Advances in Computational Mathematics (2005) 23: 55–71

A matrix decomposition MFS algorithm for axisymmetric biharmonic problems

Graeme Fairweather [a], Andreas Karageorghis [b] and Yiorgos-Sokratis Smyrlis [b]

[a] *Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, CO 80401, USA*
E-mail: gfairwea@mines.edu

[b] *Department of Mathematics and Statistics, University of Cyprus/Πανεπιστήμιο Κύπρου, P.O. Box 20537, 1678 Nicosia/Λευκωσία, Cyprus/Κύπρος*

Received 24 February 2003; accepted 10 October 2003
Communicated by Z. Wu and B.Y.C. Hon

We consider the approximate solution of axisymmetric biharmonic problems using a boundary-type meshless method, the Method of Fundamental Solutions (MFS) with fixed singularities and boundary collocation. For such problems, the coefficient matrix of the linear system defining the approximate solution has a block circulant structure. This structure is exploited to formulate a matrix decomposition method employing fast Fourier transforms for the efficient solution of the system. The results of several numerical examples are presented.

Keywords: method of fundamental solutions, axisymmetric domains, biharmonic equation

AMS subject classification: 65N38, 65F30, 65T50, 65Y99

1. Introduction

In this paper, we investigate the application of the Method of Fundamental Solutions (MFS) to axisymmetric biharmonic problems. In the MFS, the approximate solution is expressed as a linear combination of fundamental solutions of the governing differential equation, with singularities placed outside the domain of the problem. The coefficients of the fundamental solutions are determined so that the MFS solution approximates the boundary conditions either by collocation (*boundary collocation*) or by a least squares fit of the boundary data. The locations of the singularities are either *fixed*, that is, preassigned, or *moving* in which case their locations are determined during the solution process which is then nonlinear. A description of various versions of the MFS and related methods as well as a wide range of applications is given in [7,8,10].

The MFS with fixed or moving singularities has been used for the solution of various second order axisymmetric problems [13–15,19]. In these applications, the MFS was applied to the axisymmetric version of the governing equations, for which the fundamental solutions involve complete elliptic integrals and are rather complicated. Further, when the boundary conditions of the problem under consideration are not axisymmetric, this approach requires the solution of a sequence of problems in order to evaluate a finite Fourier sum. A direct MFS approach for the solution of axisymmetric harmonic problems which avoids these complications is discussed in [24]. It involves fixed singularities and boundary collocation, and leads to a linear system whose coefficient matrix has a block circulant structure. This structure is exploited via a matrix decomposition algorithm involving fast Fourier transforms (FFTs) (see, for example, [4]) to obtain an efficient method for determining the approximate solution. In the present study, these ideas are extended to the solution of axisymmetric biharmonic problems.

In three-dimensional elasticity, the so-called Galerkin vector satisfies the three-dimensional biharmonic equation (see [5,11]). This equation also arises in the solution of three-dimensional viscous incompressible flows. In such problems, one can transform the coupled biharmonic system for the vector potential into a sequence of scalar biharmonic problems as was done in [22], where finite element methods were considered for their solution. The numerical solution of three-dimensional biharmonic problems has been the subject of other studies, for example, on finite differences [1,20], finite element methods [16,21], and spectral methods [3].

This paper is organized as follows. In section 2, we describe the MFS for the axisymmetric biharmonic problem, and set up the system of linear equations to which it leads. In section 3, we formulate a matrix decomposition method employing FFTs for the solution of the system which exploits the block circulant structure of its coefficient matrix. In section 4, we present examples of three axisymmetric regions and, for each, specify the choice of singularities and boundary collocation points used in the MFS. In section 5, we describe how the singularities may be rotated, a process which often improves the accuracy of the MFS procedure. The results of numerical experiments are described in section 6 and, in the final section, section 7, we state some concluding remarks.

2. MFS formulation

We consider the boundary value problem

$$\begin{aligned} &\Delta^2 u(P) = 0, && && P \in \Omega \subset \mathbb{R}^3, \\ &u(P) = f(P), && \frac{\partial u}{\partial n_P}(P) = g(P), && P \in \partial\Omega, \end{aligned} \tag{2.1}$$

where Δ denotes the Laplace operator, $\partial/\partial n_P$ is the outward normal derivative at P, and f and g are sufficiently smooth given functions. The region Ω is axisymmetric, that is, it is formed by rotating a region $\Omega' \in \mathbb{R}^2$ about the z-axis. The boundaries of Ω and Ω' are denoted by $\partial\Omega$ and $\partial\Omega'$, respectively. The boundary $\partial\Omega'$ is also assumed to be sufficiently smooth.

As in the biharmonic MFS considered in [12], which is based on the simple layer potential representation of [11,17], we approximate the solution u of (2.1) by

$$u_{MN}(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{Q}; P) = \sum_{m=1}^{M} \sum_{n=1}^{N} \left[c_{m,n} k_1(P, Q_{m,n}) + d_{m,n} k_2(P, Q_{m,n})\right], \quad P \in \overline{\Omega}, \tag{2.2}$$

where

$$\boldsymbol{c} = (c_{11}, c_{12}, \ldots, c_{1N}, \ldots, c_{M1}, \ldots, c_{MN})^{\mathrm{T}},$$
$$\boldsymbol{d} = (d_{11}, d_{12}, \ldots, d_{1N}, \ldots, d_{M1}, \ldots, d_{MN})^{\mathrm{T}},$$

and $\boldsymbol{Q}$ is a $3MN$-vector containing the coordinates of the singularities $\{Q_{m,n}\}_{m=1,n=1}^{M,N}$, which lie outside $\overline{\Omega}$, the closure of Ω. The function $k_1(P, Q)$ is the fundamental solution of Laplace's equation in $\mathbb{R}^3$ given by

$$k_1(P, Q) = \frac{1}{4\pi|P - Q|}, \tag{2.3}$$

where $|P - Q|$ denotes the distance between the points P and Q, and

$$k_2(P, Q) = \frac{1}{8\pi}|P - Q|, \tag{2.4}$$

the fundamental solution of the biharmonic equation in $\mathbb{R}^3$ given in [5]. In the axisymmetric case considered in the present paper, the MN singularities $\{Q_{m,n}\}_{m=1,n=1}^{M,N}$ are fixed on the boundary $\partial\widetilde{\Omega}$ of a solid $\widetilde{\Omega}$ surrounding Ω which is generated by rotating about the z-axis a planar domain $\widetilde{\Omega}'$ which is similar to Ω'. These singularities are chosen in the following way. Let $\{Q_n\}_{n=1}^{N}$ be N points on the boundary $\partial\widetilde{\Omega}'$ of $\widetilde{\Omega}'$, with $Q_n = (r_{Q_n}, z_{Q_n})$ in polar coordinates. Then we take $Q_{m,n} = (x_{Q_{m,n}}, y_{Q_{m,n}}, z_{Q_{m,n}})$, where

