of the new frequencies. This technique can also be useful to avoid the numerical problems related to the ill-conditioned systems inherent to these type of mesh-free methods.

1.1. Nonhomogeneous equations

Consider a linear elliptic differential operator $\mathcal{L}$ and a bounded connected domain $\Omega \subset \mathbb{R}^d$, with a sufficiently regular boundary $\partial\Omega$. Assuming the well posedness of the problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega\,, \\ \mathcal{B}u = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathcal{B}$ defines a boundary linear operator, we find the solution u of (1) by taking the following steps:

(i) First find a particular solution u_{P}: $\mathcal{L}u_{\mathrm{P}} = f$, in Ω.

(ii) Then we solve the homogeneous problem $\mathcal{L}u_{\mathrm{H}} = 0$ in Ω, with $\mathcal{B}u_{\mathrm{H}} = g - \mathcal{B}u_{\mathrm{P}}$ on $\partial\Omega$.

(iii) $u = u_{\mathrm{H}} + u_P$ is then the solution of the problem.

In this paper, we focus on Laplace/Helmholtz operators $\mathcal{L} = \Delta - \mu$ (with $\mu \in \mathbb{C}$) in dimension $d = 2, 3$, and on Dirichlet boundary conditions $\mathcal{B}u = u$. An extension to other type of well posed elliptic problems can be made with appropriate modifications.

Our approach consists of approximating u_P using fundamental solutions of the Helmholtz equation to approximate f, and then to solve the homogeneous equation using the classical MFS to obtain u_{H}.

Let δ be the Dirac delta distribution. We denote Φ_λ the fundamental solutions of Laplace and Helmholtz equations

$$(\lambda - \Delta)\Phi_\lambda = \delta, \tag{2}$$

where

$$\Phi_\lambda(x) = \begin{cases} \dfrac{1}{2\pi} K_0(\sqrt{\lambda} r), & \text{if } \lambda > 0, \\ \dfrac{-1}{2\pi} \log(r), & \text{if } \lambda = 0, \\ \dfrac{\mathrm{i}}{4} H_0^{(1)}(\sqrt{-\lambda} r), & \text{if } \lambda < 0, \end{cases} \tag{3}$$

in 2D and

$$\Phi_\lambda(x) = \frac{\mathrm{e}^{\sqrt{\lambda} r}}{4\pi r} \tag{4}$$

in 3D. Here K_0 is the modified Bessel function of the second kind with order zero, and $H_0^{(1)}$ is the Hänkel function of the first kind with order zero. Recall that $H_0^{(1)} = J_0 + \mathrm{i}Y_0$, where J_0, Y_0 are Bessel functions of the first and second kind, respectively. The Bessel function J_0 is analytic everywhere and only K_0 and Y_0 exhibit logarithmic singular behavior at 0.

We first consider Poisson's equation. Several well known methods can be used to derive an approximation of the particular solution u_{P}. Traditionally u_{P} was evaluated by the Newtonian potential

$$u_{\mathrm{P}}(x) = \int_\Omega \Phi_0(x - y) f(y)\,\mathrm{d}y \tag{5}$$

which required tedious domain integration, since a mesh was made in the domain Ω with an integration of the singular integral. In [3], Atkinson gave three different approaches to evaluate u_{P} in (5). In particular, it is clear that the particular solution might be taken in a simpler domain containing Ω. A review for the deduction of the particular solution was given by Golberg [10]. In the literature of boundary element methods, the dual reciprocity method (DRM) [16] and the multiple reciprocity method (MRM) [15] are two very popular approaches used to approximate the particular solution. In particular, radial basis functions (RBFs) have been widely used in this respect in the DRM literature. One of the most popular choices of the RBFs is the thin plate splines (TPS) [11,12]

$$\varphi(r) = \begin{cases} r^2 \log r, & \text{in } \mathbb{R}^2, \\ r, & \text{in } \mathbb{R}^3. \end{cases} \tag{6}$$

In the DRM, we approximate f in (5) by a linear combination of basis functions $\{\varphi_j\}_{j=1}^n$; i.e.,

$$f(x) \simeq \tilde{f}(x) = \sum_{j=1}^{n} a_j \varphi_j(x), \tag{7}$$

where $\{a_j\}_{j=1}^n$ are undetermined coefficients. By collocation,

$$f(x_k) = \tilde{f}(x_k) = \sum_{j=1}^{n} a_j \varphi_j(x_k), \quad 1 \leqslant k \leqslant n, \tag{8}$$

where $\{x_k\}_{k=1}^n$ are n collocation points in $\mathbb{R}^d$. Least squares methods can also be used when the number of collocation points is larger than the number of source points (which are the points on the fictitious boundary, using the MFS approach). Notice that since the particular solution does not have to satisfy the boundary condition, the collocation points $\{x_k\}_{k=1}^n$ can be selected inside and outside the domain. For the traditional DRM, $\{x_k\}_{k=1}^n$ are chosen inside the domain. Assuming that (8) can be solved uniquely for $\{a_j\}_{j=1}^n$, the approximate particular solution $\tilde{u}_{\mathrm{P}}$ is given by

$$\tilde{u}_{\mathrm{P}}(x) = \sum_{j=1}^{n} a_j \Psi_j(x - x_k), \tag{9}$$

where

$$\Delta \Psi_j = \varphi_j, \quad 1 \leqslant j \leqslant n. \tag{10}$$

To achieve high efficiency, it is important to solve (10) analytically. For the Laplacian Δ, closed-forms for Ψ_j are usually not difficult to obtain. However, for other differential operators, the derivation of Ψ_j is not trivial. Much effort is done in deriving the closed-form particular solution for various differential operators [12].

