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A New Family of Semi-Norms Between the Berezin Radius and the Berezin Norm

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Abstract

A functional Hilbert space is the Hilbert space $\mathscr{H}$ of complex-valued functions on some set $\Theta \subseteq \mathbb{C}$ such that the evaluation functionals $\varphi_\tau(f) = f(\tau)$, $\tau \in \Theta$, are continuous on $\mathscr{H}$. The Berezin number of an operator X is defined by $\mathbf{ber}(X) = \sup_{\tau\in\Theta}\left|\widetilde{X}(\tau)\right| = \sup_{\tau\in\Theta}\left|\langle X\hat{k}_\tau, \hat{k}_\tau\rangle\right|$, where the operator X acts on the reproducing kernel Hilbert space $\mathscr{H} = \mathscr{H}(\Theta)$ over some (non-empty) set Θ. In this paper, we introduce a new family involving means $\|\cdot\|_{\sigma_t}$ between the Berezin radius and the Berezin norm. Among other results, it is shown that if $X \in \mathcal{L}(\mathscr{H})$ and f, g are two non-negative continuous functions defined on $[0, \infty)$ such that $f(t)g(t) = t$, $(t \geqslant 0)$, then

$$\|X\|_\sigma^2 \leqslant \mathbf{ber}\left(\frac{1}{4}(f^4(|X|) + g^4(|X^*|)) + \frac{1}{2}|X|^2\right)$$

and

$$\|X\|_\sigma^2 \leqslant \frac{1}{2}\sqrt{\mathbf{ber}\left(f^4(|X|) + g^2(|X|^2)\right)\mathbf{ber}\left(f^2(|X|^2) + g^4(|X^*|)\right)},$$

where σ is a mean dominated by the arithmetic mean ∇.

Keywords Reproducing kernel · Berezin number · Berezin transform · Berezin norm · Mean

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1 Introduction

Let $\mathcal{L}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators defined on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ with the identity operator $I_{\mathcal{H}}$ in $\mathcal{L}(\mathcal{H})$. When $\mathcal{H} = \mathbb{C}^n$, we identify $\mathcal{L}(\mathcal{H})$ with the algebra $\mathcal{M}_n(\mathbb{C})$ of n-by-n complex matrices.

A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Theta \subseteq \mathbb{C}$ such that the evaluation functionals $\varphi_\tau(f) = f(\tau)$, $\tau \in \Theta$, are continuous on $\mathcal{H}$. Then, by the Riesz representation theorem there is a unique element $k_\tau \in \mathcal{H}$ such that $f(\tau) = \langle f, k_\tau \rangle$ for all $f \in \mathcal{H}$ and every $\tau \in \Theta$. The function k on $\Theta \times \Theta$ defined by $k(z, \tau) = k_\tau(z)$ is called the reproducing kernel of $\mathcal{H}$, see [2, 4, 5, 17] and references therein. It was shown that $k_\tau(z)$ can be represented by

$$k_\tau(z) = \sum_{n=1}^{\infty} \overline{e_n(\tau)} e_n(z)$$

for any orthonormal basis $\{e_n\}_{n \geq 1}$ of $\mathcal{H}$, see [30]. For example, for the Hardy-Hilbert space $\mathcal{H}^2 = \mathcal{H}^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\{z^n\}_{n \geq 1}$ is an orthonormal basis, therefore the reproducing kernel of $\mathcal{H}^2$ is the function $k_\tau(z) = \sum_{n=1}^{\infty} \overline{\tau_n} z^n = (1 - \overline{\tau} z)^{-1}$, $\tau \in \mathbb{D}$. Let $\widehat{k}_\tau = \frac{k_\tau}{\|k_\tau\|}$ be the normalized reproducing kernel of the space $\mathcal{H}$. For a given a bounded linear operator X on $\mathcal{H}$, the Berezin symbol (or Berezin transform) of X is the bounded function $\widetilde{X}$ on Θ defined by

$$\widetilde{X}(\tau) = \left\langle X\widehat{k}_\tau(z), \widehat{k}_\tau(z) \right\rangle, \ \tau \in \Theta.$$

An important property of the Berezin symbol is that for all $X, Y \in \mathcal{L}(\mathcal{H})$, if $\widetilde{X}(\tau) = \widetilde{Y}(\tau)$ for all $\tau \in \Theta$, then $X = Y$ (at least when $\mathcal{H}$ consists of analytic functions, see Zhu [31]). For more details, see [3, 6, 8–10, 12–14, 16, 18–29]. So, the map $X \to \widetilde{X}$ is injective [15]. The Berezin set and the Berezin number(radius) of an operator X are defined, respectively, by

$$\mathbf{Ber}(X) = \left\{ \widetilde{X}(\tau) : \tau \in \Theta \right\} = \mathrm{Range}\left(\widetilde{X}\right)$$

and

$$\mathbf{ber}(X) = \sup\{|\gamma| : \gamma \in \mathrm{Ber}(X)\} = \sup_{\tau \in \Theta} \left| \widetilde{X}(\tau) \right|.$$

The Berezin norm of an operator $X \in \mathcal{L}(\mathcal{H})$ is defined by

$$\|X\|_{\mathrm{ber}} := \sup_{\tau \in \Theta} \left\| X\widehat{k}_\tau \right\|.$$

For $X, Y \in \mathcal{L}(\mathcal{H})$, it is clear from the above definitions of the Berezin radius (or the Berezin number) and the Berezin norm that the following properties hold:

(1) $\mathbf{ber}(tX) = |t|\mathbf{ber}(X)$ for all $t \in \mathbb{C}$;
(2) $\mathbf{ber}(X + Y) \leqslant \mathbf{ber}(X) + \mathbf{ber}(Y)$;
(3) $\mathbf{ber}(X) \leqslant \|X\|_{\mathrm{ber}}$ and $\mathbf{ber}(X) = \mathbf{ber}(X^*)$;
(4) $\|tX\|_{\mathrm{ber}} = |t| \, \|X\|_{\mathrm{ber}}$ for all $t \in \mathbb{C}$;
(5) $\|X + Y\|_{\mathrm{ber}} \leq \|X\|_{\mathrm{ber}} + \|Y\|_{\mathrm{ber}}$.

In the recent paper [7], the authors defined the t-Berezin norm on $\mathcal{L}(\mathcal{H})$ as follows:

$$\|X\|_{t-\mathbf{ber}} = \sup_{\tau\in\Theta}\sqrt{t\left|\widetilde{X}(\tau)\right|^2 + (1-t)\left\|X\hat{k}_\tau\right\|^2}.$$

The t-Berezin norm is also a norm on $\mathcal{L}(\mathcal{H})$ for $t \in [0, 1)$, and for $t = 1$ it is a norm if the functional Hilbert space has the **Ber** property, i.e., for any two operators $X, Y \in \mathcal{L}(\mathcal{H})$ such that $\widetilde{X}(\tau) = \widetilde{Y}(\tau)$ for all $\tau \in \Theta$, we have $X = Y$. Hence, the t-Berezin norm is a norm in the familiar functional for Hilbert spaces, for instance Hardy and Bergman spaces. The t-Berezin norm satisfies the following inequalities:

$$\mathbf{ber}(X) \leqslant \|X\|_{t-\mathbf{ber}} \leqslant \|X\|_{\mathbf{ber}} \quad \text{for } t \in [0, 1].$$

A binary function σ on $[0, +\infty)$ is called a mean, if the following conditions are satisfied:

(i) If $a \leqslant b$, then $a \leqslant a\,\sigma\, b \leqslant b$;
(ii) $a \leqslant c$ and $b \leqslant d$ imply $a\,\sigma\, b \leq c\,\sigma\, d$;
(iii) σ is continuous in both variables;
(iv) $t(a\,\sigma b) \leq (ta)\sigma(tb)$ $(t > 0)$.

For instance, if $\mu \in (0, 1)$, the weighted geometric mean is $a\sharp_\mu b = a^{1-\mu}b^\mu$. The case $\mu = 1/2$ gives rise to the geometric mean $a\sharp b$. A mean σ is symmetric if $a\,\sigma\, b = b\,\sigma\, a$ for all positive numbers a, b. For a symmetric mean σ, a parametrized mean σ_t, $0 \leq t \leq 1$ is called an interpolational path for σ if it satisfies

(1) $a\,\sigma_0\, b = a$, $a\,\sigma_{1/2}\, b = a\,\sigma\, b$, and $a\,\sigma_1\, b = b$;
(2) $(a\,\sigma_p\, b)\sigma(a\sigma_q\, b) = a\sigma_{\frac{p+q}{2}}\, b$ for all $p, q \in [0, 1]$;
(3) The map $t \in [0, 1] \mapsto a\,\sigma_t\, b$ is continuous for each a and b;
(4) σ_t is increasing in each of its components for $t \in [0, 1]$.

