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Nonlinear Degenerate Parabolic Equations with a Singular Nonlinearity

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Abstract
In this paper, we study the existence and regularity results for some parabolic equations with degenerate coercivity, and a singular right-hand side. The model problem is

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\left(\frac{\left(1+|\nabla u|^{-\Lambda}\right)|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta}}\right) = \frac{f}{(e^u-1)^{\gamma}} & \text{in } Q_T, \\ u(x,0)=0 & \text{on } \Omega, \\ u=0 & \text{on } \partial Q_T, \end{cases} \tag{0.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $N \geq 2$, $T>0$, $\Lambda \in [0, p-1)$, f is a non-negative function belonging to $L^m(Q_T)$, $Q_T = \Omega \times (0,T)$, $\partial Q_T = \partial\Omega \times (0,T)$, $0 \leq \theta < p-1+\frac{p}{N}+\gamma(1+\frac{p}{N})$ and $0 \leq \gamma < p-1$.

Keywords Degenerate parabolic equation · Existence and regularity of solution · Singular term · Irregular data · Fixed point theorem

Mathematics Subject Classification 35J60 · 35J70 · 35B45 · 35D30 · 35B65

1 Introduction

Let Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, Q_T is the cylinder $\Omega \times (0,T)$ $(T>0)$, ∂Q_T is the lateral surface $\partial\Omega \times (0,T)$. We consider the following double nonlinear anisotropic singular parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} + Bu = g(u)f & \text{in } Q_T, \\ u(x,0)=0 & \text{on } \Omega, \\ u=0 & \text{on } \partial Q_T, \end{cases} \tag{1.1}$$

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where $Bu = -\text{div}(b(x,t,u,\nabla u))$, f is a non-negative function belonging to a suitable Lebsgue space $L^m(Q_T)$ $(m \geq 1)$. Here, we suppose that $b : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, and satisfying for almost every (x,t) in Q_T, for every $z \in \mathbb{R}$, for all $\xi, \eta \in \mathbb{R}^N$ the following

$$b(x,t,z,\xi)\cdot\xi \geq \frac{\alpha|\xi|^p}{(1+|z|)^\theta}, \tag{1.2}$$

$$0 \leq \theta < p - 1 + \frac{p}{N} + \gamma(1 + \frac{p}{N}), \tag{1.3}$$

$$|b(x,t,z,\xi)| \leq a(x,t) + |z|^{p-1} + |\xi|^{p-1}, \tag{1.4}$$

$$(b(x,t,z,\xi) - b(x,t,z,\eta))\cdot(\xi - \eta) > 0, \quad \xi \neq \eta, \tag{1.5}$$

where α, β are strictly positive real numbers and a is a given positive function in $L^{p'}(Q_T)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, $g : [0,+\infty) \to [0,+\infty)$ is a continuous and possibly singular function with $g(0) \neq 0$ which it is finite outside the origin and such that

$$\exists\, c > 0 : \quad g(z) \leq \frac{c}{z^\gamma} \quad \text{for all } z > 0, \tag{1.6}$$

where $0 \leq \gamma < p - 1$.

In the uniform ellipticity and non singular case (i.e. $\theta = 0$ and $\gamma = 0$, it is proved the existence results for the problems (1.1) in [1–5, 7, 8, 28–30] when $f \in L^m(Q_T)$ or f is a bounded Radon measure on Q_T. We cite the paper [16], and the references therein, when $p = 2$, $\gamma = 0$, $0 \leq \theta < 1 + \frac{2}{N}$ and $f \in L^m(Q_T)$, where $m \geq 1$. In the case $\theta = 0$ and $p \geq 2$, the existence and regularity solution have been treated in [11]. Problem (1.1), in the coercive case, has been treated in [9], they have proved the existence and regularity of solutions to problem

$$\begin{cases} \frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p-2}\nabla u) = \frac{f}{u^\gamma} & \text{in } Q_T, \\ u(x,0) = u_0(x) & \text{on } \Omega, \\ u = 0 & \text{on } \partial Q_T, \end{cases}$$

with $\gamma > 0$, $p \geq 2$, $f > 0$, $f \in L^m(Q_T)$, $m \geq 1$ and $u_0 \in L^\infty(\Omega)$. If $\gamma = 0$, the problem (1.1) is studied in [12, 19, 25].

Finally, concerning the singular model case the authors in [14] studied existence and regularity of problem

$$\begin{cases} \frac{\partial u}{\partial t} - \text{div}(b(x,t,u,\nabla u)) + |u|^{s-1}u = g(u)f & \text{in } Q_T, \\ u = 0 & \text{on } \partial Q_T, \\ u(x,0) = 0 & \text{in } \Omega, \end{cases}$$

where $f \in L^m(Q_T)$ $(m \geq 1)$, $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a carathéodory function satisfying for a.e $(x,t) \in Q_T$, $\forall z \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$b(x,t,z,\xi).\xi \geq \frac{\alpha|\xi|^p}{(1+|z|)^{\theta(p-1)}} \quad \text{with} \quad 0 \leq \theta < 1,$$

and the singular term g satisfying (1.6) with $0 < \gamma < 1$. The corresponding results for parabolic equations with singularities have been developed in [13, 15, 17]. The existence and regularity results for weak solution of degenerate elliptic equation with singularities data have been proved in [18, 20, 21, 31–33].

Our main motive in this article is to investigate the results of [25] in the framework of the operator non-coercive $B(u)$. To reach this goal, we will face the following difficulties. First, let us note that (1.1) can be singular on the right-hand side in the following sense: the solution is required to be zero on the boundary of the domain but, simultaneously, the right- hand side of (1.1) could blow up. Another important feature is the lack of coercivity for positive θ, the operator $B(u)$ is not coercive as u becomes large. Due to the lack of coercivity, the classical methods can not be applied even if the data $g(u)f$ are sufficiently regular (see [27]). We will overcome these two difficulties by approximation, truncating the degenerate coercivity of the operator term and the singularity of the right-hand side (see problems (3.1)). We will prove by Schauder's theorem that these problems admit a bounded finite energy solution u_n.

The following lemma is useful when proving the boundedness of the solution u_n of problem (3.1).

Lemma 1.1 (See [6]) *Let M_1, ν, ρ, ϑ, k_0 be real positive numbers, where $\vartheta > 1$ and $\rho \in [0, 1)$. Let $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function such that*

$$\Lambda(h) \le \frac{M_1 k^{\nu\rho}}{(h-k)^{\nu}}[\Lambda(k)]^{\vartheta}, \quad \forall h > k \ge k_0.$$

Then there exists $l > 0$ such that $\Lambda(l) = 0$.

Next, we will review the results of the renowned Gagliardo-Nirenberg embedding theorem.

Lemma 1.2 (See [10]) *Let $v \in L^{\kappa}(0, T; W_0^{1,\kappa}(\Omega)) \cap L^{\infty}(0, T; L^{\varrho}(\Omega))$, $\kappa, \varrho \ge 1$. Then v belongs to $L^q(Q_T)$, where $q = \kappa\frac{N+\varrho}{N}$, and there exists a positive constant M_2 depending only on N, κ, ϱ such that*

$$\int_{Q_T} |v(x,t)|^q dxdt \le M_2 \|v\|^{\frac{\kappa}{N}}_{L^{\infty}(0,T;L^{\varrho}(\Omega))} \int_{Q_T} |\nabla v(x,t)|^{\kappa} dxdt. \tag{1.7}$$

For any $q > 1$, $q' = \frac{q}{q-1}$ is the Hölder conjugate of q. For fixed $k > 0$ we will use of the truncation T_k defined as $T_k(s) = \max(-k, \min(k, s))$ and $G_k(s) = s - T_k(s)$. We will also use the following function

$$\Xi_{\lambda}(s) = \begin{cases} 1, & \text{if } s \le \lambda, \\ \frac{\lambda - s}{\lambda}, & \text{if } \lambda < s < 2\lambda, \\ 0, & \text{if } s \ge 2\lambda. \end{cases} \tag{1.8}$$

For the sake of completeness, we recall a well-known inequality that will be useful in what follows

$$\forall a > 0, \forall \mu > 0, \exists C(\mu, a) > 0 : \quad (1+t)^{\mu} \le C t^{\mu}, \quad \forall t \in [a, +\infty). \tag{1.9}$$

2 Statements of Results

We first define the notion of a weak solution to (1.1) as follows:

Definition 2.1 We say that $u \in L^1(0,T;W_0^{1,1}(\Omega))$ is a weak solution of problem (1.1), if $b(x,t,u,\nabla u) \in (L^1(Q_T))^N$, $g(u)f \in L^1(Q_T)$ and the equality

$$\int_{Q_T} \frac{\partial u}{\partial t}\varphi dxdt + \int_{Q_T} b(x,t,u,\nabla u).\nabla\varphi dxdt = \int_{Q_T} g(u)f\varphi dxdt, \tag{2.1}$$

for every $\varphi \in C^\infty([0,T]\times\overline{\Omega})$ which is zero in a neighborhood of ∂Q_T and $\Omega\times\{T\}$.

