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Boundedness and Finite-Time Blow-up in a Chemotaxis System with Flux Limitation and Logistic Source

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Abstract
The chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u|\nabla v|^{p-2}\nabla v) + \lambda u - \mu u^{\kappa}, \\ 0 = \Delta v + u - h(u, v) \end{cases} \tag{$*$}$$

is considered in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$, where $\chi > 0$, $p > 1$, $\lambda \geq 0$, $\mu > 0$, $\kappa > 1$, and $h = v$ or $h = \frac{1}{|\Omega|}\int_\Omega u$. It is firstly proved that if $n = 1$ and $p > 1$ is arbitrary, or $n \geq 2$ and $p \in (1, \frac{n}{n-1})$, then for all continuous initial data a corresponding no-flux type initial-boundary value problem for $(*)$ admits a globally defined and bounded weak solution. Secondly, it is shown that if $n \geq 2$, $\Omega = B_R(0) \subset \mathbb{R}^n$ is a ball with some $R > 0$, $p > \frac{n}{n-1}$ and $\kappa > 1$ is small enough, then one can find a nonnegative radially symmetric function u_0 and a weak solution of $(*)$ with initial datum u_0 which blows up in finite time.

Keywords Chemotaxis · Flux limitation · Boundedness · Finite-time blow-up

Mathematics Subject Classification Primary 35B44 · Secondary 35B45 · 35D30 · 92C17

1 Introduction

A chemotaxis system, which Keller and Segel introduced ahead of others ([16]), often describes chemotactic aggregation phenomena as finite-time blow-up of solutions to the system. Their trailblazing studies on modeling chemotaxis made it possible to analyze the movement of an organism toward a higher concentration of a chemical substance mathematically, and to this date, a large variety of chemotaxis systems has been widely investigated. Naturally, in the study of these systems, one of the mathematical interests is whether the systems admit solutions which blow up in finite time, or solutions are global and bounded, though, there are still many systems that such a behavior of solutions to them is unknown or not fully investigated (cf. the survey [1]).

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In this paper we reveal global existence, boundedness and finite-time blow-up in a chemotaxis system which was remained to be clarified. Specifically, we consider the chemotaxis system with flux limitation and logistic source,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u|\nabla v|^{p-2}\nabla v) + \lambda u - \mu u^\kappa, & x \in \Omega,\ t>0, \\ 0 = \Delta v + u - h(u,v), & x \in \Omega,\ t>0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t>0, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1.1}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$), where $\chi > 0$, $p > 1$, $\lambda \ge 0$, $\mu > 0$, $\kappa > 1$,

$$h(u,v) = v \tag{1.2}$$

or

$$h(u,v) = \frac{1}{|\Omega|}\int_\Omega u, \tag{1.3}$$

ν is the outward normal vector to $\partial\Omega$, and where

$$u_0 \in C^0(\overline{\Omega}) \quad \text{and} \quad u_0 \ge 0 \text{ in } \overline{\Omega}. \tag{1.4}$$

Herein, u stands for the population density of organisms, and v is used to describe the concentration of a signal substance.

When $p = 2$, the model (1.1) turns out to be the chemotaxis-growth system,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u\nabla v) + \lambda u - \mu u^\kappa, & x \in \Omega,\ t>0, \\ 0 = \Delta v + u - h(u,v), & x \in \Omega,\ t>0, \end{cases} \tag{1.5}$$

which has been studied over decades, and there are a lot of results concerning global existence, boundedness and finite-time blow-up.

In the case that h satisfies (1.2), Tello and Winkler [27] showed that classical solutions to (1.5) are global and bounded when $\kappa = 2$ and $\mu > \max\{0, \frac{n-2}{n}\chi\}$, or $\kappa > 2$ and $\mu > 0$. Kang and Stevens [15] extended the result on global existence to the case $\mu = \frac{n-2}{n}\chi$ when $n \ge 3$ and $\kappa = 2$. Similar results for parabolic–parabolic relatives of the system can be found in [22, 30, 38], for instance.

On the other hand, in the radially symmetric case and when h satisfies (1.3), finite-time blow-up in (1.5) was firstly detected by Winkler [31] for $n \ge 5$ and $1 < \kappa < \frac{3}{2} + \frac{1}{2(n-1)}$, and then the result on blow-up in (1.5) with h satisfying (1.2) was also established in [35] when $n \ge 3$ as well as

$$1 < \kappa < \begin{cases} \frac{7}{6} & \text{if } n \in \{3,4\}, \\ 1 + \frac{1}{2(n-1)} & \text{if } n \ge 5. \end{cases} \tag{1.6}$$

Later on, these results were generalized to the case of nonlinear diffusion by Black *et al.* [4], where they also extended the conditions in [31] to the lower-dimensional as $n \ge 3$ and $1 \le \kappa < \min\{\frac{3}{2}, \frac{2(n-1)}{n}\}$. Recently, in the case that h satisfies (1.3), Fuest [9] showed that finite-time blow-up of solutions can occur at the origin when $n \ge 3$, $1 < \kappa < \min\{2, \frac{n}{2}\}$ and

$\mu > 0$, or $n \geq 5$, $\kappa = 2$ and $\mu \in (0, \frac{n-4}{n})$. In [4, 9, 35], they introduced the moment-type functional

$$\phi(t) = \int_0^t s^{-\gamma}(s_0 - s)w(s,t)\,\mathrm{d}s,$$

where w is a mass accumulation function corresponding to solutions (cf. [12]), and derived the superlinear differential inequality

$$\phi'(t) \geq c_1\phi^2(t) - c_2, \quad t > 0,$$

which ensures that the maximal existence time of solutions is finite. The same method was also used to establish finite-time blow-up in other chemotaxis systems (e.g. [7, 20, 34]).

For the system with flux limitation and logistic term,

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u(1+|\nabla v|^2)^{-\alpha}\nabla v) + \lambda u - \mu u^\kappa, & x\in\Omega,\ t>0, \\ 0 = \Delta v + u - \frac{1}{|\Omega|}\int_\Omega u(\cdot,t),\ \int_\Omega v(\cdot,t) = 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.7}$$

Marras *et al.* [19] showed that in the case $0 < \alpha < \frac{n-2}{2(n-1)}$, (1.7) admits finite-time blow-up when $n \geq 3$, $1 < \kappa < \min\{2, 1 + \frac{(n-2)^2}{4}\}$ and $\mu > 0$, or when $n \geq 5$, $\kappa = 2$, and $\mu > 0$ is small enough, whereas all radially symmetric solutions are global and bounded when $\alpha > \frac{n-2}{2(n-1)}$ and $\kappa > 1$, or when $\alpha > 0$ and $\kappa > 2$. We note that the case of the critical value $\alpha = \frac{n-2}{2(n-1)}$ in (1.7) is compatible with the case when $p = \frac{n}{n-1}$ in (1.1). Similar results for the system (1.7) with $\lambda = \mu = 0$ can be found in [37].

Now, for the system (1.1) with $\lambda = \mu = 0$, boundedness of solutions was obtained by Negreanu and Tello [21] when $p > 1$ ($n = 1$) or $1 < p < \frac{n}{n-1}$ ($n \geq 2$), whereas when h satisfies (1.3), it was shown that finite-time blow-up occurs when $p > \frac{n}{n-1}$ ($n \geq 2$) in [17, 26]. In [17], we used the technique based on the moment-type functional as in [24] and established the framework of finite-time blow-up of weak solutions in the system (see also [13, 23]). For parabolic–elliptic and fully parabolic variants, we refer to [14, 39]. However, to the best of our knowledge, boundedness and blow-up results for the system (1.1) with $\kappa > 1$ were not obtained.

Our aim of this paper is to present conditions for p and κ that weak solutions of (1.1) are globally bounded or blow up in finite time.

Global existence and boundedness of weak solutions Before we state the main results, let us first give a definition of weak solutions to (1.1).

Definition 1.1 Let u_0 satisfy (1.4) and let $T > 0$. A pair (u, v) of functions is called a *weak solution* of (1.1) in $\Omega \times (0, T)$ if

(i) $u \in C^0_{\mathrm{w}-\star}([0,T); L^\infty(\Omega)) \cap L^2_{\mathrm{loc}}([0,T); W^{1,2}(\Omega))$,
(ii) $v \in L^\infty_{\mathrm{loc}}([0,T); W^{1,2}(\Omega))$,
(iii) $u \geq 0$ a.e. on $\Omega\times(0,T)$, and $v \geq 0$ a.e. on $\Omega\times(0,T)$ if h satisfies (1.2),
(iii) $|\nabla v|^{p-2}\nabla v \in L^2_{\mathrm{loc}}(\overline{\Omega}\times[0,T))$,
(iv) for any $\varphi \in C^1_{\mathrm{c}}(\overline{\Omega}\times[0,T))$,

$$\int_0^T\int_\Omega(\nabla u\cdot\nabla\varphi - \chi u|\nabla v|^{p-2}\nabla v\cdot\nabla\varphi - (\lambda u - \mu u^\kappa)\varphi - u\varphi_t) = \int_\Omega u_0\varphi(\cdot,0), \tag{1.8}$$

$$\int_0^T\int_\Omega\nabla v\cdot\nabla\varphi - \int_0^T\int_\Omega u\varphi + \int_0^T\int_\Omega h(u,v)\varphi = 0.$$

If $(u,v):\Omega\times(0,\infty)\to\mathbb{R}^2$ is a weak solution of (1.1) in $\Omega\times(0,T)$ for all $T>0$, then (u,v) is called a *global weak solution* of (1.1), or a weak solution of (1.1) in $\Omega\times(0,\infty)$.

Our first result will then be to establish global existence and boundedness of weak solutions.

Theorem 1.1 *Let $\Omega\subset\mathbb{R}^n$ ($n\in\mathbb{N}$) be a smoothly bounded domain, and let $\chi>0$, $\lambda\geq 0$, $\mu>0$ and $\kappa>1$. Assume that p fulfills*

$$\begin{cases} p>1 & \text{if } n=1,\\ p\in\left(1,\dfrac{n}{n-1}\right) & \text{if } n\geq 2,\end{cases} \tag{1.9}$$

that h satisfies (1.2) or (1.3), and that u_0 satisfies (1.4). Then the problem (1.1) admits a global weak solution (u,v), which is bounded in the sense that there exists $C>0$ fulfilling

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}\leq C \quad \text{for all } t>0. \tag{1.10}$$

Remark 1.1 The precedent work in [19, Theorem 1.5] indicates that when $\kappa>2$, it might be able to improve the condition (1.9) as $p\in(1,2)$ if $n\geq 2$. However, due to singularity of chemotactic coefficient we could not apply their proofs in our case, and it is still unknown whether the condition (1.9) is optimal or not in the system (1.1).

Finite-time blow-up of weak solutions We next state finite-time blow-up of weak solutions to (1.1). In this part, we let $\Omega=B_R(0)\subset\mathbb{R}^n$ ($n\geq 2$) be a ball with some $R>0$, and let h satisfy (1.3). For $T\in(0,\infty]$ and a function u defined a.e. on $\Omega\times[0,T)$ which is radially symmetric with respect to $x=0$, we write $u(|x|,t)$ instead of $u(x,t)$ if necessary, and for $s_0\in(0,R^n)$ and $\gamma\in(0,1)$ we define the moment-type functional as

$$\phi(t):=\int_0^{s_0} s^{-\gamma}(s_0-s)\left(w(s,t)-\frac{s}{n}\overline{M}(t)\right)\mathrm{d}s \quad \text{for } t\in[0,T), \tag{1.11}$$

where

$$w(s,t):=\int_0^{s^{\frac{1}{n}}}\sigma^{n-1}u(\sigma,t)\,\mathrm{d}\sigma \quad \text{for } s\in[0,R^n] \text{ and } t\in[0,T)$$

and

$$\overline{M}(t):=\frac{1}{|\Omega|}\int_\Omega u(\cdot,t) \quad \text{for } t\in[0,T).$$

Within these settings, the second of our main results asserts that finite-time blow-up can occur in (1.1).

