



Qualitative and Dynamical Analysis of a Bionomic Fishery Model with Prey Refuge

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Abstract

Predation and escaping from predation through hiding are two fundamental phenomena in ecology. The most common approach to reducing the chance of predation is to use a refuge. Here, we consider a three species fishery model system with prey refuge induced by a Holling type-II functional response. These three species of fish populations are named prey, middle predator, and top predator. Harvesting is employed in most fishery models to achieve both ecological and commercial benefits. Research proves that non-linear harvesting (Michaelis–Menten type) returns more realistic outcomes. So, we have combined the Michaelis–Menten type of harvesting efforts for all populations. Uniform boundedness conditions for the solutions of the model are discussed. The existence conditions for possible equilibrium points with stability are presented. We explain the dynamical behavior at each equilibrium point through bifurcation analysis. The persistent criteria of the system are examined. Bionomic equilibrium and optimal harvesting control using Pontryagin's maximum principle are calculated. For validation of the model in the real world, we have implemented this in the freshwater ecosystem of Lake Victoria. Extraction of native fish species and ecological balances are the foremost solicitude of Lake Victoria. We may resolve this concern partially by implementing prey refuge, since it may sustain the ecology of Lake Victoria, and therefore also its economical importance. Lake Victoria is acclaimed worldwide for the trade of fishing. Also, it provides the largest employment in east-central Africa and is beneficial to fishing equipment manufacturers. So, the bionomic equilibrium and harvesting control have significant applications in the fisheries. All the analytical studies are verified by numerical simulations. We have plotted phase portraits, bifurcation diagrams, Lyapunov exponents to explore the dynamics of the proposed model.

Keywords Fishery model · Refuge · Optimal harvesting policy · Bionomic equilibrium · Bifurcation

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1 Introduction

Dynamics of an ecological system has been one of the most pre-eminent topics in ecology because of its essentiality and ubiquitousness. Predator-prey interactions were first introduced by Lotka and Volterra (Lotka 1924; Volterra 1926) which is often termed as the classical predator-prey model. After that many researchers developed many theories varying different factors which affect the dynamics of the model system such as delayed time, harvesting, competition, stochastic perturbation, group defense, diseases, functional response, refuge and many more (Kar 2006; Upadhyay et al. 2013; Maiti et al. 2019; Mishra et al. 2019; Liu et al. 2020). Functional responses are the indispensable components of predator-prey interactions which signifies the feeding relations among two species at a given unit of time. We have studied in general three types of functional responses in literature known as prey dependent, predator dependent, and lastly ratio-dependent type functional responses. Holling (1965) introduced type I, type II, and type III functional responses for species interaction.

We know that due to some external effects like overutilization, over predation and environmental factors such as pollution and famine starvation, numerous prey species have been driven toward extinction. The prey species may avoid such circumstances and predation either by guarding and defending themselves or by hiding as a refuge keeping a safe distance from predators. Many analytical and practical experiments have been performed to notice the effects of prey refuges in a predator-prey model and the result confirmed that prey refuges have a stabilizing power on prey-predator interactions. It is also observed that the extinction of the prey population can be prevented by adding the refuges (González-Olivare and Ramos-Jiliberto 2003; Kar 2006). There are enormous researches available based on the refuges. Sih (1987) reviewed the stability condition of a generalized Lotka–Volterra type prey-predator model with prey refuge and examined the effect of refuge with feeding and reproduction rate of prey. Kar (2005) and Huang et al. (2006) studied the importance of prey refuge on the stability of a predator-prey model. Ma et al. (2009) examined the consequences of induced prey refuge on a predator-prey model for multiple functional responses analytically.

Harvesting is a powerful tool to control the extinction of species and minimizing the exploitation of renewable resources, and it also has an additional influence on the population dynamics. So by adding the harvesting factor in the ecological model, the system gives a more realistic result. Clark (1979, 2005) interpreted the effects of harvesting on fisheries control. General harvesting functions studied excessively in the literature are: (i) constant rate of harvesting i.e., $h(x) = h$, where x represents the density of population (fish stock) and $h(x)$ is the harvesting rate, (ii) linear harvesting

i.e., $h(x) = qEx$, where q and E measure the catchability coefficient and harvesting efforts, respectively, with mortality rate qE , and (iii) non-linear harvesting, defined as $h(x) = qEx/(m_1E + m_2x)$ with positive values m_1 and m_2 (Clark 1979). However, the non-linear harvesting, named as the Michaelis–Menten type, is more realistic among these in all senses. For linear harvesting, $h(x) \rightarrow \infty$ with $x \rightarrow \infty$ for fixed E , and $h(x) \rightarrow \infty$ with $E \rightarrow \infty$ for fixed x , but the non-linear harvesting gives more realistic results and removes this unbounded characteristic of harvesting. For Michaelis–Menten type harvesting, we notice $h(x) \rightarrow qE/m_2$ as $x \rightarrow \infty$ and $h(x) \rightarrow qx/m_1$ as $E \rightarrow \infty$ (Das et al. 2009; Gupta et al. 2012). Hoekstra and van den Bergh (2005) reviewed a predator-prey model with the concept of conservation of the predator and harvesting. They inferred that the optimal harvesting solutions of the model are controlled by ecological along with economic parameters. A predator-prey model incorporated with constant rates of harvesting and prey refuge was examined by Ji and Wu (2010). Krishna et al. (1998) and Gupta and Chandra (2013) studied a two species predator-prey model with the Michaelis–Menten harvesting function, and the main objective of their study was to maximize economic benefits maintaining the ecological balance, and to prevent the predator from annihilation. Purohit and Chaudhuri (2004) introduced a two species bio-economic fishery model using non-linear harvesting in both species. Hu and Cao (2017) considered a predator-prey model where only the predator undergoes Michaelis–Menten harvesting. Raw et al. (2020) studied a plankton-fish model imposing non-linear harvesting, particularly on the fish species. Haque and Sarwardi (2018) studied a two species harvesting model implementing prey refuge as a Holling type-II functional response, and both the species were harvested according to a linear harvesting function. A Holling type-II functional response is generated with the assumption that the predator feeding rate is limited by its capacity to process food. Abdulghafour and Naji (2018) observed the impact of refuge in a prey-predator model and addressed the effect of some crucial parameters on the dynamics of the system.

We observe that refuge and harvesting have transcendent effects on the dynamics of prey-predator systems. In this study we consider an ecosystem of Lake Victoria to explain the application of the model. This lake is the world's second-largest freshwater lake covering a massive inland of $68,800\text{km}^2$, is situated in east-central Africa and borders the countries of Uganda, Kenya, and Tanzania (Geheb et al. 2003; NPFMP 2015b). The lake bears Africa's largest inland fishery which includes fish like Nile perch (*Lates niloticus*), Nile tilapia (*Oreochromis niloticus*), cichlids, the silver cyprinid (*Rastrineobola argentea*), airbreathing catfish, bagrid catfish, etc. In this particular study, we consider a three fish species predator-prey model with the food chain, including the Nile perch as predator one (top predator), while cichlids and the silver cyprinid as the predator two (middle predator) and prey, respectively. Here the prey population (the silver cyprinid) in nature exists by using a refuge in the presence of predators. All three species undergo non-linear harvesting efforts. The effectiveness of a Holling type-II response modelling prey refuge combined with a non-linear harvesting in a three-species model is the unique concept of this

paper. We have considered the lake ecosystem as a real-world example and prepared a comparative study based on our results with the real data. This work may inscribe a realistic approach to understand the lake ecosystem. We hope this analysis can develop a balanced ecosystem regarding the sustainability of resources with commercial fishery purposes. Though there are numerous works on this related concept (see Clark 1979; González-Olivare and Ramos-Jiliberto 2003; Gupta et al. 2012; Ghosh et al. 2017; Hu and Cao 2017; Abdulghafour and Naji 2018), there is no evidence of the existence of this particular type of model in the literature until now, to our knowledge.

In this paper we study the dynamical behaviour of a Holling type-II predator-prey fishery model with a Michaelis–Menten type of harvesting function incorporating a constant rate of refuge. This paper continues with the following four sections. In Sect. 2, the idea and development of the ecological model are introduced, and we examine the dynamics and properties like stability, boundedness, persistence, local bifurcation, and Hopf-bifurcation. Section 3 shows the numerical simulation of the model system. We validate the model with real-world situations in Sect. 4. Finally, we give a brief conclusion about our findings in Sect. 5.

2 Mathematical Model

We formulate a three species ecological system that contains a prey (X), a middle predator (Y), and a top predator population (Z) at time T with prey refuge incorporating a Holling type-II functional response and Michaelis–Menten type of harvesting function. Here, we consider a situation where the prey population X is preyed by predator Y , and the population Y serves as a favourite food for predator Z . In the absence of predator species, the prey population increases logistically. Meanwhile, the predator populations die out exponentially due to the absence of their only food (i.e., prey species). We assume that a fixed fraction of prey enters a preserved area to avoid the predators' attack, although the non-refuged prey can be attacked by the predator. Here, we take the non-refuged prey to be attacked by the middle predator following a Holling type-II functional response. The well known Holling type-II response is functionally described as, $f(X, Y) = \frac{a_1 X}{b_1 + X}$, where a_1 and b_1 are positive parameters representing the attack rate and half-saturation constant, respectively (Skalski and Gilliam 2001; Ghosh et al. 2017). Assuming the importance of prey refuge, we have imposed the refuge parameter m into the attack rate a_1 , producing $a_1(1 - m)$, $m \in [0, 1)$, and $(1 - m)X$ prey abundance is left for predation. The functional response imposing refuge turns into $f(X, Y) = \frac{a_1(1 - m)X}{b_1 + (1 - m)X}$. The model implements Holling type-II functional response to explain prey-predator feeding relations. Lastly, the model system employs a non-linear harvesting function for all three species populations. According to the above hypothesis, the model can be represented mathematically as follows:

$$\begin{aligned}
 \frac{dX}{dT} &= \underbrace{rX\left(1 - \frac{X}{K}\right)}_{\text{Logistic Growth}} - \underbrace{\frac{a_1(1-m)XY}{b_1 + (1-m)X}}_{\text{Consumption}} - \underbrace{\frac{c_1EX}{l_1E + l_2X}}_{\text{Harvesting}}, \\
 \frac{dY}{dT} &= \underbrace{-dY}_{\text{Death}} + e_1 \underbrace{\frac{a_1(1-m)XY}{b_1 + (1-m)X}}_{\text{Conversional Growth}} - \underbrace{\frac{a_2YZ}{b_2 + Y}}_{\text{Consumption}} - \underbrace{\frac{c_2EY}{l_3E + l_4Y}}_{\text{Harvesting}}, \\
 \frac{dZ}{dT} &= \underbrace{-cZ}_{\text{Death}} + \underbrace{e_2 \frac{a_2YZ}{b_2 + Y}}_{\text{Conversional Growth}} - \underbrace{\frac{c_3EZ}{l_5E + l_6Z}}_{\text{Harvesting}},
 \end{aligned}
 \tag{2.1}$$

with the initial conditions $X(0) > 0$, $Y(0) > 0$ and $Z(0) > 0$. Table 1 describes the biological meaning of the model parameters, and all are positive.

In order to reduce the number of parameters used in the model (2.1) and to determine the principal set of parameters, we take the following transformations:

$$\begin{aligned}
 t = rT, \quad X = Kx, \quad y = \frac{a_1Y}{rK}, \quad z = \frac{a_2Z}{rK}, \quad w_1 = \frac{b_1}{K}, \quad w_2 = \frac{Ec_1}{rl_2K}, \quad w_3 = \frac{l_1E}{l_2K}, \quad w_4 = \frac{d}{r}, \\
 w_5 = \frac{e_1a_1}{r}, \quad w_6 = \frac{r}{a_1}, \quad w_7 = \frac{a_1b_2}{rK}, \quad w_8 = \frac{a_1c_2E}{r^2l_4K}, \quad w_9 = \frac{a_1l_3E}{rl_4K}, \quad w_{10} = \frac{c}{r}, \quad w_{11} = \frac{e_2a_2}{r}, \\
 w_{12} = \frac{a_2c_3E}{r^2l_6K}, \quad w_{13} = \frac{a_2l_5E}{rl_6K}.
 \end{aligned}
 \tag{2.2}$$

The dimensionless system can be written as

Table 1 The biological meaning of model parameters

Parameters	Biological meaning
r	Intrinsic growth rate of the prey
K	Carrying capacity
a_1	Maximum attack rate of the middle predator
a_2	Maximum attack rate of the top predator
b_1	Half-saturation constant of the prey
b_2	Half-saturation constant of the middle predator
d	Death rate of the middle predator
c	Death rate of the top predator
$c_i; i = 1, 2, 3$	Catchability coefficients
E	Harvesting effort
$l_i; i = 1, 2, 3, 4, 5, 6$	Positive constants
e_1	The efficiency of the middle predator to convert the consumed prey into a new predator
e_2	The efficiency of the top predator to convert the consumed middle predator into a new predator
$m \in [0, 1)$	Rate of prey refuge, remaining $(1 - m)X$ non-refuged prey available for predation

$$\begin{aligned} \frac{dx}{dt} &= x \left[(1-x) - \frac{(1-m)y}{w_1 + (1-m)x} - \frac{w_2}{w_3 + x} \right] = xf_1(x, y, z), \\ \frac{dy}{dt} &= y \left[-w_4 + \frac{w_5(1-m)x}{w_1 + (1-m)x} - \frac{z}{w_6(w_7 + y)} - \frac{w_8}{w_9 + y} \right] = yf_2(x, y, z), \\ \frac{dz}{dt} &= z \left[-w_{10} + \frac{w_{11}y}{w_7 + y} - \frac{w_{12}}{w_{13} + z} \right] = zf_3(x, y, z). \end{aligned} \tag{2.3}$$

2.1 Boundedness

Boundedness of any model system assures that the system is biologically feasible and that it is realistic in representing the species population with no negative values. A mathematical problem is feasible if it has a solution, the solution is unique and it depends continuously on data and parameters.

Theorem 1 *All solutions of the system (2.3) are uniformly bounded.*

Proof From system (2.3), we take $\frac{dx}{dt} \leq x(1-x)$ with $x(0) > 0$.

Thus, solution $x(t) \leq \frac{x(0)}{e^{-t}[1-x(0)] + x(0)}$ and $\lim_{t \rightarrow \infty} x(t) \leq 1$.

Now, define the function

$$\begin{aligned} w(t) &= x(t) + \frac{1}{w_5}y(t) + \frac{1}{w_5w_6w_{11}}z(t), \\ \Rightarrow \frac{dw}{dt} &= \frac{dx}{dt} + \frac{1}{w_5}\frac{dy}{dt} + \frac{1}{w_5w_6w_{11}}\frac{dz}{dt} \\ &\leq x - \frac{w_2x}{w_3 + x} - \frac{w_4y}{w_5} - \frac{w_8y}{w_5(w_7 + y)} - \frac{w_{10}z}{w_5w_6w_{11}} - \frac{w_{12}z}{w_5w_6w_{11}(w_{13} + z)}. \end{aligned}$$

Some algebraic calculation yields $\frac{dw}{dt} \leq 2 - Lw$, where $L = \min\{1, w_4, w_{10}\}$ and $w(0) > 0$.