$$x_{Q_{m,n}} = r_{Q_n} \cos\varphi_m, \qquad y_{Q_{m,n}} = r_{Q_n} \sin\varphi_m, \qquad z_{Q_{m,n}} = z_{Q_n}, \tag{2.5}$$

with

$$\varphi_m = \frac{2(m-1)\pi}{M}, \quad m = 1, \ldots, M. \tag{2.6}$$

We choose a set of MN collocation points $\{P_{i,j}\}_{i=1,j=1}^{M,N}$ on $\partial\Omega$ by first specifying N points $\{P_j\}_{j=1}^{N}$ on $\partial\Omega'$ with $P_j = (r_{P_j}, z_{P_j})$. Then $P_{i,j} = (x_{P_{i,j}}, y_{P_{i,j}}, z_{P_{i,j}})$, where

$$x_{P_{i,j}} = r_{P_j} \cos\varphi_i, \qquad y_{P_{i,j}} = r_{P_j} \sin\varphi_i, \qquad z_{P_{i,j}} = z_{P_j}.$$

In the MFS, the coefficient vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ are determined so that the boundary conditions are satisfied at the collocation points $\{P_{i,j}\}_{i=1,j=1}^{M,N}$:

$$\begin{aligned} &u_N(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{Q}; P_{i,j}) = f(P_{i,j}), \\ &\frac{\partial u_N}{\partial n_P}(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{Q}; P_{i,j}) = g(P_{i,j}), \quad i = 1, \ldots, M, \ j = 1, \ldots, N. \end{aligned} \tag{2.7}$$

Since the problem is axisymmetric, these equations yield a $2MN \times 2MN$ linear system of the form

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\left(\frac{\boldsymbol{c}}{\boldsymbol{d}}\right) = \left(\frac{\boldsymbol{f}}{\boldsymbol{g}}\right), \tag{2.8}$$

where the $MN \times MN$ matrices A, B, C, D have a *block circulant* structure (cf. [6,24]). Specifically,

$$A = \begin{pmatrix} A_1 & A_2 & \cdots & A_M \\ A_M & A_1 & \cdots & A_{M-1} \\ \vdots & \vdots & & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{pmatrix} \equiv \operatorname{circ}(A_1, A_2, \ldots, A_M), \tag{2.9}$$

where the $N \times N$ submatrices $A_\ell = ((A_\ell)_{j,n})$, $\ell = 1, \ldots, M$, are defined by

$$(A_\ell)_{j,n} = \frac{1}{4\pi|P_{1,j} - Q_{\ell,n}|}, \quad j, n = 1, \ldots, N, \tag{2.10}$$

and

$$\begin{aligned} &B = \operatorname{circ}(B_1, B_2, \ldots, B_M), \qquad C = \operatorname{circ}(C_1, C_2, \ldots, C_M), \\ &D = \operatorname{circ}(D_1, D_2, \ldots, D_M), \end{aligned}$$

where, for $\ell = 1, \ldots, M$, $j, n = 1, \ldots, N$,

$$\begin{aligned} (B_\ell)_{j,n} &= \frac{1}{8\pi}|P_{1,j} - Q_{\ell,n}|, \\ (C_\ell)_{j,n} &= \frac{1}{4\pi}\frac{\partial}{\partial n}\left[\frac{1}{|P_{1,j} - Q_{\ell,n}|}\right] \\ &= -\frac{1}{4\pi}\left[\frac{x_{P_{1,j}} - x_{Q_{\ell,n}}}{|P_{1,j} - Q_{\ell,n}|^3}n_x + \frac{y_{P_{1,j}} - y_{Q_{\ell,n}}}{|P_{1,j} - Q_{\ell,n}|^3}n_y + \frac{z_{P_{1,j}} - z_{Q_{\ell,n}}}{|P_{1,j} - Q_{\ell,n}|^3}n_z\right], \\ (D_\ell)_{j,n} &= \frac{1}{8\pi}\left[\frac{\partial}{\partial n}|P_{1,j} - Q_{\ell,n}|\right] \\ &= \frac{1}{8\pi}\left[\frac{x_{P_{1,j}} - x_{Q_{\ell,n}}}{|P_{1,j} - Q_{\ell,n}|}n_x + \frac{y_{P_{1,j}} - y_{Q_{\ell,n}}}{|P_{1,j} - Q_{\ell,n}|}n_y + \frac{z_{P_{1,j}} - z_{Q_{\ell,n}}}{|P_{1,j} - Q_{\ell,n}|}n_z\right], \end{aligned}$$

where n_x, n_y and n_z denote the components of the outward normal vector to $\partial\Omega$ in the x, y and z directions, respectively, at the point $P_{1,j}$. The above equations only involve $P_{1,j}$ because of the circulant symmetry of the problem.

3. Matrix decomposition algorithm

Let $\mathcal{P}$ denote the $M \times M$ permutation matrix $\mathcal{P} = \operatorname{circ}(0, 1, 0, \ldots, 0)$ and $\otimes$ the matrix tensor product. Then, with $\mathcal{P}^0 = I_M$,

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} \sum_{k=1}^M \mathcal{P}^{k-1} \otimes A_k & \sum_{k=1}^M \mathcal{P}^{k-1} \otimes B_k \\ \hline \sum_{k=1}^M \mathcal{P}^{k-1} \otimes C_k & \sum_{k=1}^M \mathcal{P}^{k-1} \otimes D_k \end{array}\right)$$
$$= \sum_{k=1}^M \left(\begin{array}{c|c} \mathcal{P}^{k-1} \otimes A_k & \mathcal{P}^{k-1} \otimes B_k \\ \hline \mathcal{P}^{k-1} \otimes C_k & \mathcal{P}^{k-1} \otimes D_k \end{array}\right)$$
$$= \sum_{k=1}^M \Bigg[\left(\begin{array}{c|c} \mathcal{P}^{k-1} & 0 \\ \hline 0 & 0 \end{array}\right) \otimes A_k + \left(\begin{array}{c|c} 0 & \mathcal{P}^{k-1} \\ \hline 0 & 0 \end{array}\right) \otimes B_k$$
$$+ \left(\begin{array}{c|c} 0 & 0 \\ \hline \mathcal{P}^{k-1} & 0 \end{array}\right) \otimes C_k + \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathcal{P}^{k-1} \end{array}\right) \otimes D_k \Bigg]. \tag{3.1}$$