Theorem 1.2 *Let $\Omega=B_R(0)\subset\mathbb{R}^n$ ($n\geq 2$) be a ball with some $R>0$, and let $\chi>0$, $\lambda\geq 0$ and $\mu>0$. Assume that h satisfies (1.3), and that p and κ satisfy*

$$p>\frac{n}{n-1} \tag{1.12}$$

as well as

$$1 < \kappa < 1 + \frac{1}{(n-1)(p-1)p} \min\left\{ \frac{2+np-n-p}{n}, \frac{p(np-n-p)}{n(p-1)} \right\}. \quad (1.13)$$

Then for all $L > 0$ *and* $m > 0$, *one can find* $m_0 \in (0, m)$ *and* $r_0 \in (0, R)$ *with the following property*: *Whenever* u_0 *satisfies* (1.4) *and*

$$u_0 \text{ is radially symmetric with respect to } x = 0, \quad (1.14)$$

and is such that

$$\frac{1}{|B_r(0)|} \int_{B_r(0)} u_0 \geq \frac{1}{|\Omega|} \int_\Omega u_0 \quad \text{for all } r \in (0, R) \quad (1.15)$$

as well as

$$u_0(x) \leq L|x|^{-n(n-1)(p-1)} \quad \text{for all } x \in \Omega \quad (1.16)$$

and

$$\int_\Omega u_0 = m \quad \text{but} \quad \int_{B_{r_0}(0)} u_0 \geq m_0, \quad (1.17)$$

there exist $T_{\max} \in (0, \infty]$ *and a weak solution* (u, v) *of* (1.1) *in* $\Omega \times (0, T_{\max})$, *which satisfies*

$$\phi(t) - \phi(0) \geq C s_0^{-3+3(2-p)-\frac{2-p}{n}+(p-1)\gamma} \int_0^t \phi^p(\tau)\, d\tau - C s_0^{3-\gamma-\theta} t \quad (1.18)$$

for all $t \in (0, \min\{1, T_{\max}\})$ *with some* $s_0 \in (0, R^n)$, $\gamma \in (0, 1)$, $\theta \in (0, 2)$ *and* $C \geq 0$, *and moreover, if* $C > 0$, *then* (u, v) *blows up in finite time in the sense that* $T_{\max} < \infty$ *and*

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \quad (1.19)$$

Remark 1.2 We note that in Theorem 1.2, if (u, v) is a classical solution of (1.1) in $\Omega \times (0, T_{\max})$, then we can apply the same arguments as in Sect. 4 to obtain the moment inequality (1.18) for (u, v) with $C > 0$ directly, which ensures that $T_{\max} < \infty$ and (1.19) holds.

It might be possible to show that for any weak solutions of (1.1) the inequality (1.18) holds with $C > 0$, by choosing a suitable test function in (1.8) to construct the moment inequality for weak solutions directly. We plan to continue working on this in future.

Remark 1.3 When $p = 2$, the condition (1.13) can be reduced to (1.6), however, our method could not reach the better conditions as in [4, 9]. One of the main causes is the fact that the inequality $u_r \leq 0$ does not hold in our case, which makes a pointwise estimate for u different from the previous papers (see Lemma 4.8 for a more precise statement).

Plan of the paper The main part of our analysis in both boundedness and finite-time blow-up will be considering the regularized problem of (1.1), and establishing global boundedness or a moment inequality for approximate solutions $(u_\varepsilon, v_\varepsilon)$. Then we construct a weak solution (u, v) by approximation, and show that the same boundedness/blow-up properties are also valid for (u, v). Noting that the maximal existence time of $(u_\varepsilon, v_\varepsilon)$ depends on the parameter ε, first we will find the time $T > 0$ such that for any parameters ε the approximate solution $(u_\varepsilon, v_\varepsilon)$ exists in $\Omega \times (0, T)$ (Lemma 2.5).

Global existence and boundedness in (1.1) (Theorem 1.1) can be achieved by deriving a uniform bound for $\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}$ on $(0, \infty)$ (Lemma 3.2).

To prove finite-time blow-up (Theorem 1.2), we construct a suitable moment-type functional for approximate solutions by following the pioneering works in [37]. Then we generalize the techniques in [35] to the flux limitation case (Lemmas 4.8 and 4.9), and derive a superlinear differential inequality for its functional. The arguments for blow-up of weak solutions are based on [24].

This paper is organized as follows. In Sect. 2 we show existence of weak solutions on $\Omega \times (0, T)$ for some $T > 0$. Section 3 is devoted to the proof of Theorem 1.1, which ensures global existence and boundedness of weak solutions in (1.1). Finally, under the radially symmetric setting, in Sect. 4 we prove Theorem 1.2, establishing finite-time blow-up of weak solutions to (1.1).

2 Existence of Weak Solutions

Throughout this section, we fix a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ and u_0 satisfying (1.4). The function h is assumed to satisfy (1.2) or (1.3).

We shall show existence of weak solutions to (1.1) in $\Omega \times (0, T)$ with some $T > 0$, which we will achieve by considering a regularized problem and constructing solutions of (1.1) through an approximation procedure.

Let us start by modifying the term $-\chi\nabla \cdot (u|\nabla v|^{p-2}\nabla v)$ in the first equation therein to another term which has no singularity. Accordingly, for $\varepsilon \in (0, 1)$ we shall consider the regularized problem

$$\begin{cases} (u_\varepsilon)_t = \Delta u_\varepsilon - \chi\nabla \cdot (u_\varepsilon(|\nabla v_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}}\nabla v_\varepsilon) + \lambda u_\varepsilon - \mu u_\varepsilon^\kappa, & x \in \Omega,\ t > 0, \\ 0 = \Delta v_\varepsilon + u_\varepsilon - h(u_\varepsilon, v_\varepsilon), & x \in \Omega,\ t > 0, \\ \nabla u_\varepsilon \cdot \nu = \nabla v_\varepsilon \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x, 0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

In particular, in the case that h satisfies (1.3), for each $\varepsilon \in (0, 1)$ we consider the problem (2.1) with

$$\int_\Omega v_\varepsilon(\cdot, t) = 0, \quad t > 0. \tag{2.2}$$

A theory for local existence in (2.1) can be obtained by a standard fixed point argument and regularity theory (see e.g. [8, Lemma 1.2] or [27, Theorem 2.1]). We record the basic statement without proof.

Lemma 2.1 *Let $\chi > 0$, $p > 1$, $\lambda \ge 0$, $\mu > 0$ and $\kappa \ge 1$. Then for each $\varepsilon \in (0, 1)$ there exist $T_{\max,\varepsilon} \in (0, \infty]$ and uniquely determined functions*

$$\begin{aligned} &u_\varepsilon \in C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})) \quad \textit{and} \\ &v_\varepsilon \in \bigcap_{q>n} L^\infty_{\mathrm{loc}}([0, T_{\max,\varepsilon}); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0, T_{\max,\varepsilon})) \end{aligned} \tag{2.3}$$

such that $(u_\varepsilon, v_\varepsilon)$ *solves* (2.1) *in the classical sense in* $\Omega \times (0, T_{\max,\varepsilon})$, *that fulfills* (2.2) *in the case that h satisfies* (1.3), *and that*

$$\text{if } T_{\max,\varepsilon} < \infty, \quad \text{then} \quad \limsup_{t \nearrow T_{\max,\varepsilon}} \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.4}$$

Moreover, $u_\varepsilon \geq 0$ *in* $\Omega \times (0, T_{\max,\varepsilon})$, *and additionally,* $v_\varepsilon \geq 0$ *in* $\Omega \times (0, T_{\max,\varepsilon})$ *if h satisfies* (1.2). *Furthermore, if* $\Omega = B_R(0)$ *with some* $R > 0$ *and* u_0 *is radially symmetric with respect to* $x = 0$, *then also* $u_\varepsilon(\cdot, t)$ *and* $v_\varepsilon(\cdot, t)$ *are radially symmetric with respect to* $x = 0$ *for each* $t \in (0, T_{\max,\varepsilon})$.

In the sequel, for each $\varepsilon \in (0, 1)$ we let $T_{\max,\varepsilon}$ and $(u_\varepsilon, v_\varepsilon)$ be as accordingly provided by Lemma 2.1.

We next recall boundedness of $\|u_\varepsilon(\cdot, t)\|_{L^1(\Omega)}$, which is important as usual and will be used frequently throughout the paper.

Lemma 2.2 *Let* $\lambda \geq 0$, $\mu > 0$ *and* $\kappa > 1$. *For any* $\varepsilon \in (0, 1)$, u_ε *satisfies*

$$\int_\Omega u_\varepsilon(\cdot, t) \leq M_1 := \max\left\{\int_\Omega u_0, \left(\frac{\lambda}{\mu}\right)^{\frac{1}{\kappa-1}} |\Omega|\right\} \quad \text{for all } t \in (0, T_{\max,\varepsilon}).$$

Proof Using the first equation in (2.1) along with the Hölder inequality, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u \leq \lambda \int_\Omega u - \mu|\Omega|^{1-\kappa}\left(\int_\Omega u\right)^\kappa$$

in $(0, T_{\max,\varepsilon})$ for all $\varepsilon \in (0, 1)$. An ODE comparison argument hence proves the claim. □

Aiming to construct weak solutions of (1.1), we first have to ensure the existence of $T > 0$ such that for every $\varepsilon \in (0, 1)$, the system (2.1) admits a classical solution in $\Omega \times (0, T)$. This is sufficient to find $T > 0$ and $K > 0$ such that

$$T \leq T_{\max,\varepsilon} \quad \text{and} \quad \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq K \quad \text{for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1). \tag{2.5}$$

In order to show this, let us first state the lemma which gives an estimate for ∇v_ε.

Lemma 2.3 *Let* $\varepsilon \in (0, 1)$, $\lambda \geq 0$, $\mu > 0$, $\kappa > 1$, $q > n$ *and* $T \in (0, \infty]$. *Assume that there is* $\widetilde{M} > 0$ *such that*

$$\sup_{t\in(0,T)} \|u_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \leq \widetilde{M}.$$

Then there exists $L = L(\Omega, q, M_1, \widetilde{M}) > 0$ *such that*

$$\sup_{t\in(0,T)} \|v_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq L.$$

Proof With the aid of Lemma 2.2, this can be shown similarly in [17, Lemma 2.2]. □

Before deriving (2.5) we show the next key lemma, which implies that for $q > n$, the function $t \mapsto \|u_\varepsilon(\cdot, t)\|_{L^q(\Omega)}$ is uniformly bounded on some time interval with respect to ε. The idea of the proof is based on [11, Lemma 2.4] (see also [17, Lemma 2.3]).

Lemma 2.4 *Let $\chi > 0$, $p > 1$, $\lambda \geq 0$, $\mu > 0$, $\kappa > 1$ and $q > n$. Then there exists $T_q > 0$ such that*

$$T_q \leq T_{\max,\varepsilon} \quad \text{and} \quad \|u_\varepsilon(\cdot,t)\|^q_{L^q(\Omega)} \leq \|u_0\|^q_{L^q(\Omega)} + 1 \quad \text{for all } t \in [0, T_q) \text{ and } \varepsilon \in (0,1). \tag{2.6}$$

Proof For $\varepsilon \in (0,1)$ we put

$$\tau_\varepsilon := \sup\{\tau \in (0, T_{\max,\varepsilon}) \mid \|u_\varepsilon(\cdot,t)\|^q_{L^q(\Omega)} \leq c_1 := \|u_0\|^q_{L^q(\Omega)} + 1 \quad \text{for all } t \in (0,\tau)\}, \tag{2.7}$$

which is positive due to (2.3). We infer from (2.7) that

$$\|u_\varepsilon(\cdot,t)\|^q_{L^q(\Omega)} \leq c_1 \quad \text{for all } t \in (0,\tau_\varepsilon) \text{ and } \varepsilon \in (0,1), \tag{2.8}$$

whence Lemma 2.3 provides a constant $c_2 = c_2(\Omega, q, M_1, c_1) > 0$ such that

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq c_2 \quad \text{for all } t \in (0,\tau_\varepsilon) \text{ and } \varepsilon \in (0,1). \tag{2.9}$$

For each $\varepsilon \in (0,1)$, it suffices to consider the cases $\tau_\varepsilon = T_{\max,\varepsilon} = \infty$, and $\tau_\varepsilon < T_{\max,\varepsilon}$ with $\|u_\varepsilon(\cdot,\tau_\varepsilon)\|^q_{L^q(\Omega)} = c_1$. Indeed, if $\tau_\varepsilon = T_{\max,\varepsilon} < \infty$, then by the Moser iteration technique (cf. [25]) there is $K > 0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq K \quad \text{for all } t \in (0, T_{\max,\varepsilon}),$$

which contradicts (2.4).