Using Grönwall’s lemma (Birkhoff and Rota 1982) for differential inequalities, it is observed that

$$\begin{aligned} 0 < w(t) &\leq w(0)e^{-Lt} - \frac{2}{L}(e^{-Lt} - 1), \\ \Rightarrow \lim_{t \rightarrow \infty} w(t) &\leq \frac{2}{L}. \end{aligned}$$

Hence, it is proved that all solutions of the system (2.3) are uniformly bounded. \square

2.2 Equilibrium Points

In this section, we explore all the biological feasible equilibrium points with their existence conditions. To calculate the equilibrium points for the model (2.3), we take the zero growth isoclines as

$$xf_1(x, y, z) = 0, yf_2(x, y, z) = 0 \text{ and } zf_3(x, y, z) = 0. \tag{2.4}$$

We see that only four biologically feasible equilibrium points exist for the given model (2.3).

- (I) The trivial equilibrium point $E_0 = (0, 0, 0)$ always exists.
- (II) The predator free equilibrium point $E_1 = (\bar{x}, 0, 0)$, where

$$\bar{x} = -\frac{(w_3 - 1)}{2} + \frac{\sqrt{(w_3 - 1)^2 - 4(w_2 - w_3)}}{2},$$

exists on the positive quadrant of the x - axis if the conditions (i) $w_3 < 1$, and (ii) $(w_3 - 1)^2 > 4(w_2 - w_3)$ hold.

- (III) The equilibrium point $E_2 = (\tilde{x}, \tilde{y}, 0)$ exists, where

$$\tilde{y} = \frac{[(1 - \tilde{x})(w_3 + \tilde{x}) - w_2][w_1 + (1 - m)\tilde{x}]}{(w_3 + \tilde{x})(1 - m)}, \tag{2.5}$$

and \tilde{x} represents the positive unique root of the polynomial equation:

$$U_1x^4 + U_2x^3 + U_3x^2 + U_4x + U_5 = 0, \tag{2.6}$$

where

$$\begin{aligned} U_1 &= (w_5 - w_4)(1 - m)^2, \\ U_2 &= (1 - m)[(w_4 - w_5)(1 - m)(1 - w_3) - w_1(2w_4 - w_5)], \\ U_3 &= (1 - m)\left[\{(w_9 + w_3 - w_2)(1 - m) + (1 - w_3)w_1\}(w_4 - w_5) \right. \\ &\quad \left. + (1 - w_3)w_1w_4 + w_8(1 - m) - \frac{w_1^2w_4}{(1 - m)}\right], \\ U_4 &= (1 - m)\left[\{w_3w_9(1 - m) + (w_3 - w_2)w_1\}(w_4 - w_5) - (w_9 + w_3 - w_2)w_1w_4 \right. \\ &\quad \left. + w_8\{w_1 + (1 - m)w_3\} + \frac{(1 - w_3)w_1^2w_4}{(1 - m)}\right], \\ U_5 &= w_1[(1 - m)w_3(w_4w_9 + w_8) + w_1w_4(w_3 - w_2)]. \end{aligned}$$

$E_2 = (\tilde{x}, \tilde{y}, 0)$ exists uniquely in the xy plane provided that the given condition

$$(1 - \tilde{x})(w_3 + \tilde{x}) > w_2, \tag{2.7}$$

with one set of following conditions

$$\begin{cases} U_1 < 0, U_2 < 0, U_4 > 0, \text{ and } U_5 > 0, \\ U_i < 0; i = 1, 2, 3, 4 \text{ and } U_5 > 0, \\ U_1 < 0 \text{ and } U_i > 0; i = 2, 3, 4, 5, \\ U_1 > 0, U_2 > 0, U_4 < 0 \text{ and } U_5 < 0, \\ U_i > 0; i = 1, 2, 3, 4 \text{ and } U_5 < 0, \\ U_1 > 0 \text{ and } U_i < 0; i = 2, 3, 4, 5, \end{cases}$$

are satisfied.

(IV) Finally, for the interior equilibrium point $E_3 = (x^*, y^*, z^*)$, we have

$$\begin{cases} 1 - x^* - \frac{(1 - m)y^*}{w_1 + (1 - m)x^*} - \frac{w_2}{w_3 + x^*} = 0, \\ -w_4 + \frac{w_5(1 - m)x^*}{w_1 + (1 - m)x^*} - \frac{z^*}{w_6(w_7 + y^*)} - \frac{w_8}{w_9 + y^*} = 0, \\ -w_{10} + \frac{w_{11}y^*}{w_7 + y^*} - \frac{w_{12}}{w_{13} + z^*} = 0. \end{cases} \tag{2.8}$$

Solving the system (2.8), we get

$$y^* = \frac{R_1^* [R_2^* (1 - x^*) - w_2]}{R_2^* (1 - m)}, \tag{2.9}$$

$$z^* = \frac{w_7 R_2^* (1 - m) (w_{10} w_{13} + w_{12}) + R_1^* [w_{12} - w_{13} (w_{11} - w_{10})] [R_2^* (1 - x^*) - w_2]}{-w_{10} w_7 R_2^* (1 - m) + R_1^* [w_{11} - w_{10}] [R_2^* (1 - x^*) - w_2]}, \tag{2.10}$$

where $R_1^* = w_1 + (1 - m)x^*$ and $R_2^* = w_3 + x^*$. We can get an exact parametric solution for x^* , y^* and z^* by putting values of y^* and z^* in system (2.8) and solving toward E_3 . It is quite difficult to calculate these equilibrium coordinate values analytically due to its algebraic complexity. However, we can find $E_3 = (x^*, y^*, z^*)$ numerically later in the numerical simulation Sect. 3.

2.3 Stability and Bifurcation Analysis

In this section, we study the local stability with the help of the variational matrix for the existing equilibrium points and the local bifurcation near each equilibrium point using Sotomayor’s theorem (Sotomayor 1973). It can be noted that the existence of a non-hyperbolic equilibrium point is only a necessary condition to show the occurrence of a bifurcation. The variational matrix calculated at the point $E_i = (x, y, z)$ is

$$J(E_i) = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix}, \tag{2.11}$$

where

$$\begin{aligned} j_{11} &= x \left[-1 + \frac{(1-m)^2 y}{(w_1 + (1-m)x)^2} + \frac{w_2}{(w_3 + x)^2} \right] + \left[1 - x - \frac{(1-m)y}{w_1 + (1-m)x} - \frac{w_2}{w_3 + x} \right], \\ j_{12} &= -\frac{x(1-m)}{w_1 + (1-m)x}, \quad j_{13} = 0, \quad j_{21} = \frac{w_1 w_5 (1-m)y}{(w_1 + (1-m)x)^2}, \\ j_{22} &= y \left[\frac{w_6 z}{(w_6(w_7 + y))^2} + \frac{w_8}{(w_9 + y)^2} \right] + \left[-w_4 + \frac{w_5(1-m)x}{w_1 + x(1-m)} - \frac{z}{w_6(w_7 + y)} \right. \\ &\quad \left. - \frac{w_8}{(w_9 + y)} \right], \\ j_{23} &= -\frac{y}{w_6(w_7 + y)}, \quad j_{31} = 0, \quad j_{32} = z \left[\frac{w_7 w_{11}}{(w_7 + y)^2} \right], \quad j_{33} = \frac{w_{12} z}{(w_{13} + z)^2} \\ &\quad + \left[-w_{10} + \frac{w_{11} y}{w_7 + y} - \frac{w_{12}}{w_{13} + z} \right]. \end{aligned}$$

2.3.1 System Behaviour at $E_0 = (0, 0, 0)$

The variational matrix at the trivial equilibrium point $E_0 = (0, 0, 0)$ is

$$J_0 = \begin{bmatrix} 1 - \frac{w_2}{w_3} & 0 & 0 \\ 0 & -w_4 - \frac{w_8}{w_9} & 0 \\ 0 & 0 & -w_{10} - \frac{w_{12}}{w_{13}} \end{bmatrix}. \tag{2.12}$$

The eigenvalues of the matrix J_0 are $\lambda_{01} = 1 - \frac{w_2}{w_3}$, $\lambda_{02} = -w_4 - \frac{w_8}{w_9} < 0$, and $\lambda_{03} = -w_{10} - \frac{w_{12}}{w_{13}} < 0$. From the nature of these eigenvalues we conclude that E_0 is locally asymptotically stable for $w_3 < w_2$. If the parameter w_2 passes through its threshold value $w_2^* = w_3$, then the Jacobian matrix (2.12) becomes

$$J_0^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -w_4 - \frac{w_8}{w_9} & 0 \\ 0 & 0 & -w_{10} - \frac{w_{12}}{w_{13}} \end{bmatrix}, \tag{2.13}$$

which indicates that E_0 becomes a non-hyperbolic equilibrium point with $\lambda_{01} = 0$, and the system (2.3) encounters a bifurcation. We examine the nature of the local bifurcation using Sotomayor’s theorem. Let $v_0 = [v_{01} \ v_{02} \ v_{03}]^T$ and $w_0 = [w_{01} \ w_{02} \ w_{03}]^T$

denote the eigenvectors of the matrix J_0^* and J_0^{*T} , respectively. Then corresponding to the eigenvalue $\lambda_{01} = 0$, we obtain $V_0 = [v_{01} \ 0 \ 0]^T$ and $W_0 = [w_{01} \ 0 \ 0]^T$, where $v_{01} \neq 0$ and $w_{01} \neq 0$. Rewriting the system (2.3) in the form $N' = F(N)$, where $N = [x \ y \ z]^T$ and $F(N) = [xf_1(x, y, z) \ yf_2(x, y, z) \ zf_3(x, y, z)]^T$, we get $\frac{\partial F}{\partial w_2} = F_{w_2} = \left[-\frac{x}{w_3 + x} \ 0 \ 0\right]^T$; this implies $F_{w_2}(E_0, w_2^*) = [0 \ 0 \ 0]^T$. Hence $W_0^T F_{w_2}(E_0, w_2^*) = 0$ indicates the non-appearance of the saddle node bifurcation at this point. Now the derivative of F_{w_2} with respect to N at (E_0, w_2^*) is

$$DF_{w_2}(E_0, w_2^*) = \begin{bmatrix} -\frac{1}{w_3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow W_0^T [DF_{w_2}(E_0, w_2^*)V_0] = -\frac{w_{01}v_{01}}{w_3} \neq 0, \tag{2.14}$$

and D^2F , where $D = d/dN$ is calculated as

$$D^2F(E_0, w_2^*)(V_0, V_0) = \begin{bmatrix} -2v_{01}^2 + \frac{2}{w_3}v_{01}^2 \\ 0 \\ 0 \end{bmatrix}, \tag{2.15}$$

$$\Rightarrow W_0^T [D^2F(E_0, w_2^*)(V_0, V_0)] = 2w_{01}v_{01}^2 \left(-1 + \frac{1}{w_3}\right) \neq 0. \tag{2.16}$$

If $w_3 \neq 1$, then a transcritical bifurcation occurs near equilibrium E_0 . If the given condition doesn't satisfy, then we continue calculating the third derivative. Here

$$D^3F(E_0, w_2^*)(V_0, V_0, V_0) = \begin{bmatrix} -\frac{6}{w_3^2}v_{01}^3 \\ 0 \\ 0 \end{bmatrix}, \tag{2.17}$$

which implies

$$W_0^T [D^3F(E_0, w_2^*)(V_0, V_0, V_0)] = -\frac{6}{w_3^2}v_{01}^3w_{01} \neq 0, \tag{2.18}$$

suggesting the occurrence of a pitchfork bifurcation for the system (2.3) at the trivial equilibrium E_0 . We summarise the obtained results in the following theorem:

Theorem 2 *The system behavior around $E_0 = (0, 0, 0)$ is as follows*

- (i) E_0 is locally asymptotically stable for $w_3 < w_2$, non-hyperbolic for $w_3 = w_2$, and a saddle node for $w_3 > w_2$.

- (ii) *If the parameter w_2 passes through its threshold value $w_2^* = w_3$, then the system (2.3) at E_0 goes through a transcritical bifurcation for $w_3 \neq 1$. For $w_3 = 1$ it has a pitchfork bifurcation.*

2.3.2 System Behavior at $E_1 = (\bar{x}, 0, 0)$

The variational matrix at the equilibrium point $E_1 = (\bar{x}, 0, 0)$ is computed as

$$J_1 = \begin{bmatrix} \bar{x} \left(-1 + \frac{w_2}{(w_3 + \bar{x})^2} \right) & -\frac{(1-m)\bar{x}}{w_1 + (1-m)\bar{x}} & 0 \\ 0 & -w_4 + \frac{w_5(1-m)\bar{x}}{w_1 + (1-m)\bar{x}} - \frac{w_8}{w_9} & 0 \\ 0 & 0 & -w_{10} - \frac{w_{12}}{w_{13}} \end{bmatrix}. \tag{2.19}$$

The eigenvalues of the matrix J_1 are given by $\lambda_{11} = \bar{x} \left(-1 + \frac{w_2}{(w_3 + \bar{x})^2} \right)$, $\lambda_{12} = -w_4 + \frac{w_5(1-m)\bar{x}}{w_1 + (1-m)\bar{x}} - \frac{w_8}{w_9}$, and $\lambda_{13} = -w_{10} - \frac{w_{12}}{w_{13}} < 0$. E_1 is locally asymptotically stable if the conditions

$$\begin{cases} \frac{w_2}{(w_3 + \bar{x})^2} < 1, \\ \frac{w_5(1-m)\bar{x}}{w_1 + (1-m)\bar{x}} < w_4 + \frac{w_8}{w_9}, \end{cases} \tag{2.20}$$

hold. If the parameter w_4 passes through the value $w_4^* \equiv \frac{w_5(1-m)\bar{x}}{w_1 + (1-m)\bar{x}} - \frac{w_8}{w_9}$, then the equilibrium point $E_1 = (\bar{x}, 0, 0)$ is converted to a non-hyperbolic point with $\lambda_{12} = 0$. Hence the Jacobian matrix (2.19) becomes

$$J_1^* = \begin{bmatrix} \bar{x} \left(-1 + \frac{w_2}{(w_3 + \bar{x})^2} \right) & \frac{-(1-m)\bar{x}}{w_1 + (1-m)\bar{x}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -w_{10} - \frac{w_{12}}{w_{13}} \end{bmatrix}. \tag{2.21}$$

