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Existence Result for Solutions to Some Noncoercive Elliptic Equations

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Abstract
In this work, we study a class of degenerate Dirichlet problems, whose prototype is

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{(1+|u|)^{\gamma}} + c(x)|u|^{\theta-1}u\log^{\beta}(1+|u|)\Big) = f \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$. $0<\gamma<1$, $0<\theta\le 1$ and $0\le\beta<1$. We prove existence of bounded solutions when f and c belong to suitable Lebesgue spaces. Moreover, we investegate the existence of renormalized solutions when the function f belongs only to $L^1(\Omega)$.

Keywords Nonlinear elliptic equations · Degenerate ellipticity · Renormalized solutions · Existence results

Mathematics Subject Classification (2010) 35J60 · 35J70

1 Introduction

In this paper we are interested in the existence of solutions for some nonlinear elliptic equations whose simplest model is

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{(1+|u|)^{\gamma}} + c(x)|u|^{\theta-1}u\log^{\beta}(1+|u|)\Big) = f \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is any bounded open subset of $\mathbb{R}^N$, $N \geq 3$, $0 < \gamma < 1$, $0 < \theta \leq 1$ and $0 \leq \beta < 1$, the measurables functions c and f belong to a suitable Lebesgue spaces. It is clear that the nonlinear differential operator in the model problem (1.1) presents a strong lack of coercivity so that the classical theory for elliptic operator (see [21]) cannot be applied. In this paper, we will prove first an L^∞- estimate, when f and c belong to some Lebesgue spaces (see Theorem 3.2), and then we prove the existence of a generalized solution (the so called renormalized solution, see Definition 2.3 and Theorem 4.1 below) when the datum f is merely integrable.

When $c \equiv 0$, $f \in L^m(\Omega)$ and $m \geq 1$, there is a wide literature about problems like (1.1) (see for instance [1, 5, 9, 10, 12, 14, 17]). In these papers, existence and regularity of solutions have been proved for different ranges of the parameter γ and depending on the summability of the datum f. If $\gamma = 0$, $\beta = 0$ and $\theta = 1$, existence, uniqueness and regularity of distributional solutions of (1.1) have been proved in [6, 7] (see also [8], where the case of singular coeffecient $c(x)$ is studied). In [27] the case of $0 < \theta < 1$ was deeply studied under different summability properties of $c(x)$ and the datum f, while the case of unbounded domains was considered in [23]. For other related results, we refer to [11, 13, 15, 16, 24, 29].

When f is just an L^1 or measure data, $\theta = 1$, $\beta = 0$ and the operator $A(u) = -\operatorname{div}\Big(\frac{\nabla u}{(1+|u|)^\gamma}\Big)$ is replaced by a p-Laplacian operator, the authors in [2, 4, 18, 19] proved the existence of solutions of problem (2.1) using the framework of renomalized solutions which was introduced in [21, 22].

The main difficulty that we face in this work is due to the presence of the non-coercive operator $-\operatorname{div}\Big(\frac{\nabla u}{(1+|u|)^\gamma} - c(x)|u|^{\theta-1}u\log^\beta(1+|u|)\Big)$. In the case where the datum $f \in L^m(\Omega)$ with $m > \frac{N}{2}$, under some restriction on the parameters θ and γ that is, $\theta + \gamma \leq 1$ and for every $0 \leq \beta < 1$, we show that problem (1.1) admits at least one bounded solution (see Theorem 3.2). In order to deal with the case $m = 1$, the operator $A(u)$ is replaced by $-\operatorname{div}\Big(b(u)\frac{\nabla u}{(1+|u|)^\gamma}\Big)$, where b is a continuous function on $\mathbb{R}$ such that $b(s) \geq (1+|s|)^q$ for every $s \in \mathbb{R}$, with $q < \gamma$. Under this assumption and $\theta + \gamma \leq 1$, one can establish the existence of a renormalized solution for problem (1.1) (see Theorem 4.1).

In the case $\theta = 1$ and $\beta = 0$, one can recover the existence result of a solution in both cases ($m > \frac{N}{2}$ and $m = 1$) by adding a lower order zero term g (see Theorems 5.1 and 5.2). Indeed, under some suitable assumptions on the continuous function g (see assumptions (5.2)-(5.3) and condition at infinity (5.4)), problem (1.1) admits at least one solution.

This paper is organized as follows. In Sect. 2 we precise the assumptions on data and we give the definitions of weak solutions and renormalized solutions. Section 3 is devoted to study the existence of bounded weak solutions to problem (2.1) when $\theta + \gamma \leq 1$ and $m > \frac{N}{2}$. In Sect. 4, we establish the existence of renormalized solutions in the case where $\theta + \gamma \leq 1$ and $m = 1$. Finally, in Sect. 5 we show how the lower zero order term g will help us to insure the existence of renormalized solutions if we assume that $\theta = 1$ and $\beta = 0$.