The first theorem we state concerns with the existence of L^∞-solutions to problem (1.1), where $f \in L^m(Q_T)$ with $m > \frac{N}{p}+1$.

Theorem 2.2 *Assume that* (1.2)-(1.6) *hold true. Let* $f \in L^m(Q_T)$ *with* $m > \frac{N}{p}+1$. *Then there exists* $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ *a weak solution to problem* (1.1).

Remark 1 We apply Lemma 1.1, which requires the assumption $p \geq 2$, to obtain the L^∞-estimates for u_n the solutions of (3.1). In the case $p=2$, $\gamma=0$ the result of Theorem 2.2 coincides with the classical boundedness results for degenerate parabolic equations ([16], Theorem 1.1), furthermore if $p>2$ the results of Theorem 2.2 are similar than the regularity results of [14, 25]. To obtain the L^∞-estimate, the conditions (1.4) and (1.5) are unnecessary. However, such conditions are needed to prove the existence of u_n solution of problem (3.1).

In the following theorem we give the result of existence and regularity in the case of exact values of the summability exponent $m = \frac{N}{p}+1$.

Theorem 2.3 *Suppose that assumptions* (1.2)-(1.6) *hold,* $f \in L^m(Q_T)$ *with* $m = \frac{N}{p}+1$. *Then, for every* $r \in [p,+\infty)$ *there exists* $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^r(Q_T)$ *a weak solution to problem* (1.1).

Remark 2 Theorem 2.3 gives the result in the limit case $m = \frac{N}{p}+1$ for parabolic equations. As far as I know, the first time this case was addressed in the article [16] with $p=2$ and $\gamma=0$. The result of Theorem 2.3 has been obtained in [14, 25].

The next result deals with a given $m < \frac{N}{p}+1$, which ensures the existence of solutions in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\delta(Q_T)$.

Theorem 2.4 *Let us assume that* (1.2)-(1.6) *hold true, and that* $f \in L^m(Q_T)$, *with*

$$m_1 = \frac{p(N+\theta+2)}{(p-1)N+2p-(N-p)\theta+N\gamma} \leq m < \frac{N}{p}+1. \tag{2.2}$$

Then, there exists $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\delta(Q_T)$ *a weak solution to problem* (1.1), *such that*

$$\delta = \frac{m[p+N(p-1-\theta)+\gamma(N+p)]}{N+p-pm}. \tag{2.3}$$

Remark 3 The condition (1.3) implies that the assumption (2.2) is well defined. By (1.3) and (2.2), we have $\delta > p$, since

$$
\begin{aligned}
(1.3) \Leftrightarrow & \frac{p(N+\theta+2)}{(p-1)N+2p-(N-p)\theta+N\gamma} > \frac{p(N+p)}{p+N(p-1-\theta)+\gamma(N+p)+p^2} \\
& \Rightarrow m > \frac{p(N+p)}{p+N(p-1-\theta)+\gamma(N+p)+p^2} \\
& \Rightarrow \delta > p.
\end{aligned}
$$

If $0 \le \theta < \frac{2}{N-1} + \gamma \frac{N}{(N-1)}$, then $m_1 < p'$, so $f \notin L^{p'}(0, T; W^{-1,p'}(\Omega))$. If $\frac{2}{N-1} + \gamma \frac{N}{(N-1)} \le \theta < p - 1 + \frac{p}{N} + \gamma(1 + \frac{p}{N})$, then $m_1 \ge p'$, so $f \in L^{p'}(0, T; W^{-1,p'}(\Omega))$.

The first result deals with the case when the summability of f gives the existence of solution u belong to $L^q(0, T; W_0^{1,q}(\Omega))$, with $p - 1 < q < p$.

Theorem 2.5 *If hypotheses (1.2)-(1.6) hold and $f \in L^m(Q_T)$ with $m > 1$, such that*

$$
m_2 = \frac{N+\theta+2}{(p-1)N+p+1-\theta(N-1)+\gamma(N+p-1)} < m < m_1, \tag{2.4}
$$

then there exists $u \in L^q(0, T; W_0^{1,q}(\Omega)) \cap L^\delta(Q_T)$ a weak solution to problem (1.1), such that

$$
q = \frac{m[N(p-\theta-1)+p+\gamma(N+p)]}{N+1-(\theta+1)(m-1)+m\gamma}, \tag{2.5}
$$

where δ as defined in (2.3).

Remark 4 The hypothesis (2.5) is meaningful, because

$$
m_2 < m_1 \Leftrightarrow \theta < p - 1 + \frac{p}{N} + \gamma\left(1 + \frac{p}{N}\right).
$$

Notice that the inequality (2.4) guarantees that $p - 1 < q < p$. In Theorem 2.5, we also suppose $m > 1$,

$$
m_2 < 1 \Leftrightarrow 0 \le \theta < p - 1 + \frac{p}{N} + \gamma\left(1 + \frac{p}{N}\right) - \frac{N+\gamma+1}{N}.
$$

If $\gamma = 0$; the result of Theorem 2.5 is similar that of ([25], Theorem 2.5).

Remark 5 It will be noted to the reader that the choice of the test functions in the proof of the a priori estimates allowed us to widen the interval of variation of γ and θ compared to that in [14], with the same regularity of the solution. If we compare the results of theorems 2.2-2.5 with those of theorems in [25], we can easily see that the singular term allowed us to widen the interval of variation of θ compared to that the assumption (3) in [25].

3 Approximating Problems

Let us first consider the following approximation problems

$$\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}(b(x,t,T_n(u_n),\nabla u_n)) = g_n(u_n) f_n & \text{in } Q_T, \\ u_n(x,0) = 0 & \text{on } \Omega. \\ u_n = 0 & \text{on } \partial Q_T, \end{cases} \tag{3.1}$$

where $f_n \in L^\infty(Q_T)$ (for example, $f_n = T_n(f)$), such that

$$\begin{cases} \|f_n\|_{L^m(Q_T)} \le \|f\|_{L^m(Q_T)} \le C, \\ f_n \to f \text{ strongly in } L^m(Q_T), \quad m \ge 1, \end{cases} \tag{3.2}$$

and, we define $g(0) = \lim_{z\to 0} g(z)$, we set

$$g_n(z) = \begin{cases} T_n(g(z)) & \text{for } z > 0, \\ \min\{n, g(0)\} & \text{otherwise.} \end{cases}$$

Using (1.6), we have for all $z > 0$

$$g_n(z) = T_n(g(z)) \le g(z) \le \frac{c}{z^\gamma}. \tag{3.3}$$

Lemma 3.1 *Assume that* (1.2), (1.5) *and* (1.6) *hold true. Then, the approximating problem* (3.1) *has a non-negative solution* u_n, *such that*

$$u_n \in L^p(0,T;W_0^{1,p}(\Omega)) \cap C([0,T];L^2(\Omega)), \qquad \frac{\partial u_n}{\partial t} \in L^{p'}(0,T;W_0^{-1,p'}(\Omega)),$$

and satisfying the weak formulation

$$\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, \varphi \Big\rangle dt + \int_{Q_T} b(x,t,T_n(u_n),\nabla u_n).\nabla\varphi\, dx\,dt$$
$$= \int_{Q_T} g_n(u_n) f_n \varphi\, dx\,dt, \tag{3.4}$$

for all $n \in \mathbb{N}$ *fixed and for every* $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$, *where*

$$\Big\langle \frac{\partial u_n}{\partial t}, \varphi \Big\rangle = \int_\Omega \frac{\partial u_n}{\partial t} \varphi\, dx.$$

Proof Let $n \in \mathbb{N}$ and $v \in L^p(Q_T)$ be fixed. Consider the nonlinear parabolic problem

$$\begin{cases} \frac{\partial w}{\partial t} - \operatorname{div}(b(x,t,T_n(w),\nabla w)) = g_n(v) f_n & \text{in } Q_T, \\ w(x,0) = 0 & \text{on } \Omega, \\ w = 0 & \text{on } \partial Q_T, \end{cases} \tag{3.5}$$

it is clear that the problem (3.5) has a unique solution w with

$$w \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T) \text{ and } \frac{\partial w}{\partial t} \in L^{p'}(0,T;W_0^{-1,p'}(\Omega)) + L^1(Q_T).$$