Following [6], we denote by U the unitary $M \times M$ Fourier matrix which is the conjugate of the matrix

$$U^* = \frac{1}{M^{1/2}} \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{M-1} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{2(M-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{M-1} & \omega^{2(M-1)} & \ldots & \omega^{(M-1)(M-1)} \end{pmatrix},$$

where $\omega = \mathrm{e}^{2\pi\iota/M}$ with $\iota = \sqrt{-1}$. Then

$$\mathcal{P}^{k-1} U^* = U^* E^{k-1}, \quad k = 1, \ldots, M, \tag{3.2}$$

where $E = \operatorname{diag}(e_1, e_2, \ldots, e_M)$ with $e_j = \omega^{j-1}$. Now, premultiplying system (2.8) by

$$\left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array}\right) \otimes I_N$$

and judiciously introducing the $MN \times MN$ identity matrix yields

$$\left[\left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array}\right) \otimes I_N \right] \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \left[\left(\begin{array}{c|c} U^* & 0 \\ \hline 0 & U^* \end{array}\right) \otimes I_N \right] \left[\left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array}\right) \otimes I_N \right] \left(\frac{\boldsymbol{c}}{\boldsymbol{d}}\right)$$
$$= \left[\left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array}\right) \otimes I_N \right] \left(\frac{\boldsymbol{f}}{\boldsymbol{g}}\right). \tag{3.3}$$

Using (3.1), (3.2) and the properties of $\otimes$ (i.e., $(S_1 \otimes S_2)(T_1 \otimes T_2) = (S_1T_1) \otimes (S_2T_2)$), (3.3) becomes

$$\sum_{k=1}^{M}\left[\left(\begin{array}{c|c} E^{k-1} & 0 \\ \hline 0 & 0 \end{array}\right) \otimes A_k + \left(\begin{array}{c|c} 0 & E^{k-1} \\ \hline 0 & 0 \end{array}\right) \otimes B_k \right.$$
$$\left. + \left(\begin{array}{c|c} 0 & 0 \\ \hline E^{k-1} & 0 \end{array}\right) \otimes C_k + \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & E^{k-1} \end{array}\right) \otimes D_k \right]\left(\frac{\tilde{\boldsymbol{c}}}{\tilde{\boldsymbol{d}}}\right) = \left(\frac{\widetilde{\boldsymbol{f}}}{\tilde{\boldsymbol{g}}}\right), \tag{3.4}$$

where

$$\left(\frac{\tilde{\boldsymbol{c}}}{\tilde{\boldsymbol{d}}}\right) = \left[\left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array}\right) \otimes I_N\right]\left(\frac{\boldsymbol{c}}{\boldsymbol{d}}\right), \quad \left(\frac{\widetilde{\boldsymbol{f}}}{\tilde{\boldsymbol{g}}}\right) = \left[\left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array}\right) \otimes I_N\right]\left(\frac{\boldsymbol{f}}{\boldsymbol{g}}\right).$$

System (3.4) can then be written as

$$\sum_{k=1}^{M}\left(\begin{array}{c|c} E^{k-1} \otimes A_k & E^{k-1} \otimes B_k \\ \hline E^{k-1} \otimes C_k & E^{k-1} \otimes D_k \end{array}\right)\left(\frac{\tilde{\boldsymbol{c}}}{\tilde{\boldsymbol{d}}}\right) = \left(\frac{\widetilde{\boldsymbol{f}}}{\tilde{\boldsymbol{g}}}\right),$$

or

$$\left(\begin{array}{c|c} \sum_{k=1}^{M} E^{k-1} \otimes A_k & \sum_{k=1}^{M} E^{k-1} \otimes B_k \\ \hline \sum_{k=1}^{M} E^{k-1} \otimes C_k & \sum_{k=1}^{M} E^{k-1} \otimes D_k \end{array}\right)\left(\frac{\tilde{\boldsymbol{c}}}{\tilde{\boldsymbol{d}}}\right) = \left(\frac{\widetilde{\boldsymbol{f}}}{\tilde{\boldsymbol{g}}}\right).$$

Note that the coefficient matrix consists of four diagonal blocks and therefore the system reduces to the M independent $2N \times 2N$ linear systems

$$\left(\begin{array}{c|c} \widetilde{A}_m & \widetilde{B}_m \\ \hline \widetilde{C}_m & \widetilde{D}_m \end{array}\right)\left(\frac{\tilde{\boldsymbol{c}}_m}{\tilde{\boldsymbol{d}}_m}\right) = \left(\frac{\widetilde{\boldsymbol{f}}_m}{\tilde{\boldsymbol{g}}_m}\right), \quad m = 1, 2, \ldots, M, \tag{3.5}$$

where

$$\widetilde{A}_m = \sum_{i=1}^{M} e_m^{i-1} A_i, \tag{3.6}$$

with $\widetilde{B}_m$, $\widetilde{C}_m$ and $\widetilde{D}_m$ defined similarly, and

$$\tilde{\boldsymbol{c}}_m = [\tilde{c}_{m1}, \tilde{c}_{m2}, \ldots, \tilde{c}_{mN}]^{\mathrm{T}},$$

with $\tilde{\boldsymbol{d}}_m$, $\widetilde{\boldsymbol{f}}_m$ and $\tilde{\boldsymbol{g}}_m$ defined similarly. We thus have the following matrix decomposition algorithm for solving (2.8):

Algorithm.

Step 1. Compute $\widetilde{\boldsymbol{f}} = (U \otimes I_N)\boldsymbol{f}$ and $\tilde{\boldsymbol{g}} = (U \otimes I_N)\boldsymbol{g}$.

Step 2. For $m = 1, \ldots, M$, construct the matrices $\widetilde{A}_m$, $\widetilde{B}_m$, $\widetilde{C}_m$, $\widetilde{D}_m$, as in (3.6).

Step 3. `Solve the systems (3.5).`

Step 4. `Compute` $\boldsymbol{c} = (U^* \otimes I_N)\tilde{\boldsymbol{c}}$ `and` $\boldsymbol{d} = (U^* \otimes I_N)\tilde{\boldsymbol{d}}$.

Remarks.

(i) In *step 1*, because of the form of the matrix U, each of the two matrix–vector multiplications is equivalent to performing N FFTs of length M. This step can be done at a cost of $O(NM \log M)$ operations using the FFT routine `C06FPF` from the NAG Library [18]. Similarly, in *step 4*, because of the form of the matrix U^*, each of the two matrix–vector multiplications can be carried out via inverse FFTs at a cost of $O(NM \log M)$ operations using the NAG routine `C06FRF`.

(ii) In *step 2*, to compute an entry of any of the $N \times N$ matrices $\widetilde{A}_m$, $\widetilde{B}_m$, $\widetilde{C}_m$, $\widetilde{D}_m$, $m = 1, 2, \ldots, M$, requires an M-point inverse FFT, because $e_m = \omega^{m-1}$. The total cost of computing these matrices is then $O(N^2 M \log M)$ operations using `C06FRF`.

(iii) *Step 3* involves the solution of M complex linear systems of order $2N$. This is done at a cost of $O(MN^3)$ operations using the NAG routine `F04ADF`, which employs Gauss elimination with partial pivoting.