In this paper we propose a different method to obtain $\tilde{u}_{\mathrm{P}}$ by considering Ψ_j as fundamental solutions of the associated eigenvalue equations which avoids the effort to solve (10) analytically. Since the closed form of the fundamental solution is already known, we obtain, instead of (10),

$$\Delta \Psi_j = \lambda \Psi_j.$$

The same Ψ_j will now be used as the basis $\{\varphi_j\}_{j=1}^n$ with minor modifications.

To produce an approximation of u_{H}, we can use the classical MFS with Laplace's fundamental solution, that is

$$\tilde{u}_{\mathrm{H}}(x) = \sum_{k=1}^{N} \alpha_k \Phi_0(x - y_k), \tag{11}$$

where y_k are the source points located outside $\overline{\Omega}$.

Therefore, in this paper, we propose to consider the method of fundamental solutions in two situations:

(i) MFS-B: The classical Method of Fundamental Solutions applied to the approximation of the Boundary function.

(ii) MFS-D: The new Method of Fundamental Solutions applied to the approximation of the Domain function.

1.2. Theoretical aspects

We first present the simple idea that allows us to obtain directly a particular solution.

Suppose we can approximate f with a linear combination of fundamental solutions of Helmholtz equations,

$$\tilde{f}(x) = \sum_{i=1}^{p} \sum_{j=1}^{n} a_{ij} \Phi_{\lambda_i}(x - y_j), \tag{12}$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$ and $\{y_j\}$ are points placed in some admissible source set $\widehat{\Gamma}$ (cf. [1]). Here we will consider admissible source sets such that $\widehat{\Gamma} \subset \overline{\Omega}^c$ is the boundary of a bounded domain $\widehat{\Omega}$. In the MFS literature usually $\widehat{\Omega} \supset \Omega$ and the points on the fictitious boundary $\widehat{\Gamma}$ are called the source points.

Since $x \in \mathbb{R}^d \backslash \{0\}$, we have $\Delta \Phi_\lambda(x) = \lambda \Phi_\lambda(x)$, then

$$\tilde{u}_{\mathrm{P}}(x) = \sum_{i=1}^{p} \sum_{j=1}^{n} \frac{a_{ij}}{\lambda_i} \Phi_{\lambda_i}(x - y_j) \quad \Longrightarrow \quad \Delta \tilde{u}_{\mathrm{P}} = \tilde{f}$$

and by simply dividing a_{ij} by λ_i we obtain a particular solution of Poisson's equation $\Delta \tilde{u}_{\mathrm{P}} = \tilde{f}$. Therefore, being $\tilde{f}$ a good approximation of f then $\tilde{u}_{\mathrm{P}}$ will be a good approximation of u_{P}.

We note that the only transformation is that the coefficients a_{ij} used in the approximation of $\tilde{f}$ are now rescaled in the particular solution.

In order to justify the possibility of approximating any function $f \in L^2(\Omega)$ with a linear combination of Φ_{λ_i} in (12), we will provide some theoretical background.

Theorem 1. Let Ω be any open set and $\{y_1, \dots, y_n\} \notin \overline{\Omega}$. The functions

$$\Phi_{\lambda_1}(x - y_1), \dots, \Phi_{\lambda_p}(x - y_n)$$

are linear independent.

Proof. Suppose, by contradiction, that for some $\lambda_{i_0} \in \{\lambda_1, \dots, \lambda_p\}$ and some $y_{j_0} \in \{y_1, \dots, y_n\}$ we could write

$$\Phi_{\lambda_{i_0}}(x - y_{j_0}) = \sum_{i=1}^{p}\sum_{j=1}^{n} \alpha_{ij}\Phi_{\lambda_i}(x - y_j), \quad \text{with } \alpha_{ij} \neq 0. \tag{13}$$

Then, in Ω the Helmholtz equation is satisfied and we obtain

$$0 = (\Delta - \lambda_{i_0})\Phi_{\lambda_{i_0}}(x - y_{j_0}) = \sum_{i=1}^{p}\sum_{j=1}^{n} \alpha_{ij}(\lambda_i - \lambda_{i_0})\Phi_{\lambda_i}(x - y_j).$$

Therefore, if there exists $\lambda_{i_1} \neq \lambda_{i_0}$ this would mean that $\Phi_{\lambda_{i_1}}(x - y_{j_1})$ could be written as a linear combination, and the same procedure for $(\Delta - \lambda_{i_1})$ would imply the linear dependence until we reach a combination of a single frequency. Thus, the problem is reduced to a fixed frequency $\lambda = \lambda_{i_p}$ and in such a case, by an analytic extension from Ω to $\mathbb{R}^d \backslash \{y_1, \dots, y_n\}$,

$$0 = \sum_j \alpha_{i_p j}\Phi_\lambda(x - y_j), \quad \text{in } \Omega \quad \Longrightarrow \quad \left(\sum_j \alpha_{i_p j}\delta_{y_j}\right) * \Phi_\lambda = 0,$$

where $\delta_{y_j}(x) = \delta(x - y_j)$. This would mean $\sum_j \alpha_{i_p j}\delta_{y_j} = 0$ and therefore $\alpha_{i_p j} \neq 0$ would contradict the linear independence of the Dirac deltas. □

By proving the linear independence, we conclude that the set of functions $\{\phi_{11}, \dots, \phi_{pn}\}$, with $\phi_{ij}(x) = \Phi_{\lambda_i}(x - y_j)$, restricted to Ω, form a basis of a finite dimensional subspace of analytic functions $Q = \langle \phi_{11}, \dots, \phi_{pn} \rangle$. We will search for an approximation of f in this subspace Q. Considering a Hilbert space V, such that Q is a linear subspace of V, then the best approximation in Q to a function $f \in V$ is given by the projection $\mathbf{P}_Q f$

$$\mathbf{P}_Q f = \sum_{i=1}^{p}\sum_{j=1}^{n} \mathbf{P}_{ij} f \phi_{ij}, \tag{14}$$

where $\mathbf{P}_{ij}$ is the component in ϕ_{ij} of the projection of f to the subspace Q. These projection values $\mathbf{P}_{ij} f$ can be easily obtained by solving the least squares system

$$\big[\langle \phi_{lm}, \phi_{ij} \rangle\big]_{np \times np} [\mathbf{P}_{ij} f]_{np \times 1} = \big[\langle \phi_{lm}, f \rangle\big]_{np \times 1}. \tag{15}$$