Since the right-hand side on (3.5) belongs to $L^\infty(Q_T)$ see for instance [23, 24]. In particular, it is well defined a map $P : L^p(Q_T) \to L^p(Q_T)$ where $P(v) = w$. By the boundedness of the sequence $\{g_n(v)f_n\}_n$ in $L^\infty(Q_T)$, we have that $w \in L^\infty(Q_T)$ (see for example [14]), then, there exists $C_\infty > 0$, independents of v, w (but possibly depending in n), such that

$$\|w\|_{L^\infty(Q_T)} \le C_\infty. \tag{3.6}$$

Our aim is to prove the existence of fixed point of the map P. Using w as test function in (3.5), one gets

$$\frac{1}{2}\int_\Omega |w(T)|^2 dx + \int_{Q_T} b(x, t, T_n(w), \nabla w).\nabla w dxdt = \int_{Q_T} g_n(v) f_n w dxdt. \tag{3.7}$$

By (1.2), (1.6) and dropping a positive term on the left-hand side in (3.7)

$$\alpha \int_{Q_T} \frac{|\nabla w|^p}{(1+|T_n(w)|)^\theta} dxdt \le n^{\gamma+1} \int_{Q_T} |w| dxdt. \tag{3.8}$$

Using the Hölder's inequality on the right-hand side in (3.8), we have

$$\int_{Q_T} |\nabla w|^p dxdt \le \frac{C_1}{\alpha} n^{\gamma+1}(1+n)^\theta |Q_T|^{\frac{1}{p'}} \left(\int_{Q_T} |w|^p dxdt\right)^{\frac{1}{p}}$$

Poincaré inequality imply

$$\|w\|_{L^p(Q_T)} \le C(n, |Q_T|), \tag{3.9}$$

for some constant $C(n, |Q_T|)$ independent of v and w (possible depending on n). Let B is a ball of $L^p(Q_T)$ of radius $C(n, |Q_T|)$ is invariant for the map P. Now, we prove that the map P is continuous in B. Let $\{v_h\}_n$ be a bounded sequence in B. By (3.9) there exist a subsequence of $\{v_h\}_n$ still denoted by $\{v_h\}_n$, and a measurable function v belonging to $L^p(Q_T)$, such that

$$v_h \to v \quad \text{strongly in } L^p(Q_T). \tag{3.10}$$

Let us choose w_h as a test function in the weak formulation of the problem solved by w_h, (3.8) implies that

$$\int_0^T \|\nabla w_h\|^p_{L^p(\Omega)} dt \le C_4 \int_0^T \left(\int_\Omega |w_h|^p dx\right)^{\frac{1}{p}} dt. \tag{3.11}$$

Since the ball of $L^p(Q_T)$ is invariant for P, we have w_h belong to B and so, from the inequality (3.11), we obtain that w_h is bounded in $L^p(0, T; W_0^{1,p}(\Omega))$. The growth assumption (1.4), implies

$$\begin{aligned}\int_{Q_T} |b(x, t, w_h, \nabla w_h)|^{p'} dxdt &\le \int_{Q_T} \left[|a(x,t)| + |w_h|^{p-1} + |\nabla w_h|^{p-1}\right]^{\frac{p}{p-1}} dxdt \\ &\le \|k\|^{p'}_{L^{p'}(Q_T)} + \|w_h\|^p_{L^p(Q_T)} + \|w_h\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \\ &< +\infty.\end{aligned}$$

Using the previous inequality with (3.5), and the fact that $g_n(v)f_n \in L^1(Q_T)$ we have $\frac{\partial w_h}{\partial t}$ is bounded in $L^{p'}(0,T;W_0^{-1,p'}(\Omega)) + L^1(Q_T)$. As a result of the Corollary 4 in [34], we can conclude that w_h is relatively strongly compact in $L^1(Q_T)$. Thus, there exists a subsequence of w_h still denoted by w_h, and a measurable function w belonging to $L^1(Q_T)$ such that

$$w_h \to w \quad \text{a.e in } \; L^1(Q_T). \tag{3.12}$$

By (3.9), (3.12) and Lebesgue Theorem we have that w_h converges strongly to w in $L^p(Q_T)$, and so P is compact.

Now we prove that P is continuous. Let $w_h = P(v_h)$, (3.10) implies that $v_h \to v$ a.e in Q_T, hence $g_n(v_h)f_n$ converges to $g_n(v)f_n$ a.e in Q_T and by the dominated convergence theorem one has that $g_n(v_h)f_n$ converge strongly to $g_n(v)f_n$ in $L^p(Q_T)$. Hence, by uniqueness, one deduce that $w_h = P(v_h)$ converges to $w = P(v)$ in $L^p(Q_T)$. This gives the continuity of S. Using Schauder's fixed point theorem for every fixed n, we have there exist u_n in $L^p(0,T;W_0^{1,p}(\Omega)) \cap C([0,T];L^2(\Omega))$ and $\frac{\partial u_n}{\partial t} \in L^{p'}(0,T;W_0^{-1,p'}(\Omega)) + L^1(Q_T)$, such that $u_n = P(u_n)$.

Choosing $\varphi = -u_n^- = -u_n\chi_{\{u_n\le 0\}}$, where $\chi_{\{u_n\le 0\}}$ denotes the characteristic function of $\{(x,t)\in Q_T : u_n(x,t)\le 0\}$ as a test function in (3.1). Using (1.2), and recalling that $g_n(u_n)f_n$ is nonnegative, we obtain

$$-\frac{1}{2}\int_\Omega |u_n^-|^2 dx - \frac{\alpha}{(1+n)^\theta}\int_{Q_T} |\nabla u_n^-|^p dxdt \ge -\int_{Q_T} g_n(u_n)f_n u_n^- dxdt \ge 0,$$

dropping the term $-\frac{1}{2}\int_\Omega |u_n^-|^2 dx$, we have

$$-\int_{Q_T} |\nabla u_n^-|^p dxdt \ge 0,$$

so that $\|u_n^-\|_{L^p(0,T;W_0^{1,p}(\Omega))} = 0$, thus $u_n \ge 0$ almost everywhere in Q_T. □

4 A Priori Estimates

We shall denote by C_i, $i = 1, ..., N$ various constants depending only on the structure of p, θ, γ, T, $|\Omega|$. Let $u_n \in L^p(0,T;W_0^{1,p}(\Omega)) \cap C([0,T];L^2(\Omega))$ be a solution to problem (3.1). In this section, we prove some uniform estimates for the sequence $\{u_n\}_n$ and $\left\{\frac{\partial u_n}{\partial t}\right\}_n$.

Lemma 4.1 *Assume that (1.2)-(1.6), $p-2\le\gamma<1$ hold true. Let $f\in L^m(Q_T)$ with $m > \frac{N}{p}+1$. Then, the sequence $\{u_n\}_n$ is bounded in $L^\infty(Q_T)\cap L^p(0,T;W_0^{1,p}(\Omega))$ and $\left\{\frac{\partial u_n}{\partial t}\right\}_n$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^m(Q_T)$.*

Proof For every $\tau\in(0,T]$, we take $\varphi(u_n) = [(1+u_n)^{p-1}-1]G_k'(u_n)\chi_{(0,\tau)}$ as a test function in (3.4), we use the assumption (1.2), and the fact that

$$\Phi(u_n) = \int_0^{u_n} \left((1+y)^{p-1}-1\right) G_k'(y)dy \ge \frac{1}{p}G_k(u_n)^p G_k'(u_n),$$

we obtain

$$\int_{\Omega} \Phi(u_n)dx + \alpha(p-1)\int_0^{\tau}\int_{\Omega} \frac{|\nabla u_n|^p}{(1+u_n)^{\theta}}(1+u_n)^{p-2}G_k'(u_n)dxdt$$
$$\leq \int_0^{\tau}\int_{\Omega} f_n g_n(u_n)[(1+u_n)^{p-1}-1]G_k'(u_n)dxdt.$$