First, in the case that $\tau_\varepsilon = T_{\max,\varepsilon} = \infty$, let us choose any $\widetilde{T}_q > 0$. Then from (2.8), we find that

$$\widetilde{T}_q \leq T_{\max,\varepsilon} \quad \text{and} \quad \|u_\varepsilon(\cdot,t)\|^q_{L^q(\Omega)} \leq \|u_0\|^q_{L^q(\Omega)} + 1 \quad \text{for all } t \in [0, \widetilde{T}_q). \tag{2.10}$$

Next, in the case when $\tau_\varepsilon < T_{\max,\varepsilon}$ with $\|u_\varepsilon(\cdot,\tau_\varepsilon)\|^q_{L^q(\Omega)} = c_1$, we see on testing the first equation of (2.1) by u_ε^{q-1} in conjunction with the Young inequality and $\mu > 0$ that

$$\begin{aligned}
\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_\varepsilon^q &= -(q-1)\int_\Omega u_\varepsilon^{q-2}|\nabla u_\varepsilon|^2 + \chi(q-1)\int_\Omega u_\varepsilon^{q-1}(|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla v_\varepsilon\cdot\nabla u_\varepsilon \\
&\quad + \lambda\int_\Omega u_\varepsilon^q - \mu\int_\Omega u_\varepsilon^{q+\kappa-1} \\
&= -\frac{4(q-1)}{q^2}\int_\Omega |\nabla u_\varepsilon^{\frac{q}{2}}|^2 + \frac{2\chi(q-1)}{q}\int_\Omega u_\varepsilon^{\frac{q}{2}}(|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla v_\varepsilon\cdot\nabla u_\varepsilon^{\frac{q}{2}} \\
&\quad + \lambda\int_\Omega u_\varepsilon^q - \mu\int_\Omega u_\varepsilon^{q+\kappa-1} \\
&\leq -\frac{4(q-1)}{q^2}\int_\Omega |\nabla u_\varepsilon^{\frac{q}{2}}|^2 + \frac{q-1}{q^2}\int_\Omega |\nabla u_\varepsilon^{\frac{q}{2}}|^2 \\
&\quad + \chi^2(q-1)\int_\Omega u_\varepsilon^q(|\nabla v_\varepsilon|^2+\varepsilon)^{p-2}|\nabla v_\varepsilon|^2 + \lambda\int_\Omega u_\varepsilon^q \\
&\leq -\frac{3(q-1)}{q^2}\int_\Omega |\nabla u_\varepsilon^{\frac{q}{2}}|^2 + \chi^2(q-1)\int_\Omega u_\varepsilon^q(|\nabla v_\varepsilon|^2+1)^{p-1} + \lambda\int_\Omega u_\varepsilon^q
\end{aligned} \tag{2.11}$$

in $(0, T_{\max,\varepsilon})$, where in the last inequality we also used the facts that $\varepsilon \in (0,1)$ and $p > 1$. We combine (2.11) with (2.8) and (2.9) to estimate

$$\frac{d}{dt}\int_\Omega u_\varepsilon^q \le \chi^2 q(q-1)c_1(c_2^2+1)^{p-1} + \lambda q c_1 =: c_3 \quad \text{in } (0, \tau_\varepsilon).$$

Integrating this over $(0, \tau_\varepsilon)$ shows that

$$c_1 - \|u_0\|_{L^q(\Omega)}^q \le c_3 \tau_\varepsilon$$

and that hence

$$\frac{1}{c_3} \le \tau_\varepsilon.$$

Therefore, if we choose $\widetilde{T}_q > 0$ such that $\widetilde{T}_q \le \frac{1}{c_3}$, we obtain (2.10).

Noting that the time $\widetilde{T}_q$ is independent of ε in both cases, we thus conclude that (2.6) holds with $T_q := \frac{1}{c_3}$. □

With the above preparations at hand, we can show the existence of $T > 0$ and $K > 0$ that satisfy (2.5) by using the generalized Moser iteration technique (see [25, Appendix A]).

Lemma 2.5 *Let* $\chi > 0$, $p > 1$, $\lambda \ge 0$, $\mu > 0$ *and* $\kappa > 1$. *Then there exist* $T_0 > 0$, $K_0 > 0$ *and* $L_0 > 0$ *such that*

$$T_0 \le T_{\max,\varepsilon}, \quad \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le K_0 \quad \textit{and} \quad \|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le L_0 \tag{2.12}$$

for all $t \in (0, T_0)$ *and* $\varepsilon \in (0,1)$.

Proof Using Lemmas 2.3 and 2.4 as well as [25, Lemma A.1], this can be obtained in the same way as in [17, Lemma 2.4]. □

In the remaining part of this section, we aim to construct weak solutions of (1.1). To this end, we further give some estimates when there exist $T \in (0, \infty]$ and constants $K, L > 0$ such that

$$T \le T_{\max,\varepsilon}, \quad \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le K \quad \text{and} \quad \|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le L \tag{2.13}$$

hold for all $t \in (0, T)$ and $\varepsilon \in (0,1)$.

Lemma 2.6 *Let* $\chi > 0$, $p > 1$, $\lambda \ge 0$, $\mu > 0$ *and* $\kappa > 1$. *Assume that there are* $T > 0$, $K > 0$ *and* $L > 0$ *such that* (2.13) *holds for all* $t \in (0, T)$ *and* $\varepsilon \in (0,1)$. *Then there exists a constant* $C = C(T) > 0$ *such that*

$$\|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\Omega))}^2 \le C \quad \textit{for all } \varepsilon \in (0,1).$$

Proof Invoking the differential inequality (2.11) with $q = 2$, we readily have the claim by the same arguments as in [17, Lemma 2.5]. □

The next lemma will also be used in Sect. 3 to construct global weak solutions of (1.1). We note that unlike in Lemma 2.6, here we can treat the case $T = \infty$.

Lemma 2.7 *Suppose that u_0 satisfies (1.4), and let $\chi > 0$, $p > 1$, $\lambda \geq 0$, $\mu > 0$ and $\kappa > 1$. Assume that there exist $T \in (0, \infty]$, $K > 0$ and $L > 0$ such that (2.13) holds for all $t \in (0, T)$ and $\varepsilon \in (0, 1)$. Then there is $C > 0$ such that*

$$\|(u_\varepsilon)_t(\cdot, t)\|_{(W_0^{2,2}(\Omega))^\star} \leq C \quad \text{for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1). \tag{2.14}$$

In particular, we have

$$\|u_\varepsilon(\cdot, t_1) - u_\varepsilon(\cdot, t_2)\|_{(W_0^{2,2}(\Omega))^\star} \leq C|t_1 - t_2| \quad \text{for all } t_1, t_2 \in [0, T) \text{ and } \varepsilon \in (0, 1). \tag{2.15}$$

Proof Both the estimate (2.14) and the property (2.15) can be established in the same way as in [17, Lemma 2.6]. □

We are now in a position to show that there exist weak solutions of (1.1) in $\Omega \times (0, T)$ with some $T > 0$.

Lemma 2.8 *Suppose that u_0 satisfies (1.4), and let $\chi > 0$, $p > 1$, $\lambda \geq 0$, $\mu > 0$ and $\kappa > 1$. Assume that there exist $T > 0$, $K > 0$ and $L > 0$ such that (2.13) is valid for all $t \in (0, T)$ and $\varepsilon \in (0, 1)$. Then there exist a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0, 1)$ with $\varepsilon_j \searrow 0$ as $j \to \infty$ and functions u, v fulfilling*

$$u \in L^\infty(0, T; L^\infty(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)) \cap C^0_{w-\star}([0, T); L^\infty(\Omega)), \tag{2.16}$$

$$v \in L^\infty(0, T; W^{1,\infty}(\Omega)) \quad \text{and} \tag{2.17}$$

$$|\nabla v|^{p-2}\nabla v \in L^2(\overline{\Omega} \times (0, T)) \tag{2.18}$$

such that

$$u_{\varepsilon_j} \overset{\star}{\rightharpoonup} u \quad \text{in } L^\infty(0, T; L^\infty(\Omega)), \tag{2.19}$$

$$u_{\varepsilon_j} \to u \quad \text{a.e. in } \Omega \times (0, T), \tag{2.20}$$

$$u_{\varepsilon_j} \to u \quad \text{in } C^0_{\mathrm{loc}}([0, T); (W_0^{2,2}(\Omega))^\star), \tag{2.21}$$

$$\nabla u_{\varepsilon_j} \rightharpoonup \nabla u \quad \text{in } L^2(0, T; L^2(\Omega)), \tag{2.22}$$

$$v_{\varepsilon_j} \overset{\star}{\rightharpoonup} v \quad \text{in } L^\infty(0, T; L^\infty(\Omega)), \tag{2.23}$$

$$v_{\varepsilon_j} \to v \quad \text{a.e. in } \Omega \times (0, T), \tag{2.24}$$

$$\nabla v_{\varepsilon_j} \overset{\star}{\rightharpoonup} \nabla v \quad \text{in } L^\infty(0, T; L^\infty(\Omega)), \tag{2.25}$$

$$\nabla v_{\varepsilon_j} \to \nabla v \quad \text{a.e. in } \Omega \times (0, T) \quad \text{and} \tag{2.26}$$

$$(|\nabla v_{\varepsilon_j}|^2 + \varepsilon_j)^{\frac{p-2}{2}} \nabla v_{\varepsilon_j} \to |\nabla v|^{p-2}\nabla v \quad \text{in } L^2(\overline{\Omega} \times (0, T)) \tag{2.27}$$

as $j \to \infty$, and that (u, v) is a weak solution of (1.1) in $\Omega \times (0, T)$.

Proof Thanks to Lemmas 2.6 and 2.7, as in the proof of [17, Lemma 2.7] we can find a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0, 1)$ and functions u, v such that (2.16), (2.17), (2.19), (2.21), (2.22), (2.23) and (2.25) holds. In addition, the property (2.18) results from (2.17).

We will show (2.20). By (2.13) and Lemma 2.6, we see that

$$(u_{\varepsilon_j})_j \subset L^2(0,T;W^{1,2}(\Omega)),$$

whereas Lemma 2.7 implies that

$$((u_{\varepsilon_j})_t)_j \subset L^\infty(0,T;(W_0^{2,2}(\Omega))^\star).$$

An application of the Aubin–Lions lemma ([28, Theorem III.2.3]) therefore enables us to extract a subsequence, still denoted by $(\varepsilon_j)_{j\in\mathbb{N}}$, and to find $\xi \in L^2(0,T;L^2(\Omega))$ such that

$$u_{\varepsilon_j} \to \xi \quad \text{in } L^2(0,T;L^2(\Omega)) \text{ as } j\to\infty. \tag{2.28}$$

This together with (2.19) entails $\xi = u$. In view of (2.28), along a suitable subsequence we can furthermore achieve (2.20).

We next prove (2.24) and (2.26). Fixing $\varphi \in C^1_c(\overline{\Omega}\times[0,T))$, we test the second equation of (2.1) by φ to obtain

$$\int_0^T\int_\Omega \nabla v_{\varepsilon_j}\cdot\nabla\varphi - \int_0^T\int_\Omega u_{\varepsilon_j}\varphi + \int_0^T\int_\Omega h(u_{\varepsilon_j},v_{\varepsilon_j})\varphi = 0 \quad \text{for all } j\in\mathbb{N}. \tag{2.29}$$

By virtue of (2.23) or (2.19), we see that whenever h satisfies (1.2) or (1.3),

$$h(u_{\varepsilon_j},v_{\varepsilon_j}) \overset{\star}{\rightharpoonup} h(u,v) \quad \text{in } L^\infty(0,T;L^\infty(\Omega)) \quad \text{as } j\to\infty. \tag{2.30}$$

From (2.29) we infer on letting $j\to\infty$ and taking into account (2.19), (2.25) and (2.30) that

$$\int_0^T\int_\Omega \nabla v\cdot\nabla\varphi - \int_0^T\int_\Omega u\varphi + \int_0^T\int_\Omega h(u,v)\varphi = 0$$

holds. In particular, we have

$$\int_\Omega \nabla v\cdot\nabla\widetilde{\varphi} - \int_\Omega u\widetilde{\varphi} + \int_\Omega h(u,v)\widetilde{\varphi} = 0 \quad \text{a.e. in } (0,T)$$

for all $\widetilde{\varphi} \in W_0^{1,2}(\Omega)$. This together with standard elliptic regularity theory and (2.28) warrants that

$$v_{\varepsilon_j} \to v \quad \text{in } L^2(0,T;W^{1,2}(\Omega)) \quad \text{as } j\to\infty. \tag{2.31}$$

Thanks to (2.31), extracting a suitable subsequence we obtain (2.24) and (2.26).

Convergence (2.27) results from (2.26) and the Lebesgue convergence theorem. Finally, we argue similarly in [17, Lemma 2.7] to conclude that (u,v) is indeed a weak solution of (1.1) in $\Omega\times(0,T)$. □

Remark 2.1 If we argue similarly in Lemma 2.8 with u_ε and v_ε respectively replaced with u_{ε_j} and v_{ε_j} for a sequence $(\varepsilon_j)_{j\in\mathbb{N}}$, we can find a subsequence of $(\varepsilon_j)_{j\in\mathbb{N}}$ and functions u, v such that the assertion of Lemma 2.8 remains true for the subsequence and the functions.

3 Global Existence and Boundedness

In this section we shall consider the system (1.1) under the hypothesis that $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ is a smoothly bounded domain and u_0 satisfies (1.4). We also assume that h satisfies (1.2) or (1.3).

This section is devoted to proving Theorem 1.1, which asserts global existence and boundedness in (1.1). As a first step, we derive an L^θ-estimate for ∇v_ε with some $\theta \geq 1$.

Lemma 3.1 *Let $\lambda \geq 0$, $\mu > 0$ and $\kappa > 1$. Then for any θ satisfying*

$$\begin{cases} \theta \geq 1 & \text{if } n = 1, \\ \theta \in \left[1, \dfrac{n}{n-1}\right) & \text{if } n \geq 2, \end{cases} \tag{3.1}$$

there exists $C > 0$ such that

$$\|\nabla v_\varepsilon(\cdot, t)\|_{L^\theta(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0, 1). \tag{3.2}$$

Proof We fix θ satisfying (3.1). In the case that h satisfies (1.2), [5, Lemma 23] provides $C > 0$ such that

$$\|\nabla v_\varepsilon(\cdot, t)\|_{L^\theta(\Omega)} \leq C\|u_\varepsilon(\cdot, t)\|_{L^1(\Omega)} \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0, 1). \tag{3.3}$$

Thanks to Lemma 2.2, this immediately implies (3.2).

When h fulfills (1.3), by virtue of [6, Theorem 2.8 and Lemma 2.5] we can also find $C > 0$ such that (3.3) holds, and hence we proceed as above to obtain (3.2). □

As a consequence of Lemma 3.1 we shall acquire boundedness of u_ε in $L^\infty(\Omega)$, which will be one of the important ingredients to prove Theorem 1.1.