Using $V = L^2(\Omega)$,we have for the regular boundary $\partial\Omega$,

$$\|u_{\mathrm{P}} - \tilde{u}_{\mathrm{P}}\|_{H^2(\Omega)} \leqslant C \|f - \tilde{f}\|_{L^2(\Omega)}. \tag{16}$$

This means that it is sufficient to produce a good approximation in $L^2(\Omega)$ of f to obtain a good approximation of u_{P}. The question now is, can we produce a good approximation of a $L^2(\Omega)$ function by just using spaces like Q? We achieve that by a density result in [1], by assuming that the source points y_k lie on some admissible source set $\widehat{\Gamma} \subset \mathbb{R}^d \backslash \overline{\Omega}$. We now recall the density result.

Theorem 2. Let $\widehat{\Gamma}$ be an admissible source set and I an open interval in $]-\infty, 0]$. The space

$$\mathbf{S}_{\widehat{\Gamma},I,\Omega} = \operatorname{span}\{\Phi_\lambda(x-y)|_\Omega\colon\ y \in \widehat{\Gamma},\ \lambda \in I\} \tag{17}$$

is dense in $L^2(\Omega)$.

Proof. Let $\alpha \in L^2(\Omega)$ and define

$$v_\lambda(y) = \int_\Omega \alpha(x)\Phi_\lambda(y-x)\,\mathrm{d}x. \tag{18}$$

We must see that if $v_\lambda(y) = 0$, for all $y \in \widehat{\Gamma}$ and for all $\lambda \in I$ then $\alpha \equiv 0$. Let λ be fixed. We know that $v_\lambda = (\alpha\delta_\Omega) * \Phi_\lambda$, which means that v_λ satisfies the Helmholtz equation with null jumps on the boundary, i.e. $[v_\lambda] = [\partial_n v_\lambda] = 0$. Recall that the jump is defined by the difference between interior and exterior traces, i.e. $[v_\lambda] = v_\lambda^- - v_\lambda^+$. Thus,

$$\begin{cases} (\Delta - \lambda)v_\lambda = \alpha & \text{in } \Omega,\\ [v_\lambda] = 0 & \text{on } \Gamma,\\ [\partial_n v_\lambda] = 0 & \text{on } \Gamma, \end{cases}$$

and v_λ satisfies appropriate asymptotic conditions at infinity. For the chosen fundamental solutions, the Sommerfeld radiation condition,

$$\partial_r v_\lambda - \sqrt{\lambda}v_\lambda = \mathrm{o}\big(r^{(1-d)/2}\big), \quad \text{when } r = |x| \to \infty,$$

is verified, and the exterior problem is well posed.

Since $\widehat{\Gamma}$ is an admissible source set, $\widehat{\Gamma}$ is the boundary of $\widehat{\Omega} \supset \Omega$ and also the boundary of $\mathbb{R}^d \backslash \overline{\widehat{\Omega}}$, the well posedness of the exterior problem implies that from $v_\lambda = 0$ on $\widehat{\Gamma}$ we get $v_\lambda = 0$ in $\mathbb{R}^d \backslash \overline{\widehat{\Omega}}$. Since v_λ is analytic outside Γ, by analytic continuation we get $v_\lambda \equiv 0$ in $\mathbb{R}^d \backslash \overline{\Omega}$.

Thus, the exterior traces are null, i.e. $v_\lambda^+ = 0$, $\partial_n v_\lambda^+ = 0$, on Γ, and since we have no jumps on the boundary, the interior traces are also null i.e. $v_\lambda^- = 0$, $\partial_n v_\lambda^- = 0$ on Γ. Consider now any function w such that $\Delta w - \lambda w = 0$ in Ω. We have by Green's formula,

$$\int_\Gamma w\partial_n v_\lambda - \int_\Gamma v_\lambda \partial_n w = \int_\Omega w\Delta v_\lambda - \int_\Omega v_\lambda \Delta w,$$

and since $v_\lambda^- = 0$, $\partial_n v_\lambda^- = 0$ on Γ, we get

$$0 = \int_\Omega w(\alpha + \lambda v_\lambda) - \int_\Omega v_\lambda \lambda w \quad \Longleftrightarrow \quad 0 = \int_\Omega \alpha w,$$

which means that α is orthogonal in the L^2 norm to every function w such that $\Delta w = \lambda w$.

In particular, one can choose the functions $w(x) = \mathrm{e}^{\sqrt{\lambda} x d}$,with $d \in \partial B(0,1)$ (i.e. plane acoustic waves, with $|d| = 1$). We take, in particular,

$$w(x,\xi) = \mathrm{e}^{-\mathrm{i}x\xi}, \tag{19}$$

by choosing $\lambda = -|\xi|^2$ and $d = -\xi/|\xi|$. Notice that the values of $|\xi|$ are limited to the values of $\lambda \in I$, therefore if $I =]-b^2, -a^2[$, we have $|\xi| \in]a,b[$, which means that $\xi \in \mathbf{A}_{ab} = B(0,b) \backslash \overline{B}(0,a)$. Therefore,

$$F_\alpha(\xi) = \int_\Omega \alpha(x)\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x = 0, \quad \forall \xi \in \mathbf{A}_{ab}. \tag{20}$$

This function F_α is the Fourier transform of $\alpha\chi_\Omega$ (where χ_Ω stands for the characteristic function in Ω), thus an analytic function in the whole space. Since F_α vanishes in the open set $\mathbf{A}_{ab}$, by analytic continuation it vanishes in the whole space. By Plancherel's formula $\|\alpha\chi_\Omega\|_{L^2} = \|F_\alpha\|_{L^2} = 0$, and this implies $\alpha \equiv 0$. □

We have just shown that any $L^2(\Omega)$ function can be approximated by a sequence of functions of $\mathbf{S}_{\widehat{\Gamma},I,\Omega}$, which means that f can be approximated by a sequence of functions

$$f_k(x) = \sum_{i=1}^{p_k}\sum_{j=1}^{n_k} a_{ij}^{(k)} \Phi_{\lambda_i^{(k)}}\big(x - y_j^{(k)}\big), \tag{21}$$

each of them is in some finite subspace Q.