Let $V_1 = [v_{11} \ v_{12} \ v_{13}]^T$ and $W_1 = [w_{111} \ w_{112} \ w_{113}]^T$ denote the eigenvector of the matrix J_1^* and J_1^{*T} , respectively. We get $V_1 = [\xi_1 v_{12} \ v_{12} \ 0]^T$ and $W_1 = [0 \ w_{112} \ 0]^T$ corresponding to the eigenvalue $\lambda_{12} = 0$, where $\xi_1 = \frac{(1-m)(w_3 + \bar{x})^2}{[w_1 + (1-m)\bar{x}][w_2 - (w_3 + \bar{x})^2]} \neq 0$, $v_{12} \neq 0$, and $w_{112} \neq 0$. We get $\frac{\partial F}{\partial w_4} = F_{w_4} = [0 \ -y \ 0]^T$ and $F_{w_4}(E_1, w_4^*) = [0 \ 0 \ 0]^T$, which implies that

$$W_1^T F_{w_4}(E_1, w_4^*) = 0. \tag{2.22}$$

Hence, the saddle node bifurcation cannot appear at the equilibrium point E_1 . Now evaluating the derivative of F_{w_4} with respect to N at (E_1, w_4^*) , we get

$$DF_{w_4}(E_1, w_4^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.23}$$

$$\Rightarrow W_1^T [DF_{w_4}(E_1, w_4^*)V_1] = -w_{112}v_{12} \neq 0. \tag{2.24}$$

Further calculating D^2F , where $D = d/dN$, we get

$$D^2F(E_1, w_4^*)(V_1, V_1) = \begin{bmatrix} -2\xi_1^2v_{12}^2 + \frac{2w_2w_3}{(w_3 + \bar{x})^3}\xi_1^2v_{12}^2 - \frac{2w_2(1-m)}{(w_1 + (1-m)\bar{x})^2}\xi_1v_{12}^2 \\ \frac{2w_1w_5(1-m)}{(w_1 + (1-m)\bar{x})^2}\xi_1v_{12}^2 + \frac{2w_8}{w_9^2}v_{12}^2 \\ 0 \end{bmatrix}.$$

Hence

$$W_1^T [D^2F(E_1, w_4^*)(V_1, V_1)] = \left[\frac{2w_1w_5(1-m)}{(w_1 + (1-m)\bar{x})^2}\xi_1 + \frac{2w_8}{w_9^2} \right] w_{112}v_{12}^2 \neq 0, \tag{2.25}$$

as $\frac{2w_1w_5(1-m)}{(w_1 + (1-m)\bar{x})^2}\xi_1 + \frac{2w_8}{w_9^2} \neq 0$, which signifies that the system (2.3) evidences a transcritical bifurcation at E_1 . We further evaluate the third derivative of F with respect to N , which gives

$$D^3F(E_1, w_4^*)(V_1, V_1, V_1) = \begin{bmatrix} -\frac{6w_2w_3}{(w_3 + \bar{x})^4}\xi_1^3v_{12}^3 + \frac{6w_1(1-m)^2}{(w_1 + (1-m)\bar{x})^3}\xi_1^2v_{12}^3 \\ -\frac{6w_1w_5(1-m)^2}{(w_1 + (1-m)\bar{x})^3}\xi_1^2v_{12}^3 - \frac{6w_8}{w_9^3}v_{12}^3 \\ 0 \end{bmatrix}. \tag{2.26}$$

Thus

$$W_1^T [D^3F(E_1, w_4^*)(V_1, V_1, V_1)] = -6 \left[\frac{w_1w_5(1-m)^2}{(w_1 + (1-m)\bar{x})^3}\xi_1^2 + \frac{w_8}{w_9^3} \right] w_{112}v_{12}^3, \neq 0. \tag{2.27}$$

According to condition (2.27), the system (2.3) faces a pitchfork bifurcation at the equilibrium point E_1 . The summary of above analysis is given in the next theorem.

Theorem 3 (i) If conditions (2.20) hold, then equilibrium E_1 is locally asymptotically stable.

(ii) The system (2.3) has a transcritical bifurcation around the equilibrium point E_1 for $\frac{2w_1w_5(1-m)}{(w_1+(1-m)\tilde{x})^2}\xi_1 + \frac{2w_8}{w_9^2} \neq 0$, when the parameter w_4 crosses the bifurcation value $w_4^* \equiv \frac{w_5(1-m)\tilde{x}}{w_1+(1-m)\tilde{x}} - \frac{w_8}{w_9}$. Otherwise a pitchfork bifurcation appears for the system.

2.3.3 System Behavior at $E_2 = (\tilde{x}, \tilde{y}, 0)$

The variational matrix at the equilibrium point $E_2 = (\tilde{x}, \tilde{y}, 0)$ is evaluated as:

$$J_2 = (b_{ij})_{3 \times 3}, \tag{2.28}$$

where

$$\begin{aligned} b_{11} &= \tilde{x} \left(-1 + \frac{(1-m)^2\tilde{y}}{(w_1+(1-m)\tilde{x})^2} + \frac{w_2}{(w_3+\tilde{x})^2} \right), \quad b_{12} = -\frac{(1-m)\tilde{x}}{w_1+(1-m)\tilde{x}} < 0, \\ b_{13} &= 0, \quad b_{21} = \frac{w_1w_5(1-m)\tilde{y}}{(w_1+(1-m)\tilde{x})^2}, \quad b_{22} = \frac{w_8\tilde{y}}{(w_9+\tilde{y})^2}, \quad b_{23} = -\frac{\tilde{y}}{w_6(w_7+\tilde{y})}, \\ b_{31} &= 0, \quad b_{32} = 0, \quad b_{33} = -w_{10} - \frac{w_{12}}{w_{13}} + \frac{w_{11}\tilde{y}}{w_7+\tilde{y}}. \end{aligned}$$

One eigenvalue value of the matrix J_2 is $-w_{10} - \frac{w_{12}}{w_{13}} + \frac{w_{11}\tilde{y}}{w_7+\tilde{y}}$, and the other two eigenvalues are the roots of the sub matrix

$$J_2^* = \begin{bmatrix} \tilde{x} \left(-1 + \frac{(1-m)^2\tilde{y}}{(w_1+(1-m)\tilde{x})^2} + \frac{w_2}{(w_3+\tilde{x})^2} \right) & \frac{-(1-m)\tilde{x}}{w_1+(1-m)\tilde{x}} \\ \frac{w_1w_5(1-m)\tilde{y}}{(w_1+(1-m)\tilde{x})^2} & \frac{w_8\tilde{y}}{(w_9+\tilde{y})^2} \end{bmatrix}. \tag{2.29}$$

The eigenvalues of the matrix J_2^* have negative real parts since $tr(J_2^*) < 0$ and $det(J_2^*) > 0$. Some calculation gives

$$\begin{aligned} tr(J_2^*) &= -\tilde{x} + \frac{(1-m)^2\tilde{x}\tilde{y}}{(w_1+(1-m)\tilde{x})^2} + \frac{w_2\tilde{x}}{(w_3+\tilde{x})^2} + \frac{w_8\tilde{y}}{(w_9+\tilde{y})^2}, \\ det(J_2^*) &= \left\{ -\tilde{x} + \frac{(1-m)^2\tilde{x}\tilde{y}}{(w_1+(1-m)\tilde{x})^2} + \frac{w_2\tilde{x}}{(w_3+\tilde{x})^2} \right\} \frac{w_8\tilde{y}}{(w_9+\tilde{y})^2} + \frac{w_1w_5(1-m)^2\tilde{x}\tilde{y}}{(w_1+(1-m)\tilde{x})^3}. \end{aligned}$$

Thus the equilibrium point E_2 is locally asymptotically stable if the inequalities

$$w_{10} + \frac{w_{12}}{w_{13}} > \frac{w_{11}\tilde{y}}{w_7+\tilde{y}}, \tag{2.30}$$

$$\frac{(1 - m)^2 \tilde{x}\tilde{y}}{(w_1 + (1 - m)\tilde{x})^2} + \frac{w_2 \tilde{x}}{(w_3 + \tilde{x})^2} + \frac{w_8 \tilde{y}}{(w_9 + \tilde{y})^2} < \tilde{x}, \tag{2.31}$$

$$\left\{ \frac{(1 - m)^2 \tilde{x}\tilde{y}}{(w_1 + (1 - m)\tilde{x})^2} + \frac{w_2 \tilde{x}}{(w_3 + \tilde{x})^2} \right\} \frac{w_8 \tilde{y}}{(w_9 + \tilde{y})^2} < \frac{w_1 w_5 (1 - m)^2 \tilde{x}\tilde{y}}{(w_1 + (1 - m)\tilde{x})^3} + \frac{w_8 \tilde{x}\tilde{y}}{(w_9 + \tilde{y})^2}, \tag{2.32}$$

are satisfied. If the parameter w_{10} passes through $w_{10}^* \equiv \frac{w_{11}\tilde{y}}{w_7 + \tilde{y}} - \frac{w_{12}}{w_{13}}$, then the equilibrium point E_2 becomes a non-hyperbolic point as $\lambda_{23} = 0$. The Jacobian matrix (2.28) transforms into $J_2^* = (b_{ij})_{3 \times 3}$ with $(b_{33}) = 0$ and other (b_{ij}) the same as given in equation (2.28). Let $V_2 = [v_{21} \ v_{22} \ v_{23}]^T$ and $W_2 = [w_{211} \ w_{212} \ w_{213}]^T$ denote the eigenvectors of the matrix J_2^* and J_2^{*T} , respectively, corresponding to the eigenvalue $\lambda_{23} = 0$. Then we get $V_2 = [\xi_2 v_{23} \ \xi_3 v_{23} \ v_{23}]^T$ and $W_2 = [0 \ 0 \ w_{213}]^T$, where $\xi_2 = -\frac{b_{12}b_{23}}{b_{12}b_{21} - b_{11}b_{22}}$, $\xi_3 = \frac{b_{11}b_{23}}{b_{12}b_{21} - b_{11}b_{22}}$, $v_{23} \neq 0$, and $w_{213} \neq 0$. The partial derivative of F with respect to w_{10} is $\frac{\partial F}{\partial w_{10}} = F_{w_{10}} = [0 \ 0 \ -z]^T$, this implies

$$F_{w_{10}}(E_2, w_{10}^*) = [0 \ 0 \ 0]^T, \text{ and} \tag{2.33}$$

$$W_2^T F_{w_{10}}(E_2, w_{10}^*) = 0. \tag{2.34}$$

Conditions (2.33) and (2.34) state that the saddle node bifurcation is never possible at this equilibrium. So moving further we determine the derivative of $F_{w_{10}}$ with respect to N at (E_2, w_{10}^*) .

$$DF_{w_{10}}(E_2, w_{10}^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow W_2^T [DF_{w_{10}}(E_2, w_{10}^*)V_2] = -w_{213}v_{23}. \tag{2.35}$$

Now D^2F , where $D = d/dN$ is given by

$$D^2F(E_2, w_{10}^*)(V_2, V_2) = \begin{bmatrix} \left(-2 + \frac{2w_1(1-m)^2\tilde{y}}{(w_1 + (1-m)\tilde{x})^3} + \frac{2w_2w_3}{(w_3 + \tilde{x})^3} \right) \xi_2^2 v_{23}^2 - \frac{2w_1(1-m)}{(w_1 + (1-m)\tilde{x})^2} \xi_2 \xi_3 v_{23}^2 \\ - \frac{2w_1w_5(1-m)^2\tilde{y}}{(w_1 + (1-m)\tilde{x})^3} \xi_2^2 v_{23}^2 + \frac{2w_1w_5(1-m)}{(w_1 + (1-m)\tilde{x})^2} \xi_2 \xi_3 v_{23}^2 - \frac{2w_6w_7}{w_6(w_7 + \tilde{y})^2} \xi_3 v_{23}^2 + \frac{2w_8w_9}{(w_9 + \tilde{y})^3} \xi_3^2 v_{23}^2 \\ \frac{2w_7w_{11}}{(w_7 + \tilde{y})^2} \xi_3 v_{23}^2 + \frac{2w_{12}}{w_{13}^2} v_{23}^2 \end{bmatrix},$$

which implies

$$W_2^T [D^2F(E_2, w_{10}^*)(V_2, V_2)] = 2 \left[\frac{w_7w_{11}}{(w_7 + \tilde{y})^2} \xi_3 + \frac{w_{12}}{w_{13}^2} \right] v_{23}^2 w_{213}. \tag{2.36}$$

If $\frac{w_7 w_{11}}{(w_7 + \tilde{y})^2} \xi_3 + \frac{w_{12}}{w_{13}^2} \neq 0$, then the system (2.3) accomplishes a transcritical bifurcation at E_2 . Nevertheless, if the condition does not hold then we continue evaluating the third derivative of F with respect to N , which gives

$$D^3 F(E_2, w_{10}^*)(V_2, V_2, V_2) = \left[\begin{aligned} & - \left(\frac{6w_1(1-m)^3 \tilde{y}}{(w_1 + (1-m)\tilde{x})^4} + \frac{6w_2 w_3}{(w_3 + \tilde{x})^4} \right) \xi_2^3 v_{23}^3 + \frac{2w_1(1-m)^2}{(w_1 + (1-m)\tilde{x})^3} \xi_2^2 \xi_3 v_{23}^3 \\ & \frac{6w_1 w_5 (1-m)^3 \tilde{y}}{(w_1 + (1-m)\tilde{x})^4} \xi_2^3 v_{23}^3 - \frac{6w_1 w_5 (1-m)^2}{(w_1 + (1-m)\tilde{x})^3} \xi_2^2 \xi_3 v_{23}^3 - \frac{6w_8 w_9}{(w_9 + \tilde{y})^5} \xi_3^3 v_{23}^3 + \frac{6w_6^2 w_7}{(w_6(w_7 + \tilde{y}))^3} \xi_2^2 v_{23}^3 \\ & - \frac{6w_7 w_{11}}{(w_7 + \tilde{y})^3} \xi_2^2 v_{23}^3 - \frac{6w_{12}^2}{w_{13}^3} v_{23}^3 \end{aligned} \right].$$

Hence $W_2^T [D^3 F(E_2, w_{10}^*)(V_2, V_2, V_2)] = -6 \left[\frac{w_7 w_{11}}{(w_7 + \tilde{y})^3} \xi_2^2 + \frac{w_{12}}{w_{13}^3} \right] v_{23}^3 w_{213}$. The system (2.3) experiences a pitchfork bifurcation if $\frac{w_7 w_{11}}{(w_7 + \tilde{y})^3} \xi_2^2 + \frac{w_{12}}{w_{13}^3} \neq 0$. The system behavior around E_2 is concisely given in Theorem 4.