4. Examples of axisymmetric solids

4.1. Case I: spherical domains

Consider the case where $\Omega \subset \mathbb{R}^3$ is the sphere of radius ϱ:

$$\Omega = \left\{(x, y, z) \in \mathbb{R}^3\colon \sqrt{x^2 + y^2 + z^2} < \varrho\right\}. \tag{4.1}$$

The singularities $\{Q_{m,n}\}_{m=1,n=1}^{M,N}$ are fixed on the boundary $\partial\widetilde{\Omega}$ of the sphere

$$\widetilde{\Omega} = \left\{(x, y, z) \in \mathbb{R}^3\colon \sqrt{x^2 + y^2 + z^2} < R\right\},$$

where $R > \varrho$, and $Q_{m,n} = (x_{Q_{m,n}}, y_{Q_{m,n}}, z_{Q_{m,n}})$ with

$$x_{Q_{m,n}} = R \sin\vartheta_n \cos\varphi_m, \qquad y_{Q_{m,n}} = R \sin\vartheta_n \sin\varphi_m, \qquad z_{Q_{m,n}} = R \cos\vartheta_n,$$

where φ_m is given in (2.6) and

$$\vartheta_n = \frac{n\pi}{N+1}, \quad j = 1, \ldots, N.$$

The MN collocation points $\{P_{i,j}\}_{i=1,j=1}^{M,N}$ on $\partial\Omega$ are given by

$$x_{P_{i,j}} = \varrho \sin\vartheta_j \cos\varphi_i, \qquad y_{P_{i,j}} = \varrho \sin\vartheta_j \sin\varphi_i, \qquad z_{P_{i,j}} = \varrho \cos\vartheta_j.$$

Note that we avoid the points corresponding to $\vartheta_j = 0$ and $\vartheta_j = \pi$ as they remain invariant under rotation in the φ-direction and would lead to singular matrices.

4.2. Case II: cylindrical domains

For the cylindrical domain

$$\Omega = \{(x, y, z) \in \mathbb{R}^3\colon\ \sqrt{x^2 + y^2} < \varrho,\ -h < z < h\}, \tag{4.2}$$

the MN singularities $\{Q_{m,n}\}_{m=1,n=1}^{M,N}$ are given by

$$x_{Q_{m,n}} = r_{Q_n} \cos\varphi_m, \qquad y_{Q_{m,n}} = r_{Q_n} \sin\varphi_m, \qquad z_{Q_{m,n}} = z_{Q_n},$$

with φ_m as in (2.6) and (r_{Q_n}, z_{Q_n}), $n = 1, \ldots, N$, the polar coordinates of N points on the boundary of the rectangle $[0, R] \times [-H, H]$ with $R > \varrho$ and $H > h$. The collocation points $\{P_{i,j}\}_{i=1,j=1}^{M,N}$ are taken to be

$$x_{P_{i,j}} = r_{P_j} \cos\varphi_i, \qquad y_{P_{i,j}} = r_{P_j} \sin\varphi_i, \qquad z_{P_{i,j}} = z_{P_j},$$

with (r_{P_j}, z_{P_j}), $j = 1, \ldots, N$, the polar coordinates of N points on the boundary of the rectangle $[0, \varrho] \times [-h, h]$.

4.3. Case III: toroidal domains

Consider the torus of radii ϱ_1, ϱ_2 with $\varrho_2 > \varrho_1$:

$$\Omega = \{(x, y, z) \in \mathbb{R}^3\colon\ \left(\sqrt{x^2 + y^2} - \varrho_2\right)^2 + z^2 < \varrho_1^2\}, \tag{4.3}$$

whose boundary $\partial\Omega$ is given by the parametric equations

$$x = \varrho_2 \cos\varphi + \varrho_1 \cos\varphi \cos\theta, \qquad y = \varrho_2 \sin\varphi + \varrho_1 \sin\varphi \cos\theta, \qquad z = \varrho_1 \sin\theta,$$
$$0 \leqslant \varphi, \theta \leqslant 2\pi. \tag{4.4}$$

In this case, $\widetilde{\Omega}$ is a torus which is similar to Ω, and has boundary $\partial\widetilde{\Omega}$ defined by the parametric equations

$$x = \rho_2 \cos\varphi + R_1 \cos\varphi \cos\theta, \qquad y = \rho_2 \sin\varphi + R_1 \sin\varphi \cos\theta, \qquad z = R_1 \sin\theta,$$
$$\varrho_1 < R_1 < \varrho_2, \quad 0 \leqslant \varphi, \theta \leqslant 2\pi. \tag{4.5}$$

The singularities $\{Q_{m,n}\}_{m=1,n=1}^{M,N}$ on $\partial\widetilde{\Omega}$ have coordinates

$$x_{Q_{m,n}} = \rho_2 \cos\varphi_n + R_1 \cos\varphi_n \cos\theta_m,$$
$$y_{Q_{m,n}} = \rho_2 \sin\varphi_n + R_1 \sin\varphi_n \cos\theta_m, \qquad z_{Q_{m,n}} = R_1 \sin\theta_m,$$

where φ_n is as in (2.6) with $M = N$, and

$$\theta_m = \frac{2(m-1)\pi}{M}, \quad m = 1, \ldots, M.$$

The MN collocation points $\{P_{i,j}\}_{i=1,j=1}^{M,N}$ on $\partial\Omega$ have coordinates

$$x_{P_{i,j}} = \varrho_2 \cos\varphi_j + \varrho_1 \cos\varphi_j \cos\theta_i,$$
$$y_{P_{i,j}} = \varrho_2 \sin\varphi_j + \varrho_1 \sin\varphi_j \cos\theta_i, \quad z_{P_{i,j}} = \varrho_1 \sin\theta_i.$$