2 Assumptions and Definition of Solution

Let us consider the following nonlinear elliptic problem

$$\begin{cases} -\operatorname{div}\Big(a(x,u)\nabla u + \Phi(x,u)\Big) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is any bounded open subset of $\mathbb{R}^N$, $N \geq 3$, $a(x,s): \Omega \times \mathbb{R} \to \mathbb{R}^+$ and $\Phi(x,s):$ $\Omega \times \mathbb{R} \to \mathbb{R}^N$ are Carathéodory functions (that is, continuous with respect to s for almost every $x \in \Omega$ and measurable with respect to x for every $s \in \mathbb{R}$) satisfying

$$a(x,s) \geq \frac{b(s)}{(1+|s|)^\gamma}, \quad \text{a.e. } x \in \Omega,\ \forall s \in \mathbb{R}, \tag{2.2}$$

where b is a continous function in $\mathbb{R}$ such that $b(s) \geq \alpha_0 > 0$, $\forall s \in \mathbb{R}$ and $\gamma \in (0,1)$.

$$\sup_{|s|\leq k} |a(x,s)| \in L^\infty(\Omega),\ \forall\, k > 0, \quad \text{a.e. } x \in \Omega, \tag{2.3}$$

$$|\Phi(x,s)| \leq c(x)|s|^\theta \log^\beta(1+|s|), \ \text{ a.e. } x \in \Omega,\ \forall s \in \mathbb{R}, \tag{2.4}$$

where c belongs to $L^r(\Omega)$, $r \geq 2$, $0 < \theta \leq 1$ and $0 \leq \beta < 1$. Finally, assume that the datum f is a measurable function such that

$$f \in L^m(\Omega),\ m \geq 1. \tag{2.5}$$

Notations. Hereafter, we will make use of two truncation functions T_k and G_k: for every $k \geq 0$ and $r \in \mathbb{R}$, let

$$T_k(r) = \min(k, \max(r, -k)), \quad G_k(r) = r - T_k(r).$$

For every $s \in \mathbb{R}$, we set $\alpha(s) = \dfrac{1}{(1+|s|)^\gamma}$ and we define $\widetilde{\alpha}(s) = \displaystyle\int_0^s \alpha(r)\, dr$ which is a C^1 increasing function on $\mathbb{R}$.

For the sake of simplicity we will use, when referring to the integrals, the following notation

$$\int_\Omega f = \int_\Omega f(x)\, dx.$$

Finally, throughout this paper, C will indicate any positive constant which depends only on data and whose value may change from line to line.

Now we give the following definition of weak solutions to problem (2.1) in the sense of finite energy solutions.

Definition 2.1 A measurable function u is a weak solution of (2.1) if $a(x,u)\nabla u \in (L^2(\Omega))^N$, $\Phi(x,u) \in (L^2(\Omega))^N$, and

$$\int_\Omega a(x,u)\nabla u \nabla \varphi + \int_\Omega \Phi(x,u)\nabla\varphi = \int_\Omega f\varphi,$$

holds for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Before giving the definition of renormalized solutions to (2.1), let us first recall the definition of generalized gradient of u introduced in [3].

Definition 2.2 Let $u: \Omega \to \mathbb{R}$ be a measurable function defined on Ω which is finite almost everywhere such that $T_k(u) \in H_0^1(\Omega)$ for every $k > 0$. Then there exists a unique measurable function v defined in Ω such that

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \ \text{ a.e. in } \ \Omega,\ \forall k > 0,$$

Let us now define the renormalized solution to (2.1).

Definition 2.3 A real function u defined in Ω is a renormalized solution of problem (2.1) if

$$u \text{ is measurable and finite almost everywhere in } \Omega, \tag{2.6}$$

$$T_k(u) \in H_0^1(\Omega) \;\; \forall k > 0, \tag{2.7}$$

$$\frac{1}{m}\int_{\{|\widetilde{\alpha}(u)|\le m\}} \alpha(u)a(x,u)|\nabla u|^2 \to 0 \;\; as \;\; m \to +\infty, \tag{2.8}$$

and if, for every function $S \in W^{1,\infty}(\mathbb{R})$ such that the support of S is compact, $Supp(S) \subset [-k, k]$, u satisfies

$$\int_\Omega a(x,u)\nabla u \nabla(S(u)\varphi) + \int_\Omega \Phi(x,u)\nabla(S(u)\varphi) = \int_\Omega f S(u)\varphi, \tag{2.9}$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Remark 2.4 We notice that, since $\widetilde{\alpha}(\pm\infty) = \pm\infty$ which means that the set $\{|\widetilde{\alpha}(u)| \le k\}$ may be equivalent to $\{|u| \le k'\}$ with $k' > 0$, then, due to (2.7) and (2.3) we deduce that the condition (2.8) is well defined.