Theorem 4

- (i) *The equilibrium E_2 is locally asymptotically stable with assumptions (2.32).*
- (ii) *If w_{10} passes through the bifurcation value $w_{10}^* \equiv \frac{w_{11} \tilde{y}}{w_7 + \tilde{y}} - \frac{w_{12}}{w_{13}}$, then the system (2.3) around equilibrium point $E_2 = (\tilde{x}, \tilde{y}, 0)$ has a transcritical bifurcation, provided that $\frac{w_7 w_{11}}{(w_7 + \tilde{y})^2} \xi_3 + \frac{w_{12}}{w_{13}^2} \neq 0$.*
- (iii) *The system experiences a pitchfork bifurcation for $\frac{w_7 w_{11}}{(w_7 + \tilde{y})^3} \xi_2^2 + \frac{w_{12}}{w_{13}^3} \neq 0$. The system has no saddle node bifurcation.*

2.3.4 System Behavior at $E_3 = (x^*, y^*, z^*)$

The variational matrix at $E_3 = (x^*, y^*, z^*)$ is conferred as:

$$J_3 = (c_{ij})_{3 \times 3}, \tag{2.37}$$

where

$$\begin{aligned}
 c_{11} &= x^* \left[-1 + \frac{(1-m)^2 y^*}{(w_1 + (1-m)x^*)^2} + \frac{w_2}{(w_3 + x^*)^2} \right], \quad c_{12} = -\frac{x^*(1-m)}{w_1 + (1-m)x^*}, \\
 c_{13} &= 0, \quad c_{21} = \frac{w_1 w_5 (1-m) y^*}{(w_1 + (1-m)x^*)^2}, \quad c_{22} = y^* \left[\frac{w_6 z^*}{(w_6(w_7 + y^*))^2} + \frac{w_8}{(w_9 + y^*)^2} \right], \\
 c_{23} &= -\frac{y^*}{w_6(w_7 + y^*)}, \quad c_{31} = 0, \quad c_{32} = z^* \left[\frac{w_7 w_{11}}{(w_7 + y^*)^2} \right], \quad c_{33} = \frac{w_{12} z^*}{(w_{13} + z^*)^2}.
 \end{aligned}$$

The characteristic equation for the matrix (2.37) is

$$\lambda_*^3 + A_1 \lambda_*^2 + B_1 \lambda_* + C_1 = 0, \tag{2.38}$$

where

$$\begin{aligned}
 A_1 &= -(c_{11} + c_{22} + c_{33}), \\
 B_1 &= c_{11}(c_{22} + c_{33}) + c_{22}c_{33} - c_{12}c_{21} - c_{23}c_{32}, \\
 C_1 &= c_{11}(c_{23}c_{32} - c_{22}c_{33}) + c_{12}c_{21}c_{33}.
 \end{aligned}$$

If $A_1 > 0$, $B_1 > 0$, $C_1 > 0$ and $A_1 B_1 > C_1$, then the equation (2.38) has three roots, where the real parts of these roots are negative, as reported by Routh-Hurwitz criterion (Perko 2001). Thus the equilibrium point $E_3 = (x^*, y^*, z^*)$ is locally asymptotically stable if the following conditions hold:

$$\frac{y^* z^*}{w_6 R_3^{*2}} + \frac{w_8 y^*}{R_4^{*2}} + \frac{w_{12} z^*}{R_5^{*2}} + \frac{(1-m)^2 x^* y^*}{R_1^{*2}} + \frac{w_2 x^*}{R_2^{*2}} < x^*, \tag{2.39}$$

$$w_7 w_{11} R_4^{*2} R_5^{*2} < w_{12} R_3^* \left(z^* R_4^{*2} + w_6 w_8 R_3^{*2} \right), \tag{2.40}$$

where $R_1^* = w_1 + (1-m)x^*$, $R_2^* = w_3 + x^*$, $R_3^* = w_7 + y^*$, $R_4^* = w_9 + y^*$, and $R_5^* = w_{13} + z^*$.

Next we use a Lyapunov function to prove the global stability of E_3 . Global stability means that any trajectories of the dynamical system ultimately tend to the attractor of the system for any initial conditions.

Theorem 5 *The unique non-trivial positive equilibrium point $E_3 = (x^*, y^*, z^*)$ is globally asymptotically stable in $\chi = \{(x, y, z) \in R_+^3 : 0 \leq x \leq 1; 0 \leq y \leq \eta; 0 \leq z \leq v \text{ for } (\eta, v) > 0\}$ with respect to all the solutions with positive initial conditions, provided the following inequalities hold*

$$w_1 w_2 R_1^* + (1-m)^2 w_3 R_2^* y^* < w_1 w_3 R_1^* R_2^*, \tag{2.41}$$

$$w_5 (1-m)(w_9 + \eta) R_4^* < w_1 w_4 (w_9 + \eta) R_4^* + w_1 w_8 w_9, \tag{2.42}$$

$$w_{11} \eta (w_{13} + v) < w_7 (w_{10} (w_{13} + v) R_5^* + w_{12} w_{13}). \tag{2.43}$$

Proof Consider the Lyapunov function

$$V = \left[x - x^* - x^* \ln \left(\frac{x}{x^*} \right) \right] + \frac{\Phi}{2} [y - y^*]^2 + \frac{\Psi}{2} [z - z^*]^2, \tag{2.44}$$

where Φ and Ψ are positive constants whose values will be evaluated later. Here the function V is positive definite as $V(x^*, y^*, z^*) = 0$ and $V(x, y, z) > 0$ for all $(x, y, z) \neq (x^*, y^*, z^*)$. The time derivative of V is derived as follows:

$$\begin{aligned} \frac{dV}{dt} = & -\frac{1}{2}v_{11}(x - x^*)^2 + v_{12}(x - x^*)(y - y^*) - \frac{1}{2}v_{22}(y - y^*)^2 \\ & - \frac{1}{2}v_{11}(x - x^*)^2 + v_{13}(x - x^*)(z - z^*) - \frac{1}{2}v_{33}(z - z^*)^2 \\ & - \frac{1}{2}v_{22}(y - y^*)^2 + v_{23}(y - y^*)(z - z^*) - \frac{1}{2}v_{33}(z - z^*)^2, \end{aligned} \tag{2.45}$$

where

$$\begin{aligned} v_{11} &= 1 - \frac{(1 - m)^2 y^*}{R_1 R_1^*} - \frac{w_2}{R_2 R_2^*}, \\ v_{12} &= \frac{w_1 w_5 (1 - m) y^* \Phi}{R_1 R_1^*} - \frac{(1 - m) w_1 + (1 - m)^2 x^*}{R_1 R_1^*}, \quad v_{13} = 0, \\ v_{22} &= \Phi \left(w_4 - \frac{w_1 w_5 (1 - m) x + w_5 (1 - m)^2 x x^*}{R_1 R_1^*} + \frac{w_7 z^*}{w_6 R_3 R_3^*} + \frac{w_8 w_9}{R_4 R_4^*} \right), \\ v_{23} &= \frac{w_7 w_{11} z^* \Psi}{R_3 R_3^*} - \Phi \frac{w_7 y^* + y y^*}{w_6 R_3 R_3^*}, \quad v_{33} = \Psi \left(w_{10} + \frac{w_{12} w_{13}}{R_5 R_5^*} - \frac{w_7 w_{11} y + w_{11} y y^*}{R_3 R_3^*} \right), \end{aligned}$$

with $R_1 = w_1 + (1 - m)x$, $R_2 = w_3 + x$, $R_3 = w_7 + y$, $R_4 = w_9 + y$, $R_5 = w_{13} + z$, while the values of R_1^* , R_2^* , R_3^* , R_4^* , and R_5^* are the same as given directly after equation (2.40). $\frac{dV}{dt}$ is negative definite if the following conditions hold:

$$(i) \quad v_{11} > 0, \quad v_{22} > 0, \quad v_{33} > 0, \tag{2.46}$$

$$(ii) \quad v_{12}^2 < v_{11} v_{22}, \tag{2.47}$$

$$(iii) \quad v_{13}^2 < v_{11} v_{33}, \tag{2.48}$$

$$(iv) \quad v_{23}^2 < v_{11} v_{22}. \tag{2.49}$$

If conditions (2.41), (2.42) and (2.43) hold, then the inequalities given in equation (2.46) are automatically true. Now putting $\Phi = \frac{R_1^*}{w_1 w_5 y^*}$ and $\Psi = \frac{\Phi}{w_6 w_7 w_{11} z^*} (w_7 + \eta) y^*$, we get the conditions (2.47) and (2.49) are satisfied, while the condition (2.48) is

obvious (as $v_{13} = 0$); this shows that $\frac{dV}{dt}$ is negative definite. Hence E_3 is globally asymptotically stable, following Lyapunov’s first theorem. This completes the proof. \square

A Hopf bifurcation arises when a periodic solution or limit cycle appears or fades as a related parameter varies around an equilibrium point. Here we take w_{12} as a bifurcation parameter. In the next theorem, we state that the system (2.3) encounters a Hopf bifurcation around E_3 when w_{12} passes through a critical parameter value w_{12}^* .

Theorem 6 *A Hopf bifurcation appears near the positive equilibrium point $E_3 = (x^*, y^*, z^*)$ for the predator-prey model (2.3) whenever parameter w_{12} attains the critical value w_{12}^* , provided the following conditions*

- (i) $A_1(w_{12}^*) > 0, B_1(w_{12}^*) > 0, C_1(w_{12}^*) > 0,$
- (ii) $A_1(w_{12}^*)B_1(w_{12}^*) - C_1(w_{12}^*) = 0,$
- (iii) $\left. \frac{d\phi(w_{12})}{dw_{12}} \right|_{w_{12}=w_{12}^*} \neq 0$

hold. Here A_1, B_1 and C_1 are coefficients of the characteristic polynomial (2.38) with $\phi = A_1B_1 - C_1$.

Proof Here we choose bifurcation parameter as w_{12} . Let there exists a critical value $w_{12} = w_{12}^*$, then $A_1(w_{12}^*) > 0, B_1(w_{12}^*) > 0, C_1(w_{12}^*) > 0,$ and $\phi(w_{12}^*) = A_1(w_{12}^*)B_1(w_{12}^*) - C_1(w_{12}^*) = 0$. System (2.3) at the equilibrium point E_3 has one negative root and two purely imaginary roots if $\phi(w_{12}^*) = 0$ i.e., $A_1(w_{12}^*)B_1(w_{12}^*) - C_1(w_{12}^*) = 0$. For the Hopf-bifurcation at $w_{12} = w_{12}^*$, the characteristic equation must be of the form

$$\lambda^3(w_{12}) + A_1(w_{12})\lambda^2(w_{12}) + B_1(w_{12})\lambda(w_{12}) + C_1(w_{12}) = 0. \tag{2.50}$$

Eq. (2.50) has three roots $\lambda_1(w_{12}) = i\sqrt{B_1(w_{12})}, \lambda_2(w_{12}) = -i\sqrt{B_1(w_{12})},$ and $\lambda_3(w_{12}) = -A_1(w_{12})$. Now the main motive is to show the transversality condition $\left. \frac{d\phi(w_{12})}{dw_{12}} \right|_{w_{12}=w_{12}^*} \neq 0$.

Let $\lambda_j(w_{12}) = \alpha(w_{12}) \pm i\beta(w_{12}), j = 1, 2$. Further substituting values of $\lambda_j(w_{12})$ in the equation (2.50) and differentiating with respect to w_{12} , we get

$$\begin{cases} \phi_1(w_{12})\alpha'(w_{12}) - \phi_2(w_{12})\beta'(w_{12}) + p(w_{12}) = 0, \\ \phi_2(w_{12})\alpha'(w_{12}) + \phi_1(w_{12})\beta'(w_{12}) + q(w_{12}) = 0, \end{cases} \tag{2.51}$$

where

$$\begin{aligned} \phi_1(w_{12}) &= 3\alpha^2(w_{12}) + 2A_1(w_{12})\alpha(w_{12}) + A_2(w_{12}) - 3\beta^2(w_{12}), \\ \phi_2(w_{12}) &= 6\alpha(w_{12})\beta(w_{12}) + 2A_1(w_{12})\beta(w_{12}), \\ p(w_{12}) &= \alpha^2(w_{12})A'_1(w_{12}) + A'_2(w_{12})\alpha(w_{12}) + A'_3(w_{12}) - A'_1(w_{12})\beta^2(w_{12}), \\ q(w_{12}) &= 2\alpha(w_{12})\beta(w_{12})A'_1(w_{12}) + A'_2(w_{12})\beta(w_{12}). \end{aligned}$$

Now solving equation (2.51), we get

$$Re \left[\frac{d\lambda_j}{dw_{12}} \right]_{w_{12}=w_{12}^*} = -\frac{\phi_1 p + \phi_2 q}{\phi_1^2 + \phi_2^2} \neq 0, \tag{2.52}$$

where $\phi_1(w_{12}^*)p(w_{12}^*) + \phi_2(w_{12}^*)q(w_{12}^*) \neq 0$ and $\lambda_3(w_{12}^*) = -A_1(w_{12}^*)$. Hence the transversality condition

$$\left. \frac{d\phi(w_{12})}{dw_{12}} \right|_{w_{12}=w_{12}^*} \neq 0 \text{ holds, which implies that a Hopf-bifurcation occurs at } w_{12} = w_{12}^*. \quad \square$$

2.4 Persistence

The permanence or persistence of a system means that all the species of the given model system should exist and none of them will go to extinction. Using the average Lyapunov function (Gard and Hallam 1979), uniform persistence conditions of the system (2.3) are evaluated and given in the following theorem:

Theorem 7 *If the following conditions*

$$w_2 < w_3, \tag{2.53}$$

$$\frac{w_5(1-m)\bar{x}}{w_1 + (1-m)\bar{x}} > w_4 + \frac{w_8}{w_9}, \tag{2.54}$$

$$\frac{w_{11}\tilde{y}}{w_7 + \tilde{y}} > w_{10} + \frac{w_{12}}{w_{13}}, \tag{2.55}$$

are satisfied, system (2.3) is persistent.

Proof Consider the following function

$$\varphi(x, y, z) = x^\alpha y^\beta z^\gamma, \tag{2.56}$$

where $\alpha, \beta,$ and γ are positive constant.