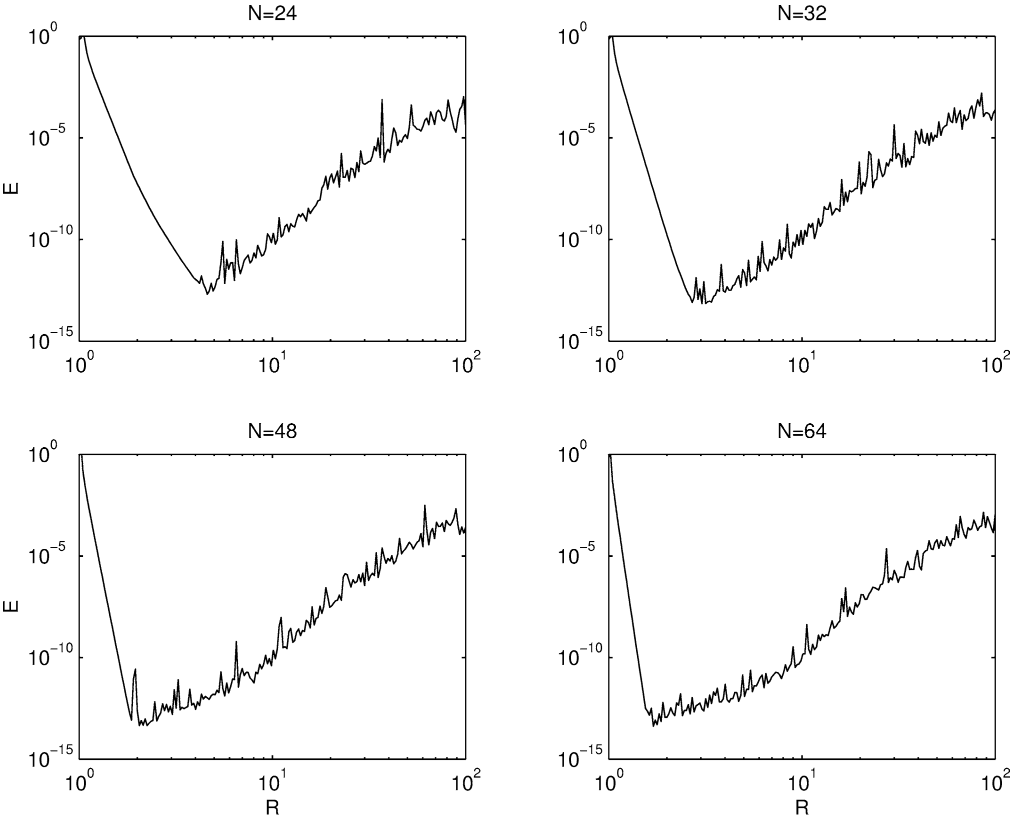

Figure 1. Results for varying distance R and various values of N in the case of the sphere for example 1.

5. Rotation of singularities

Often, improved results can be obtained by rotating the singularities in the φ direction (see [23–25]). In particular, we can place the MN singularities $\{Q_{m,n}\}_{m=1,n=1}^{M,N}$ on $\partial\widetilde{\Omega}$ by taking $Q_{m,n} = (x_{Q_{m,n}}, y_{Q_{m,n}}, z_{Q_{m,n}})$ with

$$x_{Q_{m,n}} = r_{Q_n} \cos\phi_m, \qquad y_{Q_{m,n}} = r_{Q_n} \sin\phi_m, \qquad z_{Q_{m,n}} = z_{Q_n},$$

where

$$\phi_m = \frac{2(\alpha + m - 1)\pi}{M}, \quad m = 1, \dots, M, \ \alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right). \tag{5.1}$$

With this rotation defined by α, the block circulant structure of each of the four matrices A^α, B^α, C^α and D^α corresponding to the matrices A ($\equiv A^0$), B ($\equiv B^0$), C ($\equiv C^0$) and D ($\equiv D^0$), respectively, of section 2, is preserved. In particular, we have

$$G^\alpha = \left(\begin{array}{c|c} A^\alpha & B^\alpha \\ \hline C^\alpha & D^\alpha \end{array}\right),$$

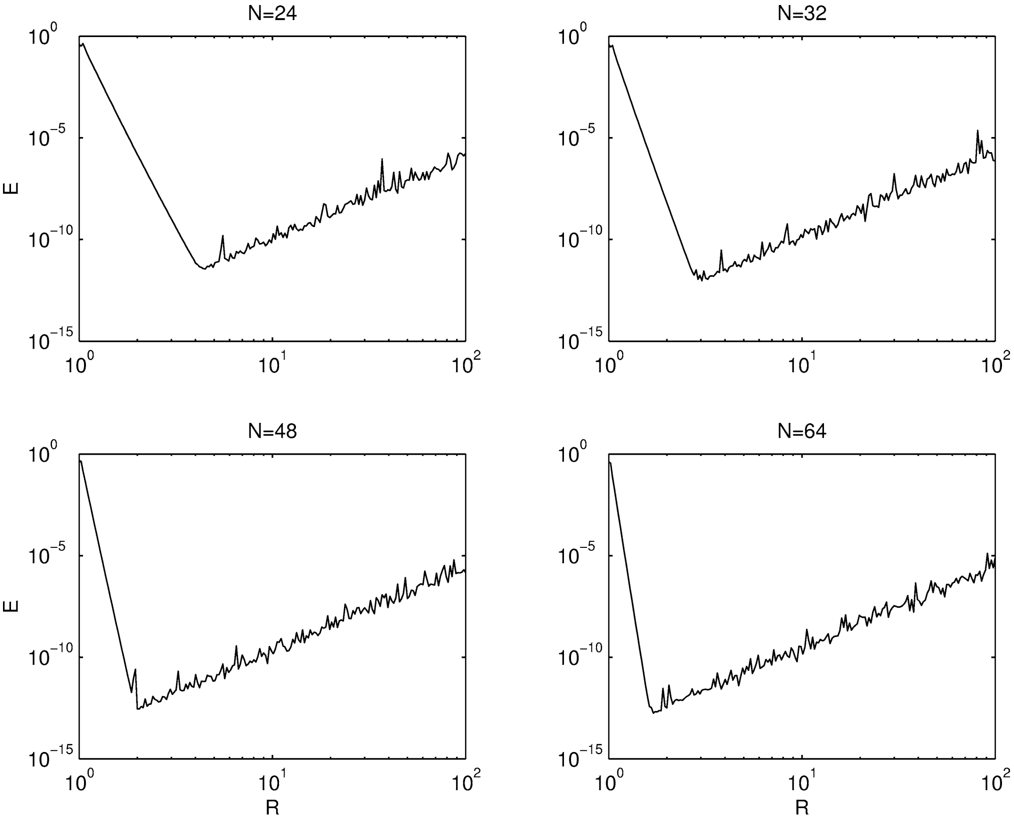

Figure 2. Results for varying distance R and various values of N in the case of the sphere for example 2.

where, for example,

$$A^{\alpha} = \begin{pmatrix} A_1^{\alpha} & A_2^{\alpha} & \dots & A_M^{\alpha} \\ A_M^{\alpha} & A_1^{\alpha} & \dots & A_{M-1}^{\alpha} \\ \vdots & \vdots & & \vdots \\ A_2^{\alpha} & A_3^{\alpha} & \dots & A_1^{\alpha} \end{pmatrix}, \tag{5.2}$$

with $N \times N$ submatrices A_{ℓ}^{α}, $\ell = 1, \dots, M$. Thus the matrix decomposition algorithm described in section 3 is also applicable in this case.