The question is now to determine if we want to proceed with the approximation in $L^2(\Omega)$, using the projection defined in this space. To calculate the projection, we must calculate the integrals $\langle\hat{\phi}_{ij}, f\rangle_{L^2(\Omega)} = \int_\Omega \hat{\phi}_{ij}(x) f(x)\,\mathrm{d}x$, and this can mean building up a mesh and leave the features of a meshless method (especially in 3D). Instead of considering the continuous inner product, we will consider a discrete inner product on prescribed points of Ω,

$$\big\langle\hat{\phi}_{ij}, f\big\rangle_{l^2(\Omega_m)} = \sum_{k=1}^{m} \hat{\phi}_{ij}(x_k) f(x_k), \tag{22}$$

where $\Omega_m = \{x_1, \ldots, x_m\} \subset \Omega$ is the set of collocation points. Notice that the approximation obtained with this discrete l^2 inner product can be easily controlled by checking the difference between the given f and the calculated $\tilde{f}$.

MFS-D. We will call the following method the Domain Method of Fundamental Solutions (MFS-D):

- take m points in the domain ($x_1, \ldots, x_m \in \Omega_m$),
- take n points in the artificial boundary ($y_1, \ldots, y_n \in \widehat{\Gamma}$),
- take p frequencies ($\lambda_1, \ldots, \lambda_p \in \mathbb{R}$),

- define the matrix

$$\mathbf{M} = \big[\big[\Phi_{\lambda_1}(x_i - y_j)\big]_{m\times n} \ldots \big[\Phi_{\lambda_p}(x_i - y_j)\big]_{m\times n}\big]_{m\times(np)}, \tag{23}$$

- solve the $(np) \times (np)$ least-squares system

$$\mathbf{M}^*\mathbf{M}\mathbf{a} = \mathbf{M}^*\mathbf{f}, \tag{24}$$

  where $\mathbf{f} = [f(x_i)]_{m\times 1}$.

The vector solution $\mathbf{a} = [a_{j,k}]_{(np)\times 1}$ will provide an approximation of f in the form

$$\tilde{f}(x) = \sum_{k=1}^{p}\sum_{j=1}^{n} a_{j,k}\Phi_{\lambda_k}(x - y_j). \tag{25}$$

Remarks. We took here the simpler approach, and there are many minor variants that could be considered:

1. One might choose different $y_1, \ldots, y_n \in \widehat{\Gamma}$ for each frequency λ_k that one considers.

2. The theoretical density result does not specify where to consider the source points on the artificial boundary $\widehat{\Gamma}$. Thus, we might consider the location of the source points also as an unknown to the minimization problem, restricting $y_j \in \widehat{\Gamma}$ (or not). However this implies to consider a nonlinear minimization problem, instead of considering a simple least squares method. To keep the simplicity of the method, we will only consider *fixed source points.* It might be a subject of future research to investigate the possibilities to consider moving source points in a nonlinear minimization procedure (as used by Fairweather and Karageorghis, e.g., [9]).

3. Using J_0 and points inside Ω.

If one wants to approximate a real function, there is no need to consider the complex fundamental solution. One might just restrict ourselves to the real part.

In fact, in 2D, since the density result is based in $H_0^{(1)} = J_0 + \mathrm{i}Y_0$ it is clear that each one of J_0 and Y_0 will be independently used to approximate the real and the complex part. Therefore, it will be sufficient to use J_0 that does not present any singular behavior. Moreover, this allows to consider the points y_j inside Ω. For instance, in the 2D-case, it will be sufficient to consider the Bessel function J_0. Moreover, since J_0 presents nonsingularities, it is also possible to consider the points $y_1, \ldots, y_n \in \Omega$.

Theorem 3. Consider the 2D case. Let $\widetilde{\Omega} \supset \overline{\Omega}$ and I be an interval in $(-\infty, 0]$. The space

$$\mathbf{R}_{I,\Omega} = \operatorname{span}\big\{J_0\big(\sqrt{-\lambda}|x - y|\big)|_\Omega\colon\ y \in \widetilde{\Omega}, \lambda \in I\big\} \tag{26}$$

is dense in $L^2(\Omega)$.

Proof. It is a simple consequence of theorem 2. Since we may just consider J_0, theorem 2 ensures that for any admissible $\widetilde{\Gamma}$ outside Ω,

$$\mathbf{S}_{\widehat{\Gamma},I,\Omega} = \operatorname{span}\left\{ J_0\left(\sqrt{-\lambda}|x-y|\right)|_\Omega\colon\ y \in \widetilde{\Gamma},\ \lambda \in I \right\}$$

is dense in $L^2(\Omega)$. Now, since $\widetilde{\Omega} \supset \overline{\Omega}$, we just have to consider any admissible $\widetilde{\Gamma} \subset \widetilde{\Omega}\backslash\overline{\Omega}$, which is always possible – it suffices to take $\widetilde{\Gamma} = \partial\Omega^*$, where Ω^* is a domain such that $\overline{\Omega} \subset \Omega^* \subset \widetilde{\Omega}$. □