$\varphi(x, y, z) > 0$ for all $(x, y, z) \in \text{int}(\mathbb{R}_+^3)$ and $\varphi(x, y, z) \rightarrow 0$ when x or y or $z \rightarrow 0$. Differentiating (2.56) w. r. t. time (t), we get

$$\begin{aligned}
 \frac{\varphi'}{\varphi} &= \frac{\alpha}{x} \frac{dx}{dt} + \frac{\beta}{y} \frac{dy}{dt} + \frac{\gamma}{z} \frac{dz}{dt} \\
 &= \alpha \left[(1-x) - \frac{(1-m)y}{w_1 + (1-m)x} - \frac{w_2}{w_3 + x} \right] \\
 &\quad + \beta \left[-w_4 + \frac{w_5(1-m)x}{w_1 + (1-m)x} - \frac{z}{w_6(w_7 + y)} - \frac{w_8}{w_9 + y} \right] \\
 &\quad + \gamma \left[-w_{10} + \frac{w_{11}y}{w_7 + y} - \frac{w_{12}}{w_{13} + z} \right].
 \end{aligned} \tag{2.57}$$

For the system (2.3) to be persistent, the values of $\frac{\varphi'}{\varphi}$ must be positive at all possible equilibrium points of the system for some positive values of α , β , and γ . Now

$$\begin{aligned}
 \frac{\varphi'}{\varphi}(E_0) &= \alpha \left[1 - \frac{w_2}{w_3} \right] + \beta \left[-w_4 - \frac{w_8}{w_9} \right] + \gamma \left[-w_{10} - \frac{w_{12}}{w_{13}} \right], \\
 \frac{\varphi'}{\varphi}(E_1) &= \beta \left[-w_4 + \frac{w_5(1-m)\bar{x}}{w_1 + (1-m)\bar{x}} - \frac{w_8}{w_9} \right] + \gamma \left[-w_{10} - \frac{w_{12}}{w_{13}} \right], \\
 \frac{\varphi'}{\varphi}(E_2) &= \gamma \left[-w_{10} + \frac{w_{11}\tilde{y}}{w_7 + \tilde{y}} - \frac{w_{12}}{w_{13}} \right].
 \end{aligned} \tag{2.58}$$

$\frac{\varphi'}{\varphi}(E_0) > 0$, if the condition (2.53) holds for β and γ with a sufficiently large α . If the condition (2.54) is satisfied then $\frac{\varphi'}{\varphi}(E_1) > 0$ with sufficiently large values of β with respect to γ , and $\frac{\varphi'}{\varphi}(E_2) > 0$ under the condition (2.55). This completes the proof. □

2.5 Bionomic Equilibrium

In the previous section we have archived the biological equilibrium points for the model system (2.3). Here the economic equilibrium is evaluated for the model system (2.1) and it is calculated when the net revenue earned by trading the harvested biomass (TR) is equal to the inclusive cost of effort employed for harvesting (TC) (Krishna et al. 1998; Kar et al. 2006). Here the total cost is proportional to the harvesting effort, i.e., $TC=CE$, where C is the fishing cost. Also, the total revenue earned from harvesting biomass is proportional to harvesting yield, written as $TR = \frac{p_1 c_1 X}{l_1 E + l_2 X} + \frac{p_2 c_2 Y}{l_3 E + l_4 Y} + \frac{p_3 c_3 Z}{l_5 E + l_6 Z}$, where $\frac{c_1 X}{l_1 E + l_2 X} + \frac{c_2 Y}{l_3 E + l_4 Y} + \frac{c_3 Z}{l_5 E + l_6 Z}$ is the harvesting yield function (Clark 2005) and $p_i, i = 1, 2, 3$ are prices per unit biomass for X, Y and Z fish populations, respectively, with fishing cost C .

The economic revenue (rent) at any time is assumed as

$$\pi(X, Y, Z, E) = TR - TC = \left[\frac{p_1 c_1 X}{l_1 E + l_2 X} + \frac{p_2 c_2 Y}{l_3 E + l_4 Y} + \frac{p_3 c_3 Z}{l_5 E + l_6 Z} - C \right] E. \tag{2.59}$$

Now

$$\begin{aligned} \frac{dX}{dt} = 0 &\Rightarrow X = 0 \text{ or } r \left(1 - \frac{X}{K} \right) - \frac{a_1(1-m)Y}{b_1 + (1-m)X} = \frac{c_1 E}{l_1 E + l_2 X} \\ &\Rightarrow E = \frac{l_2 X \{ (a_1 Y(1-m)/(b_1 + (1-m)X)) - r(1 - X/K) \}}{l_1 \{ r(1 - X/K) - (a_1 Y(1-m)/(b_1 + (1-m)X)) \} - c_1}. \end{aligned}$$

Here the harvesting effort (E) is positive if

$$\frac{a_1 Y(1-m)}{b_1 + (1-m)X} < r \left(1 - \frac{X}{K} \right) < \frac{a_1 Y(1-m)}{b_1 + (1-m)X} + \frac{c_1}{l_1}. \tag{2.60}$$

Again

$$\begin{aligned} \frac{dY}{dt} = 0 &\Rightarrow Y = 0 \text{ or } -d + \frac{e_1 a_1(1-m)X}{b_1 + (1-m)X} - \frac{a_2 Z}{b_2 + Y} = \frac{c_2 E}{l_3 E + l_4 Y} \\ &\Rightarrow E = \frac{l_4 Y \{ d + (a_2 Z/(b_2 + Y)) - (e_1 a_1(1-m)X/(b_1 + (1-m)X)) \}}{l_3 \{ (e_1 a_1(1-m)X/(b_1 + (1-m)X)) - d - (a_2 Z/(b_2 + Y)) \} - c_2}. \end{aligned}$$

For this case, E is positive if

$$d + \frac{a_2 Z}{b_2 + Y} < \frac{e_1 a_1(1-m)X}{b_1 + (1-m)X} < d + \frac{a_2 Z}{b_2 + Y} + \frac{c_2}{l_3}. \tag{2.61}$$

Similarly

$$\begin{aligned} \frac{dZ}{dt} = 0 &\Rightarrow Z = 0 \text{ or } -c + \frac{e_2 a_2 Y}{b_2 + Y} = \frac{c_3 E}{l_5 E + l_6 Z} \\ &\Rightarrow E = \frac{l_6 Z \{ c - (e_2 a_2 Y/(b_2 + Y)) \}}{l_5 \{ (e_2 a_2 Y/(b_2 + Y)) - c \} - c_3}. \end{aligned}$$

For the above equation, E is positive if

$$c < \frac{e_2 a_2 Y}{b_2 + Y} < c + \frac{c_3}{l_5}. \tag{2.62}$$

Hence the bionomic equilibrium solution exists at a point on the following curve

$$\begin{aligned}
 & \frac{l_2 X \{ (a_1 Y(1 - m) / (b_1 + (1 - m)X)) - r(1 - X/K) \}}{l_1 \{ r(1 - X/K) - (a_1 Y(1 - m) / (b_1 + (1 - m)X)) \} - c_1} \\
 &= \frac{l_4 Y \{ d + (a_2 Z / (b_2 + Y)) - (e_1 a_1 (1 - m)X / (b_1 + (1 - m)X)) \}}{l_3 \{ (e_1 a_1 (1 - m)X / (b_1 + (1 - m)X)) - d - (a_2 Z / (b_2 + Y)) \} - c_2} \quad (2.63) \\
 &= \frac{l_6 Z \{ c - (e_2 a_2 Y / (b_2 + Y)) \}}{l_5 \{ (e_2 a_2 Y / (b_2 + Y)) - c \} - c_3}.
 \end{aligned}$$

The bionomic equilibrium $(X_\infty, Y_\infty, Z_\infty, E_\infty)$ of the fishery model is determined by the Eq. (2.63) with the constraint

$$\pi(X, Y, Z, E) = TR - TC = 0 \Rightarrow \frac{p_1 c_1 X}{l_1 E + l_2 X} + \frac{p_2 c_2 Y}{l_3 E + l_4 Y} + \frac{p_3 c_3 Z}{l_5 E + l_6 Z} = C, \quad (2.64)$$

where $X_\infty, Y_\infty, Z_\infty$ and E_∞ are the positive solution of Eqs. (2.63) and (2.64).

2.6 Optimal Harvesting Policy

Here we discuss about the optimal control of fish harvesting with the motivation to gain maximum profit in fisheries. Let the continuous stream function be

$$\zeta = \int_0^\infty \pi(X, Y, Z, E, t) e^{-\delta t} dt, \quad (2.65)$$

where δ is annual rate of discount and $\pi(X, Y, Z, E)$ is given in (2.59). The main objective of this analysis is to find $\max \zeta$ subject to model system (2.1) using Pontryagin’s maximum principle (Pontryagin et al. 1962). Here the control variable $E(t)$ is subject to $0 < E(t) < E_{max}$, where E_{max} is the upper limit of harvesting effort E .

The Hamiltonian function is

$$\begin{aligned}
 H = & \left[\frac{p_1 c_1 X}{l_1 E + l_2 X} + \frac{p_2 c_2 Y}{l_3 E + l_4 Y} + \frac{p_3 c_3 Z}{l_5 E + l_6 Z} - C \right] E e^{-\delta t} \\
 & + \lambda_1 \left[rX \left(1 - \frac{X}{K} \right) - \frac{a_1 (1 - m)XY}{b_1 + (1 - m)X} - \frac{c_1 EX}{l_1 E + l_2 X} \right] \\
 & + \lambda_2 \left[-dY + \frac{e_1 a_1 (1 - m)XY}{b_1 + (1 - m)X} - \frac{a_2 YZ}{b_2 + Y} - \frac{c_2 EY}{l_3 E + l_4 Y} \right] \\
 & + \lambda_3 \left[-cZ + \frac{e_2 a_2 YZ}{b_2 + Y} - \frac{c_3 EZ}{l_5 E + l_6 Z} \right]. \quad (2.66)
 \end{aligned}$$

Here $\lambda_i, i = 1, 2, 3$ represent the adjoint variables, and the adjoint equations are

$$\begin{aligned} \frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial X} = & - \left[\frac{p_1 c_1 l_1 E}{(l_1 E + l_2 X)^2} E e^{-\delta t} + \lambda_1 \left\{ r - \frac{2Xr}{K} - \frac{a_1 b_1 (1-m)Y}{(b_1 + (1-m)X)^2} \right. \right. \\ & \left. \left. - \frac{c_1 l_1 E^2}{(l_1 E + l_2 X)^2} \right\} + \lambda_2 \left\{ \frac{e_1 a_1 b_1 (1-m)Y}{(b_1 + (1-m)X)^2} \right\} \right], \end{aligned} \tag{2.67}$$

$$\begin{aligned} \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial Y} = & - \left[\frac{p_2 c_2 l_3 E}{(l_3 E + l_4 Y)^2} E e^{-\delta t} + \lambda_1 \left\{ -\frac{a_1 b_1 (1-m)X}{(b_1 + (1-m)X)^2} \right\} + \lambda_2 \left\{ \right. \\ & - d + \frac{e_1 a_1 (1-m)X}{b_1 + (1-m)X} \\ & \left. - \frac{a_2 b_2 Z}{(b_2 + Y)^2} - \frac{c_2 l_3 E}{(l_3 E + l_4 Y)^2} \right\} + \lambda_3 \left\{ \frac{e_2 a_2 b_2 Z}{(b_2 + Y)^2} \right\} \right], \end{aligned} \tag{2.68}$$

$$\begin{aligned} \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial Z} = & - \left[\frac{p_3 c_3 l_5 E^2}{(l_5 E + l_6 Z)^2} E e^{-\delta t} + \lambda_2 \left\{ -\frac{a_2 Y}{b_2 + Y} \right\} \right. \\ & \left. + \lambda_3 \left\{ -c + \frac{e_2 a_2 Y}{b_2 + Y} - \frac{c_3 l_5 E^2}{(l_5 E + l_6 Z)^2} \right\} \right]. \end{aligned} \tag{2.69}$$

Now the main motive is to derive an optimal equilibrium solution for the problem. Using bionomic equilibrium solution curve (2.63) in adjoint equations (2.67)–(2.69), we get

$$\begin{aligned} \frac{d\lambda_1}{dt} = \lambda_1 \left[\frac{Xr}{K} - \frac{a_1 (1-m)^2 XY}{(b_1 + (1-m)X)^2} - \frac{c_1 l_2 EX}{(l_1 E + l_2 X)^2} \right] - \lambda_2 \frac{e_1 a_1 b_1 (1-m)Y}{(b_1 + (1-m)X)^2} \\ - \frac{p_1 c_1 l_1 E^2}{(l_1 E + l_2 X)^2} e^{-\delta t}, \end{aligned} \tag{2.70}$$

$$\begin{aligned} \frac{d\lambda_2}{dt} = \lambda_1 \frac{a_1 b_1 (1-m)X}{(b_1 + (1-m)X)^2} - \lambda_2 \left[\frac{a_2 YZ}{(b_2 + Y)^2} + \frac{c_2 l_4 EY}{(l_3 E + l_4 Y)^2} \right] - \lambda_3 \frac{e_2 a_2 b_2 Z}{(b_2 + Y)^2} \\ - \frac{p_2 c_2 l_3 E^2}{(l_3 E + l_4 Y)^2} e^{-\delta t}, \end{aligned} \tag{2.71}$$

$$\begin{aligned} \frac{d\lambda_3}{dt} = \lambda_2 \frac{a_2 Y}{b_2 + Y} - \lambda_3 \frac{c_3 l_6 EZ}{(l_5 E + l_6 Z)^2} \\ - \frac{p_3 c_3 l_5 E^2}{(l_5 E + l_6 Z)^2} e^{-\delta t}. \end{aligned} \tag{2.72}$$

Further rearranging the above system, we get

$$\begin{cases} \left(H_1 - \frac{d}{dt}\right)\lambda_1 - H_2\lambda_2 = He^{-\delta t}, \\ I_1\lambda_1 - \left(I_2 + \frac{d}{dt}\right)\lambda_2 - I_3\lambda_3 = Ie^{-\delta t}, \\ F_2\lambda_2 - \left(F_3 + \frac{d}{dt}\right)\lambda_3 = Fe^{-\delta t}, \end{cases} \tag{2.73}$$

where $H_1 = \frac{Xr}{K} - \frac{a_1(1-m)^2XY}{(b_1 + (1-m)X)^2} - \frac{c_1l_2EX}{(l_1E + l_2X)^2}$, $H_2 = \frac{e_1a_1b_1(1-m)Y}{(b_1 + (1-m)X)^2}$, $H = \frac{p_1c_1l_1E^2}{(l_1E + l_2X)^2}$,
 $I_1 = \frac{a_1b_1(1-m)X}{(b_1 + (1-m)X)^2}$, $I_2 = \frac{a_2YZ}{(b_2 + Y)^2} + \frac{c_2l_4EY}{(l_3E + l_4Y)^2}$, $I_3 = \frac{e_2a_2b_2Z}{(b_2 + Y)^2}$, $I = \frac{p_2c_2l_3E^2}{(l_3E + l_4Y)^2}$, $F_2 = \frac{a_2Y}{b_2 + Y}$,
 $F_3 = \frac{c_3l_6EZ}{(l_5E + l_6Z)^2}$ and $F = \frac{p_3c_3l_5E^2}{(l_5E + l_6Z)^2}$.