It is easy to see that the set of all $r \in [0, 1]$ satisfying

$$(a\sigma_p b)\sigma_r(a\sigma_q b) = a\sigma_{rp+(1-r)q} b \tag{1.1}$$

for all p, q is a convex subset of $[0, 1]$ including 0 and 1. For instance, the power means

$$a\, m_r b = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}} \qquad (r \in [-1, 1])$$

are some typical interpolational means. Their interpolational paths are

$$a\, m_{r,t} b = ((1-t)a^r + tb^r)^{\frac{1}{r}} \qquad (t \in [0, 1]).$$

In particular, $a\, m_{1,t} b = a\nabla_t b = (1-t)a + tb$ is the weighted arithmetic mean, $a\, m_{0,t} b = a\sharp_t b = a^{1-t}b^t$ is the weighted geometric mean and $a\, m_{-1,t} b = a!_t b = \left((1-t)a^{-1} + tb^{-1}\right)^{-1}$ is the weighted harmonic mean. It is well-known that $a!_t b \leqslant a\sharp_t b \leqslant a\nabla_t b$ for positive numbers a and b and $t \in [0, 1]$. For more information about means, see [25] and references therein.

In this paper, we define a new quantity and establish some related results. The main ideas of this paper are stimulated by [7] and [11].

2 Main Results

We begin this section with the following definition.

Definition 2.1 Let $X \in \mathcal{L}(\mathcal{H})$ and σ_t be an interpolational path of a symmetric mean σ. We define

$$\|X\|_{\sigma_t} = \sup_{\tau \in \Theta} \left\{ \sqrt{\left|\widetilde{X}(\tau)\right|^2 \sigma_t \left\|X\widehat{k_\tau}\right\|^2} \right\} \text{ for } 0 \leqslant t \leqslant 1.$$

Example 2.2 We consider the Hardy-Hilbert space, $\mathcal{H}^2(\mathbb{D})$, defined over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ as follows:

$$\mathcal{H}^2(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } \sum_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

The inner product on $\mathcal{H}^2(\mathbb{D})$ is defined by $\langle f, g\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$, for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ and $\{z^n\}_{n\geq 1}$ forms an orthonormal basis. We can identify $\mathcal{H}^2(\mathbb{D})$ with $l^2(\mathbb{N})$, since

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \leftrightarrow (a_0, a_1, a_2, \ldots).$$

Therefore, the reproducing kernel of $\mathcal{H}^2(\mathbb{D})$ is given by the function $k_\tau(z) = \sum\limits_{n=1}^{\infty} \overline{\tau}^n z^n = (1 - \overline{\tau} z)^{-1}$, $\tau \in \mathbb{D}$ and

$$\|k_\tau\|^2 = \langle k_\tau, k_\tau\rangle = k_\tau(\tau) = (1 - \overline{\tau}\tau)^{-1} = \left(1 - |\tau|^2\right)^{-1},$$

for any $\tau \in \mathbb{D}$.

On $l^2(\mathbb{N})$, we consider the unilateral shift operator U defined by

$$U(a_0, a_1, a_2, a_3, \ldots) = (0, a_0, a_1, a_2, \ldots)$$

for $(a_0, a_1, a_2, a_3, \ldots) \in l^2(\mathbb{N})$. Thus, for any $\tau \in \mathbb{D}$, we have

$$\begin{aligned}
|\widetilde{U}(\tau)|^2 &= \left|\left\langle U\widehat{k_\tau}(z), \widehat{k_\tau}(z)\right\rangle\right|^2 = \frac{1}{\|k_\tau\|^4} |\langle Uk_\tau(z), k_\tau(z)\rangle|^2 \\
&= \frac{1}{\|k_\tau\|^4} \left|\left\langle U(1, \overline{\tau}, \overline{\tau}^2, \ldots), (1, \overline{\tau}, \overline{\tau}^2, \ldots)\right\rangle\right|^2 \\
&= \frac{1}{\|k_\tau\|^4} \left|\left\langle (0, 1, \overline{\tau}, \overline{\tau}^2, \ldots), (1, \overline{\tau}, \overline{\tau}^2, \ldots)\right\rangle\right|^2 \\
&= \frac{1}{\|k_\tau\|^4} \left|\sum_{j=0}^{\infty} \overline{\tau}^j \tau^{j+1}\right|^2 = \frac{1}{\|k_\tau\|^4} \left(\frac{1}{1 - |\tau|^2}|\tau|\right)^2 = |\tau|^2,
\end{aligned}$$

and due to the fact that U is an isometry, we conclude that

$$\|U\widehat{k_\tau}\|^2 = \|\widehat{k_\tau}\|^2 = 1.$$

Then, we obtain that

$$|\widetilde{U}(\tau)|^2 \leq |\tau|^2 < 1 = \|U\widehat{k}_\tau\|^2, \tag{2.1}$$

and by the monotonicity of σ_t, we have

$$1 = \sup_{\tau\in\mathbb{D}}|\tau| = \sup_{\tau\in\mathbb{D}}\left|\widetilde{U}(\tau)\right| \leq \sup_{\tau\in\mathbb{D}}\left\{\sqrt{\left|\widetilde{U}(\tau)\right|^2 \sigma_t \ \|U\widehat{k}_\tau\|^2}\right\} \leq \sup_{\tau\in\mathbb{D}}\left\|U\widehat{k}_\tau\right\| = 1 \tag{2.2}$$

for $t \in [0, 1]$.

In conclusion, we have that $\mathbf{ber}(U) \leqslant \|U\|_{\sigma_t} \leqslant \|U\|_{\mathbf{ber}}$, and in particular, $\|U\|_{\sigma_t} = 1$ for any $t \in [0, 1]$.

Following the ideas from the previous example and given that $\left|\widetilde{X}(\tau)\right| \leq \left\|X\widehat{k}_\tau\right\|$, from the Cauchy-Schwartz inequality, we have that

$$\mathbf{ber}(X) \leqslant \|X\|_{\sigma_t} \leqslant \|X\|_{\mathbf{ber}}$$

for $t \in [0, 1]$ and $X \in \mathcal{L}(\mathcal{H})$. It is easy to see that for the special case $\sigma_t = \nabla_t$ $(0 \leqslant t \leqslant 1)$, we have $\|\cdot\|_{\sigma_t} = \|\cdot\|_{(1-t)-\mathbf{ber}}$.

The next proposition shows some properties of $\|\cdot\|_{\sigma_t}$.

Proposition 2.3 *Let $X \in \mathcal{L}(\mathcal{H})$ and σ_t, τ_μ be interpolational paths of symmetric means σ and τ. Then*

(1) $\|X\|_{\sigma_0} = \mathbf{ber}(X)$ *and* $\|X\|_{\sigma_1} = \|X\|_{\mathbf{ber}}$;
(2) $\|X\|_{\sigma_t} \leqslant \sqrt{\mathbf{ber}^2(X)\ \sigma_t\ \|X\|^2_{\mathbf{ber}}}$ *for* $t \in [0, 1]$;
(3) $\|\tau X\|_{\sigma_t} = |\tau|\|X\|_{\sigma_t}$ *for all* $\tau \in \mathbb{C}$;
(4) *If the functional Hilbert space has the* ***Ber*** *property and* $t \in [0, 1)$, *then* $\|X\|_{\sigma_t} = 0$ *if and only if* $X = 0$;
(5) *If* $\sigma_t \leqslant \tau_s$, *then* $\|X\|_{\sigma_t} \leqslant \|X\|_{\tau_s}$ *for* $s, t \in [0, 1]$.