Lemma 3.2 *Let $\chi > 0$, $\lambda \geq 0$, $\mu > 0$ and $\kappa > 1$. Suppose that p satisfies (1.9). Then there exists $C > 0$ such that*

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0, 1). \tag{3.4}$$

In particular, we have

$$T_{\max,\varepsilon} = \infty \quad \text{for all } \varepsilon \in (0, 1). \tag{3.5}$$

Proof As p satisfies (1.9), Lemma 3.1 asserts the existence of $q > n$ and $c_1 > 0$ such that abbreviating $P_\varepsilon(\cdot, t) := -\chi(|\nabla v_\varepsilon(\cdot, t)|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla v_\varepsilon(\cdot, t)$ for $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0, 1)$, we have

$$\|P_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \leq c_1 \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0, 1). \tag{3.6}$$

Since u_ε satisfies

$$(u_\varepsilon)_t \leq \Delta u_\varepsilon + \nabla \cdot (u_\varepsilon P_\varepsilon) + c_2$$

in $\Omega \times (0, T_{\max,\varepsilon})$ for all $\varepsilon \in (0,1)$ with $c_2 := \lambda(\frac{\lambda}{\mu\kappa})^{\frac{1}{\kappa-1}} + 1 > 0$, invoking semigroup estimates for the Neumann heat semigroup (cf. [29, Lemma 1.3]) together with Lemma 2.2 and (3.6), we find $\iota \in (0,1)$ and $c_3 = c_3(\|u_0\|_{L^\infty(\Omega)}, c_1, c_2, M_1, \iota) > 0$ such that

$$\begin{aligned} u_\varepsilon(\cdot,t) &\le \left\| e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon(\cdot,s)P_\varepsilon(\cdot,s))\,\mathrm{d}s + \int_0^t e^{(t-s)\Delta}c_2\,\mathrm{d}s \right\|_{L^\infty(\Omega)} \\ &\le c_3 + c_3 \sup_{\tau\in(0,t)} \|u_\varepsilon(\cdot,\tau)\|^{\iota}_{L^\infty(\Omega)} \end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. By nonnegativity of u_ε, we conclude that (3.4) and (3.5) hold. □

Let us close this section by showing existence of global bounded weak solutions to (1.1).

Proof of Theorem 1.1 Noting that u_ε is continuous on $\overline{\Omega} \times [0,\infty)$, we see from Lemma 3.2 that the function $t \mapsto u_\varepsilon(\cdot,t)$ is measurable as $L^\infty(\Omega)$-valued function, and there exists $K > 0$ such that

$$\|u_\varepsilon\|_{L^\infty(0,\infty;L^\infty(\Omega))} \le K \quad \text{for all } \varepsilon \in (0,1), \tag{3.7}$$

whence Lemma 2.3 applies so as to ensure the existence of $L = L(\Omega, n, M_1, K) > 0$ such that

$$\|v_\varepsilon\|_{L^\infty(0,\infty;W^{1,\infty}(\Omega))} \le L \quad \text{for all } \varepsilon \in (0,1). \tag{3.8}$$

In view of (3.7), (3.8) and Lemma 2.7, an extraction procedure on the basis of the Arzelà–Ascoli theorem enable us to obtain a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ with $\varepsilon_j \searrow 0$ as $j \to \infty$, as well as functions

$$u \in L^\infty(0,\infty;L^\infty(\Omega)) \quad \text{and} \quad v \in L^\infty(0,\infty;W^{1,\infty}(\Omega))$$

such that

$$u_{\varepsilon_j} \stackrel{\star}{\rightharpoonup} u \quad \text{in } L^\infty(0,\infty;L^\infty(\Omega)), \tag{3.9}$$

$$u_{\varepsilon_j} \to u \quad \text{in } C^0_{\mathrm{loc}}([0,\infty);(W_0^{2,2}(\Omega))^\star), \tag{3.10}$$

$$v_{\varepsilon_j} \stackrel{\star}{\rightharpoonup} v \quad \text{in } L^\infty(0,\infty;L^\infty(\Omega)) \quad \text{and} \tag{3.11}$$

$$\nabla v_{\varepsilon_j} \stackrel{\star}{\rightharpoonup} \nabla v \quad \text{in } L^\infty(0,\infty;L^\infty(\Omega)) \tag{3.12}$$

as $j \to \infty$. We claim that (u, v) is a global weak solution of (1.1).

To verify this, we fix an arbitrary $T > 0$. According to (3.7) and (3.8), we infer from Lemma 2.8 and Remark 2.1 that there exist a subsequence, still denoted by $(\varepsilon_j)_{j\in\mathbb{N}}$, and functions $\widetilde{u}$, $\widetilde{v}$ such that

$$u_{\varepsilon_j} \stackrel{\star}{\rightharpoonup} \widetilde{u} \quad \text{in } L^\infty(0,T;L^\infty(\Omega)), \tag{3.13}$$

$$v_{\varepsilon_j} \stackrel{\star}{\rightharpoonup} \widetilde{v} \quad \text{in } L^\infty(0,T;L^\infty(\Omega)) \quad \text{and} \tag{3.14}$$

$$\nabla v_{\varepsilon_j} \stackrel{\star}{\rightharpoonup} \nabla\widetilde{v} \quad \text{in } L^\infty(0,T;L^\infty(\Omega)) \tag{3.15}$$

as $j \to \infty$, and that $(\widetilde{u}, \widetilde{v})$ is a weak solution of (1.1) in $\Omega \times (0, T)$. According to (3.9) and (3.13), we observe that $\widetilde{u} = u$ in $L^\infty(0, T; L^\infty(\Omega))$. Similarly, from (3.11), (3.12), (3.14) and (3.15) we deduce that $\widetilde{v} = v$ in $L^\infty(0, T; W^{1,\infty}(\Omega))$. In consequence, these would show that (u, v) is a weak solution of (1.1) in $\Omega \times (0, T)$ for all $T > 0$, and that hence the claim holds.

Thereupon, boundedness of u as in (1.10) results from (3.7), (3.9) and (3.10) in the same way as detailed in [18, Theorem 1.2] (see also [33, Lemma 4.2]). □

4 Finite-Time Blow-up

We henceforth assume $n \geq 2$ and $\Omega := B_R(0) \subset \mathbb{R}^n$ for some $R > 0$. We also suppose that u_0 satisfies (1.4) and (1.14), and that h satisfies (1.3).

The goal of this section will be to establish the integrated version of moment inequality (1.18) and to finally prove Theorem 1.2, which ensures existence of weak solutions to (1.1) that could blow up in finite time.

The main part of our analysis will be focused on deriving a moment inequality for approximate solutions. We note that since u_0 is radially symmetric with respect to $x = 0$, by Lemma 2.1 we observe that for every $\varepsilon \in (0, 1)$, the functions $u_\varepsilon(\cdot, t)$ and $v_\varepsilon(\cdot, t)$ are also radially symmetric with respect to $x = 0$ for each $t \in (0, T_{\max,\varepsilon})$.

4.1 Moment-Type Functional for Approximate Solutions

Following [12], for each $\varepsilon \in (0, 1)$ we define the mass accumulation function

$$w_\varepsilon(s, t) := \int_0^{s^{\frac{1}{n}}} \sigma^{n-1} u_\varepsilon(\sigma, t)\, d\sigma \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, T_{\max,\varepsilon}).$$

Since u_ε is nonnegative for any $\varepsilon \in (0, 1)$, it follows that

$$(w_\varepsilon)_s(s, t) = \frac{1}{n} u_\varepsilon(s^{\frac{1}{n}}, t) \geq 0 \quad \text{for all } s \in (0, R^n),\ t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0, 1). \tag{4.1}$$

Moreover, for all $\varepsilon \in (0, 1)$ the function w_ε transforms (2.1) into the Dirichlet problem

$$\begin{aligned}
(w_\varepsilon)_t &= n^2 s^{2-\frac{2}{n}} (w_\varepsilon)_{ss} \\
&\quad + n\chi \left(w_\varepsilon - \frac{s}{n} \overline{M_\varepsilon}(t) \right) (w_\varepsilon)_s \left(s^{\frac{2}{n}-2} \left(w_\varepsilon - \frac{s}{n} \overline{M_\varepsilon}(t) \right)^2 + \varepsilon \right)^{\frac{p-2}{2}} \\
&\quad + \lambda w_\varepsilon - n^{\kappa-1} \mu \int_0^s (w_\varepsilon)_s^\kappa(\xi, t)\, d\xi, \quad s \in (0, R^n),\ t \in (0, T_{\max,\varepsilon}),
\end{aligned} \tag{4.2}$$

$$w_\varepsilon(0, t) = 0, \quad w_\varepsilon(R^n, t) = \frac{R^n}{n} \overline{M_\varepsilon}(t), \quad t \in (0, T_{\max,\varepsilon}), \tag{4.3}$$

$$w_\varepsilon(s, 0) = w_0(s) := \int_0^{s^{\frac{1}{n}}} \sigma^{n-1} u_0(\sigma) d\sigma, \quad s \in (0, R^n), \tag{4.4}$$

where

$$\overline{M_\varepsilon}(t) := \frac{1}{|\Omega|} \int_\Omega u_\varepsilon(\cdot, t) \quad \text{for } t \in [0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0, 1). \tag{4.5}$$

The next lemma is important in constructing a moment-type functional for approximate solutions. The idea is based on [37, Lemma 3.1], in which they considered the system (1.7) with $\lambda = \mu = 0$.

Lemma 4.1 *Let $\chi > 0$, $p > 1$, $\lambda \geq 0$, $\mu > 0$ and $\kappa > 1$. Suppose that u_0 satisfies (1.4), (1.14) and (1.15). Then*

$$w_\varepsilon(s,t) \geq \frac{s}{n}\overline{M_\varepsilon}(t) \quad \textit{for all } s \in (0, R^n),\ t \in (0, T_{\max,\varepsilon}) \textit{ and } \varepsilon \in (0,1).$$

Proof For $\varepsilon \in (0,1)$ we put

$$\underline{w_\varepsilon}(s,t) := \frac{s}{n}\overline{M_\varepsilon}(t) \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, T_{\max,\varepsilon}). \tag{4.6}$$

Then we obtain from (4.5) and (2.1) that

$$(\underline{w_\varepsilon})_t(s,t) = \frac{s}{n|\Omega|}\left(\lambda \int_\Omega u_\varepsilon(\cdot,t) - \mu \int_\Omega u_\varepsilon^\kappa(\cdot,t)\right) \tag{4.7}$$

for all $s \in (0, R^n)$, $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$, whereas we deduce from (4.6) and (4.5) that

$$\begin{aligned} &\lambda(\underline{w_\varepsilon})(s,t) - n^{\kappa-1}\mu \int_0^s (\underline{w_\varepsilon})_s^\kappa(\xi,t)\mathrm{d}\xi \\ &= \frac{\lambda s}{n|\Omega|}\int_\Omega u_\varepsilon(\cdot,t) - \frac{\mu s}{n|\Omega|^\kappa}\left(\int_\Omega u_\varepsilon(\cdot,t)\right)^\kappa \end{aligned} \tag{4.8}$$

for all $s \in (0, R^n)$, $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. Furthermore, (4.6) implies that

$$(\underline{w_\varepsilon})_{ss}(s,t) = 0 \quad \text{and} \quad \underline{w_\varepsilon}(s,t) - \frac{s}{n}\overline{M_\varepsilon}(t) = 0 \tag{4.9}$$

for all $s \in (0, R^n)$, $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. In light of (4.7), (4.8), (4.9) and the Hölder inequality, we thus see that

$$\begin{aligned} &(\underline{w_\varepsilon})_t - n^2 s^{2-\frac{2}{n}}(\underline{w_\varepsilon})_{ss} \\ &- n\chi\left(\underline{w_\varepsilon} - \frac{s}{n}\overline{M_\varepsilon}(t)\right)(\underline{w_\varepsilon})_s\left(s^{\frac{2}{n}-2}\left(\underline{w_\varepsilon} - \frac{s}{n}\overline{M_\varepsilon}(t)\right)^2 + \varepsilon\right)^{\frac{p-2}{2}} \\ &- \left(\lambda\underline{w_\varepsilon} - n^{\kappa-1}\mu\int_0^s (\underline{w_\varepsilon})_s^\kappa(\xi,t)\mathrm{d}\xi\right) \\ &= \frac{\mu s}{n|\Omega|^\kappa}\left(\int_\Omega u_\varepsilon(\cdot,t)\right)^\kappa - \frac{\mu s}{n|\Omega|}\int_\Omega u_\varepsilon^\kappa(\cdot,t) \\ &\leq 0 \end{aligned}$$

for all $s \in (0, R^n)$, $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. In addition, $\underline{w_\varepsilon}$ satisfies

$$\underline{w_\varepsilon}(0,t) = 0 \quad \text{and} \quad \underline{w_\varepsilon}(R^n,t) = \frac{R^n}{n}\overline{M_\varepsilon}(t)$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. Besides, the assumption (1.15) allows us to derive

$$w_0(s) = \frac{s}{n}\frac{1}{|B_{s^{\frac{1}{n}}}(0)|}\int_{B_{s^{\frac{1}{n}}}(0)} u_0 \geq \frac{s}{n}\frac{1}{|\Omega|}\int_\Omega u_0 = \underline{w_\varepsilon}(s,0)$$

for all $s \in (0, R^n)$ and $\varepsilon \in (0,1)$. As a consequence, an application of a comparison principle (cf. [3, Lemma 5.1]) to (4.2), (4.3) and (4.4) yields the result. □