Elimination of λ_2 and λ_3 from system of equations (2.73) yield

$$\begin{aligned} \frac{d^3\lambda_1}{dt^3} + (I_2 + F_3 - H_1)\frac{d^2\lambda_1}{dt^2} + (H_2I_1 + I_2F_3 + I_3F_2 - I_2H_1 - F_3H_1)\frac{d\lambda_1}{dt} \\ + (F_3H_3I_1 - I_2F_3H_1 - I_3F_2H_1)\lambda_1 = S_a e^{-\delta t}, \end{aligned} \tag{2.74}$$

where $S_a = (I_2 + F_3)H\delta + (F_3I - I_3F)H_2 - (I_2F_3 + I_3F_2)H - H\delta^2 - H_2I\delta$. The auxiliary equation for Eq. (2.74) is

$$\begin{aligned} n^3 + (I_2 + F_3 - H_1)n^2 + (H_2I_1 + I_2F_3 + I_3F_2 - I_2H_1 - F_3H_1)n \\ + (F_3H_3I_1 - I_2F_3H_1 - I_3F_2H_1) = 0. \end{aligned} \tag{2.75}$$

If $I_2 + F_3 > H_1$, $H_2I_1 + I_2F_3 + I_3F_2 > I_2H_1 + F_3H_1$, $F_3H_3I_1 > I_2F_3H_1 + I_3F_2H_1$ and $(I_2 + F_3 - H_1)(H_2I_1 + I_2F_3 + I_3F_2 - I_2H_1 - F_3H_1) > F_3H_3I_1 - I_2F_3H_1 - I_3F_2H_1$, then the roots of Eq. (2.75) are real and negative or complex conjugate with negative real parts. Hence the general solution of Eq. (2.74) is in the form

$$\lambda_1(t) = A_{11}e^{n_1t} + B_{11}e^{n_2t} + C_{11}e^{n_3t} + \frac{S_a}{P}e^{-\delta t}. \tag{2.76}$$

Here $P = \delta^3 + (I_2 + F_3 - H_1)\delta^2 + (H_2I_1 + I_2F_3 + I_3F_2 - I_2H_1 - F_3H_1)\delta + (F_3H_3I_1 - I_2F_3H_1 - I_3F_2H_1)$. We see that $\lambda_1(t) \rightarrow 0$ as $t \rightarrow \infty$. Similarly, we may show $\lambda_2(t) \rightarrow 0$ and $\lambda_3(t) \rightarrow 0$ as $t \rightarrow \infty$. The shadow prices of populations are

$$\lambda_1(t)e^{\delta t} = A_{11}e^{(n_1+\delta)t} + B_{11}e^{(n_2+\delta)t} + C_{11}e^{(n_3+\delta)t} + \frac{S_a}{P}, \tag{2.77}$$

$$\lambda_2(t)e^{\delta t} = A_{22}e^{(n_1+\delta)t} + B_{22}e^{(n_2+\delta)t} + C_{22}e^{(n_3+\delta)t} + \frac{S_b}{P}, \tag{2.78}$$

$$\lambda_3(t)e^{\delta t} = A_{33}e^{(n_1+\delta)t} + B_{33}e^{(n_2+\delta)t} + C_{33}e^{(n_3+\delta)t} + \frac{S_c}{P}, \tag{2.79}$$

where $S_b = (IF_3 - I\delta - I_3F)H_1 + IF_3\delta - I\delta^2 - I_3F\delta - I_1F_3H + HI_1\delta$ and $S_c = (I_2F - F\delta + F_2I)H_1 + I_2F\delta - F\delta^2 + F_2I\delta - H_2I_1F - HI_1F_2$. The transversality condition at $t \rightarrow \infty$ requires that the shadow prices $\lambda_i(t)e^{\delta t}$ of the populations remain bounded. To satisfy this, we take $A_{ii} = B_{ii} = C_{ii} = 0, i = 1, 2, 3$. Thus, we have

$$\lambda_1(t)e^{\delta t} = \frac{S_a}{P} = \text{constant.} \tag{2.80}$$

Similarly, $\lambda_2(t)e^{\delta t} = \frac{S_b}{P}$ and $\lambda_3(t)e^{\delta t} = \frac{S_c}{P}$ are constant, hence bounded. We know that the Hamiltonian function (see Eq. (2.66)) must be maximized for $E \in [0, E_{max}]$. Therefore assuming the optimal equilibrium doesn't exist at $E = 0$ or $E = E_{max}$, we have the singular control as

$$\begin{aligned} \frac{\partial H}{\partial E} &= \left[\frac{p_1c_1l_2X^2}{(l_1E + l_2X)^2} + \frac{p_2c_2l_4Y^2}{(l_3E + l_4Y)^2} + \frac{p_3c_3l_6Z^2}{(l_5E + l_6Z)^2} - C \right] e^{-\delta t} \\ &\quad - \frac{\lambda_1c_1l_2X^2}{(l_1E + l_2X)^2} - \frac{\lambda_2c_2l_4Y^2}{(l_3E + l_4Y)^2} - \frac{\lambda_3c_3l_6Z^2}{(l_5E + l_6Z)^2} = 0, \end{aligned} \tag{2.81}$$

which implies

$$\frac{\lambda_1c_1l_2X^2}{(l_1E + l_2X)^2} + \frac{\lambda_2c_2l_4Y^2}{(l_3E + l_4Y)^2} + \frac{\lambda_3c_3l_6Z^2}{(l_5E + l_6Z)^2} = e^{-\delta t} \frac{\partial \pi}{\partial E}. \tag{2.82}$$

Equation (2.82) signifies that the total cost of harvest is equal to the total discount of a marginal profit of the static effort. Also, we get the fishing cost C from Eq. (2.81) as

$$\frac{c_1l_2X^2}{(l_1E + l_2X)^2} \left[p_1 - \frac{S_a}{P} \right] + \frac{c_2l_4Y^2}{(l_3E + l_4Y)^2} \left[p_2 - \frac{S_b}{P} \right] + \frac{c_3l_6Z^2}{(l_5E + l_6Z)^2} \left[p_3 - \frac{S_c}{P} \right] = C. \tag{2.83}$$

Equations (2.83) and (2.59) together give the optimum equilibrium solution $(X_\delta, Y_\delta, Z_\delta)$. For $\delta \rightarrow \infty$, we see that $\frac{S_a}{P}, \frac{S_b}{P}$ and $\frac{S_c}{P}$ tend to zero. Hence Eq. (2.83) leads to

$$\frac{c_1l_2X_\infty^2}{(l_1E + l_2X_\infty)^2} + \frac{c_2l_4Y_\infty^2}{(l_3E + l_4Y_\infty)^2} + \frac{c_3l_6Z_\infty^2}{(l_5E + l_6Z_\infty)^2} = C, \tag{2.84}$$

$$\Rightarrow \frac{\partial \pi}{\partial E}(X_\infty, Y_\infty, Z_\infty, E) = 0. \tag{2.85}$$

Equation (2.85) signifies that the fishing cost per unit effort vanishes at high discount rate. Now

$$\frac{\partial \pi}{\partial E} = \frac{p_1c_1l_2X^2}{(l_1E + l_2X)^2} + \frac{p_2c_2l_4Y^2}{(l_3E + l_4Y)^2} + \frac{p_3c_3l_6Z^2}{(l_5E + l_6Z)^2} - C. \tag{2.86}$$

Further, using Eq. (2.83) in Eq. (2.86), we get

$$\frac{\partial \pi}{\partial E} = \frac{S_a}{P} \frac{c_1 l_2 X^2}{(l_1 E + l_2 X)^2} + \frac{S_b}{P} \frac{c_2 l_4 Y^2}{(l_3 E + l_4 Y)^2} + \frac{S_c}{P} \frac{c_3 l_6 Z^2}{(l_5 E + l_6 Z)^2}. \quad (2.87)$$

Here S_a , S_b and S_c are $o(\delta^2)$, whereas P is $o(\delta^3)$. Hence we see that $\frac{\partial \pi}{\partial E}$ is $o(\delta^{-1})$. Thus $\frac{\partial \pi}{\partial E}$ is a decreasing function of δ (≥ 0). Therefore, we can say that $\frac{\partial \pi}{\partial E}$ has its maximum at $\delta = 0$. Hence, it is proved that $\delta = 0$ leads to $\max \zeta$.

3 Numerical Simulations

The prime objective of this section is to determine the complex dynamics including chaos for the dynamical system (2.3) and to explore the global stability conditions numerically. Theoretical studies can never be justified without proper numerical validation. We have conducted several numerical simulations considering multiple data sets and found the biological feasible parameter set (3.1) to be the most appropriate. This parameter set satisfies all the conditions and restrictions of analysis set out in Sect. 2. To analyze the deterministic behaviour and to show the complex dynamics of the model, we have generated phase portraits, time evolutions, Lyapunov exponents w.r.t. time, and bifurcation diagrams. It is noticed that the model system (2.3) shows numerous diverse dynamics varying the parameter w_7 , related to the middle predator's maximum attack rate and half-saturation constant with intrinsic growth rate and carrying capacity of the prey populations, for the following parameter set:

$$\begin{cases} w_1 = 0.4, w_2 = 1.3, w_3 = 2.7, w_4 = 0.09, w_5 = 0.85, w_6 = 3.2, w_7 = 0.3, \\ w_8 = 0.02, w_9 = 0.65, w_{10} = 0.2, w_{11} = 0.69, w_{12} = 0.2, w_{13} = 2, m = 0.5. \end{cases} \quad (3.1)$$

The model (2.3) yields chaotic dynamics for the parameter set with $w_7 = 0.11$ in phase space $\mathbb{R}^3(x, y, z)$ as seen in Fig. 1a. To ensure this strange behavior of system concerning time, we plot its time evolution in Fig. 1b. Chaotic dynamics are very sensitive to initial conditions, which means a small change in initial conditions leads to very diverse dynamics. Studies say that the Lyapunov exponent provides a detailed characterization of chaotic dynamics within dynamical systems (Wolf et al. 1985; Mishra et al. 2019). A system is chaotic if it has a positive Lyapunov exponent, provided that it is not asymptotic to an unstable periodic solution. The values of the Lyapunov exponents are 0.55176, 0.19155 and -0.23735 . Since two values are positive the system is chaotic. This result is shown in Fig. 1c. The dynamics of the Lyapunov exponents are calculated using the Wolf's scheme with the parameter set (3.1) at $w_7 = 0.11$, time $t = 2000$ and time step $h = 0.5$. A stable limit cycle is visible for the parameter value $w_7 = 0.3$ as shown in Fig. 2a. Time evolution concerning the species is given in Fig. 2b. Increasing the parameter value w_7 , i.e. $w_7 = 0.5$ and keeping the other parameter values the same (see Eq. (3.1)), we can see that the dynamics of the system changes towards a stable focus (see Fig. 3a). The time evolutions for the stable focus for the three variables are plotted in Fig. 3b.

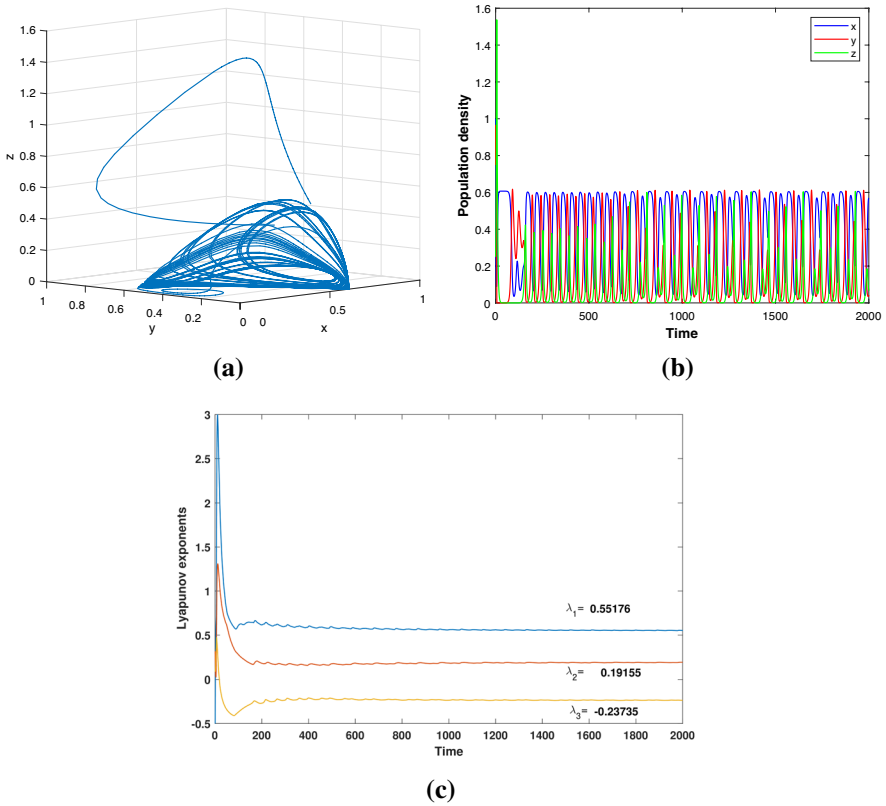


Fig. 1 For parameter set (3.1) with $w_7 = 0.11$, the system (2.3) displays **a** a chaotic attractor in 3D plane, **b** its time evolution, **c** time evolution w.r.t. Lyapunov exponents

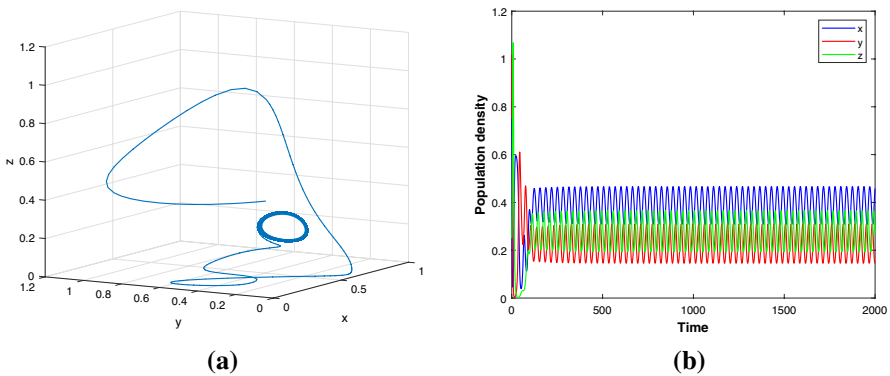


Fig. 2 Attractor of the system in 3D plane for the parameter set (3.1) with $w_7 = 0.3$ exhibits **a** a stable limit cycle, **b** time evolution with respect to populations density

Numerical simulations of any model are required to analyze the behavior of the system whose mathematical model is too complicated to produce analytical solutions. Now we calculate the feasible equilibrium points of the system for precise parameter values. The system approaches towards the stable equilibrium E_0 for the value $w_2 = 1.4$ and $w_3 = 1.3 < w_2$ (other parameter values fixed as in (3.1)) as simulated in Fig. 4a. The prey existing equilibrium $E_1 = (0.2, 0, 0)$ is stable for the parameter values $w_1 = 0.7$, $w_2 = w_3 = 0.8$ (see Fig. 4b). For $w_2 = 1.1$ and $w_3 = 1.3$ with parameter set (3.1), we have calculated the equilibrium point E_2 as $(0.1283, 0.2317, 0)$. Here $tr(J_2^*) = -0.0427 < 0$ and $det(J_2^*) = 0.025 > 0$. Hence this parameter set satisfies the stability conditions as given in Eqs. (2.30)–(2.32) and we have verified this result graphically in Fig. 4c. The system (2.3) is persistent for the parameter set (3.1) as given in Theorem 7. The system approaches towards a globally asymptotically stable interior equilibrium point $E_3 = (0.1304, 0.3812, 0.0276)$ for $w_7 = 0.5$, and with other parameter values as in (3.1). To confirm our global stability conditions, we plot the phase portrait diagram setting different initial conditions as given in Fig. 5a and b. These two diagrams witness the existence of a globally asymptotic stable point which we have found analytically in Theorem 5.