Remark 2.4 If $X \in \mathcal{L}(\mathcal{H})$, then

$$\begin{aligned}\||X|\|^2_{\mathbf{ber}} &= \sup_{\tau\in\Theta}\left\||X|\hat{k}_\tau\right\|^2 = \sup_{\tau\in\Theta}\langle|X|\hat{k}_\tau, |X|\hat{k}_\tau\rangle \\ &= \sup_{\tau\in\Theta}\langle X^*X\hat{k}_\tau, \hat{k}_\tau\rangle = \sup_{\tau\in\Theta}\langle X\hat{k}_\tau, X\hat{k}_\tau\rangle = \sup_{\tau\in\Theta}\left\|X\hat{k}_\tau\right\|^2 = \|X\|^2_{\mathbf{ber}}\end{aligned}$$

and for a semi-hyponormal operator X, i.e. $|X^*| \leqslant |X|$, the mixed Cauchy-Schwarz inequality $\left|\langle X\hat{k}_\tau, \hat{k}_\tau\rangle\right|^2 \leqslant \langle|X|\hat{k}_\tau, \hat{k}_\tau\rangle\langle|X^*|\hat{k}_\tau, \hat{k}_\tau\rangle$ implies that $\left|\langle X\hat{k}_\tau, \hat{k}_\tau\rangle\right|^2 \leqslant \langle|X|\hat{k}_\tau, \hat{k}_\tau\rangle^2$ for all $\hat{k}_\tau \in \mathcal{H}$, and then

$$\mathbf{ber}(X) = \sup_{\tau\in\Theta}\left|\langle X\hat{k}_\tau, \hat{k}_\tau\rangle\right| \leqslant \sup_{\tau\in\Theta}\left\langle|X|\hat{k}_\tau, \hat{k}_\tau\right\rangle = \mathbf{ber}(|X|).$$

Using the definition of $\|\cdot\|_{\sigma_t}$ and the monotonicity of σ_t, we have the next result.

Theorem 2.5 *Let $X \in \mathcal{L}(\mathcal{H})$ and σ_t be an interpolational path of a symmetric mean σ for all $t \in [0, 1]$. Then*

(1) *If X is hyponormal, i.e., $XX^* \leqslant X^*X$, then $\|X^*\|_{\sigma_t} \leqslant \|X\|_{\sigma_t}$.*

(2) *If X is co-hyponormal, i.e.,* $X^*X \leqslant XX^*$, *then* $\|X\|_{\sigma_t} \leqslant \|X^*\|_{\sigma_t}$.
(3) *If X is semi-hyponormal, i.e.,* $|X^*| \leqslant |X|$, *then* $\|X\|_{\sigma_t} \leqslant \|\,|X|\,\|_{\sigma_t}$.
(4) *If X is* (α, β)*-normal, i.e.,* $\alpha X^*X \leqslant XX^* \leqslant \beta X^*X$ *for some positive real numbers* α *and* β *with* $\alpha \leqslant 1 \leqslant \beta$, *then*

$$\alpha\|X\|_{\sigma_t} \leqslant \|X^*\|_{\sigma_t} \leqslant \beta\|X\|_{\sigma_t}.$$

(5) *If X is normal, then* $\|X^*\|_{\sigma_t} = \|X\|_{\sigma_t}$.

Proof (1) It follows from the hyponormality of X that $\|X^*\widehat{k}_\tau\| \leqslant \|X\widehat{k}_\tau\|$ for all $\tau \in \Theta$. Moreover, $|\langle X\widehat{k}_\tau, \widehat{k}_\tau\rangle| = |\langle X^*\widehat{k}_\tau, \widehat{k}_\tau\rangle|$ for all $\tau \in \Theta$. Hence, by the monotonicity of σ_t, we get

$$\left|\langle X^*\widehat{k}_\tau, \widehat{k}_\tau\rangle\right|^2 \sigma_t \|X^*\widehat{k}_\tau\|^2 \leqslant \left|\langle X\widehat{k}_\tau, \widehat{k}_\tau\rangle\right|^2 \sigma_t \|X\widehat{k}_\tau\|^2 \quad \text{for all } \tau \in \Theta.$$

Then, by the definition of $\|\cdot\|_{\sigma t}$, we have $\|X^*\|_{\sigma t} \leqslant \|X\|_{\sigma t}$.
(2) The proof is similar to that of part (1).
(3) The condition of semi-hyponormality and the mixed Cauchy-Schwarz inequality imply that $\left|\langle X\hat{k}_\tau, \hat{k}_\tau\rangle\right|^2 \leqslant \langle |X|\hat{k}_\tau, \hat{k}_\tau\rangle^2$ for all $\hat{k}_\tau \in \mathcal{H}$. Also, $\|X\hat{k}_\tau\| = \|\,|X|\,\hat{k}_\tau\|$ for all $\hat{k}_\tau \in \mathcal{H}$. Therefore,

$$\left|\langle X\widehat{k}_\tau, \widehat{k}_\tau\rangle\right|^2 \sigma_t \|X\widehat{k}_\tau\|^2 \leqslant \langle |X|\widehat{k}_\tau, \widehat{k}_\tau\rangle^2 \sigma_t \|\,|X|\,\widehat{k}_\tau\|^2 \quad \text{for all } \tau \in \Theta.$$

By taking the supremum over all $\tau \in \Theta$, we get $\|X\|_{\sigma t} \leqslant \|\,|X|\,\|_{\sigma t}$.
(4) Since X is (α, β)-normal, we have $\alpha\|X\hat{k}_\tau\| \leqslant \|X^*\hat{k}_\tau\| \leqslant \beta\|X\hat{k}_\tau\|$ for all $\tau \in \Theta$. It follows from the fact that σ_t is increasing in its both variables that

$$\alpha^2\left|\langle X\widehat{k}_\tau, \widehat{k}_\tau\rangle\right|^2 \sigma_t \alpha^2\|X\widehat{k}_\tau\|^2 \leqslant \left|\langle X^*\widehat{k}_\tau, \widehat{k}_\tau\rangle\right|^2 \sigma_t \|X^*\widehat{k}_\tau\|^2 \leqslant \beta^2\left|\langle X\widehat{k}_\tau, \widehat{k}_\tau\rangle\right|^2 \sigma_t \beta^2\|X\widehat{k}_\tau\|^2$$

for all $\tau \in \Theta$. Hence, $\alpha\|X\|_{\sigma t} \leqslant \|X^*\|_{\sigma t} \leqslant \beta\|X\|_{\sigma t}$.
(5) It follows from the normality of X that X is both hyponormanl and co-hyponormanl, and then by the parts (1) and (2) we have the desired result. □

Theorem 2.6 *Let* $X \in \mathcal{L}(\mathcal{H})$ *and* $t \in [0, 1]$. *Then the following conditions are equivalent.*

(1) $\|X\|^2_{\sigma_t} = \mathbf{ber}^2(X)\,\sigma_t\,\|X\|^2_{\mathbf{ber}}$.
(2) *There exists a sequence* $\{\widehat{k}_{\tau_n}\}$ *in* $\mathcal{H}$ *such that*

$$\lim_{n\to\infty}\left|\widetilde{X}(\tau_n)\right| = \mathbf{ber}(X) \quad \textit{and} \quad \lim_{n\to\infty}\left\|X\widehat{k}_{\tau_n}\right\| = \|X\|_{\mathbf{ber}}.$$

Proof We first prove that (1) implies (2). By the definition of the supremum, there exists a sequence $\left\{\widehat{k}_{\tau_n}\right\}$ in $\mathcal{H}$ such that

$$\|X\|^2_{\sigma_t} = \lim_{n\to\infty}\left|\widetilde{X}(\tau_n)\right|^2 \sigma_t \left\|X\widehat{k}_{\tau_n}\right\|^2.$$

It follows from the boundedness of the sequences $\left\{\left|\widetilde{X}(\tau_n)\right|\right\}$ and $\left\{\left\|X\widehat{k}_{\tau_n}\right\|\right\}$ that there exists a subsequence $\left\{\widehat{k}_{\tau_{nk}}\right\}$ such that $\left\{\left|\widetilde{X}(\tau_{nk})\right|\right\}$ and $\left\{\left\|X\widehat{k}_{\tau_{nk}}\right\|\right\}$ are convergent. Then, we have

$$\mathbf{ber}^2(X)\,\sigma_t\,\|X\|^2_{\mathbf{ber}} = \|X\|^2_{\sigma_t}$$

$$= \lim_{n\to\infty} \left(|\widetilde{X}(\tau_{nk})|^2 \sigma_t \|X\widehat{k}_{\tau_{nk}}\|^2\right)$$
$$\leqslant \mathbf{ber}^2(X)\sigma_t\|X\|^2_{\mathbf{ber}}.$$