By virtue of Lemma 4.1, under the assumption (1.15), for any $\varepsilon \in (0,1)$ the function z_ε defined as

$$z_\varepsilon(s,t) := w_\varepsilon(s,t) - \frac{s}{n}\overline{M_\varepsilon}(t) \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, T_{\max,\varepsilon}) \tag{4.10}$$

is nonnegative on $(0, R^n) \times (0, T_{\max,\varepsilon})$. Moreover, we infer from (4.2) and (4.7) that for any $\varepsilon \in (0,1)$ the function z_ε satisfies

$$\begin{aligned}(z_\varepsilon)_t &= n^2 s^{2-\frac{2}{n}}(z_\varepsilon)_{ss} + n\chi z_\varepsilon (w_\varepsilon)_s (s^{\frac{2}{n}-2} z_\varepsilon^2 + \varepsilon)^{\frac{p-2}{2}} + \lambda w_\varepsilon \\ &\quad - n^{\kappa-1}\mu \int_0^s (w_\varepsilon)_s^\kappa(\xi,t)\,\mathrm{d}\xi \\ &\quad - \frac{s}{n|\Omega|}\left(\lambda \int_\Omega u_\varepsilon(\cdot,t) - \mu \int_\Omega u_\varepsilon^\kappa(\cdot,t)\right)\end{aligned} \tag{4.11}$$

for all $s \in (0, R^n)$ and $t \in (0, T_{\max,\varepsilon})$. In accordance with the results, let us introduce the moment-type functional for approximate solutions similar as in [17, 37], namely, for $\varepsilon \in (0,1)$, $s_0 \in (0, R^n)$ and $\gamma \in (0,1)$ we define

$$\phi_\varepsilon(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) z_\varepsilon(s,t)\,\mathrm{d}s \quad \text{for } t \in [0, T_{\max,\varepsilon}). \tag{4.12}$$

We note that in light of Lemma 2.1, for any $\varepsilon \in (0,1)$ the property

$$\phi_\varepsilon \in C^0([0, T_{\max,\varepsilon})) \cap C^1((0, T_{\max,\varepsilon}))$$

holds.

Lemma 4.2 *Let $\chi > 0$, $p > 1$, $\lambda \geq 0$, $\mu > 0$ and $\kappa \geq 1$. Then for all $\varepsilon \in (0,1)$, $s_0 \in (0, R^n)$ and $\gamma \in (0,1)$, the function ϕ_ε as in (4.12) satisfies the inequality*

$$\begin{aligned}\phi_\varepsilon'(t) &\geq n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s)(z_\varepsilon)_{ss}(s,t)\,\mathrm{d}s \\ &\quad + n\chi \int_0^{s_0} s^{-\gamma}(s_0 - s) z_\varepsilon(s,t)(w_\varepsilon)_s(s,t)(s^{\frac{2}{n}-2} z_\varepsilon^2(s,t) + \varepsilon)^{\frac{p-2}{2}}\,\mathrm{d}s \\ &\quad - n^{\kappa-1}\mu \int_0^{s_0} s^{-\gamma}(s_0 - s)\left(\int_0^s (w_\varepsilon)_s^\kappa(\xi,t)\,\mathrm{d}\xi\right)\mathrm{d}s \\ &\quad - \frac{\lambda}{n|\Omega|}\left(\frac{1}{2-\gamma} - \frac{1}{3-\gamma}\right) s_0^{3-\gamma}\int_\Omega u_\varepsilon(\cdot,t) \\ &=: I_{1,\varepsilon}(t) + I_{2,\varepsilon}(t) + I_{3,\varepsilon}(t) + I_{4,\varepsilon}(t)\end{aligned} \tag{4.13}$$

for all $t \in (0, T_{\max,\varepsilon})$.

Proof According to (4.11), we have

$$\begin{aligned}\phi_\varepsilon'(t) &= n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0-s)(z_\varepsilon)_{ss}(s,t)\,\mathrm{d}s \\ &\quad + n\chi \int_0^{s_0} s^{-\gamma}(s_0-s)z_\varepsilon(s,t)(w_\varepsilon)_s(s,t)(s^{\frac{2}{n}-2}z_\varepsilon^2(s,t)+\varepsilon)^{\frac{p-2}{2}}\,\mathrm{d}s \\ &\quad + \lambda \int_0^{s_0} s^{-\gamma}(s_0-s)w_\varepsilon(s,t)\,\mathrm{d}s \\ &\quad - n^{\kappa-1}\mu \int_0^{s_0} s^{-\gamma}(s_0-s)\left(\int_0^s (w_\varepsilon)_s^\kappa(\xi,t)\,\mathrm{d}\xi\right)\mathrm{d}s \\ &\quad - \frac{1}{n|\Omega|}\left(\lambda\int_\Omega u_\varepsilon(\cdot,t) - \mu\int_\Omega (u_\varepsilon)^\kappa(\cdot,t)\right)\int_0^{s_0} s^{1-\gamma}(s_0-s)\,\mathrm{d}s \end{aligned} \tag{4.14}$$

for all $t \in (0, T_{\max,\varepsilon})$, $\varepsilon \in (0,1)$, $s_0 \in (0, R^n)$ and $\gamma \in (0,1)$. Furthermore, for all $s_0 \in (0, R^n)$ and $\gamma \in (0,1)$ we compute

$$\int_0^{s_0} s^{1-\gamma}(s_0-s)\,\mathrm{d}s = \left(\frac{1}{2-\gamma} - \frac{1}{3-\gamma}\right)s_0^{3-\gamma} \tag{4.15}$$

Inserting (4.15) into (4.14), and in view of the fact that $\lambda \ge 0$ and $\mu > 0$, we arrive at the conclusion. □

4.2 Estimating the Term $I_{2,\varepsilon}$ in (4.13)

Our derivation begins with giving estimates for the second integral on the right of (4.13), which arises from the chemotactic term. The idea of the estimations for $I_{2,\varepsilon}$ is based on [37], however, in order to construct moment solutions of (1.1), we have to estimate this term uniformly with respect to $\varepsilon \in (0,1)$. The next lemma makes it possible to deal with the term independently of $\varepsilon \in (0,1)$.

Lemma 4.3 *Let $\chi > 0$ and $p > 1$. Suppose that u_0 satisfies (1.4), (1.14) and (1.15). Then for any $\varepsilon \in (0,1)$, $s_0 \in (0, R^n)$, $\gamma \in (0,1)$ and $\beta \in (0,1]$,*

$$\begin{aligned} I_{2,\varepsilon}(t) &\ge n\chi \int_0^{s_0} s^{(1-\frac{1}{n})(2-p)-\gamma}(s_0-s)z_\varepsilon^{p-1}(s,t)(w_\varepsilon)_s(s,t)\,\mathrm{d}s \\ &\quad - \frac{n\chi(2-p)_+}{2\beta}\int_0^{s_0} s^{(1-\frac{1}{n})(2-p+2\beta)-\gamma}(s_0-s)z_\varepsilon^{p-1-2\beta}(s,t)(z_\varepsilon)_s(s,t)\,\mathrm{d}s \\ &\quad - \frac{\chi(2-p)_+}{2\beta}\overline{M_\varepsilon}(t)\int_0^{s_0} s^{(1-\frac{1}{n})(2-p+2\beta)-\gamma}(s_0-s)z_\varepsilon^{p-1-2\beta}(s,t)\,\mathrm{d}s \\ &=: J_{1,\varepsilon}(t) - J_{2,\varepsilon}(t) - J_{3,\varepsilon}(t) \end{aligned} \tag{4.16}$$

for all $t \in (0, T_{\max,\varepsilon})$, where $(2-p)_+ := \max\{0, 2-p\}$.

Proof We first consider the case when $p \in (1,2)$. Then we observe that

$$(s^{\frac{2}{n}-2}z_\varepsilon^2(s,t)+\varepsilon)^{\frac{p-2}{2}} \ge (s^{\frac{2}{n}-2}z_\varepsilon^2(s,t)+1)^{\frac{p-2}{2}} \tag{4.17}$$

for all $s \in (0, R^n)$, $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$, whence

$$I_{2,\varepsilon}(t) \geq n\chi \int_0^{s_0} s^{-\gamma}(s_0 - s) z_\varepsilon(s,t)(w_\varepsilon)_s(s,t)(s^{\frac{2}{n}-2} z_\varepsilon^2(s,t) + 1)^{\frac{p-2}{2}}\, ds \tag{4.18}$$

for all $t \in (0, T_{\max,\varepsilon})$, $\varepsilon \in (0,1)$, $s_0 \in (0, R^n)$ and $\gamma \in (0,1)$. Fixing $\beta \in (0,1]$, and using the inequality $(1+\xi)^{-\alpha} \geq 1 - \frac{\alpha}{\beta}\xi^\beta$ for all $\alpha > 0$ and $\xi \geq 0$ (cf. [37, Lemma 3.4]), we further compute

$$\begin{aligned}
&\int_0^{s_0} s^{-\gamma}(s_0 - s) z_\varepsilon(s,t)(w_\varepsilon)_s(s,t)(s^{\frac{2}{n}-2} z_\varepsilon^2(s,t) + 1)^{\frac{p-2}{2}}\, ds \\
&\quad = \int_0^{s_0} s^{(1-\frac{1}{n})(2-p)-\gamma}(s_0 - s) z_\varepsilon^{p-1}(s,t)(w_\varepsilon)_s(s,t)(1 + s^{2-\frac{2}{n}} z_\varepsilon^{-2}(s,t))^{-\frac{2-p}{2}}\, ds \\
&\quad \geq \int_0^{s_0} s^{(1-\frac{1}{n})(2-p)-\gamma}(s_0 - s) z_\varepsilon^{p-1}(s,t)(w_\varepsilon)_s(s,t)\, ds \\
&\qquad - \frac{2-p}{2\beta}\int_0^{s_0} s^{(1-\frac{1}{n})(2-p+2\beta)-\gamma}(s_0 - s) z_\varepsilon^{p-1-2\beta}(s,t)(w_\varepsilon)_s(s,t)\, ds
\end{aligned} \tag{4.19}$$

for all $t \in (0, T_{\max,\varepsilon})$, $\varepsilon \in (0,1)$, $s_0 \in (0, R^n)$ and $\gamma \in (0,1)$. Moreover, we infer from (4.10) that

$$(w_\varepsilon)_s(s,t) = (z_\varepsilon)_s(s,t) + \frac{1}{n}\overline{M_\varepsilon}(t) \quad \text{for all } s \in (0, R^n),\ t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \tag{4.20}$$

Combining (4.18) with (4.19) and (4.20), we obtain (4.16).

Now, we can also obtain (4.16) in the complementary case $p \geq 2$, by using the estimate

$$(s^{\frac{2}{n}-2} z_\varepsilon^2(s,t) + \varepsilon)^{\frac{p-2}{2}} \geq s^{(1-\frac{1}{n})(2-p)} z_\varepsilon^{p-2}(s,t)$$

for all $s \in (0, R^n)$, $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$, instead of (4.17) in (4.18). □

Next, we give an estimate for $J_{1,\varepsilon}$.

Lemma 4.4 *Let $\chi > 0$, and let $p > 1$ and $\gamma \in (0,1)$ be such that*

$$\gamma > \left(1 - \frac{1}{n}\right)(2-p). \tag{4.21}$$

Then there exists $C > 0$ such that whenever u_0 satisfies (1.4), (1.14) and (1.15), for all $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$,

$$\begin{aligned}
J_{1,\varepsilon}(t) &\geq C\int_0^{s_0} s^{(1-\frac{1}{n})(2-p)-\gamma-1}(s_0 - s) z_\varepsilon^p(s,t)\, ds \\
&\quad + C\int_0^{s_0} s^{(1-\frac{1}{n})(2-p)-\gamma} z_\varepsilon^p(s,t)\, ds \\
&=: H_{1,\varepsilon}(t) + H_{2,\varepsilon}(t)
\end{aligned}$$

holds for all $t \in (0, T_{\max,\varepsilon})$.

Proof It follows from (4.20) that

$$(w_\varepsilon)_s \ge (z_\varepsilon)_s \quad \text{on } (0, R^n) \times (0, T_{\max,\varepsilon}) \text{ for all } \varepsilon \in (0,1),$$

whence

$$J_{1,\varepsilon}(t) \ge n\chi \int_0^{s_0} s^{(1-\frac{1}{n})(2-p)-\gamma}(s_0 - s) z_\varepsilon^{p-1}(s,t)(z_\varepsilon)_s(s,t)\,\mathrm{d}s$$

for all $t \in (0, T_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$. The conclusion thereby follows from [37, Lemma 3.6] with $\alpha = \frac{2-p}{2}$. □

The term $H_{1,\varepsilon}$ would be a good term in deriving the moment inequality as follows:

Lemma 4.5 *Let* $p > 1$ *and* $\gamma \in (0,1)$ *satisfy*

$$(p-1)\gamma < 2 - 2(2-p) + \frac{2-p}{n}. \tag{4.22}$$

Then there is $C > 0$ *such that for any* $\varepsilon \in (0,1)$ *and* $s_0 \in (0, R^n)$, *whenever* u_0 *satisfies* (1.4), (1.14) *and* (1.15) *it follows that*

$$H_{1,\varepsilon}(t) \ge C s_0^{-3+3(2-p)-\frac{2-p}{n}+(p-1)\gamma} \phi_\varepsilon^p(t)$$

for all $t \in (0, T_{\max,\varepsilon})$.