The prey refuge m is an important parameter for our model system. So to understand its effects on the dynamics of the model, we have simulated the phase portrait and bifurcation diagram by taking it as a control parameter and found the following results. Figure 6 describes the effects of prey refuge m on the dynamics of the fishery model (2.3). Prey and middle predator populations display a certain range of oscillation whereas the top predator species goes for extinction at $m = 0.01$. All the species populations show oscillatory behavior for $m = 0.3$. Both the predator species go for extinction for the prey refuge value $m = 0.9$ but the prey population remains stable. Since the variation of m affects the top predator z most we have plotted the bifurcation diagram of population z w.r.t. parameter m in Fig. 7. The system (2.3) exhibits a Hopf bifurcation point (H), a limit point (LP) and two branching point (BP) for the parameter set (3.1). All the generated bifurcation points with eigenvalues are given in Table 2. For the prey refuge parameter value $m < 0.859254$, the

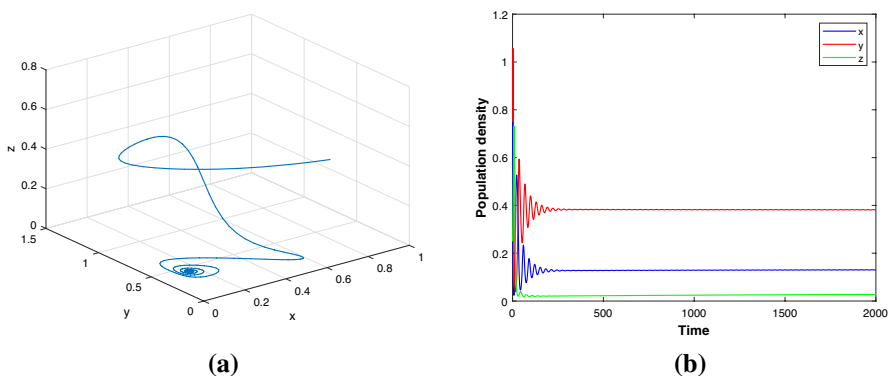


Fig. 3 Attractor of the system in 3D plane for the parameter set (3.1) with $w_7 = 0.5$, the system (2.3) displays **a** a stable focus, **b** time evolution with respect to populations density

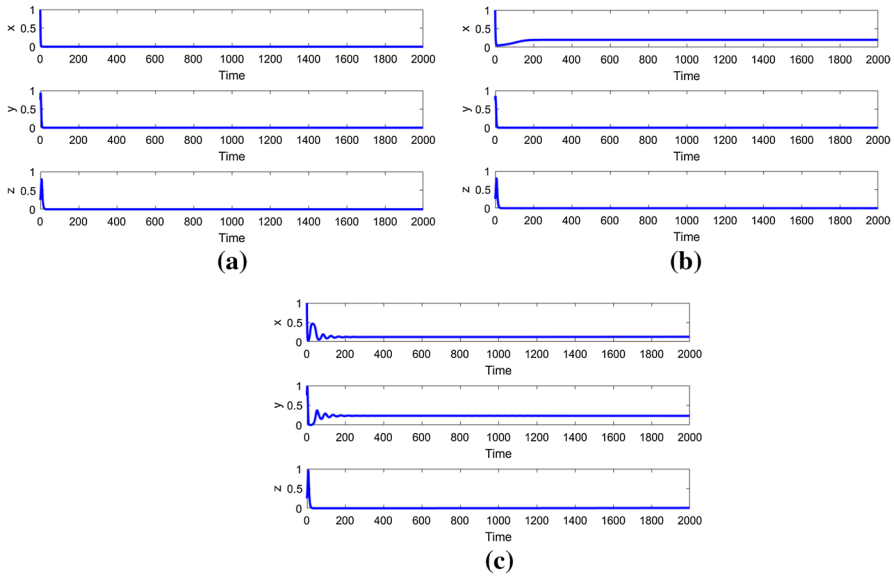


Fig. 4 System (2.3) approaching towards **a** E_0 for $w_2 = 1.4$, $w_3 = 1.3$, **b** $E_1 = (0.2, 0, 0)$ for $w_1 = 0.7$, $w_2 = 0.8$, $w_3 = 0.8$, **c** $E_2 = (0.1283, 0.2317, 0)$ for $w_2 = 1.1$, $w_3 = 1.64$, and other parameter values as given in (3.1)

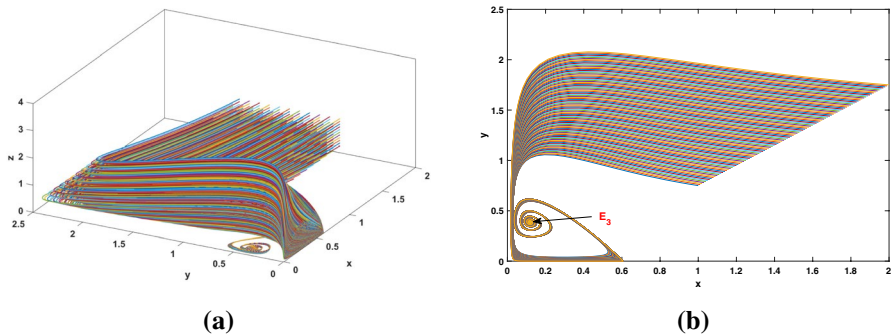


Fig. 5 **a** Phase portrait showing the system (2.3) is globally asymptotically stable around the equilibrium point $E_3(0.1304, 0.3812, 0.0276)$ for the parameter set (3.1) with $w_7 = 0.5$, **b** 2D view of global stability

positive equilibrium point is unstable, whereas for $m \geq 0.859254$ the equilibria proceed towards stability but the extinction of z happens.

We have plotted bifurcation diagrams concerning different parameters for the system (2.3) as exhibited in Figs. 8, 9, 10, 11, 12 and 13. In Fig. 8, bifurcation diagrams for prey-predator populations x , y and z concerning to the parameter w_2 are generated in the range $0.00 \leq w_2 \leq 2.0$. Here w_2 is related to harvesting effort, prey growth rate and the carrying capacity of prey. The top predator concedes toward extinction for $w_2 > 1.78$ and the dynamics of the system approaches toward E_2 .

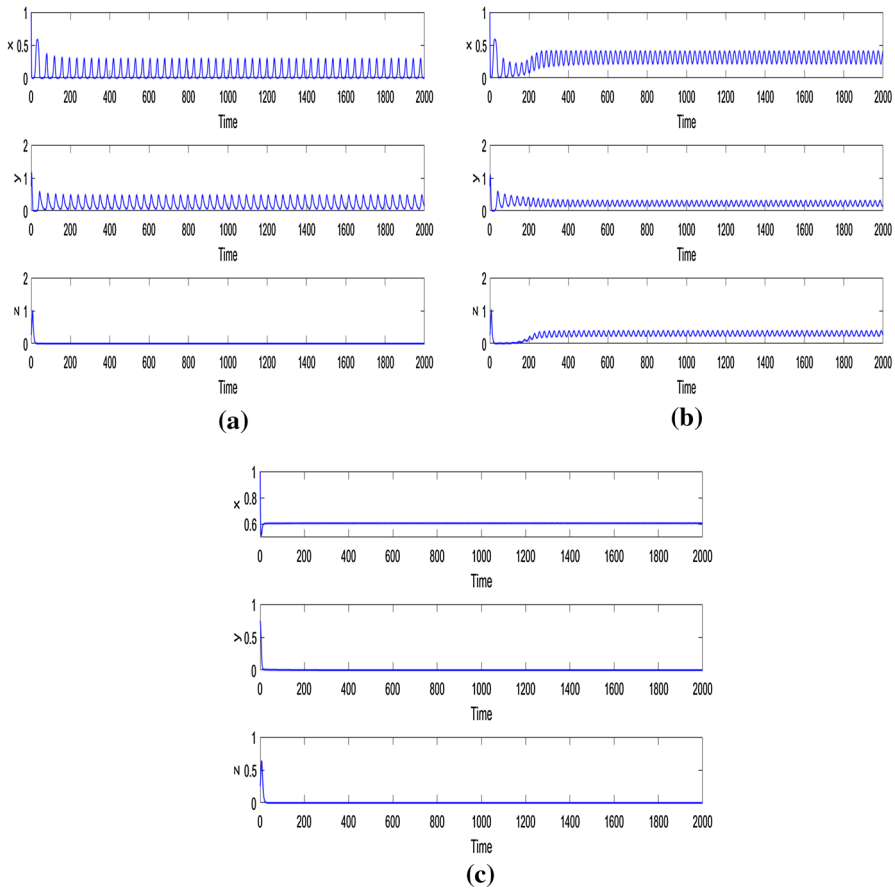


Fig. 6 Dynamical changes of fish populations for prey refuge (m) values, **a** 0.01, **b** 0.3, **c** 0.9. All parameters are given in equation (3.1)

Fig. 9 portrays the bifurcation diagrams for prey-predator populations x , y and z concerning to the parameter w_3 for the interval $[2.0, 4.0]$. This bifurcation diagram shows that the dynamics of the proposed model changes from stable focus to chaos with increasing parameter value of w_3 . Bifurcation diagrams for successive maxima of prey, middle and top predator population densities are generated in the interval $[0.0, 0.6]$, $[0.0, 1.0]$, and $[0.0, 0.6]$ respectively, with respect to parameter w_4 in the range $0.0 \leq w_4 \leq 0.2$ as displayed in Fig. 10. The system depicts larger periodic and chaotic oscillations for $w_4 \in (0, 0.1)$, whereas for $w_4 \in (0.1, 0.2)$ it shows stable behavior. In Fig. 11 bifurcation diagrams are drawn for the fish populations x , y and z in the interval $[0.0, 0.7]$ varying the parameter w_7 in $0.05 \leq w_7 \leq 0.5$. Fig. 12 illustrates the bifurcation diagrams of x , y and z in the interval $[0.0, 0.7]$, $[0.0, 0.65]$, and $[0.0, 1.1]$ respectively, with respect to the parameter w_{10} in the range $0.001 \leq w_{10} \leq 0.3$.

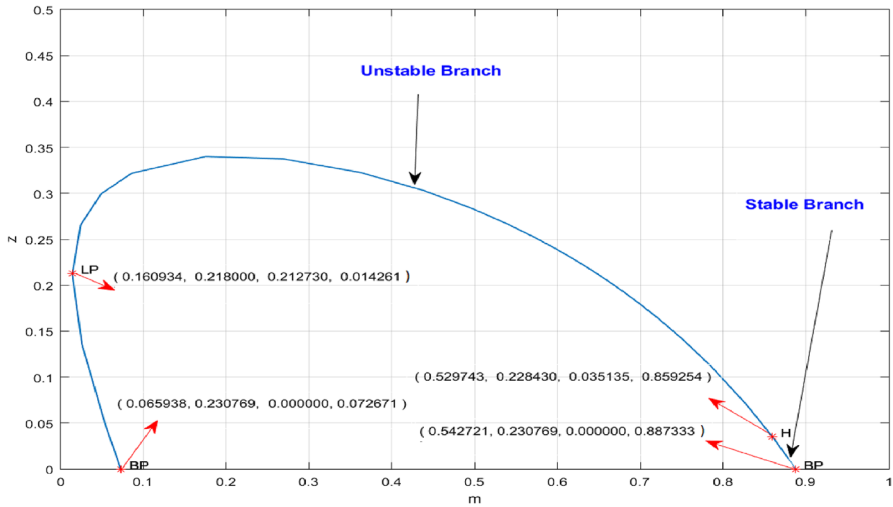


Fig. 7 Bifurcation diagram of z w.r.t. refuge parameter m for the parameter set (3.1)

Table 2 Bifurcation and stability analysis of Fig. 7 for the parameter set (3.1)

Label	Bifurcation point	Equilibrium $E^* = (x^*, y^*, z^*)$	Eigenvalues
H	$m = 0.859254$	$(0.529743, 0.228430, 0.035135)$	$\lambda_1 = -0.436479$ $\lambda_{2,3} = -2.18594e - 07 \pm 0.060406i$
BP	$m = 0.887333$	$(0.542721, 0.230769, 0.000000)$	$\lambda_1 = -0.456222$ $\lambda_2 = -0.005977$ $\lambda_3 = -3.20932e - 13$
LP	$m = 0.014261$	$(0.160934, 0.218000, 0.212730)$	$\lambda_1 = 1.62694e - 07$ $\lambda_{2,3} = 0.021175 \pm 0.29385i$
BP	$m = 0.072671$	$(0.065938, 0.230769, 0.000000)$	$\lambda_1 = -5.5902e - 17$ $\lambda_{2,3} = 0.006374 \pm 0.212996i$

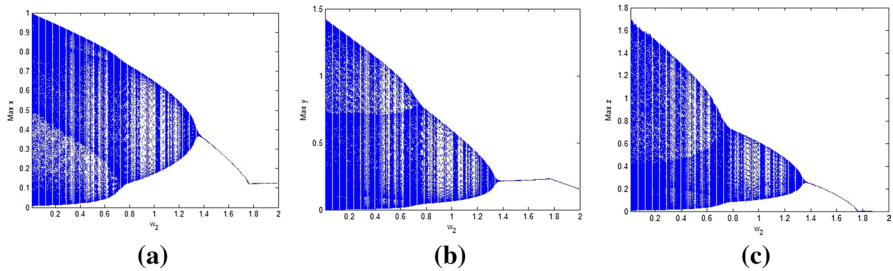


Fig. 8 Bifurcation diagram of max x , max y and max z for the system (2.3) with respect to w_2 is plotted in the parameter space $(0, 2)$ for the parameters given in equation (3.1)

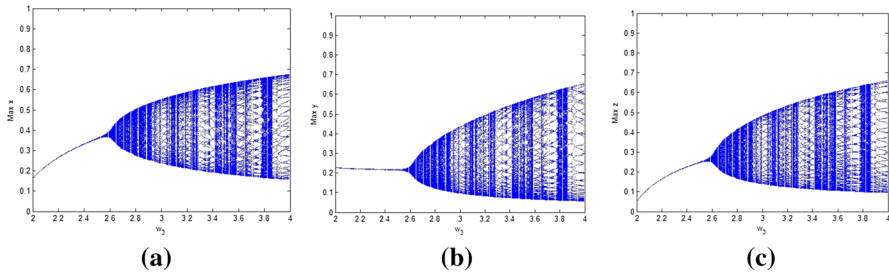


Fig. 9 Bifurcation diagram of max x , max y and max z concerning w_3 for the system (2.3)