Therefore,

$$\lim_{n\to\infty} |\widetilde{X}(\tau_{nk})| = \mathbf{ber}(X) \quad \text{and} \quad \lim_{n\to\infty} \|X\widehat{k}_{\tau_{nk}}\| = \|X\|_{\mathbf{ber}}.$$

Now, we prove that (2) implies (1). We have

$$\begin{aligned}\|X\|^2_{\sigma_t} &= \sup_{\tau\in\Theta}\left\{|\widetilde{X}(\tau)|^2 \ \sigma_t \ \|X\widehat{k}_\tau\|^2\right\}\\ &\geqslant \lim_{n\to\infty}\left\{|\widetilde{X}(\tau_n)|^2 \ \sigma_t \ \|X\widehat{k}_{\tau_n}\|^2\right\}\\ &= \mathbf{ber}^2(X) \ \sigma_t \ \|X\|^2_{\mathbf{ber}}.\end{aligned}$$

Hence, $\|X\|^2_{\sigma_t} = \mathbf{ber}^2(X)\,\sigma_t\|X\|^2_{\mathbf{ber}}$. □

We have seen in Proposition 2.3 that $\|\cdot\|_{\sigma_t}$ $(0 \leqslant t \leqslant 1)$ fulfills the semi-norm properties, except possibly for the triangle inequality. In particular, when $\sigma = \nabla_t$ $(0 \leqslant t \leqslant 1)$, we have the next proposition.

Proposition 2.7 *Let* $X, Y \in \mathcal{L}(\mathcal{H})$ *and* $0 \leqslant t \leqslant 1$. *Then*

$$\|X+Y\|_{\nabla_t} \leqslant \|X\|_{\nabla_t} + \|Y\|_{\nabla_t}.$$

Proof Let $\tau \in \Theta$ be a unit vector. Then

$$\begin{aligned}&t|\widetilde{(X+Y)}(\tau)|^2 + (1-t)\|(X+Y)\widehat{k}_\tau\|^2\\ &\leqslant t\left(|\widetilde{X}(\tau)| + |\widetilde{Y}(\tau)|\right)^2 + (1-t)\left(\|X\widehat{k}_\tau\| + \|Y\widehat{k}_\tau\|\right)^2\\ &= t\left(|\widetilde{X}(\tau)|^2 + |\widetilde{Y}(\tau)|^2 + 2|\widetilde{X}(\tau)||\widetilde{Y}(\tau)|\right)\\ &\qquad + (1-t)\left(\|X\widehat{k}_\tau\|^2 + \|Y\widehat{k}_\tau\|^2 + 2\|X\widehat{k}_\tau\|\|Y\widehat{k}_\tau\|\right)\\ &= t|\widetilde{X}(\tau)|^2 + (1-t)\|X\widehat{k}_\tau\|^2 + t|\widetilde{Y}(\tau)|^2 + (1-t)\|Y\widehat{k}_\tau\|^2\\ &\qquad + 2\left(t|\widetilde{X}(\tau)||\widetilde{Y}(\tau)| + (1-t)\|X\widehat{k}_\tau\|\|Y\widehat{k}_\tau\|\right).\end{aligned}$$

Moreover, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}&t|\widetilde{X}(\tau)||\widetilde{Y}(\tau)|+(1-t)\|X\widehat{k}_\tau\|\|Y\widehat{k}_\tau\|\\ &\qquad \leqslant \sqrt{t|\widetilde{X}(\tau)|^2 + (1-t)\|X\widehat{k}_\tau\|^2}\sqrt{t|\widetilde{Y}(\tau)|^2 + (1-t)\|Y\widehat{k}_\tau\|^2}.\end{aligned}$$

Combining the above inequalities, we have

$$\begin{aligned}&t|\widetilde{(X+Y)}(\tau)|^2 + (1-t)\|(X+Y)\widehat{k}_\tau\|^2\\ &\leqslant t|\widetilde{X}(\tau)|^2 + (1-t)\|X\widehat{k}_\tau\|^2 + t|\widetilde{Y}(\tau)|^2 + (1-t)\|Y\widehat{k}_\tau\|^2\end{aligned}$$

$$+ 2\sqrt{t|\widetilde{X}(\tau)|^2 + (1-t)\|X\widehat{k}_\tau\|^2}\sqrt{t|\widetilde{Y}(\tau)|^2 + (1-t)\|Y\widehat{k}_\tau\|^2}$$
$$\leqslant \|X\|_{\nabla_t}^2 + \|Y\|_{\nabla_t}^2 + 2\|X\|_{\nabla_t}\|X\|_{\nabla_t}.$$

Therefore,

$$\|X+Y\|_{\nabla_t}^2 = \sup_{\tau\in\Theta}\left\{t|\widetilde{(X+Y)}(\tau)|^2 + (1-t)\|(X+Y)\widehat{k}_\tau\|^2\right\} \leqslant \left(\|X\|_{\nabla_t} + \|Y\|_{\nabla_t}\right)^2.$$

□

In the following theorem, we give an equivalent condition that $\|X+Y\|_{\nabla_t} = \|X\|_{\nabla_t} + \|Y\|_{\nabla_t}$ for all $X, Y \in \mathcal{L}(\mathcal{H})$.

Theorem 2.8 *Let $X, Y \in \mathcal{L}(\mathcal{H})$ and $0 < t < 1$. Then the following conditions are equivalent.*

(1) $\|X+Y\|_{\nabla_t} = \|X\|_{\nabla_t} + \|Y\|_{\nabla_t}$.
(2) *There exists a sequence $\{\widehat{k}_{\tau_n}\}$ in $\mathcal{H}$ such that*

$$\lim_{n\to\infty}\Re\left(t\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right) = \|X\|_{\nabla_t}\|Y\|_{\nabla_t},$$

where $\Re(z)$ denotes the real part of a complex number z.

Proof (1) $\Rightarrow$ (2) Using the definition of the supremum and the hypothesis, there exists a sequence $\{\widehat{k}_{\tau_n}\}$ in $\mathcal{H}$ such that

$$\lim_{n\to\infty}\left(t|\widetilde{(X+Y)}(\tau_n)|^2 + (1-t)\|(X+Y)\widehat{k}_{\tau_n}\|^2\right) = \left(\|X\|_{\nabla_t} + \|Y\|_{\nabla_t}\right)^2.$$

Hence,

$$\begin{aligned}
&t|\widetilde{(X+Y)}(\tau_n)|^2 + (1-t)\|(X+Y)\widehat{k}_{\tau_n}\|^2\\
&= t\left(|\widetilde{X}(\tau_n)|^2 + |\widetilde{Y}(\tau_n)|^2 + 2\Re\left(\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n)\right)\right)\\
&\qquad + (1-t)\left(\|X\widehat{k}_{\tau_n}\|^2 + \|Y\widehat{k}_{\tau_n}\|^2 + 2\Re\left(\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right)\right)\\
&= t|\widetilde{X}(\tau_n)|^2 + (1-t)\|X\widehat{k}_{\tau_n}\|^2 + t|\widetilde{Y}(\tau_n)|^2 + (1-t)\|Y\widehat{k}_{\tau_n}\|^2\\
&\qquad + 2\Re\left(t\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right)\\
&\leqslant \|X\|_{\nabla_t}^2 + \|Y\|_{\nabla_t}^2 + 2\Re\left(t\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right)\\
&\leqslant \|X\|_{\nabla_t}^2 + \|Y\|_{\nabla_t}^2 + 2\left(t|\widetilde{X}(\tau_n)||\widetilde{Y}(\tau_n)| + (1-t)|\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle||\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle|\right)\\
&\leqslant \|X\|_{\nabla_t}^2 + \|Y\|_{\nabla_t}^2 + 2\left(t\|X\widehat{k}_{\tau_n}\|\|Y\widehat{k}_{\tau_n}\| + (1-t)|\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle||\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle|\right)\\
&\leqslant \|X\|_{\nabla_t}^2 + \|Y\|_{\nabla_t}^2 + 2\sqrt{t|\widetilde{X}(\tau_n)|^2 + (1-t)\|X\widehat{k}_{\tau_n}\|^2}\sqrt{t|\widetilde{Y}(\tau_n)|^2 + (1-t)\|Y\widehat{k}_{\tau_n}\|^2}\\
&\qquad\qquad\text{(by the Cauchy-Schwarz inequality)}\\
&\leqslant \left(\|X\|_{\nabla_t} + \|Y\|_{\nabla_t}\right)^2.
\end{aligned}$$