Note that, when M is even, the matrix $G^{1/2}$ is singular. The proof of this is similar to that of the corresponding result in [24], and is based on the fact that each of the matrices $A^{1/2}$, $B^{1/2}$, $C^{1/2}$ and $D^{1/2}$ has a special block circulant structure. For example, if $M = 2\mu$ with μ a positive integer, then

$$A^{1/2} = \operatorname{circ}\bigl(A_1^{1/2}, A_2^{1/2}, \dots, A_{\mu-1}^{1/2}, A_{\mu}^{1/2}, A_{\mu}^{1/2}, A_{\mu-1}^{1/2}, \dots, A_2^{1/2}, A_1^{1/2}\bigr).$$

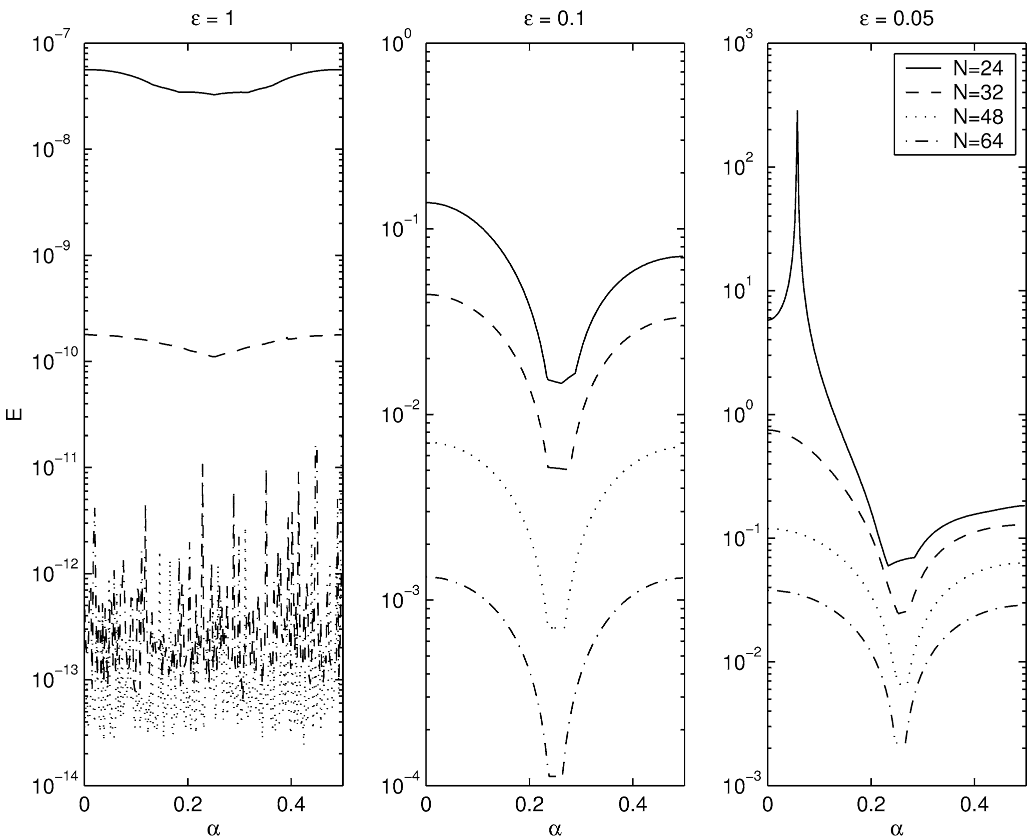

Figure 3. Results for varying parameter α and various values of ε in the case of the sphere for example 1.

6. Numerical results

We consider two test problems of the form (2.1) with Ω a sphere, a cylinder and a torus, and the functions f and g corresponding to the following exact solutions:

- **Problem 1.** $u = (x^2 + y^2 + z^2)\cosh(0.3x)\cosh(0.4y)\cos(0.5z)$.
- **Problem 2.** $u = x^4 - 2y^4 + z^4$.

In each of the numerical examples, we take $N = M = 24, 32, 48, 64$, and calculate E, the maximum relative error at the nodes of a partition of $\overline{\Omega}$. In the case of a sphere, this partition is

$$x_{i,j,k} = \varrho_k \sin\hat{\vartheta}_j \cos\hat{\varphi}_i, \qquad y_{i,j,k} = \varrho_k \sin\hat{\vartheta}_j \sin\hat{\varphi}_i, \qquad z_{i,j,k} = \varrho_k \cos\hat{\vartheta}_j, \quad i, j, k = 1, \dots, L,$$

where

$$\hat{\varphi}_i = \frac{2(i-1)\pi}{L}, \qquad \hat{\vartheta}_j = \frac{j\pi}{L+1}, \qquad \varrho_k = \frac{k\varrho}{L};$$

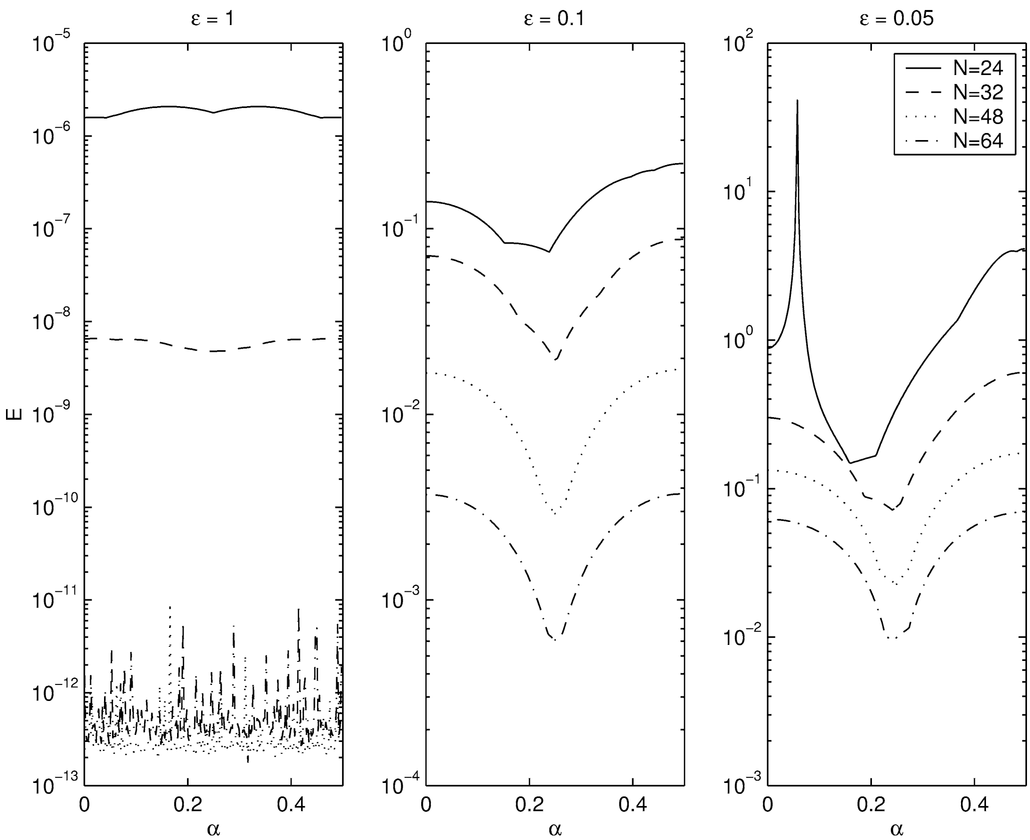

Figure 4. Results for varying parameter α and various values of ε in the case of the sphere for example 2.