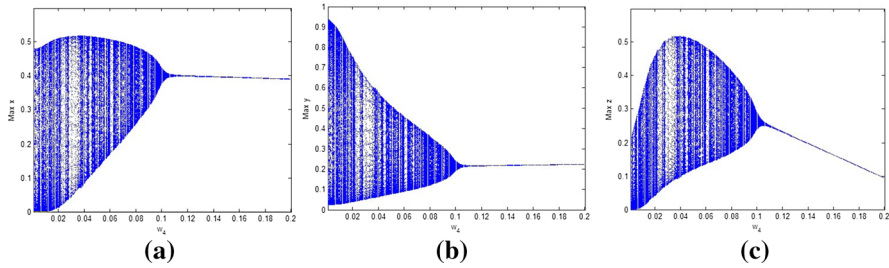


Fig. 10 Bifurcation diagram of max x , max y and max z concerning parameter w_4 for the system (2.3)

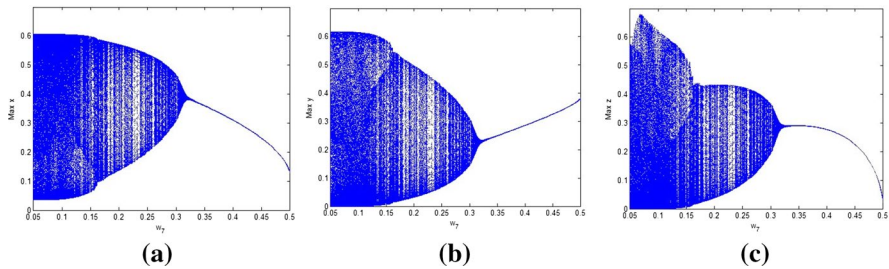


Fig. 11 Bifurcation diagram of max x , max y and max z concerning parameter w_7 for the system (2.3) is illustrated in the range $0.05 \leq w_7 \leq 0.5$

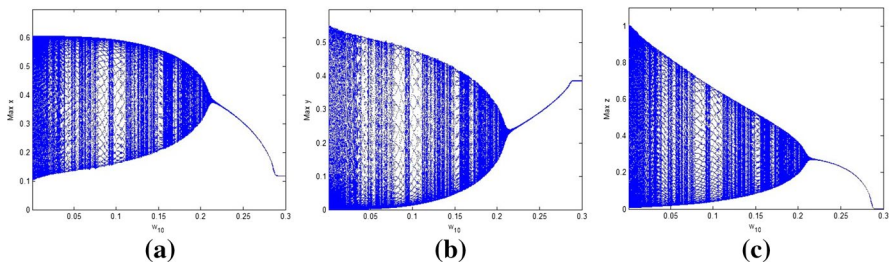


Fig. 12 Bifurcation diagram of max x , max y and max z with respect to w_{10} for the system (2.3) is drawn in the range $0.001 \leq w_{10} \leq 0.3$

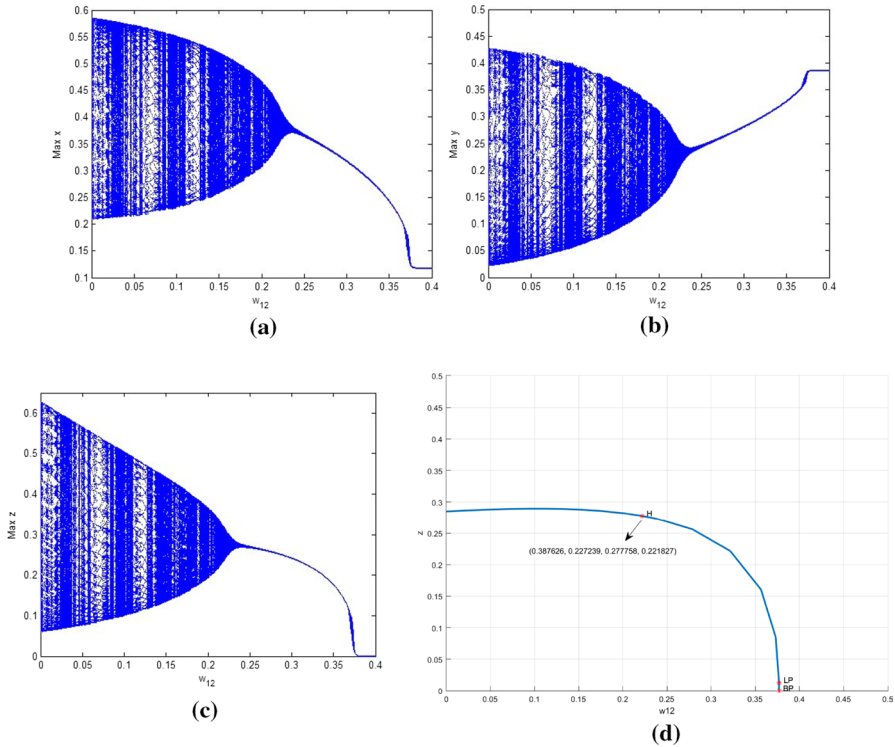


Fig. 13 Bifurcation diagrams for the model system (2.3) with respect to w_{12} for the parameters given in equation (3.1)

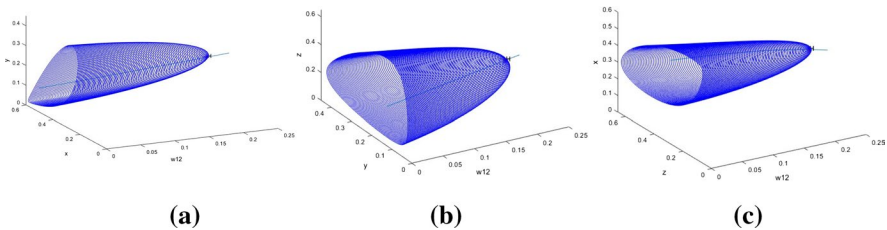


Fig. 14 Hopf-Bifurcation diagram starting from Hopf point $(0.387626, 0.227239, 0.277758, 0.221827)$ with respect to w_{12} in **a** $x - y$ plane, **b** $y - z$ plane, **c** $x - z$ plane

The model system (2.3) encounters a Hopf bifurcation at the critical value $w_{12} = w_{12}^* \approx 0.221827$ as given in Fig. 13d. The other highlighted points in this figure have negative coordinate values, so we have omitted their existence and significance. The Hopf-bifurcation diagram for the model system (2.3) with respect to the parameter w_{12} is shown in Fig. 14. Here the Hopf point (x, y, z, w_{12}) is found at $(0.387626, 0.227239, 0.277758, 0.221827)$, assuming the interior equilibrium point E_3 as the starting point, with the other parameter values the same as in (3.1).

The first Lyapunov coefficient is $-8.000308e - 01$, hence the Hopf-bifurcation is of super-critical type. This numeric outcome supports the analytical result obtained in Theorem 6. From the bifurcation diagram (see Fig. 13), it is clear that the system is stable for the upper threshold value of w_{12}^* , that is for $w_{12} > 0.24$, and chaotic below the critical threshold value of the parameter $w_{12}^* = w_{12} < 0.24$. The top predator (z) goes extinct for the value $w_{12} > 0.38$. We have observed that for value $w_{12} > 0.24$, the prey and the top predator population decrease, whereas the middle predator population increases to a certain value and then proceeds to a fixed growth.

4 Real-World Application

Lake Victoria occupies 43% of the surface area of Uganda, 6% of Kenya, and 51% of Tanzania (Geheb et al. 2003; NPFMP 2015b). This lake is the home for various fish species with diverse feeding habits. For example, Nile perch are piscivore with demersal habitat while Nile tilapias are omnivores. Silver cyprinids are zooplanktivore with pelagic habitat and haplochromines are benthopelagic with variable feeding mode. Silver cyprinid (locally known as Dagua) and haplochromines are native fish of the lake while perch and Nile tilapia were introduced later in 1950s and 1960s. These non-native fish populations dominated the food web of Lake Victoria along with endemic cyprinids. However, the growth of these species disrupted the lake biodiversity. Although Silver cyprinid fishery exploded, the real concern is the fluctuations of perch from 2011 (Glaser et al. 2019). This decline of the catch is maybe due to overfishing or a raise in the fishing efforts which directly or indirectly affects the ecosystem of the lake. The annual catch of Nile perch is noted in Fig. 15 (source LVFO NPFMP 2015a). We know that increase in catches doesn't render the population rise of species but it shows the tread-off efficiency of fishing. Figures 8 and 12 showcase the fluctuations of the top predator concerning harvesting efforts.

Silver cyprinid is the only native fish in Lake Victoria that persisted and turned out to be a sustainable and profitable fishery after 1980s ecological revolutions (LVFO 2016). This endurance of the silver cyprinid in the lake may be due to the

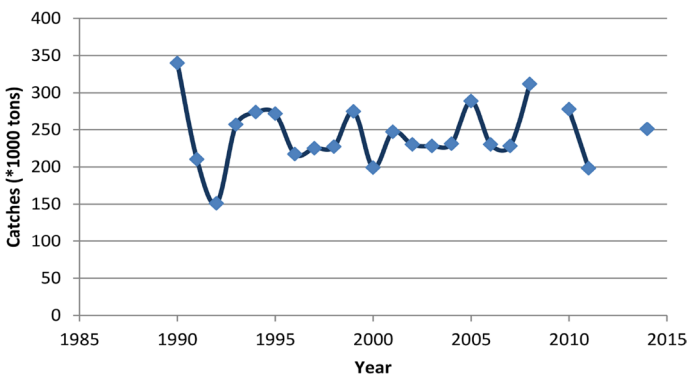


Fig. 15 Annual catches of Nile Perch in Lake Victoria (1990–2014). Source LVFO, NPFMP (2015a)

factor of prey refuge. Figure 6 also clarifies that the refuge parameter continuously enhances the sustainability of the prey population. Dagua production exhibits a steady surge in the volume and Tanzania was the largest producer of dagaa with 289,873 tons in 2010, extending to 433,845 tons in 2015 (Fig. 16, source LVFO 2016). Fishing provides maximum employment and livelihood for these African countries. The most challenging fact is to maximize economic gains by maintaining the ecological structure. According to the LVFO reports (NPFMP 2015a, b), 1 million fish are harvested every year worth of US \$ 600–850 million by selling the fish only. Nile perch, tilapias, cyprinid, and haplochromine cichlids are the most commercial fishes in Lake Victoria. So this type of harvesting model can be valuable for the fishing industry. Also, by inflicting this model, we can predict the expected synopses according to our necessities.

5 Conclusions

In this paper, a three species predator-prey fishery model system consisting of a Holling type-II functional response function is formulated and examined analytically as well as numerically. We have studied the dynamics of the model considering external factors such as prey refuge and non-linear harvesting. Also we have checked the existence, uniqueness, and uniformly boundedness conditions of the solutions of model (2.3). Stability criteria for the existing equilibrium points are determined. The persistence of the system is studied using the average Lyapunov method. The local bifurcation analysis at each equilibrium point is implemented according to conditions given in Sotomayor's theorem. The existence of a Hopf-bifurcation about the internal equilibrium point E_3 is investigated analytically and numerically by choosing the bifurcation parameter as w_{12} . The proposed model (2.3) is numerically illustrated by taking a feasible parameter set (3.1), and further analytical findings are justified. The non-linear or Michaelis–Menten type

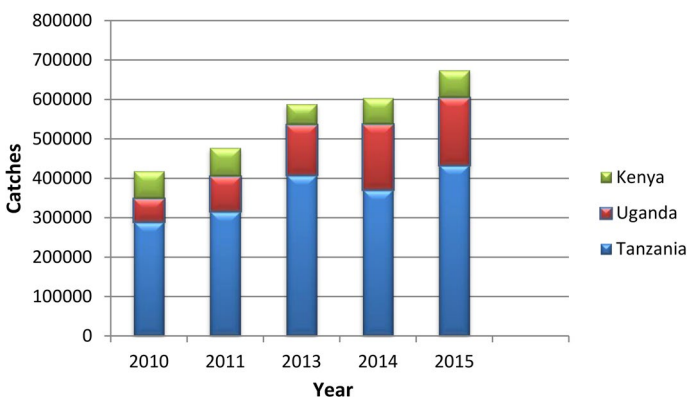


Fig. 16 Catches of Silver cyprinid in tonnes over year 2010–2015 on Lake Victoria. Data source LVFO (2016)

of harvesting is used by many researchers (see Krishna et al. 1998; Purohit and Chaudhuri 2004; Clark 2005; Das et al. 2009; Gupta et al. 2012). However, a three-species predator-prey model with all the populations having non-linear harvesting are new and also complicated. Nevertheless, we have used this type of harvesting in our proposed model and obtained conditions for the existence of a bionomic equilibrium point as well as for optimal harvesting control.

The key element of our proposed model is parameter w_7 , which is a combination of factors like the maximum attack rate and half-saturation constant of the middle predator, the growth rate of the prey, and the carrying capacity of the environment. From the above observations we can say that these four parameters influence the dynamics of the model system. Concerning this parameter, we have plotted the bifurcation diagrams and observed that for $w_7 < 0.3$ the system oscillates in a certain range showing chaotic behavior; for $w_7 > 0.3$ the dynamics of the system proceed towards stability (see Fig. 11). One more important analysis is that for $w_7 > 0.3$ populations of prey and top predator decrease continuously, whereas the middle predator population increases. The dynamics of the proposed model (2.3) has witnessed the period-doubling route to chaos in Fig. 9, and period-halving route to a limit cycle in Figs. 8, 10, 11, 12 and 13. The parameter w_3 , which is related to the harvesting effort and the carrying capacity, also has a great impact on the dynamics of the model. Increasing the value of w_3 by increasing the harvesting effort while keeping the carrying capacity fixed (as well as the positive constants l_1 and l_2), makes the system loose its stability, and the model dynamics changes from stable to chaotic (see Fig. 9). The parameter w_{12} puts the top predator population down. As we can see, increasing this parameter value decreases the population density z . Finally, the population tends towards extinction for $w_{12} > 0.37$ as seen in Fig. f13c. The stable positive equilibrium can never be transformed into an unstable state by increasing a predetermined amount of prey refuge, which agrees with Kar (2006) and González-Olivare and Ramos-Jiliberto (2003). Figure 6 signifies that for the larger values of refuge parameter m (with $0 < m < 1$), the state of the system switches to prey existing equilibrium E_1 . Gradually increasing the coefficient of refuge, we can see that the predator goes for extinction. This result shows the realistic picture of nature that due to certainty of prey species and deduction in predation, the prey population increases whereas the predator volume decreases towards extinction, which is in close agreement with former results (Gupta et al. 2012; Haque and Sarwardi 2018).

Fishing industries in Lake Victoria export millions of dollars of fish, where the majority of the catch is Nile perch (LVFO, NPFMP 2015b). We have observed the variation in catches of perch in Fig. 15. Overfishing or over-harvesting may cause some species to disappear from the lake ecology (Geheb et al. 2003). Our model also expresses the same concern as we see that higher harvesting efforts diminish the growth of Nile perch. We have discerned the variations in catches of this fish in Fig. 15. The vital observation regarding this study is that the non-linear harvesting and refuge factors have accomplished our model more pragmatic and realistic. The researchers may be motivated by this work to ascertain more realistic models and explore their qualitative dynamics.

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