Now, if we let $n \to \infty$, we get

$$\lim_{n\to\infty}\Re\left(t\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right) = \|X\|_{\nabla_t}\|Y\|_{\nabla_t}.$$

(2) $\Rightarrow$ (1) Suppose that there exists a sequence $\{\widehat{k}_{\tau_n}\}$ in $\mathcal{H}$ such that

$$\lim_{n\to\infty} \Re\left(t\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right) = \|X\|_{\nabla_t}\|Y\|_{\nabla_t}.$$

Then, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\Re^2\left(t\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right)\\
&\quad\leqslant \left|(1-t)\widetilde{X}(\tau_n)\widetilde{Y}(\tau_n) + t\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right|^2\\
&\qquad - \Im^2\left(t\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right)\\
&\quad\leqslant \left|(1-t)\widetilde{X}(\tau_n)\widetilde{Y}(\tau_n) + t\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right|^2,
\end{aligned}$$

where $\Im(z)$ denotes the imaginary part of a complex number z. Hence,

$$\begin{aligned}
&\Re\left(t\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right)\\
&\quad\leqslant \left|(1-t)\widetilde{X}(\tau_n)\widetilde{Y}(\tau_n) + t\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right|\\
&\quad\leqslant (1-t)|\widetilde{X}(\tau_n)||\widetilde{Y}(\tau_n)| + t|\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle||\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle|\\
&\quad\leqslant \left((1-t)\|X\widehat{k}_{\tau_n}\|\|Y\widehat{k}_{\tau_n}\| + t|\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle||\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle|\right)\\
&\quad\leqslant \sqrt{t|\widetilde{X}(\tau_n)|^2 + (1-t)\|X\widehat{k}_{\tau_n}\|^2}\sqrt{t|\widetilde{Y}(\tau_n)|^2 + (1-t)\|Y\widehat{k}_{\tau_n}\|^2}\\
&\qquad\qquad\text{(by the Cauchy-Schwarz inequality)}\\
&\quad\leqslant \|X\|_{\nabla_t}\|Y\|_{\nabla_t}.
\end{aligned}\tag{2.3}$$

It follows from $t|\widetilde{X}(\tau_n)|^2 + (1-t)\|X\widehat{k}_{\tau_n}\|^2 \leqslant \|X\|^2_{\nabla_t}$ and $t|\widetilde{Y}(\tau_n)|^2 + (1-t)\|Y\widehat{k}_{\tau_n}\|^2 \leqslant \|Y\|^2_{\nabla_t}$ that

$$\lim_{n\to\infty}\left(t|\widetilde{X}(\tau_n)|^2 + (1-t)\|X\widehat{k}_{\tau_n}\|^2\right) = \|X\|^2_{\nabla_t}$$

and

$$\lim_{n\to\infty}\left(t|\widetilde{Y}(\tau_n)|^2 + (1-t)\|Y\widehat{k}_{\tau_n}\|^2\right) = \|Y\|^2_{\nabla_t}.$$

Therefore,

$$\begin{aligned}
\left(\|X\|_{\nabla_t} + \|Y\|_{\nabla_t}\right)^2 &= \|X\|^2_{\nabla_t} + \|Y\|^2_{\nabla_t} + 2\|X\|_{\nabla_t}\|Y\|_{\nabla_t}\\
&= \lim_{n\to\infty}\left(t|\widetilde{X}(\tau_n)|^2 + (1-t)\|X\widehat{k}_{\tau_n}\|^2\right)\\
&\qquad + \lim_{n\to\infty}\left(t|\widetilde{Y}(\tau_n)|^2 + (1-t)\|Y\widehat{k}_{\tau_n}\|^2\right)\\
&\qquad + 2\lim_{n\to\infty}\Re\left(t\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right)\\
&= \lim_{n\to\infty}\Big[t\left(|\widetilde{X}(\tau_n)|^2 + |\widetilde{Y}(\tau_n)|^2 + 2\Re\left(\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad + (1-t)\left(\|X\widehat{k}_{\tau_n}\|^2 + \|Y\widehat{k}_{\tau_n}\|^2 + 2\Re\left(\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right)\right)\Big]\\
&= \lim_{n\to\infty}\left(t|\widetilde{(X+Y)}(\tau_n)|^2 + (1-t)\|(X+Y)\widehat{k}_{\tau_n}\|^2\right)\\
&= \|X+Y\|_{\nabla_t}^2\\
&\leqslant \left(\|X\|_{\nabla_t} + \|Y\|_{\nabla_t}\right)^2\\
&\qquad \text{(by Proposition 2.7)}.
\end{aligned}
$$

Hence, $\|X+Y\|_{\nabla_t} = \|X\|_{\nabla_t} + \|Y\|_{\nabla_t}$. □

Recently, Altwaijry et al. in [1], introduced the following generalization of $\|\cdot\|_{t-\mathbf{ber}}$. Given non-negative real scalars α and β such that $(\alpha,\beta)\neq(0,0)$ and $X\in\mathcal{L}(\mathcal{H})$, let

$$
\|X\|_{\alpha,\beta}^{\mathbf{ber}} = \sup_{\tau\in\Theta}\left\{\sqrt{\beta\left|\widetilde{X}(\tau)\right|^2 + \alpha\left\|X\widehat{k}_\tau\right\|^2}\right\}. \tag{2.4}
$$

Then, we have

$$
\begin{aligned}
\frac{1}{\sqrt{\alpha+\beta}}\|X\|_{\alpha,\beta}^{\mathbf{ber}} &= \sup_{\tau\in\Theta}\left\{\sqrt{\frac{\beta\left|\widetilde{X}(\tau)\right|^2 + \alpha\left\|X\widehat{k}_\tau\right\|^2}{\alpha+\beta}}\right\}\\
&= \sup_{\tau\in\Theta}\left\{\sqrt{\frac{\alpha}{\alpha+\beta}\left\|X\widehat{k}_\tau\right\|^2 + \frac{\beta}{\alpha+\beta}\left|\widetilde{X}(\tau)\right|^2}\right\} = \|X\|_{\nabla_{t_0}} = \|X\|_{\nabla_{t_1}},
\end{aligned} \tag{2.5}
$$

where $t_0 = \frac{\alpha}{\alpha+\beta}$ and $t_1 = \frac{\beta}{\alpha+\beta}$.

As a consequence of Theorem 2.8 and the previous identity, we derive the following characterization of the equality in the triangle inequality for the norm $\|\cdot\|_{\alpha,\beta}^{\mathbf{ber}}$.

Corollary 2.9 [1, *Theorem* 10] *Let* $X, Y\in\mathcal{L}(\mathcal{H})$ *and non-negative real scalars* α *and* β *such that* $(\alpha,\beta)\neq(0,0)$. *Then the following conditions are equivalent.*

(1) $\|X+Y\|_{\alpha,\beta}^{ber} = \|X\|_{\alpha,\beta}^{ber} + \|Y\|_{\alpha,\beta}^{ber}$.

(2) *There exists a sequence* $\{\widehat{k}_{\tau_n}\}$ *in* $\mathcal{H}$ *such that*

$$
\lim_{n\to\infty}\left(t_0\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t_0)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right) = \|X\|_{\nabla_{t_0}}\|Y\|_{\nabla_{t_0}}, \tag{2.6}
$$

where $t_0 = \frac{\alpha}{\alpha+\beta}$.