Proof This is a direct consequence of [37, Lemma 3.12]. □

The term $J_{2,\varepsilon}$ and $J_{3,\varepsilon}$ can be treated as well when $p \in (1,2)$.

Lemma 4.6 *Let* $\chi > 0$, $p \in (1,2)$ *and* $\gamma \in (0,1)$. *Then for all* $\beta \in (0, \frac{p-1}{2})$ *and* $\eta > 0$, *there are* $c_1 > 0$ *and* $c_2 > 0$ *such that for any* $\varepsilon \in (0,1)$ *and* $s_0 \in (0, R^n)$ *we have*

$$J_{2,\varepsilon}(t) \le \eta H_{1,\varepsilon}(t) + \eta H_{2,\varepsilon}(t) + c_1 s_0^{3-\frac{2}{n}-\gamma} \tag{4.23}$$

and

$$J_{3,\varepsilon}(t) \le \eta H_{1,\varepsilon}(t) + c_2 s_0^{\frac{3-\frac{2-p}{n}-\gamma+2\beta(3-\frac{2}{n}-\gamma)}{1+2\beta}} \tag{4.24}$$

for all $t \in (0, T_{\max,\varepsilon})$.

Proof Noting that

$$\overline{M_\varepsilon}(t) \le \frac{M_1}{|\Omega|} \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0,1)$$

by Lemma 2.2, the estimate (4.23) results from [37, Lemmas 3.7 and 3.8], whereas (4.24) is a consequence of [37, Lemma 3.9]. □

4.3 Controlling the Term $I_{3,\varepsilon}$ Arising from the Logistic Source in (4.13)

In order to estimate $I_{3,\varepsilon}$ in (4.13), we shall derive a pointwise estimate for u_ε. We start with an upper bound for $(v_\varepsilon)_r$ therein.

Lemma 4.7 *Let $\chi > 0$, $p > 1$, $\lambda \geq 0$, $\mu > 0$ and $\kappa > 1$. Then for all $m > 0$ there exists $C > 0$ with the following property: Whenever u_0 satisfies (1.4) and (1.14) as well as*

$$\int_\Omega u_0 \leq m, \tag{4.25}$$

we have

$$-\frac{R^n}{n}\overline{M_\varepsilon}(t)r^{-(n-1)} \leq (v_\varepsilon)_r(r,t) \leq \frac{1}{n}\overline{M_\varepsilon}(t)r \tag{4.26}$$

for all $r \in (0, R)$, $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0, 1)$. In particular,

$$|(v_\varepsilon)_r(r,t)| \leq Cr^{-(n-1)} \tag{4.27}$$

for all $r \in (0, R)$, $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0, 1)$.

Proof The estimate (4.26) can be established similarly in [2, Lemma 2.5]. We next prove (4.27). We infer from Lemma 2.2 and (4.25) that

$$\int_\Omega u_\varepsilon(\cdot,t) \leq \max\left\{m, \left(\frac{\lambda}{\mu}\right)^{\frac{1}{\kappa-1}}\right\} =: M_2 \tag{4.28}$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0, 1)$. Moreover, we observe that

$$\frac{R^n}{n|\Omega|}r^{-(n-1)} \geq \frac{1}{n|\Omega|}r \quad \text{for all } r \in (0, R). \tag{4.29}$$

The estimate (4.27) thereby results from (4.26), (4.28) and (4.29) with $C = \frac{R^n M_2}{n|\Omega|}$. □

An application of Lemma 4.7 yields a pointwise estimate for u_ε. The idea is based on [35, Lemma 3.3].

Lemma 4.8 *Let $\chi > 0$, $p \geq \frac{n}{n-1}$, $\lambda \geq 0$, $\mu > 0$ and $\kappa > 1$. Then for all $m > 0$, $L > 0$ and $\delta > 0$ there is $C > 0$ such that whenever u_0 satisfies (1.4), (1.14), (1.16) and (4.25), it follows that*

$$u_\varepsilon(r,t) \leq Cr^{-n(n-1)(p-1)-\delta} \tag{4.30}$$

for all $r \in (0, R)$, $t \in (0, \widetilde{T}_{\max,\varepsilon})$ and $\varepsilon \in (0, 1)$, where $\widetilde{T}_{\max,\varepsilon} := \min\{1, T_{\max,\varepsilon}\}$.

Proof We will apply [36, Theorem 1.1] to derive the estimate (4.30). To see this, we first infer from Lemma 4.7 that there is $c_1 > 0$ such that

$$|(v_\varepsilon)_r(r,t)| \leq c_1 r^{-(n-1)} \tag{4.31}$$

for all $r \in (0,R)$, $t \in (0,T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. Noting that $(n-1)(p-1) \geq 1$ holds, for any $\delta > 0$ we choose $q > n$ satisfying

$$\alpha := n(n-1)(p-1)\delta > \frac{n(n-1)(p-1)q}{q-n}, \tag{4.32}$$

and we utilize (4.31) so that we can derive

$$\begin{aligned}
&\int_\Omega |x|^{(n-1)(p-1)q} \left|-\chi(|\nabla v_\varepsilon(x,t)|^2+\varepsilon)^{\frac{p-2}{2}}\nabla v_\varepsilon(x,t)\right|^q \mathrm{d}x \\
&\le \frac{n|\Omega|\chi^q}{R^n}\int_0^R r^{n-1+(n-1)(p-1)q}(|(v_\varepsilon)_r(r,t)|^2+1)^{\frac{(p-1)q}{2}}\,\mathrm{d}r \\
&\le \frac{n|\Omega|\chi^q}{R^n}\int_0^R r^{n-1+(n-1)(p-1)q}(c_1^2 r^{-2(n-1)}+1)^{\frac{(p-1)q}{2}}\,\mathrm{d}r \\
&\le 2^{\frac{(p-1)q}{2}} c_1^{(p-1)q}\frac{n|\Omega|\chi^q}{R^n}\int_0^R r^{n-1+(n-1)(p-1)q-(n-1)(p-1)q}\,\mathrm{d}r \\
&\quad + 2^{\frac{(p-1)q}{2}}\frac{n|\Omega|\chi^q}{R^n}\int_0^R r^{n-1+(n-1)(p-1)q}\,\mathrm{d}r \\
&= (\sqrt{2}c_1)^{(p-1)q}|\Omega|\chi^q + 2^{\frac{(p-1)q}{2}}\frac{n|\Omega|R^{(n-1)(p-1)q}\chi^q}{n+(n-1)(p-1)q}
\end{aligned} \tag{4.33}$$

for all $t \in (0,T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. Now for each $\varepsilon \in (0,1)$, we put

$$U_\varepsilon(x,t) := e^{-\lambda t}u_\varepsilon(x,t) \quad \text{for } (x,t) \in \overline{\Omega}\times[0,T_{\max,\varepsilon}),$$

and from (2.1) we see that

$$(U_\varepsilon)_t \le \Delta U_\varepsilon + \nabla\cdot(-\chi U_\varepsilon(|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla v_\varepsilon) \quad \text{in } \Omega\times(0,T_{\max,\varepsilon}),$$

with $\nabla U_\varepsilon\cdot\nu = 0$ on $\partial\Omega\times(0,T_{\max,\varepsilon})$. Therefore, due to (4.32) and (4.33), we can apply [36, Theorem 1.1] and obtain some $c_2 > 0$ such that $U_\varepsilon(x,t) \le c_2|x|^{-\alpha}$ for every $x \in \Omega$, $t \in (0,T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$, which readily proves (4.30) with $C := c_2e^\lambda$. □

We also give a pointwise estimate for z_ε in the next lemma, which is important to control $I_{3,\varepsilon}$ in (4.13). The idea is based on [35, Lemma 4.2].

Lemma 4.9 *Let $\chi > 0$, $\lambda \ge 0$, $\mu > 0$, $\kappa > 1$ and $s_0 \in (0,R^n)$. Assume that $p > 1$ and $\gamma \in (0,1)$ satisfy (4.21). Then*

$$z_\varepsilon(s,t) \le p^{\frac{1}{p}} s^{\frac{1}{p}(\gamma-(1-\frac{1}{n})(2-p))}(s_0-s)^{-\frac{1}{p}} J_{1,\varepsilon}^{\frac{1}{p}}(t)$$

for all $s \in (0,s_0)$, $t \in (0,T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$.

Proof Noting that (2.3) and (4.1) imply $z_\varepsilon(\cdot,t) \in C^1([0,R^n])$ for arbitrary $\varepsilon \in (0,1)$ and $t \in (0,T_{\max,\varepsilon})$, for each $\varepsilon \in (0,1)$ we let

$$\psi_\varepsilon(s,t) := \frac{1}{p}s^{(1-\frac{1}{n})(2-p)-\gamma}(s_0-s)z_\varepsilon^p(s,t) \quad \text{for } s \in [0,R^n] \text{ and } t \in [0,T_{\max,\varepsilon}),$$

so that $\psi_\varepsilon(\cdot,t)\in C^1([0,s_0])$ for all $t\in[0,T_{\max,\varepsilon})$, and thanks to (4.21),

$$\begin{aligned}
\psi_\varepsilon(s,t) &= \int_0^s (\psi_\varepsilon)_s(\sigma,t)\,\mathrm{d}\sigma\\
&= \int_0^s \sigma^{(1-\frac{1}{n})(2-p)-\gamma}(s_0-\sigma)z_\varepsilon^{p-1}(\sigma,t)(z_\varepsilon)_s(\sigma,t)\,\mathrm{d}\sigma\\
&\quad -\frac{1}{p}\left(\gamma-\left(1-\frac{1}{n}\right)(2-p)\right)\int_0^s \sigma^{(1-\frac{1}{n})(2-p)-1-\gamma}(s_0-\sigma)z_\varepsilon^{p}(\sigma,t)\,\mathrm{d}\sigma\\
&\quad -\frac{1}{p}\int_0^s \sigma^{(1-\frac{1}{n})(2-p)-\gamma}z_\varepsilon^{p}(\sigma,t)\,\mathrm{d}\sigma\\
&\le \int_0^s \sigma^{(1-\frac{1}{n})(2-p)-\gamma}(s_0-\sigma)z_\varepsilon^{p-1}(\sigma,t)(w_\varepsilon)_s(\sigma,t)\,\mathrm{d}\sigma\\
&\le \int_0^{s_0} \sigma^{(1-\frac{1}{n})(2-p)-\gamma}(s_0-\sigma)z_\varepsilon^{p-1}(\sigma,t)(w_\varepsilon)_s(\sigma,t)\,\mathrm{d}\sigma
\end{aligned}$$

for all $s\in(0,s_0)$, $t\in(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$. Hence we arrive at the conclusion. □

We are now in a position to give an estimate for $I_{3,\varepsilon}$.

Lemma 4.10 *Let $\chi>0$, $\lambda\ge 0$ and $\mu>0$. Moreover, we assume that $p\ge\frac{n}{n-1}$, $\kappa>1$ and $\gamma\in(0,1)$ satisfy (4.21) as well as*

$$(n-1)(p-1)(\kappa-1)<1, \tag{4.34}$$

and

$$\frac{1}{p}\left(1-\frac{1}{n}\right)(2-p)+(n-1)(p-1)(\kappa-1)<\frac{\gamma}{p}. \tag{4.35}$$

Then for all $m>0$, $L>0$ and $\eta>0$ there is $C>0$ such that whenever u_0 satisfies (1.4), (1.14), (1.15), (1.16) and (4.25), for any $s_0\in(0,R^n)$ and $\varepsilon\in(0,1)$,

$$\begin{aligned}
I_{3,\varepsilon}(t) &\ge -Cs_0^{-\frac{1}{p}(1-\frac{1}{n})(2-p)-(n-1)(p-1)(\kappa-1)+\frac{2p-1-(p-1)\gamma}{p}-\eta}J_{1,\varepsilon}^{\frac{1}{p}}(t)\\
&\quad -Cs_0^{-(n-1)(p-1)(\kappa-1)+3-\gamma-\eta}
\end{aligned}$$

holds for all $t\in(0,\widetilde{T}_{\max,\varepsilon})$.