By (3.3), (1.9), $(1+u_n)^{p-1}-1 \leq (1+u_n)^{p-1}$ and the fact that

$$\int_{Q_T\cap\{u_n=0\}} f_n g_n(u_n)[(1+u_n)^{p-1}-1]G_k'(u_n)dxdt$$
$$\leq \int_{Q_T} f_n \lim_{z\to 0} g(z)[(1+0)^{p-1}-1]G_k'(0)dxdt = 0,$$

we have

$$\int_{\Omega} \Phi(u_n)dx + \alpha(p-1)\int_0^{\tau}\int_{\Omega} \frac{|\nabla u_n|^p}{(1+u_n)^{\theta}}(1+u_n)^{p-2}G_k'(u_n)dxdt$$
$$\leq \int_{Q_T\cap\{u_n>0\}} f_n \frac{(1+u_n)^{p-1}}{u_n^{\gamma}} G_k'(u_n)dxdt$$
$$+ \int_{Q_T\cap\{u_n=0\}} f_n g_n(u_n)[(1+u_n)^{p-1}-1]G_k'(u_n)dxdt$$
$$\leq \int_0^T\int_{\Omega} f_n(1+u_n)^{p-1-\gamma}G_k'(u_n)dxdt.$$

Using Hölder's inequality on the right-hand side of the previous inequality, (3.2) and the fact that $1+u_n \leq 2(k+G_k(u_n))$ as $k \geq 1$, one has

$$\int_{E_{k,n}(\tau)} G_k(u_n(\tau))^p dx + \int_0^{\tau}\int_{E_{k,n}(t)} \frac{|\nabla u_n|^p}{(1+u_n)^{\theta-p+2}}dxdt$$
$$\leq C_1\|f\|_{L^m(Q_T)}\left(\int_0^T\int_{E_{k,n}(t)} (k+G_k(u_n))^{(p-1-\gamma)m'}dxdt\right)^{\frac{1}{m'}},$$

where $E_{k,n}(t) = \{x \in \Omega : u_n(x,t) > k\}$, $t \in (0,T)$. Hence

$$\|G_k(u_n)\|^p_{L^{\infty}(0,T;L^p(E_{k,n}))} + \int_0^T\int_{E_{k,n}(t)} \frac{|\nabla u_n|^p}{(1+u_n)^{\theta-p+2}}dxdt$$
$$\leq C_1\|f\|_{L^m(Q_T)}\left(\int_0^T\int_{E_{k,n}(t)} (k+G_k(u_n))^{(p-1-\gamma)m'}dxdt\right)^{\frac{1}{m'}}. \tag{4.1}$$

The proof is divided into two cases.

Case 1: Suppose that

$$p-2 < \theta < p-1+\frac{p}{N}+\gamma(1+\frac{p}{N}).$$

For all $1 \le p-1 < \sigma < p$, Writing

$$\int_{Q_T} |\nabla G_k(u_n)|^\sigma \,dxdt = \int_{Q_T} \frac{|\nabla u_n|^\sigma}{(1+u_n)^{\frac{(\theta-p+2)\sigma}{p}}} (1+u_n)^{\frac{(\theta-p+2)\sigma}{p}} \,dxdt.$$

Using (3.2), (4.1) and Hölder's inequality, we have

$$\begin{aligned}
&\int_{Q_T} |\nabla G_k(u_n)|^\sigma \,dxdt \\
&\le \left(\int_0^T \int_{E_{k,n}(t)} \frac{|\nabla G_k(u_n)|^p}{(1+u_n)^{\theta-p+2}} dxdt \right)^{\frac{\sigma}{p}} \\
&\quad \times \left(\int_0^T \int_{E_{k,n}(t)} (1+u_n)^{\frac{(\theta-p+2)\sigma}{p-\sigma}} \,dxdt \right)^{\frac{p-\sigma}{p}} \\
&\le C_2 \left(\int_0^T \int_{E_{k,n}(t)} (k+G_k(u_n))^{m'(p-1-\gamma)} dxdt \right)^{\frac{\sigma}{pm'}} \\
&\quad \times \left(\int_0^T \int_{E_{k,n}(t)} (k+G_k(u_n))^{\frac{(\theta-p+2)\sigma}{p-\sigma}} \,dxdt \right)^{\frac{p-\sigma}{p}}.
\end{aligned} \tag{4.2}$$

Choosing $\sigma = \frac{2pN-2N+p^2-N\theta}{N+p}$, this choice of σ, implies that $p-1 < \sigma < p$ and $0 < \frac{(\theta-p+2)\sigma}{p-\sigma} = \frac{(N+p)\sigma}{N}$. By (4.2), we deduce that

$$\begin{aligned}
&\int_{Q_T} |\nabla G_k(u_n)|^\sigma \,dxdt \\
&\le C_2 \left(\int_0^T \int_{E_{k,n}(t)} (k+G_k(u_n))^{m'(p-1-\gamma)} dxdt \right)^{\frac{\sigma}{pm'}} \\
&\quad \times \left(\int_0^T \int_{E_{k,n}(t)} (k+G_k(u_n))^{\frac{(N+p)\sigma}{N}} \,dxdt \right)^{\frac{p-\sigma}{p}}.
\end{aligned} \tag{4.3}$$

From Lemma 1.2 (here $v = G_k(u_n)$, $\kappa = \sigma$, $\varrho = p$), (4.1) and (4.3), we obtain

$$\begin{aligned}
&\int_0^T \int_{E_{k,n}(t)} G_k(u_n)^{\frac{(N+p)\sigma}{N}} \,dxdt \\
&\le \left(\|G_k(u_n)\|^p_{L^\infty(0,T;L^p(E_{k,n}))} \right)^{\frac{\sigma}{N}} \int_0^T \int_{E_{k,n}(t)} |\nabla G_k(u_n)|^\sigma \,dxdt \\
&\le C_2 \left(\int_0^T \int_{E_{k,n}(t)} (k+G_k(u_n))^{m'(p-1-\gamma)} dxdt \right)^{\frac{(N+p)\sigma}{pNm'}} \\
&\quad \times \left(\int_0^T \int_{E_{k,n}(t)} (k+G_k(u_n))^{\frac{(N+p)\sigma}{N}} \,dxdt \right)^{\frac{p-\sigma}{p}}.
\end{aligned} \tag{4.4}$$

Since

$$m > \frac{N}{p} + 1, \quad \text{and} \quad \sigma > p - 1 > p - 1 - \gamma, \tag{4.5}$$

then we have $\frac{\sigma(N+p)}{m'N(p-1-\gamma)} > 1$. Thus, using Hölder's inequality, we have

$$\begin{aligned} &\int_0^T \int_{E_{k,n}(t)} (k + G_k(u_n))^{(p-1-\gamma)m'} dxdt \\ &\leq \left(\int_0^T \int_{E_{k,n}(t)} (k + G_k(u_n))^{\frac{\sigma(N+p)}{N}} dxdt \right)^{\frac{(p-1-\gamma)m'N}{(N+p)\sigma}} \\ &\quad \times \left(\int_0^T |E_{k,n}(t)| \, dt \right)^{1 - \frac{(p-1-\gamma)m'N}{(N+p)\sigma}}. \end{aligned} \tag{4.6}$$

We denote by

$$\Lambda_n(k) = \int_0^T |E_{k,n}(t)| dt, \quad \mathbf{G}_{nk} = \int_0^T \int_{E_{k,n}(t)} G_k(u_n)^{\frac{(N+p)\sigma}{N}} dxdt.$$

From (4.6) and (4.4), we can write for all $k \geq 1$

$$\begin{aligned} \mathbf{G}_{nk} &\leq C_3 \mathbf{G}_{nk}^{\frac{2p-1-\sigma-\gamma}{p}} \Lambda_n(k)^{\frac{(N+p)\sigma}{pNm'} - \frac{p-1-\gamma}{p}} \\ &\quad + C_3 k^{\frac{(N+p)\sigma}{N}\left(\frac{2p-1-\sigma-\gamma}{p}\right)} \Lambda_n(k)^{\frac{(N+p)\sigma}{pNm'} + \frac{p-\sigma}{p}}, \end{aligned} \tag{4.7}$$

(4.5) implies that $\frac{2p-1-\sigma-\gamma}{p} < 1$, then, by Young's inequality for all $\varepsilon > 0$,

$$\begin{aligned} \mathbf{G}_{nk}^{\frac{2p-1-\sigma-\gamma}{p}} \Lambda_n(k)^{\frac{(N+p)\sigma}{pNm'} - \frac{p-1-\gamma}{p}} &\leq C(\varepsilon) \Lambda_n(k)^{\left(\frac{(N+p)\sigma}{Nm'} - (p-1-\gamma)\right)\frac{1}{\sigma+1-p+\gamma}} \\ &\quad + \varepsilon \mathbf{G}_{nk}. \end{aligned} \tag{4.8}$$