The renormalized equation (2.9) is formally obtained through a pointwise multiplication of (2.1) by $S(u)\varphi$. Let us observe that by (2.7) and the propreties of S, every term in (2.9) makes sense.

3 Existence of a Bounded Weak Solution to Problem (2.1)

In this section, we will prove the existence of a bounded weak solution to problem (2.1). We begin by recalling the following technical Lemma proved in [26] (see also the Appendix of [10]).

Lemma 3.1 *Let $a > 0$ and let $\varphi : [a, +\infty[\to \mathbb{R}^+$ be a nonincreasing function which satisfies*

$$\varphi(h) \le \frac{\omega(k)^\rho}{(h-k)^\rho}\varphi(k)^{1+\nu} \;\; \forall h > k \ge a,$$

where $\lim_{k\to+\infty} \frac{\omega(k)}{k} = 0$ *and* $\rho, \nu > 0$. *Then, there exist* $k^*, k_0 > a$ *such that* $k^* = k_0 + d$ *and* $\varphi(k^*) = 0$, *where*

$$d^\rho = M[\varphi(k_0)]^\nu 2^{\frac{(1+\nu)\rho}{\nu}},$$

with $M > 0$.

Now we state the main result of this section.

Theorem 3.2 *Assume* (2.2)-(2.4), *with* $r > N$, *and assume that* f *belongs to* $L^m(\Omega)$, $m > \frac{N}{2}$. *Furthermore, we suppose that* $\gamma + \theta \leq 1$ *so that*

$$\lim_{|s|\to+\infty} \frac{|s|^\theta}{1+|\widetilde{\alpha}(s)|} = \ell \in \mathbb{R}^+. \tag{3.1}$$

Then there exists a weak solution u *for* (2.1) *in the sense of Definition* 2.1.

Remark 3.3 We point out that from the limit condition (3.1), we derive the existence of a nonnegative constant C and real number $k_0 > 0$ such that for every $|s| > k_0$, one has

$$|s|^\theta \leq C(1+|\widetilde{\alpha}(s)|). \tag{3.2}$$

Proof For $n \in \mathbb{N}$ let us define

$$\Phi_n(x,s) = T_n(\Phi(x,s)),$$
$$a_n(x,s) = a(x, T_n(s))$$

and

$$f_n = T_n(f).$$

Let us consider the following Dirichlet approximate problems

$$\begin{cases} -\operatorname{div}\Big(a_n(x,u_n)\nabla u_n + \Phi_n(x,u_n)\Big) = f_n \ \text{ in } \Omega, \\ u_n = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{3.3}$$

Note that the existence of weak solutions $u_n \in H_0^1(\Omega)$ follows from the classical results of ([21]) and Schauder's fixed point theorem. Moreover, thanks to Stampacchia's boundedness theorem (see [28]), the solutions u_n belong to $L^\infty(\Omega)$.

In order to prove Theorem 3.2, we have to distinguish two cases.

The case $\gamma + \theta = 1$. In this case, it easy to check, by Hôpital's rule that

$$\lim_{|s|\to+\infty} \frac{|s|^\theta}{1+|\widetilde{\alpha}(s)|} = \theta,$$

so that $\ell = \theta$, where ℓ is defined in (3.1). Next, we define the nonnegative function Ψ

$$\Psi(s) = \begin{cases} 0 \ \text{ if } \ |s| \leq k, \\ \dfrac{s}{1+s} - \dfrac{k}{1+k} \ \text{ if } \ s > k, \\ \dfrac{-s}{1-s} - \dfrac{k}{1+k} \ \text{ if } \ s < -k, \end{cases}$$

and let us take $\Psi(\widetilde{\alpha}(u_n))$ as test function in (3.3), using assumptions (2.2), (2.4) and since $|T_n(s)| \leq |s|$, $|\Psi(s)| \leq 1$ for every $s \in \mathbb{R}$ and for $k \geq k_0$, using (3.2), we obtain