Proof We fix arbitrary $\eta>0$. Then from (4.34) and (4.35), there exists $\delta>0$ such that

$$\frac{\delta}{n}(\kappa-1)\le\min\{\eta,1\}, \tag{4.36}$$

$$(n-1)(p-1)(\kappa-1)+\frac{\delta}{n}(\kappa-1)<1 \quad \text{and} \tag{4.37}$$

$$\frac{1}{p}\left(1-\frac{1}{n}\right)(2-p)+(n-1)(p-1)(\kappa-1)+\frac{\delta}{n}(\kappa-1)<\frac{\gamma}{p}, \tag{4.38}$$

whence Lemma 4.8 provides $c_1 > 0$ such that

$$u_\varepsilon(r,t) \le c_1 r^{-n(n-1)(p-1)-\delta}$$

for all $r \in (0,R)$, $t \in (0,\widetilde{T}_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$, and hence

$$(w_\varepsilon)_s^{\kappa-1}(s,t) \le \left(\frac{c_1}{n}\right)^{\kappa-1} s^{-(n-1)(p-1)(\kappa-1)-\frac{\delta}{n}(\kappa-1)} \tag{4.39}$$

for all $s \in (0,R^n)$, $t \in (0,\widetilde{T}_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. By virtue of the Fubini theorem and (4.39), we entail that

$$\begin{aligned} I_{3,\varepsilon}(t) &= -n^{\kappa-1}\mu \int_0^{s_0} \left(\int_\xi^{s_0} s^{-\gamma}(s_0-s)\,\mathrm{d}s\right)(w_\varepsilon)_s^\kappa(\xi,t)\,\mathrm{d}\xi \\ &\ge -\frac{\mu c_1^{\kappa-1}}{1-\gamma} s_0^{1-\gamma} \int_0^{s_0} s^{-(n-1)(p-1)(\kappa-1)-\frac{\delta}{n}(\kappa-1)}(s_0-s)(w_\varepsilon)_s(s,t)\,\mathrm{d}s \end{aligned} \tag{4.40}$$

for all $t \in (0,\widetilde{T}_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0,R^n)$, whereas integrating by parts yields

$$\begin{aligned} &\int_0^{s_0} s^{-(n-1)(p-1)(\kappa-1)-\frac{\delta}{n}(\kappa-1)}(s_0-s)(w_\varepsilon)_s(s,t)\,\mathrm{d}s \\ &\quad= -\liminf_{\xi\searrow 0}\left(\xi^{-(n-1)(p-1)(\kappa-1)-\frac{\delta}{n}(\kappa-1)}(s_0-\xi)w_\varepsilon(\xi,t)\right) \\ &\qquad - \int_0^{s_0} \frac{\partial}{\partial s}(s^{-(n-1)(p-1)(\kappa-1)-\frac{\delta}{n}(\kappa-1)}(s_0-s))w_\varepsilon(s,t)\,\mathrm{d}s \\ &\quad\le -\int_0^{s_0} \frac{\partial}{\partial s}(s^{-(n-1)(p-1)(\kappa-1)-\frac{\delta}{n}(\kappa-1)}(s_0-s))w_\varepsilon(s,t)\,\mathrm{d}s \end{aligned} \tag{4.41}$$

for any $t \in (0,\widetilde{T}_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0,R^n)$. Now, in light of (4.36) we further compute

$$\begin{aligned} &-\frac{\partial}{\partial s}(s^{-(n-1)(p-1)(\kappa-1)-\frac{\delta}{n}(\kappa-1)}(s_0-s)) \\ &= \left((n-1)(p-1)(\kappa-1)+\frac{\delta}{n}(\kappa-1)\right)s^{-(n-1)(p-1)(\kappa-1)-1-\frac{\delta}{n}(\kappa-1)}(s_0-s) \\ &\quad + s^{-(n-1)(p-1)(\kappa-1)-\frac{\delta}{n}(\kappa-1)} \\ &\le c_2 s_0 s^{-(n-1)(p-1)(\kappa-1)-1-\frac{\delta}{n}(\kappa-1)} \end{aligned}$$

for all $s_0 \in (0,R^n)$ and $s \in (0,s_0)$ with $c_2 := (n-1)(p-1)(\kappa-1)+2$, and this together with (4.41) shows that

$$\begin{aligned} &\int_0^{s_0} s^{-(n-1)(p-1)(\kappa-1)-\frac{\delta}{n}(\kappa-1)}(s_0-s)(w_\varepsilon)_s(s,t)\,\mathrm{d}s \\ &\quad\le c_2 s_0 \int_0^{s_0} s^{-(n-1)(p-1)(\kappa-1)-1-\frac{\delta}{n}(\kappa-1)} w_\varepsilon(s,t)\,\mathrm{d}s \end{aligned} \tag{4.42}$$

for all $t \in (0,\widetilde{T}_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0,R^n)$. Combining (4.40) with (4.42), we deduce

$$I_{3,\varepsilon}(t) \ge -c_3 s_0^{2-\gamma} \int_0^{s_0} s^{-(n-1)(p-1)(\kappa-1)-1-\frac{\delta}{n}(\kappa-1)} w_\varepsilon(s,t)\,\mathrm{d}s \tag{4.43}$$

for all $t \in (0, \widetilde{T}_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$, with $c_3 := \frac{c_1^{\kappa-1}\mu}{1-\gamma} c_2$. We now estimate the integral on the right of (4.43). In light of (4.10), we split the integral as

$$\begin{aligned}
&\int_0^{s_0} s^{-(n-1)(p-1)(\kappa-1)-1-\frac{\delta}{n}(\kappa-1)} w_\varepsilon(s,t)\,\mathrm{d}s \\
&= \int_0^{s_0} s^{-(n-1)(p-1)(\kappa-1)-1-\frac{\delta}{n}(\kappa-1)} z_\varepsilon(s,t)\,\mathrm{d}s + \frac{1}{n}\overline{M_\varepsilon}(t)\int_0^{s_0} s^{-(n-1)(p-1)(\kappa-1)-\frac{\delta}{n}(\kappa-1)}\,\mathrm{d}s \\
&=: I_{A,\varepsilon}(t) + I_{B,\varepsilon}(t) \qquad (4.44)
\end{aligned}$$

for all $t \in (0, \widetilde{T}_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$. By virtue of Lemma 4.9 in conjunction with [4, Lemma 3.3], we confirm that

$$\begin{aligned}
I_{A,\varepsilon}(t) &\le p^{\frac{1}{p}} J_{1,\varepsilon}^{\frac{1}{p}}(t) \int_0^{s_0} s^{\frac{1}{p}(\gamma-(1-\frac{1}{n})(2-p))-(n-1)(p-1)(\kappa-1)-1-\frac{\delta}{n}(\kappa-1)} (s_0-s)^{-\frac{1}{p}}\,\mathrm{d}s \\
&= p^{\frac{1}{p}} J_{1,\varepsilon}^{\frac{1}{p}}(t) s_0^{\frac{1}{p}(\gamma-1-(1-\frac{1}{n})(2-p))-(n-1)(p-1)(\kappa-1)-\frac{\delta}{n}(\kappa-1)} \\
&\quad \times B\left(\frac{\gamma}{p} - \frac{1}{p}\left(1-\frac{1}{n}\right)(2-p) - (n-1)(p-1)(\kappa-1) - \frac{\delta}{n}(\kappa-1), \frac{p-1}{p}\right) \qquad (4.45)
\end{aligned}$$

for all $t \in (0, \widetilde{T}_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$, where B is the beta function, noting that this is well-defined thanks to (4.38). On the other hand, in light of (4.28) and (4.37), it follows that for any choice of $s_0 \in (0, R^n)$,

$$I_{B,\varepsilon}(t) \le \frac{M_2}{n|\Omega|} c_4 s_0^{1-(n-1)(p-1)(\kappa-1)-\frac{\delta}{n}(\kappa-1)} \qquad (4.46)$$

for all $t \in (0, \widetilde{T}_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$, where $c_4 := (1-(n-1)(p-1)(\kappa-1)-\frac{\delta}{n}(\kappa-1))^{-1}$. Inserting (4.45) and (4.46) into (4.44), we thus obtain

$$\begin{aligned}
&\int_0^{s_0} s^{-(n-1)(p-1)(\kappa-1)-1-\frac{\delta}{n}(\kappa-1)} w_\varepsilon(s,t)\,\mathrm{d}s \\
&\le c_5 R^{n\eta-\delta(\kappa-1)} s_0^{\frac{1}{p}(\gamma-1-(1-\frac{1}{n})(2-p))-(n-1)(p-1)(\kappa-1)-\eta} J_{1,\varepsilon}^{\frac{1}{p}}(t) \\
&\quad + c_6 R^{n\eta-\delta(\kappa-1)} s_0^{1-(n-1)(p-1)(\kappa-1)-\eta} \qquad (4.47)
\end{aligned}$$

for all $t \in (0, \widetilde{T}_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$, where

$$c_5 := p^{\frac{1}{p}} B\left(\frac{\gamma}{p} - \frac{1}{p}\left(1-\frac{1}{n}\right)(2-p) - (n-1)(p-1)(\kappa-1) - \frac{\delta}{n}(\kappa-1), \frac{p-1}{p}\right)$$

and

$$c_6 := \frac{M_2}{n|\Omega|} c_4.$$

Combining (4.43) with (4.47), we finally arrive at the conclusion. □

4.4 Estimating the Terms $I_{1,\varepsilon}$ and $I_{4,\varepsilon}$ in (4.13)

Quite in the style of [37], we can estimate the term $I_{1,\varepsilon}$ in (4.13) as in the next lemma.

Lemma 4.11 *Let $p > \frac{2n-2}{2n-3}$ and $\gamma \in (0,1)$ satisfy*

$$(p-1)\gamma < 2 - \frac{4}{n} - 2(2-p) + \frac{3(2-p)}{n}. \tag{4.48}$$

Then for any choice of $\eta > 0$ there is $C > 0$ such that whenever u_0 satisfies (1.4), (1.14) and (1.15), for all $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$ the function $I_{1,\varepsilon}$ satisfies

$$I_{1,\varepsilon}(t) \geq -\eta H_{1,\varepsilon}(t) - \eta H_{2,\varepsilon}(t) - C s_0^{\frac{3-\frac{4}{n}-3(2-p)+\frac{3}{n}(2-p)-(p-1)\gamma}{p-1}}$$

for all $t \in (0, T_{\max,\varepsilon})$.

Proof Following [37, Lemma 3.3], we observe that

$$\begin{aligned} I_{1,\varepsilon}(t) \geq & -n^2\left(2 - \frac{2}{n} - \gamma\right)\left(\gamma - 1 + \frac{2}{n}\right)\int_0^{s_0} s^{-\frac{2}{n}-\gamma}(s_0 - s) z_\varepsilon(s,t)\,\mathrm{d}s \\ & - 2n^2\left(2 - \frac{2}{n} - \gamma\right)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma} z_\varepsilon(s,t)\,\mathrm{d}s \end{aligned} \tag{4.49}$$

for all $t \in (0, T_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$. Besides, [37, Lemma 3.11] warrants that for any $\eta > 0$ there is $C > 0$ such that

$$\int_0^{s_0} s^{-\frac{2}{n}-\gamma}(s_0 - s) z_\varepsilon(s,t)\,\mathrm{d}s \leq \eta H_{1,\varepsilon}(t) + C s_0^{\frac{3-\frac{4}{n}-3(2-p)+\frac{3}{n}(2-p)-(p-1)\gamma}{p-1}}$$

and

$$\int_0^{s_0} s^{1-\frac{2}{n}-\gamma} z_\varepsilon(s,t)\,\mathrm{d}s \leq \eta H_{2,\varepsilon}(t) + C s_0^{\frac{3-\frac{4}{n}-3(2-p)+\frac{3}{n}(2-p)-(p-1)\gamma}{p-1}}$$

for all $t \in (0, T_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$. These together with (4.49) prove the claim. □

Finally, the estimation for $I_{4,\varepsilon}$ is already obtained throughout our analysis.

Lemma 4.12 *Let $\chi > 0$, $p > 1$, $\lambda \geq 0$, $\mu > 0$ and $\kappa > 1$. Suppose that u_0 satisfy (1.4), (1.14), (1.15) and (4.25). Then for all $s_0 \in (0, R^n)$, $\gamma \in (0,1)$ and $m > 0$ there exists $C > 0$ such that*

$$I_{4,\varepsilon}(t) \geq -C s_0^{3-\gamma} \quad \textit{for all } t \in (0, T_{\max,\varepsilon}) \textit{ and } \varepsilon \in (0,1).$$

Proof This can be immediately derived from (4.28). □

4.5 Moment Inequality for Approximate Solutions

With all these preparations in our hand, our analysis will reach the important lemma which gives a superlinear differential inequality for ϕ_ε.

Lemma 4.13 *Let $\chi > 0$, $\lambda \geq 0$ and $\mu > 0$. Assume that p and κ satisfy (1.12) and (1.13). Then there exist $\gamma \in (0,1)$ and $\theta \in (0,2)$ such that for all $m > 0$ and $L > 0$ one can find $C > 0$ with the following property: Whenever u_0 satisfies (1.4), (1.14), (1.15), (1.16) and (4.25), for all $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$ the function ϕ_ε as in (4.12) satisfies*

$$\phi_\varepsilon'(t) \geq C s_0^{-3+3(2-p)-\frac{2-p}{n}+(p-1)\gamma} \phi_\varepsilon^p(t) - C s_0^{3-\gamma-\theta}$$

for all $t \in (0, \widetilde{T}_{\max,\varepsilon})$.