Proof We note that from the equality (2.5), the condition (1) is equivalent to

$$
\|X+Y\|_{\nabla_{t_0}} = \|X\|_{\nabla_{t_0}} + \|Y\|_{\nabla_{t_0}}, \tag{2.7}
$$

with $t_0 = \frac{\alpha}{\alpha+\beta}$. Now, by Theorem 2.8, the condition (2.7) is equivalent to the existence of a sequence $\{\widehat{k}_{\tau_n}\}$ in $\mathcal{H}$ such that

$$
\lim_{n\to\infty}\Re\left(t_0\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t_0)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n}\rangle\right) = \|X\|_{\nabla_{t_0}}\|Y\|_{\nabla_{t_0}}. \tag{2.8}
$$

To finish the proof, it is enough to show that (2.8) is equivalent to

$$\lim_{n\to\infty}\left(t_0\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n)+(1-t_0)\langle X\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\right)=\|X\|_{\nabla_{t_0}}\|Y\|_{\nabla_{t_0}}. \tag{2.9}$$

Indeed, if (2.9) holds, then

$$\begin{aligned}&\lim_{n\to\infty}\left(t_0\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n)+(1-t_0)\langle X\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\right)\\&=\|X\|_{\nabla_{t_0}}\|Y\|_{\nabla_{t_0}}=\frac{\|X\|_{\nabla_{t_0}}\|Y\|_{\nabla_{t_0}}+\overline{\|X\|_{\nabla_{t_0}}\|Y\|_{\nabla_{t_0}}}}{2}\\&=\lim_{n\to\infty}\Re\left(t_0\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n)+(1-t_0)\langle X\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\right).\end{aligned} \tag{2.10}$$

On the other hand, by (2.3) for any $n\in\mathbb{N}$, we have

$$\begin{aligned}&\Re^2\left(t_0\widetilde{X}(\tau_n)\widetilde{Y}(\tau_n)+(1-t_0)\langle X\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\right)\\&\le\left(t_0|\widetilde{X}(\tau_n)||\widetilde{Y}(\tau_n)|+(1-t_0)|\langle X\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle||\langle Y\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle|\right)^2\\&\leqslant\|X\|^2_{\nabla_{t_0}}\|Y\|^2_{\nabla_{t_0}}.\end{aligned}$$

Thus, if we assume that condition (2.8) is fulfilled, then we can conclude that

$$\lim_{n\to\infty}\Im\left(t_0\widetilde{X}(\tau_n)\widetilde{Y}(\tau_n)+(1-t_0)\langle X\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\right)=0,$$

and

$$\lim_{n\to\infty}\left(t_0\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n)+(1-t_0)\langle X\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\right)=\|X\|_{\nabla_{t_0}}\|Y\|_{\nabla_{t_0}},$$

and this completes the proof. □

Remark 2.10 We note that condition (2.6) is equivalent to
(2') There exists a sequence $\{\widehat{k}_{\tau_n}\}$ in $\mathscr{H}$ such that

$$\lim_{n\to\infty}\left(\alpha\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n)+\beta\langle X\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\right)=\|X\|^{\mathbf{ber}}_{\alpha,\beta}\|Y\|^{\mathbf{ber}}_{\alpha,\beta}.$$

Indeed, if we denote by $t_0=\frac{\alpha}{\alpha+\beta}$, then by (2.6)

$$\begin{aligned}\|X\|^{\mathbf{ber}}_{\alpha,\beta}\|Y\|^{\mathbf{ber}}_{\alpha,\beta}&=\sqrt{\alpha+\beta}\|X\|_{\nabla_{t_0}}\sqrt{\alpha+\beta}\|Y\|_{\nabla_{t_0}}\\&=\lim_{n\to\infty}(\alpha+\beta)\left(t_0\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n)+(1-t_0)\langle X\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\right)=\\&=\lim_{n\to\infty}\left(\alpha\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n)+\beta\langle X\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\langle Y\widehat{k}_{\tau_n},\widehat{k}_{\tau_n}\rangle\right).\end{aligned}$$

Finally, we remark that in [1, Theorem 10], the authors obtained a similar characterization of the equality $\|X+Y\|^{\mathbf{ber}}_{\alpha,\beta}=\|X\|^{\mathbf{ber}}_{\alpha,\beta}+\|Y\|^{\mathbf{ber}}_{\alpha,\beta}$.

3 Some Estimations for $\|\cdot\|_{\sigma_t}$

In this section, we present some upper and lower bounds for $\|\cdot\|_{\sigma_t}$. The following well-known lemmas will be essential to prove our results.

Lemma 3.1 [25] *Let $X \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator with spectrum in an interval J and $\tau \in \Theta$.*

(1) *If $f : J \to \mathbb{R}$ is convex, then $f\big(\langle X\widehat{k}_\tau, \widehat{k}_\tau\rangle\big) \leqslant \langle f(X)\widehat{k}_\tau, \widehat{k}_\tau\rangle$.*
(2) *If $f : J \to \mathbb{R}$ is concave, then $f\big(\langle X\widehat{k}_\tau, \widehat{k}_\tau\rangle\big) \geqslant \langle f(X)\widehat{k}_\tau, \widehat{k}_\tau\rangle$.*

Lemma 3.2 [24] *Let $X \in \mathcal{L}(\mathcal{H})$ and f, g be two non-negative continuous functions defined on $[0, \infty)$ such that $f(t)g(t) = t$ for $t \geqslant 0$. Then*

$$|\widetilde{X}(\tau)| \leqslant |\widetilde{f^2(|X|)}(\tau)||\widetilde{g^2(|X^*|)}(\tau)|$$

for all $\tau \in \Theta$.

For an operator $X \in \mathcal{L}(\mathcal{H})$, the Crawford Berezin number $\widetilde{c}(X)$ is defined as $\widetilde{c}(X) = \inf_{\tau\in\Theta}|\widetilde{X}(\tau)|$. In the following theorem, we obtain a lower bound for $\|\cdot\|_{\sigma_t}$ in the terms of $\widetilde{c}(\cdot)$.

Theorem 3.3 *Let $X \in \mathcal{L}(\mathcal{H})$ and $0 \leqslant t \leqslant 1$. Then,*

$$\|X\|_{\sigma_t} \geqslant \max\left\{\sqrt{\boldsymbol{ber}^2(X)\sigma_t\widetilde{c}^2(X^*X)},\ \sqrt{\widetilde{c}^2(X)\sigma_t\|X\|^2_{\boldsymbol{ber}}}\right\}.$$

Proof Let $\tau \in \Theta$ and $0 \leqslant t \leqslant 1$. Then

$$\begin{aligned}\|X\|^2_{\sigma_t} &= \sup_{\tau\in\Theta}\{|\widetilde{X}(\tau)|^2\sigma_t\|X(\widehat{k}_\tau)\|^2\}\\ &\geqslant |\widetilde{X}(\tau)|^2\sigma_t\|X(\widehat{k}_\tau)\|^2\\ &= |\widetilde{X}(\tau)|^2\sigma_t|\widetilde{X^*X}(\tau)|^2\\ &\geqslant |\widetilde{X}(\tau)|^2\sigma_t\widetilde{c}^2(X^*X).\end{aligned}$$

Taking the supremum over all vectors $\tau \in \Theta$, we get $\|X\|^2_{\sigma_t} \geqslant \mathbf{ber}^2(X)\sigma_t\widetilde{c}^2(X^*X)$. Similarly, we have

$$\|X\|^2_{\sigma_t} = \sup_{\tau\in\Theta}\{|\widetilde{X}(\tau)|^2\sigma_t\|X(\widehat{k}_\tau)\|^2\} \geqslant |\widetilde{X}(\tau)|^2\sigma_t\|X(\widehat{k}_\tau)\|^2 \geqslant \widetilde{c}^2(X)\sigma_t\|X(\widehat{k}_\tau)\|^2,$$

whence $\|X\|^2_{\sigma_t} \geqslant \widetilde{c}^2(X)\sigma_t\|X\|^2_{\mathbf{ber}}$. Combining the above inequalities, we get the desired result. □

In the next result, we get some special case of Theorem 3.3.