Taking $\varepsilon = \frac{1}{2C_3}$ in (4.8) and applying (4.7) to (4.8), we get

$$\begin{aligned} \mathbf{G}_{nk} &\leq C_4 \Lambda_n(k)^{\frac{(N+p)\sigma - Nm'(p-1-\gamma)}{Nm'(\sigma+1-p+\gamma)}} \\ &\quad + C_4 k^{\frac{(N+p)\sigma}{N}\left(\frac{2p-1-\sigma-\gamma}{p}\right)} \Lambda_n(k)^{\frac{(N+p)\sigma + Nm'(p-\sigma)}{pNm'}}. \end{aligned} \tag{4.9}$$

The assumption $m > \frac{N}{p} + 1$ implies

$$\frac{(N+p)\sigma - Nm'(p-1-\gamma)}{Nm'(\sigma+1-p+\gamma)} > \frac{(N+p)\sigma + Nm'(p-\sigma)}{pNm'} > 1.$$

We note that $|\Lambda_n(k)| \leq T|\Omega|$, $k \geq 1$, and so

$$\mathbf{G}_{nk} \leq C_5 k^{\frac{\sigma(N+p)}{N}\left(\frac{2p-1-\sigma-\gamma}{p}\right)} \Lambda_n(k)^{\frac{(N+p)\sigma + Nm'(p-\sigma)}{pNm'}}. \tag{4.10}$$

Since $G_k(u_n) > h-k$ on $E_{h,n}(t)$ if $h > k$ and $E_{h,n}(t) \subset E_{k,n}(t)$. By virtue of $\frac{2p-1-\sigma-\gamma}{p} < 1$, (4.10) can be written as

$$\Lambda_n(h) \le \frac{C_5 k^{\frac{\sigma(N+p)}{N}\left(\frac{2p-1-\sigma-\gamma}{p}\right)} \Lambda_n(k)^{\frac{(N+p)\sigma}{pNm'}+\frac{p-\sigma}{p}}}{(h-k)^{\frac{\sigma(N+p)}{N}}}, \quad \forall h > k \ge 1. \tag{4.11}$$

Lemma 1.1 applied to

$$\rho = \frac{2p-1-\sigma-\gamma}{p}, \quad \nu = \frac{(N+p)\sigma}{N}, \quad \text{and} \quad \vartheta = \frac{(N+p)\sigma}{pNm'} + \frac{p-\sigma}{p},$$

we have, there exists a positive constant l such that $\Lambda_n(l) = 0$. By the fact that $|\Lambda_n(k)| \le T|\Omega|$ (see the proof of Lemma A.1 of [6]), there exists a positive constant d_0 independent of n such that $l \le d_0$, so that

$$\Lambda_n(d_0) = 0. \tag{4.12}$$

Therefore, from (4.12), it follows that the sequence $\{u_n\}_n$ is bounded in $L^\infty(Q_T)$.

Case 2: Suppose that $0 \le \theta \le p-2$. By (4.1), we can write

$$\begin{aligned}\int_{Q_T} |\nabla G_k(u_n)|^p dxdt &\le \int_0^T \int_{E_{n,k}(t)} |\nabla u_n|^p (1+u_n)^{p-2-\theta} dxdt \\ &\le C_6 \left(\int_0^T \int_{E_{n,k}(t)} (k + G_k(u_n))^{(p-1-\gamma)m'} dxdt \right)^{\frac{1}{m'}}.\end{aligned}$$

From Lemma 1.2 (here $v = G_k(u_n)$, $\kappa = \varrho = p$), the previous inequality and (4.1) gives

$$\begin{aligned}&\int_0^T \int_{E_{k,n}(t)} G_k(u_n)^{\frac{(N+p)p}{N}} dxdt \\ &\le \left(\|G_k(u_n)\|^p_{L^\infty(0,T;L^p(E_{k,n}))} \right)^{\frac{p}{N}} \int_0^T \int_{E_{k,n}(t)} |\nabla G_k(u_n)|^p dxdt \\ &\le C_7 \left(\int_0^T \int_{E_{k,n}(t)} (k + G_k(u_n))^{m'(p-1-\gamma)} dxdt \right)^{\frac{(N+p)}{Nm'}}.\end{aligned}$$

By Hölder's inequality with exponent $\frac{(N+p)p}{Nm'(p-1-\gamma)} > 1$ (since $m > \frac{N}{p} + 1$), we have

$$\begin{aligned}&\int_0^T \int_{E_{k,n}(t)} G_k(u_n)^{\frac{(N+p)p}{N}} dxdt \\ &\le C_8 \left(\int_0^T \int_{E_{k,n}(t)} (k + G_k(u_n))^{\frac{(N+p)p}{N}} dxdt \right)^{\frac{p-1-\gamma}{p}} \Lambda_n(k)^{\frac{p+N}{Nm'} - \frac{p-1-\gamma}{p}}.\end{aligned}$$

Therefore, (4.11) holds true for $\sigma = p$

$$\Lambda_n(h) \le \frac{C_9 k^{\frac{(N+p)p}{N} \cdot \frac{p-1-\gamma}{p}} \Lambda_n(k)^{\frac{(N+p)}{Nm'}}}{(h-k)^{\frac{(N+p)p}{N}}}, \quad \forall h > k \ge 1.$$

Using that $\frac{p-1-\gamma}{p} \in (0,1)$ (since $p-2 \le \gamma < 1$) and that $\frac{(N+p)}{Nm'} > 1$, thus u_n is bounded in $L^\infty(Q_T)$.

Now, choosing u_n as a test function for problem (3.4). Using (1.2) and (1.6), we obtain

$$\frac{1}{2}\int_\Omega u_n(T)^2 dx + \alpha \int_{Q_T} \frac{|\nabla u_n|^p}{(1+u_n)^\theta} dxdt \le \int_{Q_T} f_n u_n^{1-\gamma} dxdt. \tag{4.13}$$

Dropping the non-negative term, by Hölder's inequality on the right-hand side of the inequality (4.13), and the boundedness of the sequence $\{u_n\}_n$ in $L^\infty(Q_T)$, we obtain

$$\begin{aligned}\int_{Q_T} \frac{|\nabla u_n|^p}{(1+u_n)^\theta} dxdt &\le \|f\|_{L^m(Q_T)} |Q_T|^{\frac{1}{m'}} \|u_n^{1-\gamma}\|_{L^\infty(Q_T)} \\ &\le C_{10} \|f\|_{L^m(Q_T)}.\end{aligned} \tag{4.14}$$

Therefore, from (4.14) and since the sequence $\{u_n\}_n$ is bounded in $L^\infty(Q_T)$, we get

$$\begin{aligned}\int_{Q_T} |\nabla u_n|^p dxdt &= \int_{Q_T} \frac{|\nabla u_n|^p}{(1+u_n)^\theta} (1+u_n)^\theta dxdt \\ &\le \left(1 + \|u_n\|_{L^\infty(Q_T)}\right)^\theta \int_{Q_T} \frac{|\nabla u_n|^p}{(1+u_n)^\theta} dxdt \\ &\le C_{11}.\end{aligned}$$

Consequently the sequence $\{u_n\}_n$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$. By (3.1), (3.2) and the boundedness of the sequence $\{u_n\}_n$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we obtain the sequence $\left\{\frac{\partial u_n}{\partial t}\right\}_n$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^m(Q_T)$. □

Lemma 4.2 *Let $m = \frac{N}{p} + 1$ and let (1.2)-(1.6) hold. Then, the sequence $\{u_n\}_n$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^r(Q_T)$ for every $r \in [p,+\infty)$.*

Proof For $\tau \in (0,T]$, if we take $\varphi(u_n) = (1+u_n)^{(p-1)\mu} - 1$ as a test function in (3.4), with

$$\mu \ge \frac{1+\theta}{p-1}. \tag{4.15}$$

By assumptions (1.2), (1.6), we obtain

$$\begin{aligned}&\int_\Omega \Psi(u_n(x,\tau))dx + (p-1)\mu\alpha \int_0^\tau \int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^\theta} (1+u_n)^{(p-1)\mu-1} dxdt \\ &\le \int_0^\tau \int_\Omega f_n u_n^{(p-1)\mu-\gamma} dxdt,\end{aligned} \tag{4.16}$$

where

$$\Psi(s) = \int_0^\zeta \varphi(y)dy \ge \frac{\zeta^{(p-1)\mu+1}}{(p-1)\mu+1} \quad \forall \zeta > 0,\ \mu > 1. \tag{4.17}$$