Obviously, in the 3D case, it suffices to consider

$$\mathbf{R}_{I,\Omega} = \operatorname{span}\left\{ \frac{\sin(\sqrt{-\lambda}|x-y|)}{|x-y|}\Big|_\Omega\colon\ y \in \widetilde{\Omega},\ \lambda \in I \right\}.$$

1.3. Nonhomogeneous Helmholtz equation

A similar technique can be applied to find a particular solution of the following nonhomogeneous Helmholtz-type equations,

$$\Delta u - \mu u = f$$

for any $\mu \in \mathbb{C}$. In fact, assuming that we have an approximation of f given by $\tilde{f}$ in formula (25), then from

$$\tilde{u}_{\mathrm{P}}(x) = \sum_{k=1}^{p}\sum_{j=1}^{n} \frac{a_{j,k}}{\lambda_k - \mu}\Phi_{\lambda_k}(x - y_j) \tag{27}$$

with $\lambda_1, \ldots, \lambda_p \neq \mu$, we get

$$\begin{aligned}(\Delta - \mu)\tilde{u}_{\mathrm{P}}(x) &= \sum_{k=1}^{p}\sum_{j=1}^{n} \frac{a_{j,k}}{\lambda_k - \mu}(\Delta - \mu)\Phi_{\lambda_k}(x - y_j)\\ &= \sum_{k=1}^{p}\sum_{j=1}^{n} \frac{a_{j,k}}{\lambda_k - \mu}(\lambda_k - \mu)\Phi_{\lambda_k}(x - y_j)\\ &= \sum_{k=1}^{p}\sum_{j=1}^{n} a_{j,k}\Phi_{\lambda_k}(x - y_j) = \tilde{f}(x).\end{aligned}$$

The arguments used in the proofs for the Poisson problem can also be adapted for a general $(\Delta - \mu)u = f$ equation, and we get in a simple manner a general procedure to approach the solutions of the nonhomogeneous equations associated with Helmholtz or Laplace operators.

The coefficients $a_{j,k}$ used in the approximation $\tilde{f}$ are independently rescaled in the particular solution with the factor $1/(\lambda_k - \mu)$. Therefore an approximation $\tilde{f}$ provides useful information on separate components that are rescaled accordingly to the PDE in a simple fashion.

In the case of the Dirichlet problem for the nonhomogeneous Helmholtz equation, $(\Delta - \mu)u = f$, once we have obtained the approximation of the particular solution, $\tilde{u}_{\mathrm{P}}$, we must then solve the problem

$$\begin{cases} \Delta u_{\mathrm{H}} - \mu u_{\mathrm{H}} = 0 & \text{in } \Omega, \\ \mathcal{B} u_{\mathrm{H}} = g - \mathcal{B} \tilde{u}_{\mathrm{P}} & \text{on } \partial\Omega, \end{cases} \tag{28}$$

and this can be done using the classical method of fundamental solutions on the boundary (MFS-B), with the same (or other points) on the artificial boundary $\widehat{\Gamma}$. We must keep in mind that the condition on the boundary is now given with an approximated $\tilde{u}_{\mathrm{P}}$ and a poor approximation of u_{P} might carry more significant errors to $\tilde{u}_{\mathrm{H}}$,

$$\|u_{\mathrm{H}} - \tilde{u}_{\mathrm{H}}\|_{H^1(\Omega)} \leqslant c \big\| g - u_{\mathrm{P}} - (g - \tilde{u}_{\mathrm{P}}) \big\|_{H^{1/2}(\partial\Omega)} \leqslant C \|u_{\mathrm{P}} - \tilde{u}_{\mathrm{P}}\|_{H^1(\Omega)}.$$

We recall that the solution obtained by the MFS-B will now be written in the form

$$\tilde{u}_{\mathrm{H}}(x) = \sum_{j=1}^{m} a_j \Phi_\mu(x - y_j).$$

In the case of the Helmholtz equation, it has been proposed to use only the nonsingular part of $H_0^{(1)}$, given by the Bessel function J_0, which allows us to consider the points $y_j \in \partial\Omega$, called boundary knot method (BKM) [6]. The use of the space $\mathbf{R}_{I,\Omega}$ inside the domain Ω, might also be seen as an extension of the BKM to approach nonhomogeneous problems.

2. Numerical results

Throughout this section, we denote as usual $B(0, r) = \{(x, y) : x^2 + y^2 \leqslant r^2\}$ and $\partial B(0, r) = \{(x, y) : x^2 + y^2 = r^2\}$.

2.1. Example 1

Poisson equation

First, we test our proposed approach for the Poisson problem where the known solution is $u_1(x, y) = xy \sin(xy)$ and the function on the right-hand side is given by

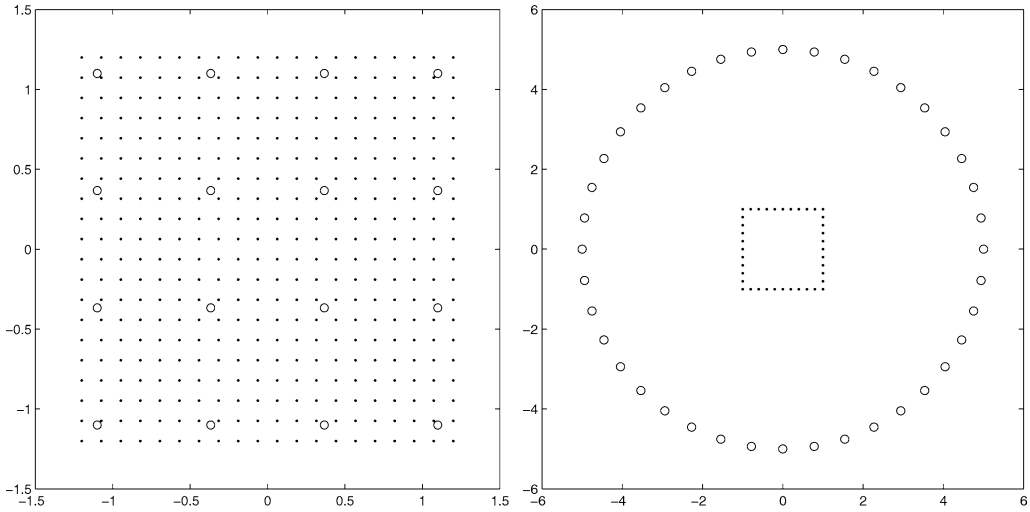
$$f_1(x, y) = 2\big(x^2 + y^2\big) \cos(xy) - \big(x^3 y + x y^3\big) \sin(xy)$$

on the square $[-1, 1]^2 = \Omega$.