$$\alpha_0 \int_{\{|\widetilde{\alpha}(u_n)|>k\}} \frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2} \leq C \int_{\{|\widetilde{\alpha}(u_n)|>k\}} c(x)\log^\beta(1+|u_n|)\frac{|\nabla\widetilde{\alpha}(u_n)|}{1+|\widetilde{\alpha}(u_n)|}$$
$$+ \int_{\{|\widetilde{\alpha}(u_n)|>k\}} |f|,$$

where the positive constant C does not depend on k. On the other hand, applying again (3.2), one has for every $|s| > k_0$

$$\log(1+|s|) \le \log(1+C(1+|\widetilde{\alpha}(s)|)^{\frac{1}{\theta}}) \le \log((1+C)(1+|\widetilde{\alpha}(s)|)^{\frac{1}{\theta}}) \tag{3.4}$$

$$\le C(1+\log(1+|\widetilde{\alpha}(s)|)),$$

which then implies that

$$\alpha_0 \int_{\{|\widetilde{\alpha}(u_n)|>k\}} \frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2} \le C \int_{\{|\widetilde{\alpha}(u_n)|>k\}} c(x)\log^{\beta}(1+|\widetilde{\alpha}(u_n)|)\frac{|\nabla\widetilde{\alpha}(u_n)|}{1+|\widetilde{\alpha}(u_n)|}$$

$$+C\int_{\{|\widetilde{\alpha}(u_n)|>k\}} c(x)\frac{|\nabla\widetilde{\alpha}(u_n)|}{1+|\widetilde{\alpha}(u_n)|} + \int_{\{|\widetilde{\alpha}(u_n)|>k\}} |f|. \tag{3.5}$$

Now we deal with the first term in the right hand side of (3.5), we have

$$\int_{\{|\widetilde{\alpha}(u_n)|>k\}} c(x)\log^{\beta}(1+|\widetilde{\alpha}(u_n)|)\frac{|\nabla\widetilde{\alpha}(u_n)|}{1+|\widetilde{\alpha}(u_n)|}$$

$$\le \frac{1}{\log^{1-\beta}(1+k)} \int_{\{|\widetilde{\alpha}(u_n)|>k\}} c(x)\log(1+|\widetilde{\alpha}(u_n)|)\frac{|\nabla\widetilde{\alpha}(u_n)|}{1+|\widetilde{\alpha}(u_n)|}.$$

Let us notice that $|s| = k + |G_k(s)|$ in $\{|s| > k\}$, which gives

$$\int_{\{|\widetilde{\alpha}(u_n)|>k\}} c(x)\log^{\beta}(1+|\widetilde{\alpha}(u_n)|)\frac{|\nabla\widetilde{\alpha}(u_n)|}{1+|\widetilde{\alpha}(u_n)|}$$

$$\le \frac{1}{\log^{1-\beta}(1+k)} \int_{\{|\widetilde{\alpha}(u_n)|>k\}} c(x)\Big|\log(1+k+|G_k(\widetilde{\alpha}(u_n))|) - \log(1+k)\Big|\frac{|\nabla\widetilde{\alpha}(u_n)|}{1+|\widetilde{\alpha}(u_n)|}$$

$$+\log^{\beta}(1+k)\int_{\{|\widetilde{\alpha}(u_n)|>k\}} c(x)\frac{|\nabla\widetilde{\alpha}(u_n)|}{1+|\widetilde{\alpha}(u_n)|}.$$

Then, using Hölder and Young inequalities, we obtain

$$\int_{\{|\widetilde{\alpha}(u_n)|>k\}} c(x)\log^{\beta}(1+|\widetilde{\alpha}(u_n)|)\frac{|\nabla\widetilde{\alpha}(u_n)|}{1+|\widetilde{\alpha}(u_n)|}$$

$$\le \frac{\|c\|_{L^N(\Omega)}}{\log^{1-\beta}(1+k)}\Big(\int_{\Omega} |\log(1+k+|G_k(\widetilde{\alpha}(u_n))|) - \log(1+k)|^{2^*}\Big)^{\frac{1}{2^*}}$$

$$\times\Big(\int_{\{|\widetilde{\alpha}(u_n)|>k\}} \frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2}\Big)^{\frac{1}{2}}$$

$$+C\log^{2\beta}(1+k)\int_{\{|\widetilde{\alpha}(u_n)|>k\}} c^2(x) + \frac{\alpha_0}{4}\int_{\{|\widetilde{\alpha}(u_n)|>k\}} \frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2},$$

where $2^* = \frac{2N}{N-2}$, so, with the help of Sobolev inequality, it yields that

$$\int_{\{|\widetilde{\alpha}(u_n)|>k\}} c(x)\log^{\beta}(1+|\widetilde{\alpha}(u_n)|)\frac{|\nabla\widetilde{\alpha}(u_n)|}{1+|\widetilde{\alpha}(u_n)|} \leq \frac{C\|c\|_{L^N(\Omega)}}{\log^{1-\beta}(1+k)}\Big(\int_{\{|\widetilde{\alpha}(u_n)|>k\}}\frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2}\Big)$$
$$+C\log^{2\beta}(1+k)\int_{\{|\widetilde{\alpha}(u_n)|>k\}} c^2(x) + \frac{\alpha_0}{4}\int_{\{|\widetilde{\alpha}(u_n)|>k\}}\frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2}.$$