Thus, (4.16), (4.17), (3.2), and Hölder's inequality on the last integral, we get

$$\frac{1}{(p-1)\mu+1}\int_\Omega u_n(x,\tau)^{(p-1)\mu+1}dx + (p-1)\mu\alpha\int_0^\tau\int_\Omega |\nabla u_n|^p(1+u_n)^{(p-1)\mu-1-\theta}dxdt \le \|f\|_{L^m(Q_T)}\left(\int_{Q_T} u_n^{((p-1)\mu-\gamma)m'}dxdt\right)^{\frac{1}{m'}}. \tag{4.18}$$

The condition (4.15) implies that $(p-1)\mu+p-1-\theta \ge p$, so (4.18) and (3.2) yield

$$ess\sup_{t\in[0,T]}\int_\Omega\left[u_n(x,\tau)^{\frac{(p-1)\mu+p-1-\theta}{p}}\right]^{\frac{p((p-1)\mu+1)}{(p-1)\mu+p-1-\theta}}dx + \int_{Q_T}\left|\nabla u_n^{\frac{(p-1)\mu+p-1-\theta}{p}}\right|^p dxdt \le C_{12}\left(\int_{Q_T} u_n^{((p-1)\mu-\gamma)m'}dxdt\right)^{\frac{1}{m'}} + C_{12}. \tag{4.19}$$

Thus, by the Gagliardo-Nirenberg inequality (1.7), where

$$v(x,t) = u_n(x,t)^{\frac{(p-1)\mu+p-1-\theta}{p}}, \quad \varrho = \frac{p((p-1)\mu+1)}{(p-1)\mu+p-1-\theta}, \quad \kappa = p,$$

we have

$$\int_{Q_T}\left[u_n^{\frac{(p-1)\mu+p-1-\theta}{p}}\right]^{p\frac{N+\frac{p((p-1)\mu+1)}{(p-1)\mu+p-1-\theta}}{N}}dxdt \le M_2\left(ess\sup_{0\le t\le T}\int_\Omega\left[u_n(x,t)^{\frac{(p-1)\mu+p-1-\theta}{p}}\right]^{\frac{p((p-1)\mu+1)}{(p-1)\mu+p-1-\theta}}dx\right)^{\frac{p}{N}} \times \int_{Q_T}\left|\nabla u_n^{\frac{(p-1)\mu+p-1-\theta}{p}}\right|^p dxdt. \tag{4.20}$$

By (4.19)-(4.20) and taking

$$s = \frac{p((p-1)\mu+1)+N((p-1)\mu+p-1-\theta)}{N},$$

we obtain

$$\int_{Q_T} u_n^s dxdt \le C_{13}\left(\int_{Q_T} u_n^{((p-1)\mu-\gamma)m'}dxdt\right)^{\frac{N+p}{Nm'}} + C_{13}. \tag{4.21}$$

Since $\frac{N+p}{Nm'} = 1$ and by (1.3), we obtain

$$((p-1)\mu-\gamma)m' = ((p-1)\mu-\gamma)\frac{N+p}{N} < s. \tag{4.22}$$

From (4.22), Hölder's inequality and Young's inequality with $\varepsilon > 0$, we deduce that

$$\begin{aligned}\int_{Q_T} u_n^{\frac{((p-1)\mu-\gamma)(N+p)}{N}} dxdt &\leq C_{14}\left(\int_{Q_T} u_n^s dxdt\right)^{\frac{((p-1)\mu-\gamma)(N+p)}{sN}} \\ &\leq \varepsilon \int_{Q_T} |u_n|^s dxdt + c(\varepsilon).\end{aligned} \tag{4.23}$$

By (4.21), (4.23) and letting $\varepsilon = \frac{1}{2C_{13}}$, we get

$$\int_{Q_T} |u_n|^s dxdt \leq C_{15}. \tag{4.24}$$

On the other hand, since

$$\mu \geq \frac{1+\theta}{p-1} \geq \frac{N(\theta+1)-p}{(p-1)(p+N)},$$

and

$$s \geq p \Leftrightarrow \mu \geq \frac{N(\theta+1)-p}{(p-1)(p+N)},$$

therefore, from (4.24) with $r = s$, it follows that the sequence $\{u_n\}_n$ is bounded in $L^r(Q_T)$. Inequalities (4.18), (4.23), and the boundedness of the sequence $\{u_n\}_n$ in $L^r(Q_T)$, imply then

$$\int_{Q_T} |\nabla u_n|^p dxdt \leq \int_{Q_T} |\nabla u_n|^p (1+u_n)^{(p-1)\mu-1-\theta} dxdt \leq C_{16}. \tag{4.25}$$

Thus from (4.25) immediately follows the boundedness of the sequence $\{u_n\}_n$ in $L^p(0, T; W_0^{1,p}(\Omega))$. □

Lemma 4.3 *Let $f \in L^m(Q_T)$, with m satisfies (2.2), and (1.2)-(1.6) hold. Then, the sequence $\{u_n\}_n$ is bounded in $L^\delta(Q_T) \cap L^p(0, T; W_0^{1,p}(\Omega))$, where δ as in (2.3).*

Proof We put

$$((p-1)\mu-\gamma)m' = \frac{p((p-1)\mu+1)+N((p-1)\mu+p-1-\theta)}{N}, \tag{4.26}$$

in the proof of Lemma 4.2. By (2.2) and (4.26) we get

$$\mu = \frac{(m-1)(p+N(p-1-\theta))+\gamma m N}{(p-1)(N+p-pm)} \geq \frac{1+\theta}{p-1}.$$

Consequently

$$\delta = ((p-1)\mu-\gamma)\frac{m}{m-1} = \frac{m(p+N(p-1-\theta)+\gamma(N+p))}{N+p-pm}. \tag{4.27}$$

Note that $\frac{N+p}{Nm'} < 1$. The Young's inequality gives

$$\left(\int_{Q_T} u_n^{((p-1)\mu-\gamma)m'} dxdt\right)^{\frac{N+p}{Nm'}} \leq \varepsilon \int_{Q_T} u_n^{((p-1)\mu-\gamma)m'} dxdt + c(\varepsilon). \tag{4.28}$$

Taking (4.28) in (4.21) and letting $\varepsilon = \frac{1}{2C_{13}}$, by (4.26) and (4.27), we deduce that the sequence $\{u_n\}_n$ is bounded in $L^{\delta}(Q_T)$. The rest of the proof is the same way in proof of Lemma 4.2. □

Lemma 4.4 *Let f belongs to $L^m(Q_T)$, with m satisfies (2.4), and (1.2)-(1.6) hold. Then, the sequence $\{u_n\}_n$ is bounded in $L^{\delta}(Q_T) \cap L^q(0, T; W_0^{1,q}(\Omega))$, where δ and q are defined in Theorem 2.5.*

Proof Suppose that

$$0 < \mu < \frac{1+\theta}{p-1}.$$

Let φ and Ψ as in (4.17). Choosing $\varphi(u_n(x,t))\chi_{(0,\tau)}(t)$ as a test function in (3.4). Using the fact that

$$\Psi(\zeta) \geq c\zeta^{\mu(p-1)+1} - c \quad \forall s \in \mathbb{R}_+,$$

we have

$$\begin{aligned} &ess \sup_{0\leq t\leq \tau} \int_{\Omega} u_n(x,\tau)^{(p-1)\mu+1} dx + \int_{Q_T} \frac{|\nabla u_n|^p}{(1+u_n)^{\theta+1-(p-1)\mu}} dxdt \\ &\leq C_{17} \left(\int_{Q_T} u_n^{((p-1)\mu-\gamma)m'} dxdt\right)^{\frac{1}{m'}} + C_{17}. \end{aligned} \tag{4.29}$$

Using Hölder's inequality with the exponents $\frac{p}{q}$ and (4.29), we obtain

$$\begin{aligned} \int_{Q_T} |\nabla u_n|^q dxdt &\leq \left(\int_{Q_T} \frac{|\nabla u_n|^p}{(1+u_n)^{\theta+1-(p-1)\mu}} dxdt\right)^{\frac{q}{p}} \\ &\quad \times \left(\int_{Q_T} (1+u_n)^{\frac{(\theta+1-(p-1)\mu)q}{p-q}} dxdt\right)^{\frac{p-q}{p}} \\ &\leq C_{18} \left(\left(\int_{Q_T} u_n^{((p-1)\mu-\gamma)m'} dxdt\right)^{\frac{1}{m'}} + 1\right)^{\frac{q}{p}} \\ &\quad \times \left(\int_{Q_T} u_n^{\frac{(\theta+1-(p-1)\mu)q}{p-q}} dxdt + 1\right)^{\frac{p-q}{p}}. \end{aligned} \tag{4.30}$$