In figure 1, we show the distribution of the collocation and source points:

(i) Points in a domain $W \supset \Omega$ (we considered $W = [-1.2, 1.2]^2$ with $20^2 = 400$ points).

(ii) Source points (we took 16 points in the square $[-1.1, 1.1]^2$).



(a) (b)

Figure 1. Source points (circle) and collocation points (dot) using MFS-D (a) and MFS-B (b).

(iii) List of frequencies (we took $\{-1, -4, -9, -16, -25, -36, -49, -64\}$) are what we need to obtain the approximate function $\tilde{f}_1$ and approximate particular solution $\tilde{u}_P$. Note that we consider the domain W larger than Ω to produce a better approximation of f_1 in Ω.

We also visualize the other points used to obtain $\tilde{u}_H$ and therefore $\tilde{u}_1 = \tilde{u}_H + \tilde{u}_P$:

(iv) Points on the square boundary $\partial\Omega$ (we considered 40 points).

(v) Points on the fictitious boundary $\partial\widetilde{\Omega}$ (we took also 40 points on the circle $\partial B(0, 5)$).

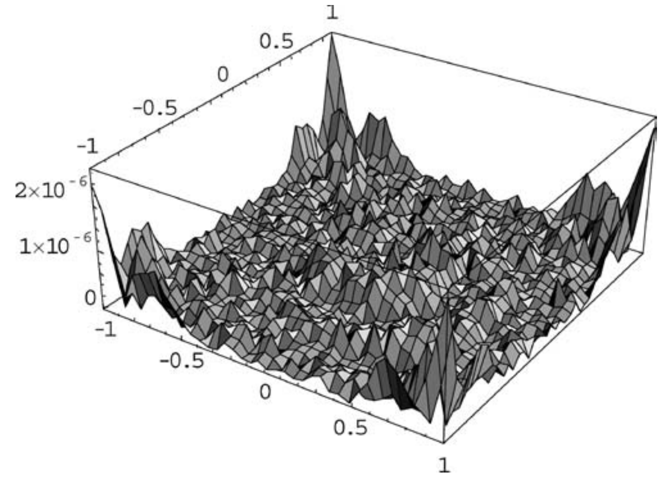
In figure 2 we plot the results obtained by comparing the exact function and the function approximation by the MFS-D in Ω. For the given data, we obtain the following results

$$\|f_1 - \tilde{f}_1\|_\infty < 3 \times 10^{-6} \quad \text{and} \quad \|u_1 - \tilde{u}_1\|_\infty < 7 \times 10^{-7}$$

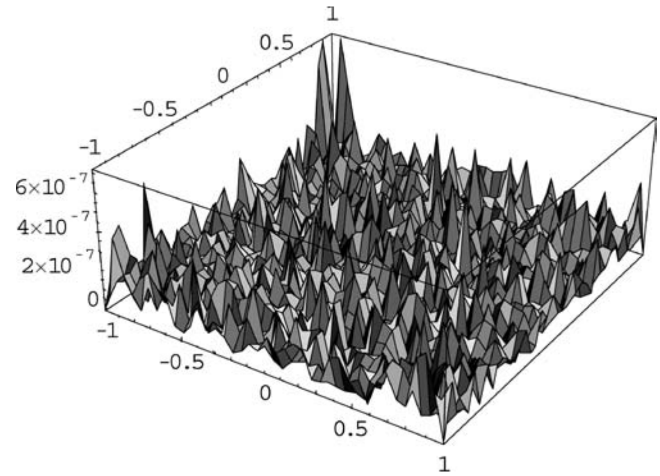
In the above case we considered source points inside the domain, for the approximation of the particular solution, and this does not present a singularity problem because we have used the nonsingular part of the Bessel function J_0 as the basis. In fact, we could have also placed the 16 source points on a circle $\partial B(0, 4)$ outside the domain (see figure 3). The absolute error of the function approximation $\|f_1 - \tilde{f}_1\|_\infty$ is about ten times worse than the previous case (see figure 4). We obtain similar results for the PDE solution.

Nonhomogeneous Helmholtz equation

It is clear that the same approach can also be applied to modified Helmholtz equation $(\Delta - \mu)u = f$. In fact, one may use the same frequencies as in the previous example



(a)



(b)

Figure 2. Absolute error for f_1 (a) and u_1 (b) using interior points sources.

to approach right-hand side except $\lambda_k \neq \mu$. Therefore, with the numerical scheme used in the previous example, we could still be able to derive a particular solution for a non-homogeneous Helmholtz equation.