Corollary 3.4 *Let $X \in \mathcal{L}(\mathcal{H})$, $r \in [-1, 1]$ and $0 \leqslant \mu \leqslant 1$. Then*

$$\|X\|_{m_{r,\mu}} \geqslant \max\left\{\sqrt[2r]{\frac{(1-\mu)\boldsymbol{ber}^r(X) + \mu\widetilde{c}^r(X^*X)}{2}},\ \sqrt[2r]{\frac{(1-\mu)\widetilde{c}^r(X) + \mu\|X\|^r_{\boldsymbol{ber}}}{2}}\right\}.$$

In particular,

$$\|X\|_{\nabla_\mu} \geqslant \max\left\{\sqrt{(1-\mu)\boldsymbol{ber}^2(X) + \mu\widetilde{c}^2(X^*X)},\ \sqrt{(1-\mu)\widetilde{c}^2(X) + \mu\|X\|^2_{\boldsymbol{ber}}}\right\}$$

and

$$\|X\|_{\sharp_\mu} \geqslant \max\left\{\boldsymbol{ber}^{(1-\mu)}(X)\widetilde{c}^{\mu}(X^*X)\,,\,\widetilde{c}^{(1-\mu)}(X)\|X\|^{\mu}_{\boldsymbol{ber}}\right\}.$$

Proof Letting σ_t be the interpolational paths of the power means $m_{r,\mu}$ for $r \in [-1,1]$ and $0 \leqslant \mu \leqslant 1$ in Theorem 3.3, we have the first inequality. If we take the weighted arithmetic mean ∇_μ and the weighted geometric mean $\sharp_\mu$, $(0 \leqslant \mu \leqslant 1)$ in Theorem 3.3, then we have the second and the third inequalities, respectively. □

Theorem 3.5 *Let $X \in \mathcal{L}(\mathcal{H})$, and let f, g be two non-negative continuous functions defined on $[0,\infty)$ such that $f(t)g(t) = t$ for $t \geqslant 0$. If σ is a mean dominated by the arithmetic mean ∇, then*

$$\|X\|^2_\sigma \leqslant \boldsymbol{ber}\left(\frac{1}{4}(f^4(|X|)+g^4(|X^*|))+\frac{1}{2}|X|^2\right)$$

and

$$\|X\|^2_\sigma \leqslant \frac{1}{2}\sqrt{\boldsymbol{ber}\left(f^4(|X|)+g^2(|X|^2)\right)\boldsymbol{ber}\left(f^2(|X|^2)+g^4(|X^*|)\right)}.$$

Proof Let $\tau \in \Theta$. Then

$$\begin{aligned}
|\widetilde{X}(\tau)|^2\sigma\|X\widehat{k}_\tau\|^2 &= |\widetilde{X}(\tau)|^2\sigma|\widetilde{X^*X}(\tau)| \\
&\leqslant |\widetilde{f^2(|X|)}(\tau)||\widetilde{g^2(|X^*|)}(\tau)|\sigma|\widetilde{X^*X}(\tau)| \\
&\qquad \text{(by Lemma 3.2)} \\
&\leqslant \frac{1}{2}\left(|\widetilde{f^2(|X|)}(\tau)|^2+|\widetilde{g^2(|X^*|)}(\tau)|^2\right)\sigma|\widetilde{X^*X}(\tau)| \\
&\leqslant \frac{1}{2}\left(|\widetilde{f^4(|X|)}(\tau)|+|\widetilde{g^4(|X^*|)}(\tau)|\right)\sigma|\widetilde{X^*X}(\tau)| \\
&\qquad \text{(by Lemma 3.1)} \\
&= \frac{1}{2}\left(|f^4(|\widetilde{X|)+g}^4(|X^*|)(\tau)|\right)\sigma||\widetilde{X}|^2(\tau)|.
\end{aligned}$$

It follows from $\sigma \leqslant \nabla$ and the above inequalities that

$$\begin{aligned}
|\widetilde{X}(\tau)|^2\sigma\|X\widehat{k}_\tau\|^2 &\leqslant \frac{1}{2}\left(|f^4(|\widetilde{X|)+g}^4(|X^*|)(\tau)|\right)\sigma||\widetilde{X}|^2(\tau)| \\
&\leqslant \frac{1}{2}\left[\frac{1}{2}\left(|f^4(|\widetilde{X|)+g}^4(|X^*|)(\tau)|\right)+||\widetilde{X}|^2(\tau)|\right] \\
&\leqslant \left|\widetilde{\left(\frac{1}{4}\left(f^4(|X|)+g^4(|X^*|)\right)+\frac{1}{2}|X|^2\right)}(\tau)\right| \\
&\leqslant \mathbf{ber}\left(\frac{1}{4}\left(f^4(|X|)+g^4(|X^*|)\right)+\frac{1}{2}|X|^2\right).
\end{aligned}$$

Then, by taking the supremum over $\tau \in \Theta$, we get the first result. For the second inequality, we have

$$
\begin{aligned}
|\widetilde{X}(\tau)|^2 \sigma \|X\widehat{k}_\tau\|^2 &= |\widetilde{X}(\tau)|^2 \sigma |\widetilde{|X|^2}(\tau)| \\
&\leqslant |\widetilde{f^2(|X|)}(\tau)||\widetilde{g^2(|X^*|)}(\tau)| \sigma \sqrt{|\widetilde{f^2(|X|^2)}(\tau)||\widetilde{g^2(|X|^2)}(\tau)|} \\
&= \sqrt{|\widetilde{f^2(|X|)}(\tau)|^2 |\widetilde{g^2(|X^*|)}(\tau)|^2} \sigma \sqrt{|\widetilde{f^2(|X|^2)}(\tau)||\widetilde{g^2(|X|^2)}(\tau)|} \\
&\leqslant \sqrt{|\widetilde{f^4(|X|)}(\tau)||\widetilde{g^4(|X^*|)}(\tau)|} \sigma \sqrt{|\widetilde{f^2(|X|^2)}(\tau)||\widetilde{g^2(|X|^2)}(\tau)|} \\
&\leqslant \frac{1}{2}\left(\sqrt{|\widetilde{f^4(|X|)}(\tau)||\widetilde{g^4(|X^*|)}(\tau)|} + \sqrt{|\widetilde{f^2(|X|^2)}(\tau)||\widetilde{g^2(|X|^2)}(\tau)|}\right) \\
&\leqslant \frac{1}{2}\left(\sqrt{|\widetilde{f^4(|X|)}(\tau)| + |\widetilde{g^2(|X|^2)}(\tau)|} \sqrt{|\widetilde{f^2(|X|^2}(\tau)| + |\widetilde{g^4(|X^*|)}(\tau)|)}\right) \\
&\qquad \text{(by the Cauchy-Schwarz inequality } \sqrt{ab}+\sqrt{cd} \leqslant \sqrt{a+c}\sqrt{b+d}) \\
&\leqslant \frac{1}{2}\left(\sqrt{|\big(\widetilde{f^4(|X|) + g^2(|X|^2)}\big)(\tau)||\widetilde{f^2(|X|^2) + g^4(|X^*|)}(\tau)|}\right) \\
&\leqslant \frac{1}{2}\sqrt{\mathbf{ber}\left(f^4(|X|) + g^2(|X|^2)\right)\mathbf{ber}\left(f^2(|X|^2) + g^4(|X^*|)\right)}.
\end{aligned}
$$

Therefore,

$$
\|X\|_\sigma^2 \leqslant \frac{1}{2}\sqrt{\mathbf{ber}\left(f^4(|X|) + g^2(|X|^2)\right)\mathbf{ber}\left(f^2(|X|^2) + g^4(|X^*|)\right)} \qquad \square
$$

For the special case $f(t) = g(t) = \sqrt{t}$, we have the following result.

Corollary 3.6 *Let $X \in \mathcal{L}(\mathcal{H})$ and let σ is a mean dominated by the arithmetic mean ∇. Then*

$$
\|X\|_\sigma^2 \leqslant \boldsymbol{ber}\left(\frac{3}{4}|X|^2 + \frac{1}{4}|X^*|^2\right) \tag{3.1}
$$

and

$$
\|X\|_\sigma^2 \leqslant \frac{1}{2}\sqrt{2\boldsymbol{ber}\left(|X|^2\right)\boldsymbol{ber}\left(|X|^2 + |X^*|^2\right)}.
$$

Remark 3.7 If $X \in \mathcal{L}(\mathcal{H})$ and $\sigma \leqslant \nabla$, then by the inequality (3.1) and also the subadditivity of the Berezin radius, we have

$$
\begin{aligned}
\|X\|_\sigma^2 &\leqslant \mathbf{ber}\left(\frac{3}{4}|X|^2 + \frac{1}{4}|X^*|^2\right) \\
&\leqslant \frac{3}{4}\mathbf{ber}\left(|X|^2\right) + \frac{1}{4}\mathbf{ber}\left(|X^*|^2\right).
\end{aligned}
$$

Now, if X is normal, then by the definition of the Berezin radius, we get $\mathbf{ber}(|X^*|) = \mathbf{ber}(|X|)$. Hence, we have

$$
\|X\|_\sigma^2 \leqslant \frac{3}{4}\mathbf{ber}\left(|X|^2\right) + \frac{1}{4}\mathbf{ber}\left(|X^*|^2\right)
$$

$$\leqslant \mathbf{ber}\left(|X|^2\right)$$
$$\leqslant \mathbf{ber}^2\left(|X|\right) \quad \text{(by Lemma 3.1)} \tag{3.2}$$

Moreover, if X is positive, then the inequality (3.2) and the fact that $\mathbf{ber}(X) \leqslant \|X\|_\sigma$ imply that $\|X\|_\sigma = \mathbf{ber}(X)$.