Now we take μ, such that

$$\mu = \frac{(\theta+1)q + \gamma m'(p-q)}{(p-1)((p-q)m'+q)} < \frac{1+\theta}{p-1},$$

hence

$$((p-1)\mu-\gamma)m' = \frac{(\theta+1-(p-1)\mu)q}{p-q}. \tag{4.31}$$

Then, by the inequality (4.30), we get

$$\int_{Q_T} |\nabla u_n|^q dxdt \le C_{19} \left(\int_{Q_T} u_n^{((p-1)\mu-\gamma)m'dxdt} \right)^{\frac{q}{pm'}+\frac{p-q}{p}} + C_{19}. \tag{4.32}$$

Applying Lemma 1.2 (here $v(x,t) = u_n(x,t)$, $\varrho = (p-1)\mu+1$, $\kappa = q$) and from (4.29), (4.32), we have

$$\begin{aligned}
&\int_{Q_T} u_n^{\frac{(N+(p-1)\mu+1)q}{N}} dxdt \\
&\le \left(ess \sup_{0\le t\le T} \int_\Omega u_n(x,t)^{(p-1)\mu+1} dx \right)^{\frac{q}{N}} \int_{Q_T} |\nabla u_n|^q dxdt \\
&\le C_{20} \left(\int_{Q_T} u_n^{((p-1)\mu-\gamma)m'} dxdt \right)^{\frac{q(N+p)}{pNm'}+\frac{p-q}{p}} + C_{20}.
\end{aligned} \tag{4.33}$$

Set

$$((p-1)\mu-\gamma)\frac{m}{m-1} = \frac{(N+(p-1)\mu+1)q}{N},$$

then, by (4.31) we obtain

$$\mu = \frac{(m-1)(p+N(p-1-\theta))+m\gamma N}{(p-1)(N+p-pm)}, \tag{4.34}$$

and

$$q = \frac{m[N(p-\theta-1)+p+\gamma(N+p)]}{N+1-(\theta+1)(m-1)+m\gamma}. \tag{4.35}$$

By (4.34), (4.35) and (4.33), we have $(p-1)\mu m' = \delta$ and

$$\int_{Q_T} u_n^\delta dxdt \le C_{20} \left(\int_{Q_T} u_n^\delta dxdt \right)^{\frac{q(N+p)}{pNm'}+\frac{p-q}{p}} + C_{20}. \tag{4.36}$$

Since $\frac{q(N+p)}{pNm'} + \frac{p-q}{p} < 1$, then from (4.36), we deduce that the sequence $\{u_n\}_n$ is bounded in $L^\delta(Q_T)$. Going back to (4.32) and (4.35), this in turn implies that the sequence $\{u_n\}_n$ is bounded in $L^q(0,T;W_0^{1,q}(\Omega))$. □

5 Proof of Main Results

5.1 Proof of Theorem 2.2

In virtue of Lemma 4.1, we have the sequence $\{u_n\}_n$ is bounded in $L^\infty(Q_T) \cap L^p(0,T; W_0^{1,p}(\Omega))$. Then, there exists a function $u \in L^\infty(Q_T) \cap L^p(0,T;W_0^{1,p}(\Omega))$, such that, up to subsequence,

$$u_n \rightharpoonup u \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)),$$
$$u_n \rightharpoonup u \quad \text{weakly}^*\text{in } L^\infty(Q_T) \text{ for } \sigma^*(L^\infty(Q_T),L^1(Q_T)).$$

In view of Lemma 4.1, we have that the sequence $\left\{\frac{\partial u_n}{\partial t}\right\}_n$ is bounded in $L^1(Q_T)\cap L^{p'}(0,T;W_0^{-1,p'}(\Omega))$. So, using compactness results (corollary 4 of [34]) we obtain $\{u_n\}_n$ is relatively compact in $L^1(Q_T)$. This implies that

$$u_n \to u \quad \text{strongly in } L^1(Q_T), \text{ and a.e. in } Q_T. \tag{5.1}$$

To carry on the proof, we need the following Lemma.

Lemma 5.1 [23] *For all $k>0$, there exists a function θ_k such that for all $\varepsilon>0$, we have*

$$\limsup_n \int_{\{|u_n-u^k|\le\varepsilon\}} a(x,t,T_n(u_n),\nabla u_n)(\nabla u_n-\nabla u^k)dxdt \le \theta_k(\varepsilon),$$

with $\lim \theta_k(\varepsilon)=0$, $u^k=\phi_k(u)$.

By Lemma 5.1, we can adopt the approach of [22, 26], we deduce that there exists a subsequence, still denoted by $\{u_n\}_n$, such that

$$\nabla u_n \to \nabla u \quad \text{almost everywhere in } Q_T. \tag{5.2}$$

From (5.1), (5.2) and the fact that b is Carathéodory function, we obtain

$$b(x,t,u_n,\nabla u_n) \to b(x,t,u,\nabla u) \quad \text{almost everywhere in } Q_T. \tag{5.3}$$

By (5.3), and Vitali's theorem, one has

$$a(x,t,u_n,\nabla u_n) \to a(x,t,u,\nabla u) \quad \text{strongly in } L^{p'}(Q_T). \tag{5.4}$$

We begin by proving an important lemma useful to prove of Theorem 2.2-2.5.

Lemma 5.2 *Let u_n be a weak solution of (3.1). Then*

$$\lim_{n\to+\infty}\int_{Q_T} g_n(u_n)f_n\psi\,dxdt=\int_{Q_T} g(u)f\psi\,dxdt, \tag{5.5}$$

for all $\psi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q_T)$.

Proof Let $\psi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q_T)$ as a test function in (3.1), we obtain

$$\int_Q \frac{\partial u_n}{\partial t}\psi\,dxdt+\int_{Q_T} b(x,t,T_n(u_n),\nabla u_n).\nabla\psi\,dxdt$$
$$=\int_{Q_T} g_n(u_n)f_n\psi\,dxdt. \tag{5.6}$$

If $g(0) < +\infty$, we obtain (5.5) hold true. Suppose that $g(0) = \lim_{z\to 0} g(z)$. Let ψ be a non-negative function in $L^\infty(Q_T)\cap L^p(0,T;W_0^{1,p}(\Omega))$ as a test function in the weak formulation (5.6), using (1.4) and Young's inequality, we obtain

$$\begin{aligned}\int_{Q_T} g_n(u_n) f_n \psi\, dxdt &\le \frac{1}{p}\int_{Q_T} |u_n|^p dxdt + \frac{1}{p'}\int_{Q_T}\left|\frac{\partial\psi}{\partial t}\right|^{p'} dxdt \\ &\quad + \frac{1}{p}\int_{Q_T} |\nabla\psi|^p dxdt \\ &\quad + \frac{1}{p'}\int_{Q_T}\left(a(x,t) + |T_n(u_n)|^{p-1} + |\nabla u_n|^{p-1}\right)^{p'} dxdt \\ &\le C\left(\|u_n\|_{L^p(Q_T)} + \left\|\frac{\partial\psi}{\partial t}\right\|_{L^{p'}(Q_T)} + \|k\|_{L^{p'}(Q_T)}\right) \\ &\quad + C\left(\|u_n\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|\psi\|_{L^p(0,T;W_0^{1,p}(\Omega))}\right) \\ &\le C. \end{aligned} \tag{5.7}$$

Therefore (5.7) implies that $\{f_n g_n(u_n)\}_n$ is bounded in $L^1(Q_T)$. Passing to the limit as $n\to+\infty$ in (5.5), Fatou's lemma implies

$$\int_{Q_T} f g(u)\psi\, dxdt \le C \quad \forall n, \tag{5.8}$$

then we have

$$\int_{\{u=0\}} \lim_{z\to 0} g(z) f\varphi\, dxdt < +\infty,$$

so that, $f\varphi = 0$ a.e. on $\{u=0\}$ for all nonnegative $\varphi \in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q_T)$. Yielding

$$f = 0 \text{ a.e. on } \{u=0\}. \tag{5.9}$$

For every fixed $\lambda > 0$, we can write

$$\begin{aligned}\int_{Q_T} f_n g_n(u_n)\psi\, dxdt &= \int_{Q_T\cap\{u_n>\lambda\}} f_n g_n(u_n)\psi\, dxdt \\ &\quad + \int_{Q_T\cap\{u_n\le\lambda\}} f_n g_n(u_n)\psi\, dxdt \\ &= \mathcal{I}^1_{n,\lambda} + \mathcal{I}^2_{n,\lambda}. \end{aligned} \tag{5.10}$$