We now take another example, where the exact solution is given by

$$u_2(x, y) = \frac{1}{1 + x^4 + y^2}$$

and therefore the RHS is given by

$$f_2(x, y) = \frac{-\mu}{1 + x^4 + y^2} - \frac{2 + 2x^4 - 20x^6 - 6y^2 + 12x^2(1 + y^2)}{(1 + x^4 + y^2)^3}.$$

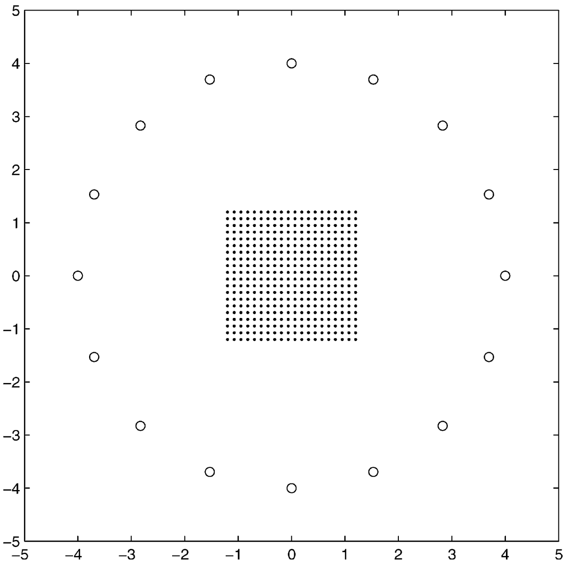


Figure 3. The source points are placed outside the domain using MFS-D.

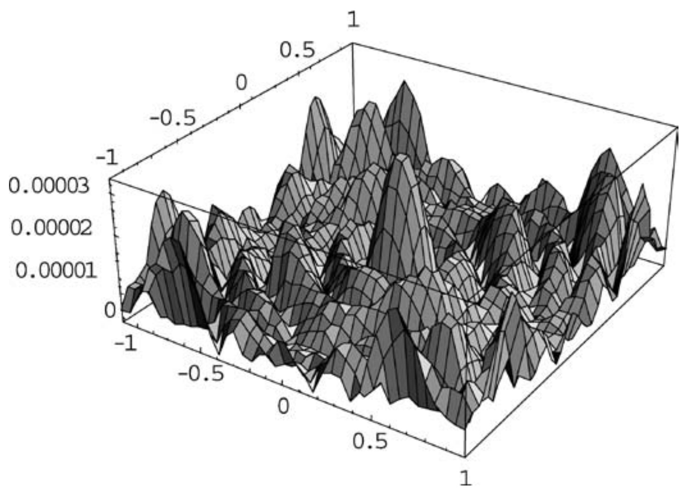
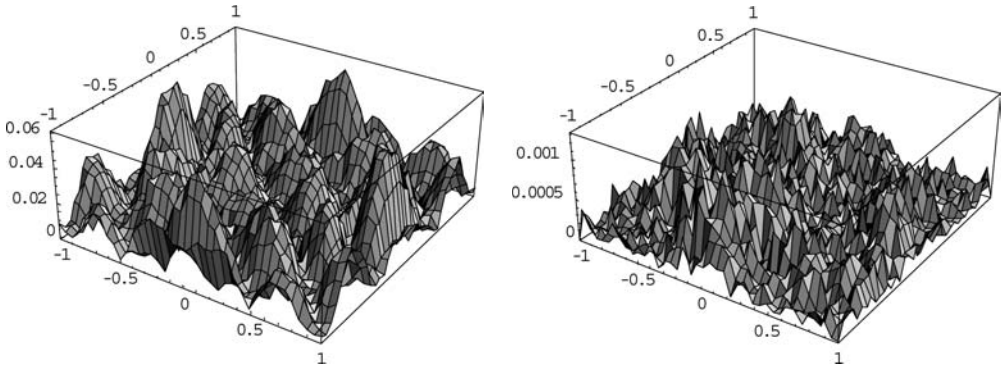


Figure 4. Profile of absolute error of f_1.

Now consider the Helmholtz equation with $\mu = -18$. Using the previous data with interior point sources, we obtain about 0.5% relative errors in the function approximation and about 0.1% relative errors in the approximation of the solution (see figure 5 where we plot the absolute error of f_2 and u_2).

These are also good results, and we emphasize that, in this smooth example, we obtained the same order of relative errors even for higher frequencies (we tested up to $\mu = -10000$ with the same data). The computation time is quite small (less than 1 second on a Pentium 4 running a Fortran code).



(a) (b)

Figure 5. Absolute error of f_2 (a) and u_2 (b).

2.2. Example 2

We now test the approximation for nonrectangular domains.

Poisson equation

We consider the exact solution of the given Poisson equation to be $u_3(x, y) = \sin(y - x^2)$ and therefore the function on the right-hand side is

$$f_3(x, y) = -2\cos\bigl(x^2 - y\bigr) + \bigl(1 + 4x^2\bigr)\sin\bigl(x^2 - y\bigr).$$

The domain is the disk $B(0, 3/2) = \Omega$. To evaluate the approximate particular solution, we choose the extended domain $W = [-2, 2]^2$ with 20^2 collocation points and 25 source points on the fictitious boundary $\partial\widetilde{\Omega} = \partial B(0, 3)$. To evaluate the approximate homogeneous solution, we choose 80 collocation points on the physical boundary $\partial\Omega$, and 25 source points on the fictitious boundary $\partial\widetilde{\Omega} = \partial B(0, 3.25)$. We added a few more frequencies such as -81 and -100 to the list from the last example. In general, the frequencies can be chosen in an arbitrary fashion. However, the effect of a certain choice of frequencies on the accuracy is not completely clear, as it happens with the choice of source points in the MFS. The distribution of source and collocation points in this example is shown in figure 6.

In figure 7, we plot the absolute errors of f_3 and u_3. The results using the MFS-D is quite good, with less than 0.5% relative error in approximating both f_3 and u_3.