Proof Since p and κ satisfy (1.12) and (1.13), we see that the condition (4.34) is valid. Also, the assumption (1.12) ensures that $p - 1 > \frac{1}{2n-3}$, which implies

$$k_1 := \frac{3}{n} + \frac{1}{n(p-1)} < 2. \tag{4.50}$$

In addition, [37, Lemma 3.13] warrants that there is $\gamma \in (0,1)$ such that (4.21), (4.22), (4.35) and (4.48) hold. Based on (4.34) and (4.35), we can choose $\eta > 0$ satisfying

$$(n-1)(p-1)(\kappa-1) + \eta < 1$$

and

$$\frac{1}{p-1}\left(1-\frac{1}{n}\right)(2-p) + p(n-1)(\kappa-1) + \frac{p}{p-1}\eta < \frac{\gamma}{p-1} < \frac{1}{p-1} = 1 - \frac{p-2}{p-1},$$

whence

$$k_2 := \max\left\{(n-1)(p-1)(\kappa-1)+\eta, -\frac{2-p}{n(p-1)} + p(n-1)(\kappa-1) + \frac{p}{p-1}\eta\right\} < 1 \tag{4.51}$$

holds. Now, Lemma 4.10 together with the Young inequality provides $c_1, c_2 > 0$ such that

$$\begin{aligned} I_{3,\varepsilon}(t) &\geq -c_1 s_0^{-\frac{1}{p}(1-\frac{1}{n})(2-p)-(n-1)(p-1)(\kappa-1)+\frac{2p-1-(p-1)\gamma}{p}-\eta} J_{1,\varepsilon}^{\frac{1}{p}}(t) \\ &\quad - c_1 s_0^{-(n-1)(p-1)(\kappa-1)+3-\gamma-\eta} \\ &\geq -\frac{1}{2} J_{1,\varepsilon}(t) - c_2 s_0^{-\frac{1}{p-1}(1-\frac{1}{n})(2-p)-p(n-1)(\kappa-1)+\frac{2p-1-(p-1)\gamma}{p-1}-\frac{p}{p-1}\eta} \\ &\quad - c_1 s_0^{-(n-1)(p-1)(\kappa-1)+3-\gamma-\eta} \end{aligned} \tag{4.52}$$

for all $t \in (0, \widetilde{T}_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$. On the other hand, we infer from Lemma 4.12 that with $c_3 > 0$ we have

$$I_{4,\varepsilon}(t) \geq -c_3 s_0^{3-\gamma} \tag{4.53}$$

for all $t \in (0, T_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$. Combining (4.13) with (4.16), (4.52) and (4.53), we deduce that

$$\begin{aligned}\phi_\varepsilon'(t) \geq\ & I_{1,\varepsilon}(t) + \frac{1}{2}J_{1,\varepsilon}(t) - J_{2,\varepsilon}(t) - J_{3,\varepsilon}(t)\\ & - c_1 s_0^{-(n-1)(p-1)(\kappa-1)+3-\gamma-\eta}\\ & - c_2 s_0^{-\frac{1}{p-1}(1-\frac{1}{n})(2-p)-p(n-1)(\kappa-1)+\frac{2p-1-(p-1)\gamma}{p-1}-\frac{p}{p-1}\eta} - c_3 s_0^{3-\gamma}\end{aligned} \tag{4.54}$$

for all $t \in (0, \widetilde{T}_{\max,\varepsilon})$, $\varepsilon \in (0,1)$, $s_0 \in (0, R^n)$ and $\beta \in (0,1]$. We further employ Lemma 4.4 to find $c_4 > 0$ such that

$$J_{1,\varepsilon}(t) \geq c_4 H_{1,\varepsilon}(t) + c_4 H_{2,\varepsilon}(t) \tag{4.55}$$

for all $t \in (0, T_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$, whereas in the case that $p < 2$, we fix $\beta \in (0, \frac{p-1}{2})$ and apply Lemma 4.6 to find $c_5, c_6 > 0$ such that

$$J_{2,\varepsilon}(t) \leq \frac{c_4}{8}H_{1,\varepsilon}(t) + \frac{c_4}{8}H_{2,\varepsilon}(t) + c_5 s_0^{3-\frac{2}{n}-\gamma}, \tag{4.56}$$

and

$$J_{3,\varepsilon}(t) \leq \frac{c_4}{8}H_{1,\varepsilon}(t) + c_6 s_0^{\frac{3-\frac{2-p}{n}-\gamma+2\beta(3-\frac{2}{n}-\gamma)}{1+2\beta}} \tag{4.57}$$

for all $t \in (0, T_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$. In addition, Lemma 4.11 entails that with $c_7 > 0$ we have

$$I_{1,\varepsilon}(t) \geq -\frac{c_4}{8}H_{1,\varepsilon}(t) - \frac{c_4}{8}H_{2,\varepsilon}(t) - c_7 s_0^{\frac{3-\frac{4}{n}-3(2-p)+\frac{3}{n}(2-p)-(p-1)\gamma}{p-1}} \tag{4.58}$$

for all $t \in (0, T_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$. Taking account of (4.54) together with (4.55), (4.56), (4.57) and (4.58), we thus find $c_8 > 0$ such that

$$\phi_\varepsilon'(t) \geq \frac{c_4}{8}H_{1,\varepsilon}(t) - c_8 s_0^{3-\gamma-\theta} \tag{4.59}$$

for all $t \in (0, \widetilde{T}_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$ with $\theta := \max\{k_1, k_2\}$, noting that $\theta \in (0,2)$ due to (4.50) and (4.51). Since Lemma 4.5 provides $c_9 > 0$ such that

$$H_{1,\varepsilon}(t) \geq c_9 s_0^{-3+3(2-p)-\frac{2-p}{n}+(p-1)\gamma}\phi_\varepsilon^p(t)$$

for all $t \in (0, T_{\max,\varepsilon})$, $\varepsilon \in (0,1)$ and $s_0 \in (0, R^n)$, this together with (4.59) leads to the conclusion. □

4.6 Proof of Theorem 1.2

Before we prove Theorem 1.2, we introduce the framework of weak solutions, so-called *moment solutions* and *maximal moment solutions*, which satisfy a suitable integral form of moment inequality. A similar concept was initially introduced in [24].

Definition 4.1 Let p and κ satisfy (1.12) as well as (1.13), and let $\gamma \in (0,1)$ and $\theta \in (0,2)$ as in Lemma 4.13. Suppose that u_0 satisfies (1.4) and (1.14). Let $T \in (0,\infty]$, and let (u,v) be a weak solution of (1.1) in $\Omega \times (0,T)$ which is radially symmetric with respect to $x=0$. Then (u,v) will be called a *moment solution* of (1.1) on $[0,T)$ if there exists $C = C(R,n,\chi,p,\lambda,\mu,\kappa,\gamma) \geq 0$ such that for any $s_0 \in (0,R^n)$ the function ϕ defined in (1.11) with u satisfies

$$\phi(t) - \phi(0) \geq C s_0^{-3+3(2-p)-\frac{2-p}{n}+(p-1)\gamma} \int_0^t \phi^p(\tau)\,d\tau - C s_0^{3-\gamma-\theta} t \tag{4.60}$$

for all $t \in (0, \min\{1,T\})$.

Definition 4.2 Define

$$\mathcal{S} := \{(T,u,v) \mid T \in (0,\infty],\ (u,v) \text{ is a moment solution of (1.1) on } [0,T)\}. \tag{4.61}$$

Moreover, when $\mathcal{S}$ is nonempty, we introduce the order relation $\preceq$ on $\mathcal{S}$ given by

$$(T_1,u_1,v_1) \preceq (T_2,u_2,v_2) \ :\Longleftrightarrow\ T_1 \leq T_2,\ u_2|_{(0,T_1)} = u_1,\ v_2|_{(0,T_1)} = v_1.$$

If there is a maximal element $(T_{\max},u,v) \in \mathcal{S}$, then (u,v) is called a *maximal moment solution* of (1.1) on $[0,T_{\max})$.

We first state that moment solutions of (1.1) exist.

Lemma 4.14 *Let $\chi > 0$, $\lambda \geq 0$ and $\mu > 0$. Assume that the function h satisfies* (1.3), *and suppose that p and κ fulfill* (1.12) *and* (1.13). *Then for all $L > 0$ and $m > 0$, whenever u_0 satisfies* (1.4), (1.14), (1.15) *as well as* (1.16), *there is $T > 0$ such that* (1.1) *admits a moment solution on* $[0,T)$.

Proof We regard $T_0 > 0$, $K_0 > 0$ and $L_0 > 0$ as in Lemma 2.5, which satisfy (2.12). We claim that (1.1) admits a moment solution on $[0,T_0)$. To see this, first we infer that with $\gamma \in (0,1)$ and $\theta \in (0,2)$ as in Lemma 4.13, there exists $C > 0$, independently of T_0, such that for any $s_0 \in (0,R^n)$, it follows that

$$\phi_\varepsilon(t) - \phi_\varepsilon(0) \geq C s_0^{-3+3(2-p)-\frac{2-p}{n}+(p-1)\gamma} \int_0^t \phi_\varepsilon^p(\tau)\,\mathrm{d}\tau - C s_0^{3-\gamma-\theta} t \tag{4.62}$$

for all $t \in (0,\min\{1,T_0\})$ and $\varepsilon \in (0,1)$. Letting $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and the function u as in Lemma 2.8, we argue as in [17, Lemma 3.4] (see also [10, Lemma 6.1]) to find a constant $\alpha = \alpha(n) \in (0,1)$ and a subsequence $(\varepsilon_{j_i})_i$ such that

$$u_{\varepsilon_{j_i}} \to u \quad \text{in } C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times [\delta,T]) \text{ as } i \to \infty$$

for all δ and T with $0 < \delta < T < T_0$. By the same argument as in [17, Lemma 3.5], this along with (4.62) and Lemma 2.8 thus proves the claim. □

We also ensure existence of maximal moment solutions to (1.1).

Lemma 4.15 *Let $\chi > 0$, $\lambda \geq 0$ and $\mu > 0$. Assume that the function h satisfies (1.3), and suppose that p and κ fulfill (1.12) and (1.13). Then for all $L > 0$ and $m > 0$, whenever u_0 satisfies (1.4), (1.14), (1.15) as well as (1.16), there exist $T_{\max} \in (0, \infty]$ and a pair of functions (u, v) such that (u, v) is a maximal moment solution of (1.1) on $[0, T_{\max})$, and that if $T_{\max} < \infty$ and (u, v) satisfies (1.18) with $C > 0$, then*

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{4.63}$$

Proof Thanks to Lemma 4.14, the set $\mathcal{S}$ in (4.61) is nonempty and inductive, whence the Zorn lemma guarantees the existence of $T_{\max} \in (0, \infty]$ and a maximal moment solution (u, v) to (1.1) on $[0, T_{\max})$. The remaining statement of this lemma can be derived by the same way as in [17, Lemma 3.7]. □

We are finally in a position to prove Theorem 1.2, ensuring that finite-time blow-up of weak solutions to (1.1) can occur.

Proof of Theorem 1.2 Lemma 4.15 guarantees that there are $T_{\max} \in (0, \infty]$ and (u, v) such that (u, v) is a weak solution of (1.1) in $\Omega \times (0, T_{\max})$, that there are $\gamma \in (0, 1)$, $\theta \in (0, 2)$ and $C \geq 0$ satisfying (1.18), and that (4.63) holds if $C > 0$. Therefore, we only have to show that $T_{\max} < \infty$ in the case when $C > 0$. We note that as in the proof of [37, Lemma 3.15 and Theorem 1.1], for $s_0 \in (0, \frac{R^n}{4})$ we let $m_0 := \frac{m}{2}$ and $r_0 := (\frac{s_0}{2})^{\frac{1}{n}}$ to see that from (1.17) we find $c_1 = c_1(n, m, R, \gamma) > 0$ fulfilling

$$\phi(0) \geq c_1 s_0^{2-\gamma}.$$

This together with (1.18) shows that there is $c_2 > 0$ such that

$$\phi(t) \geq c_1 s_0^{2-\gamma} + c_2 s_0^{-3+3(2-p)-\frac{2-p}{n}+(p-1)\gamma} \int_0^t \phi^p(\tau)\, d\tau - c_2 s_0^{3-\gamma-\theta} t \tag{4.64}$$

for any $t \in (0, \min\{1, T_{\max}\})$. Now, abbreviating

$$a(s_0) := c_1 s_0^{2-\gamma} \quad \text{and} \quad b(s_0) := c_2 s_0^{-3+3(2-p)-\frac{2-p}{n}+(p-1)\gamma},$$

we observe that $a^{p-1}b \to \infty$ as $s_0 \searrow 0$, due to the fact that

$$(p-1)(2-\gamma) - 3 + 3(2-p) + \frac{2-p}{n} + (p-1)\gamma < 0$$

holds according to (1.12). We therefore can choose $s_0 \in (0, \frac{R^n}{4})$ small enough so that $\frac{2}{(p-1)a^{p-1}b} < 1$, and since $\phi \in C^0([0, T_{\max}))$, thanks to [32, Lemma 4.9] we conclude from (4.64) that

$$T_{\max} \leq \frac{2}{(p-1)a^{p-1}(s_0)b(s_0)} < 1,$$

and hence (4.63) implies that (u, v) blows up in finite time $T_{\max}$. □

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Declarations

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