For $\mathcal{I}^1_{n,\lambda}$, we have

$$0 \le g_n(u_n) f_n \chi_{\{u_n>\lambda\}}\varphi \le \sup_{z\in[\lambda,+\infty)} [g(z)] f\varphi \in L^1(Q_T). \tag{5.11}$$

Using Lebesgue's dominated convergence theorem and that the sequence $\left\{\chi_{\{u_n>\lambda\}}\right\}_n$ converges to $\chi_{\{u\geq\lambda\}}$ a.e. in Q_T, we get

$$\lim_{n\to\infty}\mathcal{I}^1_{n,\lambda}=\int_{Q_T\cap\{u\geq\lambda\}}g(u)f\psi dxdt.$$

Since $g(u)f\varphi\in L^1(Q_T)$, Lebesgue's theorem, with respect to λ, imply that

$$\lim_{\lambda\to0^+}\lim_{n\to\infty}\mathcal{I}^1_{n,\lambda}=\int_{Q_T\cap\{u\geq0\}}g(u)f\psi dxdt=\int_{Q_T}g(u)f\psi dxdt.$$

By (5.9), it follows that

$$\lim_{\lambda\to0^+}\lim_{n\to\infty}\mathcal{I}^1_{n,\lambda}=\int_{Q_T\cap\{u>0\}}g(u)f\psi dxdt=\int_{Q_T}g(u)f\psi dxdt. \tag{5.12}$$

Now in order to get rid of $\mathcal{I}^2_{n,\lambda}$. We take $\Xi_\lambda(u_n)\psi$ as test function in (3.1), where Ξ_λ is defined in (1.8), we obtain

$$\begin{aligned}&\int_{Q_T}\frac{\partial u_n}{\partial t}\Xi_\lambda(u_n)\psi dxdt+\int_{Q_T}b(x,t,T_n(u_n),\nabla u_n)\nabla(\Xi_\lambda(u_n)\psi)dxdt\\&=\int_{Q_T}f_ng_n(u_n)\Xi_\lambda(u_n)\psi dxdt.\end{aligned}\tag{5.13}$$

Using integration by parties and definition of Ξ_λ, we have

$$\int_{Q_T}\frac{\partial u_n}{\partial t}\Xi_\lambda(u_n)\psi dxdt=-\int_{Q_T}\Theta(u_n)\frac{\partial\psi}{\partial t}dxdt, \tag{5.14}$$

where

$$\Theta(\zeta)=\int_0^\zeta\Xi_\lambda(y)dy.$$

Using (1.2), $\Xi'_\lambda(u_n)=-\frac{1}{\lambda}$ and the fact that

$$\nabla(\Xi_\lambda(u_n)\psi)=\Xi_\lambda(u_n)\nabla\psi-\frac{1}{\lambda}(u_n)\psi\nabla u_n,$$

we get

$$\begin{aligned}&\int_{Q_T}b(x,t,T_n(u_n),\nabla u_n)\nabla(\Xi_\lambda(u_n)\psi)dxdt\\&\leq\int_{Q_T}b(x,t,T_n(u_n),\nabla u_n)\Xi_\lambda(u_n).\nabla\psi dxdt.\end{aligned}\tag{5.15}$$

On the other hand

$$\begin{aligned}\int_{Q_T} f_n g_n(u_n)\Xi_\lambda(u_n)\psi dxdt &= \int_{Q_T\cap\{u_n\le\lambda\}} f_n g_n(u_n)\Xi_\lambda(u_n)\psi dxdt\\ &\quad+\int_{Q_T\cap\{\lambda<u_n<2\lambda\}} f_n g_n(u_n)\Xi_\lambda(u_n)\psi dxdt\\ &\le \int_{Q_T\cap\{u_n\le\lambda\}} f_n g_n(u_n)\Xi_\lambda(u_n)\psi dxdt. \end{aligned}\tag{5.16}$$

Combining (5.13)-(5.15) and (5.16), we obtain

$$\begin{aligned}\int_{Q_T\cap\{u_n\le\lambda\}} f_n g_n(u_n)\Xi_\lambda(u_n)\psi dxdt &\le \int_{Q_T} b(x,t,T_n(u_n),\nabla u_n)\Xi_\lambda(u_n).\nabla\psi dxdt\\ &\quad-\int_{Q_T}\Theta(u_n)\frac{\partial\psi}{\partial t}dxdt.\end{aligned}$$

Using that Ξ_λ is bounded and Θ is continue we deduce that as n tends to infinity

$$\Theta(u_n)\frac{\partial\psi}{\partial t}\to\Theta(u)\frac{\partial\psi}{\partial t}\quad\text{strongly in } L^1(Q_T),$$

and

$$b(x,t,T_n(u_n),\nabla u_n)\Xi_\lambda(u_n)\rightharpoonup b(x,t,u,\nabla u)\Xi_\lambda(u)\quad\text{weakly in } L^{p'}(Q_T).$$

This implies that

$$\begin{aligned}&\limsup_{n\to+\infty}\int_{Q_T\cap\{u_n\le\lambda\}} f_n g_n(u_n)\Xi_\lambda(u_n)\psi dxdt\\ &\le -\int_{\{u=0\}}\Theta(u)\frac{\partial\psi}{\partial t}dxdt+\int_{\{u=0\}} b(x,t,u,\nabla u)\Xi_\lambda(u).\nabla\psi dxdt,\end{aligned}$$

then

$$\lim_{\lambda\to0}\lim_{n\to\infty}\mathcal{I}^2_{n,\lambda}=0.\tag{5.17}$$

By (5.12) and (5.17) we deduce that, for all nonnegative $\psi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q_T)$

$$\lim_{n\to\infty}\int_{Q_T} f_n g_n(u_n)\psi dxdt=\int_{Q_T} fg(u)\psi dxdt.\tag{5.18}$$

Moreover, by decomposing any $\psi=\psi^+-\psi^-$ with $\psi^+=\max\{\psi,0\}$ and $\psi^-=-\min\{\varphi,0\}$, we deduce that (5.18) holds for every $\psi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q_T)$. This concludes (5.5).

Let $n\to+\infty$ in (5.6), by (5.1), (5.4) and (5.5) we get

$$\int_{Q_T}\frac{\partial u}{\partial t}\psi dxdt+\int_{Q_T} b(x,t,u,\nabla u).\nabla\psi dxdt=\int_{Q_T} g(u)f\psi dxdt,\tag{5.19}$$

for every $\psi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q_T)$. □

5.2 Proof of the Theorem 2.5

From Lemma 4.4, we have the sequence $\{u_n\}_n$ is bounded in $L^q(0,T;W_0^{1,q}(\Omega))\cap L^\delta(Q_T)$ and $\left\{\frac{\partial u_n}{\partial t}\right\}_n$ is bounded in $L^{q'}(0,T;W^{-1,q'}(\Omega))+L^m(Q_T)$. Thus, the sequence $\left\{\frac{\partial u_n}{\partial t}\right\}_n$ is bounded in $L^1(0,T,W^{-1,\epsilon}(\Omega))$ for every $\epsilon<\min\left\{\frac{N}{N-1},q'\right\}$. So, by corollary 4 of [34], we get the sequence $\{u_n\}_n$ is relatively compact in $L^1(Q_T)$. This implies that we can extract a subsequence (denote again by $\{u_n\}_n$) such that the sequence $\{u_n\}_n$ converges to u strongly in $L^1(Q_T)$. From (5.1), (5.2), we obtain

$$b(x,t,u_n,\nabla u_n)\to b(x,t,u,\nabla u)\quad \text{a.e. in } Q_T. \tag{5.20}$$

Using Lemma 4.4, (5.20), $\frac{q}{p-1}>1$ and Vitali's theorem, one has

$$b(x,t,u_n,\nabla u_n)\to b(x,t,u,\nabla u)\quad \text{strongly in } L^{\frac{q}{p-1}}(Q_T).$$

Thus, it is possible to pass to the limit in (5.6) as $n\to+\infty$, obtaining (5.19).

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Declarations

Competing Interests I declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

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