Nonhomogeneous Helmholtz equation

We now consider the case of a simple connected domain Ω where

$$\partial\Omega = \left\{\bigl(x(t), y(t)\bigr)\colon\ x(t) = \frac{6}{4}\cos(t),\ y(t) = \frac{1}{4}\bigl(6 - \cos(3t)\bigr)\sin(t), t \in (0, 2\pi]\right\}.$$

Let W be an extended rectangular domain of Ω. We choose 25 source points and 400 collocation points in W for the evaluation of the approximate particular solution. The

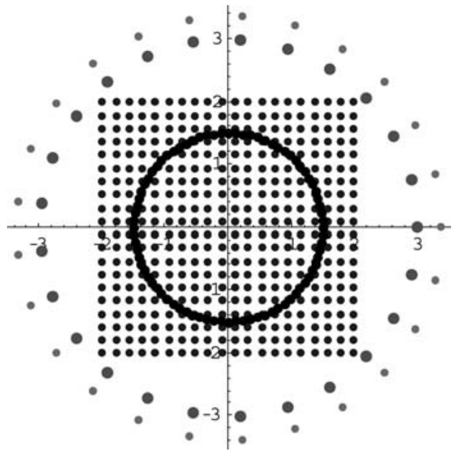
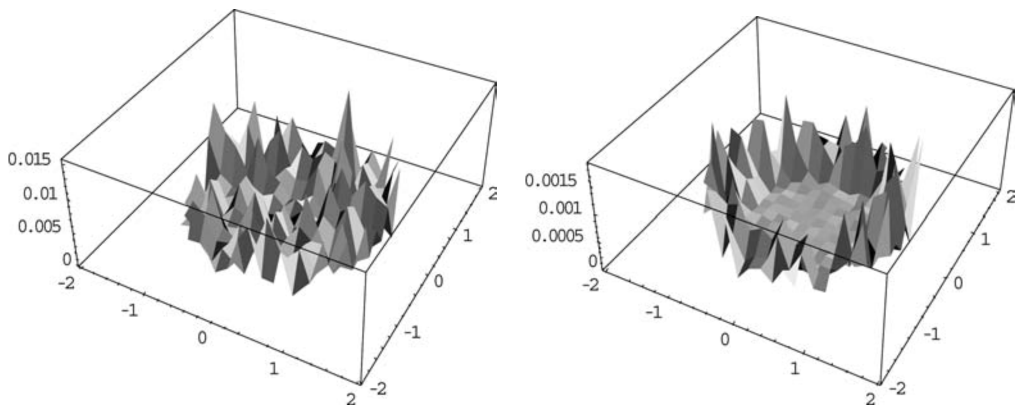


Figure 6. Distribution of collocation and source points.



(a) (b)

Figure 7. Absolute error of f_3 (a) and u_3 (b).

list of frequencies is the same as before. To approximate the homogeneous solution, we choose 60 collocation points on $\partial\Omega$ and 30 source points on the fictitious boundary $\partial\widetilde{\Omega}$ ($\widetilde{\Omega}$ has the same parametrization as Ω with an expansion factor of 2). The profile of the distribution of collocation and source points is shown in figure 8.

We consider the same exact solution as the function $u_4 = u_3$ and the only difference, by taking $\mu = -7$, is in the right-hand side

$$f_4(x, y) = -2\cos(x^2 - y) - (6 - 4x^2)\sin(x^2 - y).$$

The distribution of these two sets of collocation and source points are shown in figure 8. The absolute error of f_4 and u_4 using the MSF-D are shown in figure 9 in which the relative errors for f_4 and u_4 are less than 0.2% and 0.03%, respectively.

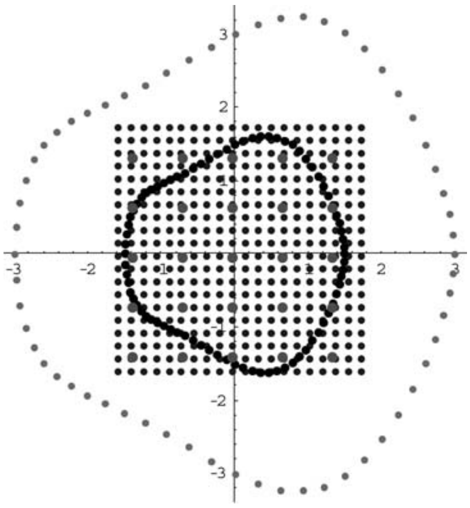


Figure 8. distribution of collocation and source points.

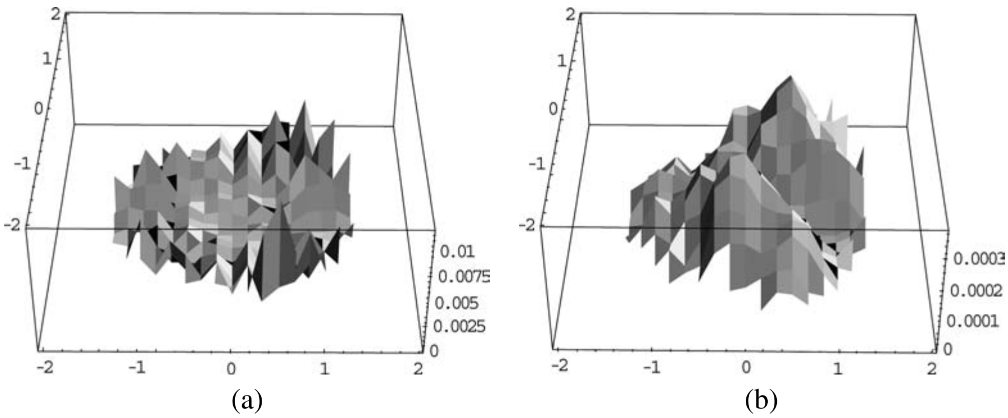


Figure 9. Absolute error of f_4 (a) and u_4 (b).

3. Conclusions

In the past the MFS has been used to solve homogeneous equations. In this paper we proposed to use an extension of the MFS to approximate the forcing term directly. As a result, an approximation for the particular solution of the nonhomogeneous solution can be derived easily. The preliminary numerical results show that this extended MFS is a valuable simple method for solving certain nonhomogeneous elliptic problems. In this method the coefficients obtained for each test frequency are directly and independently rescaled giving a simple approximation to a particular solution. The application of this approach to multiscale problems and other differential equations (including nonlinear and time-dependent problems) is under investigation.

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