Example 3.8 Consider for $\mathbb{C}^2$ the standard orthonormal basis $\{e_1, e_1\}$ as a RKHS on the set $\{1, 2\}$. Then for the self-adjoint matrix $X = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$, which is not positive, we have

$$\mathbf{ber}(X) = 2 \lneqq \|X\|_\sigma = \sqrt{5}.$$

Theorem 3.9 *Let $X \in \mathcal{L}(\mathcal{H})$, and let σ is a mean dominated by the arithmetic mean ∇ and $0 \leqslant \mu \leqslant 1$. Then*

$$\|X\|_\sigma^2 \leqslant \frac{1}{2}\boldsymbol{ber}\big((1+\mu)|X|^2 + (1-\mu)|X^*|^2\big).$$

In particular,

$$\|X\|_\sigma^2 \leqslant \frac{1}{2}\boldsymbol{ber}\big(|X|^2 + |X^*|^2\big).$$

Proof Let $\tau \in \Theta$. Then

$$
\begin{aligned}
|\widetilde{X}(\tau)|^2 \sigma \, \|X\widehat{k}_\tau\|^2 &= |\widetilde{X}(\tau)|^2 \sigma \, ||\widetilde{X}|^2(\tau)| \\
&\leqslant |\widetilde{|X|^{2\mu}}(\tau)|\widetilde{||X^*|^{2(1-\mu)}}(\tau)|\sigma \, ||\widetilde{X}|^2(\tau)| \\
&\leqslant |\widetilde{|X|^2}(\tau)|^{\mu} |\widetilde{|X^*|^2}(\tau)|^{(1-\mu)}\sigma \, ||\widetilde{X}|^2(\tau)| \\
&\qquad \text{(by Lemma 3.1)} \\
&\leqslant \left|(\widetilde{\mu|X|^2 + (1-\mu)|X^*|^2})(\tau)\right|\sigma \, ||\widetilde{X}|^2(\tau)| \\
&\qquad \text{(by the weighted arithmetic-geometric mean inequality)} \\
&\leqslant \frac{1}{2}\left(\left|(\widetilde{\mu|X|^2 + (1-\mu)|X^*|^2})(\tau)\right| + ||\widetilde{X}|^2(\tau)|\right) \\
&= \frac{1}{2}\left(\left|(\widetilde{(1+\mu)|X|^2 + (1-\mu)|X^*|^2})(\tau)\right|\right) \\
&\leqslant \frac{1}{2}\mathbf{ber}\big((1+\mu)|X|^2 + (1-\mu)|X^*|^2\big).
\end{aligned}
$$

Taking the supremum over all $\tau \in \Theta$, we get

$$\|X\|_\sigma^2 \leqslant \frac{1}{2}\mathbf{ber}\big((1+\mu)|X|^2 + (1-\mu)|X^*|^2\big).$$

If we put $\mu = 0$, then we have the second inequality. □

Theorem 3.10 *Let $X \in \mathcal{L}(\mathcal{H})$ and $0 \leqslant t \leqslant 1$. Then*

$$\|X\|_{\nabla_t} \leqslant \inf_{\lambda\in[0,1]} \sqrt{\lambda \|X\|_{\text{ber}}^2 + (1-\lambda)\|X\|_{\text{ber}} \Big((1-t)\|X\|_{\text{ber}} + t\,\boldsymbol{ber}(X)\Big)}.$$

Proof Given $u, v \in \mathcal{H}$ and $\lambda \in [0,1]$, we have the following refinement of the classical Cauchy-Schwarz inequality:

$$\begin{aligned}|\langle u, v\rangle|^2 &= [(1-\lambda)+\lambda]|\langle u, v\rangle|^2 \\ &\le (1-\lambda)|\langle u, v\rangle|^2 + \lambda\|u\|^2\|v\|^2 \\ &\le (1-\lambda)\|u\|\|v\|\,|\langle u, v\rangle| + \lambda\|u\|^2\|v\|^2. \end{aligned} \tag{3.3}$$

Utilizing the inequality (3.3), yields

$$(1-t)|\langle u, v\rangle|^2 \le (1-t)(1-\lambda)\|u\|\|v\|\,|\langle u, v\rangle| + (1-t)\lambda\|u\|^2\|v\|^2, \tag{3.4}$$

and

$$t|\langle u, w\rangle|^2 \le t(1-\lambda)\|u\|\|w\|\,|\langle u, w\rangle| + t\lambda\|u\|^2\|w\|^2, \tag{3.5}$$

for all $u, v, w \in \mathcal{H}$ and $t \in [0,1]$. Adding the relations (3.4), (3.5) and replacing v with $\frac{u}{\|v\|}$, we obtain

$$\begin{aligned}(1-t)\|u\|^2 + t|\langle u, w\rangle|^2 &\le \lambda\|u\|^2\left((1-t)+t\|w\|^2\right) \\ &\quad + (1-\lambda)\|u\|\left((1-t)\|u\| + t\|w\|\,|\langle u, w\rangle|\right). \end{aligned} \tag{3.6}$$

By substituting u for $X\hat{k}_\tau$ and w for $\hat{k}_\tau$, we have

$$\begin{aligned}(1-t)\|X\hat{k}_\tau\|^2 + t|\langle X\hat{k}_\tau, \hat{k}_\tau\rangle|^2 &\le \lambda\|X\hat{k}_\tau\|^2 \\ &\quad + (1-\lambda)\|X\hat{k}_\tau\|\Big((1-t)\|X\hat{k}_\tau\| + t|\langle X\hat{k}_\tau, \hat{k}_\tau\rangle|\Big). \end{aligned} \tag{3.7}$$

Taking the supremum over all $\tau \in \Theta$, we have

$$\begin{aligned}\|X\|_{\nabla_t}^2 &= \sup_{\tau\in\Theta}\left\{(1-t)\|X\hat{k}_\tau\|^2 + t|\langle X\hat{k}_\tau, \hat{k}_\tau\rangle|^2\right\} \\ &\le \sup_{\tau\in\Theta}\left\{\lambda\|X\hat{k}_\tau\|^2 + (1-\lambda)\|X\hat{k}_\tau\|\Big((1-t)\|X\hat{k}_\tau\| + t|\langle X\hat{k}_\tau, \hat{k}_\tau\rangle|\Big)\right\} \\ &\le \lambda\|X\|_{\text{ber}}^2 + (1-\lambda)\|X\|_{\text{ber}}\Big((1-t)\|X\|_{\text{ber}} + t\,\mathbf{ber}(X)\Big), \end{aligned}$$

for any $\lambda \in [0,1]$. □

Remark 3.11 Taking $\lambda = \frac{1}{3}$ in Theorem 3.10, we have a refinement of [1, Theorem 6]. Moreover, from the relation (2.5) and Theorem 3.10, we obtain for any pair of non-negative real numbers α, β such that $(\alpha, \beta) \neq (0,0)$,

$$\|X\|_{\alpha,\beta}^{\mathbf{ber}} = \sqrt{\alpha+\beta}\,\|X\|_{\nabla_{t_1}}$$

$$\leq \sqrt{\alpha+\beta} \inf_{\lambda\in[0,1]} \sqrt{\lambda \left\|X\right\|_{\text{ber}}^2 + (1-\lambda) \left\|X\right\|_{\text{ber}} \Big((1-t_1) \left\|X\right\|_{\text{ber}} + t_1 \,\mathbf{ber}\,(X)\Big)}$$

$$= \inf_{\lambda\in[0,1]} \sqrt{(\alpha+\beta)\lambda \left\|X\right\|_{\text{ber}}^2 + (1-\lambda) \left\|X\right\|_{\text{ber}} \Big(\alpha \left\|X\right\|_{\text{ber}} + \beta \,\mathbf{ber}\,(X)\Big)}$$

where $t_1 = \frac{\beta}{\alpha+\beta}$.

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