for a cylinder,

$$x_{i,j,k} = r_j \cos\hat{\varphi}_i, \qquad y_{i,j,k} = r_j \sin\hat{\varphi}_i, \qquad z_{i,j,k} = z_k,$$
$$i, j = 1, \dots, L, \quad k = 1, \dots, L+1,$$

where

$$r_j = \frac{j\varrho}{L}, \qquad z_k = -h + \frac{2(k-1)}{L}h,$$

and for a torus,

$$x_{i,j,k} = \varrho_2 \cos\hat{\varphi}_i + \varrho_j \cos\hat{\varphi}_i \cos\hat{\theta}_k, \qquad y_{i,j,k} = \varrho_2 \sin\hat{\varphi}_i + \varrho_j \sin\hat{\varphi}_i \cos\hat{\theta}_k,$$
$$z_{i,j,k} = \varrho_j \sin\hat{\theta}_k,$$

where $i, j, k = 1, \dots, L$, and

$$\varrho_j = \frac{j\varrho}{L}\varrho_1, \qquad \hat{\theta}_k = \frac{2(k-1)\pi}{L}.$$

In the numerical experiments, we took $L = 16$.

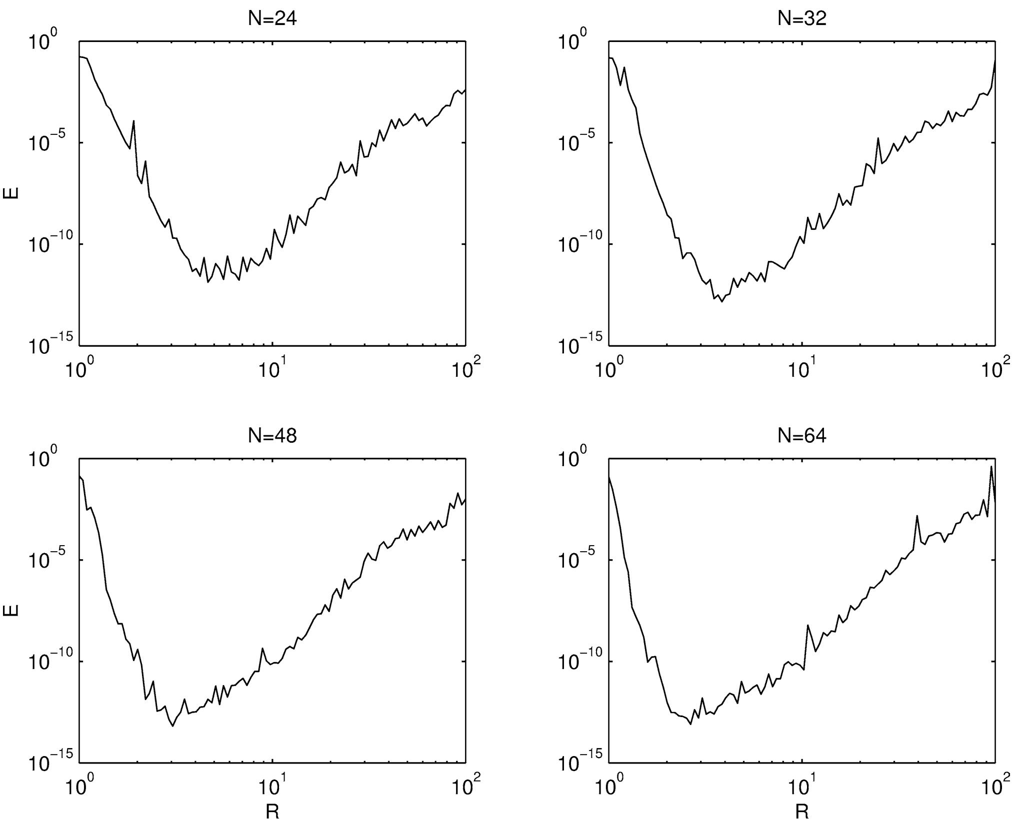

Figure 5. Results for varying distance R and various values of N in the case of the cylinder for example 1.

In the following figures, the plots of E versus R are on a log–log scale and those of E versus α on a semi-log scale.

6.1. Case I

For problem 1 with the unit sphere, that is, $\varrho = 1$, we present in figure 1 plots of E versus R when $\alpha = 0$. This figure reveals that the MFS approximation is poor when $\varepsilon = R - \varrho$ is either very small or very large. Similar results are obtained for problem 2; see figure 2.

For problem 1, plots of E versus α are given in figure 3. Because of the symmetry of the problem about $\alpha = 0$, we choose $\alpha \in [0, 1/2)$. We consider three cases, $\varepsilon = R - \varrho = 1, 0.1, 0.05$. For $\varepsilon = 0.05$ (and smaller as was observed from numerical experiments) and $\varepsilon = 0.1$, as N increases, the error E is optimized for $\alpha \approx 1/4$ whereas for $\varepsilon = 1$ (and larger values of ε as was observed from numerical experiments) E becomes independent of α as N increases. Again the results for problem 2 are similar; see figure 4. In order to give an indication of the computing time required, in table 1 we

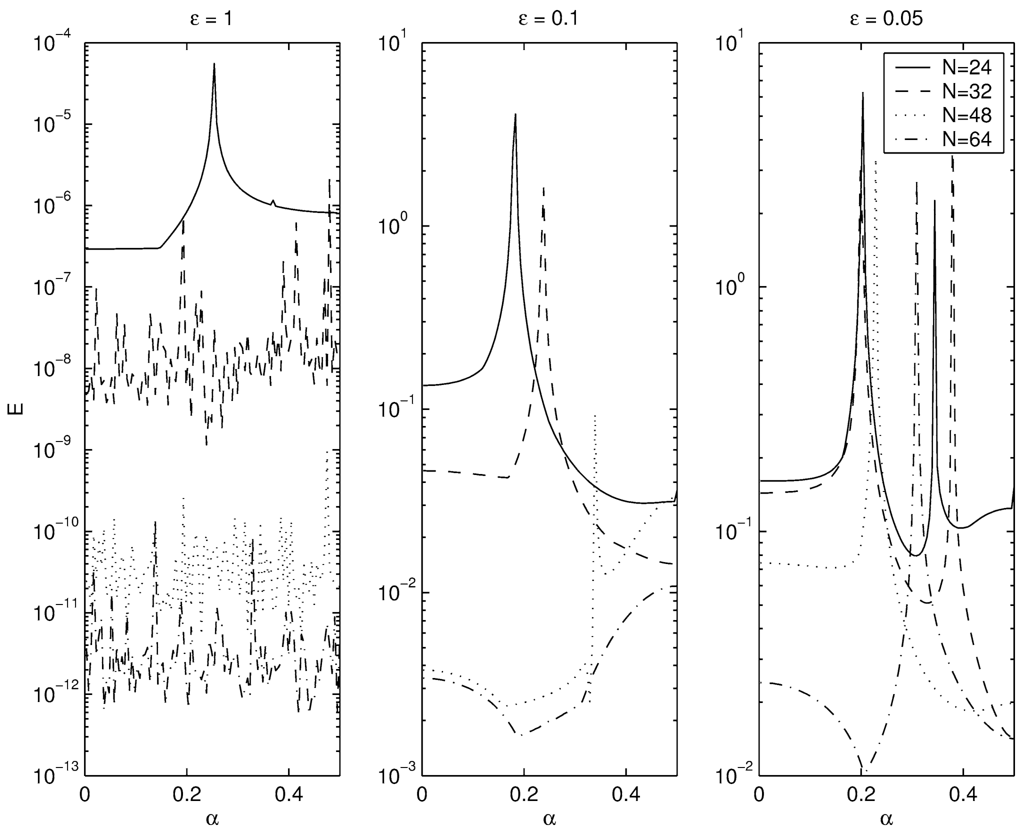

Figure 6. Results for varying parameter α and various values of ε in the case of the sphere for example 1.

Table 1
CPU times (in seconds) for case I.

$M = N$	16	24	32	48	64	96
Secs	0.09	0.27	0.66	2.48	8.28	34.84

present the CPU times for the solution of case I. These times were recorded on an IBM RS6000 (375 MHz).

6.2. Case II

For the cylinder with $h = 1$, $\varrho = 1$, we present results for example 1 only as those for example 2 are similar. In figure 5, we plot E versus R for $\alpha = 0$, and, as in the case of the sphere, the MFS approximation is poor when $\varepsilon = R - \varrho$ $(= H - h)$ is either very small or very large. In figure 6 are plots of E versus α, $\alpha \in [0, 1/2)$ for $\varepsilon = R - \varrho = H - h = 1, 0.1, 0.05$. We observe that the behaviour of the error as α varies is rather erratic. For $\varepsilon = 0.1$ and 0.05, as N increases the error E is optimized

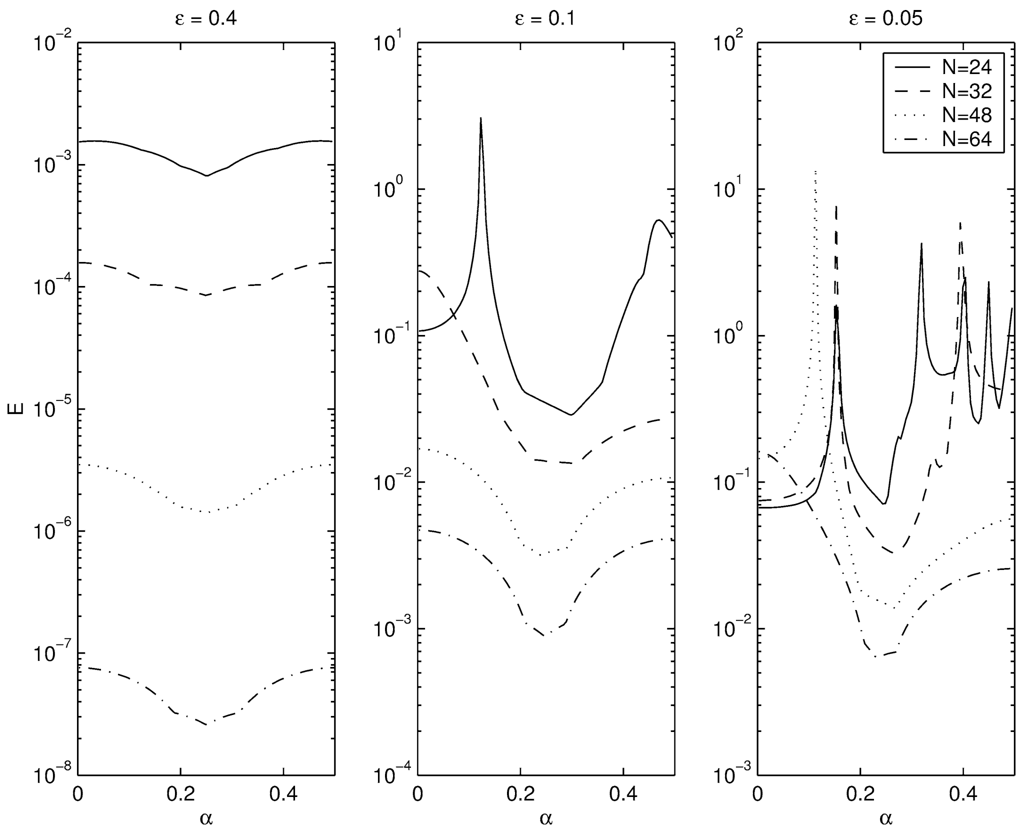

Figure 7. Results for varying parameter α and various values of ε in the case of the torus for example 1.

for $\alpha \approx 1/4$. For $\varepsilon = 1$ (and larger values of ε), E becomes independent of α as N increases. In particular, the larger the distance ε, the less dependent E is on α.

6.3. *Case III*

For the torus with $\varrho_1 = 1/2$, $\varrho_2 = 1$, we again present results for example 1 only. In figure 7, we plot E versus α for $\alpha \in [0, 1/2)$, and $\varepsilon = R_1 - \varrho_1 = 0.4, 0.1, 0.02$. Note that, in this case, we require that $\varepsilon \in (0, 1/2)$. For $\varepsilon = 0.1$ and 0.02, as N increases, the error E is optimized for $\alpha \approx 1/4$ whereas, for $\varepsilon = 0.4$, E again becomes independent of α as N increases.

The optimal value of $\alpha \approx 1/4$ was also observed for the case of the two-dimensional harmonic problem on a disk (see [23,25]) and for three-dimensional axisymmetric harmonic problems (see [24]). This indicates that the error is minimized around that value of α when $\partial\widetilde{\Omega}$ is close to the boundary $\partial\Omega$.

The poor accuracy when using either small or large ε has been observed in other applications of the MFS (see [2,9,23]).

7. Concluding remarks

An efficient MFS algorithm is formulated for the solution of biharmonic problems in axisymmetric domains. The algorithm exploits the circulant structure of the matrices resulting from the MFS discretization. The ideas developed in this work can also be used when boundary element methods (BEMs) are applied to axisymmetric problems, as the structure of the matrices arising in these methods is similar to that of the MFS discretization matrices. It is worth noting that the MFS has several advantages over BEMs. The MFS is easy to implement and requires little data preparation. Moreover, it does not require an elaborate discretization of the boundary, nor does it involve potentially troublesome integrations such as are present in BEMs. By using the method of particular solutions (see [9]), the proposed algorithm could also be used to solve problems governed by inhomogeneous equations.

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