Then, from the previous inequality, (3.5) and using again Young's inequality, we obtain

$$\Big(\frac{\alpha_0}{2} - \frac{C\|c\|_{L^N(\Omega)}}{\log^{1-\beta}(1+k)}\Big)\int_{\{|\widetilde{\alpha}(u_n)|>k\}}\frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2}$$
$$\leq C(1+\log^{2\beta}(1+k))\int_{\{|\widetilde{\alpha}(u_n)|>k\}} c^2(x) + \int_{\{|\widetilde{\alpha}(u_n)|>k\}}|f|.$$

We remark that there exists $k_1 > 0$ such that for every $k \geq k_1$

$$\frac{\alpha_0}{2} - \frac{C\|c\|_{L^N(\Omega)}}{\log^{1-\beta}(1+k)} \geq \frac{\alpha_0}{4}.$$

Thus we have if $k \geq k_1$,

$$\frac{\alpha_0}{4}\int_{\{|\widetilde{\alpha}(u_n)|>k\}}\frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2}$$
$$\leq C(1+\log^{2\beta}(1+k))\int_{\{|\widetilde{\alpha}(u_n)|>k\}} c^2(x) + \int_{\{|\widetilde{\alpha}(u_n)|>k\}}|f|.$$

Now putting $k = e^h - 1$, $w = \log(1+|\widetilde{\alpha}(u_n)|)$, $A_h = \{w > h\}$ and applying Poincaré inequality we obtain

$$\int_{\{w>h\}}|G_h(w)|^2 \leq C\Big((1+h^{2\beta})\int_{\{w>h\}} c^2(x) + |f|\Big).$$

Let us take $l > h > 0$, then

$$A_l \subset A_h, \quad |G_h(w)| \geq l-h \text{ in } A_l,$$

so, using Sobolev and Hölder inequalities, it follows that

$$|A_l| \leq \frac{C}{(l-h)^{2^*}}\Big((1+h^{2\beta})\|c\|_{L^{2m}(\Omega)} + \|f\|_{L^m(\Omega)}\Big)^{\frac{2^*}{2}}|A_h|^{\frac{2^*}{2}(1-\frac{1}{m})}\ \forall l > h.$$

Denoting $\omega(h) = \Big((1+h^{2\beta})\|c\|_{L^{2m}(\Omega)} + \|f\|_{L^m(\Omega)}\Big)^{\frac{1}{2}}$, since $m > \frac{N}{2}$, which means $\frac{2^*}{2}(1-\frac{1}{m}) > 1$, and since $\lim_{h\to+\infty}\frac{\omega(h)}{h} = 0$, then, Lemma 3.1 implies that there exists $k^*(\omega, N, \alpha_0, \beta, m, f) > 0$ such that $|\{w > k^*\}| = 0$ and $\widetilde{\alpha}(u_n)$ is bounded as desired. Moreover, since $\widetilde{\alpha}(\pm\infty) = \pm\infty$, we deduce that u_n is bounded as well.

The case $\gamma + \theta < 1$. In this case, one can easily check that

$$\lim_{|s|\to+\infty}\frac{|s|^{\theta}}{1+|\widetilde{\alpha}(s)|}\log^{\beta}(1+|s|) = 0.$$

Then, using as above the test function $\Psi(\widetilde{\alpha}(u_n))$ in (3.3) and by Young's inequality it results

$$\frac{\alpha_0}{2}\int_{\{|\widetilde{\alpha}(u_n)|>k\}}\frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2}\leq C\int_{\{|\widetilde{\alpha}(u_n)|>k\}}c^2(x)+\int_{\{|\widetilde{\alpha}(u_n)|>k\}}|f|.$$

Then, by following the proof of the previous case, we deduce that

$$|A_l|\leq\frac{C}{(l-h)^{2^*}}\Big(\|c\|_{L^{2m}(\Omega)}+\|f\|_{L^m(\Omega)}\Big)^{\frac{2^*}{2}}|A_h|^{\frac{2*}{2}(1-\frac{1}{m})}\ \forall l>h.$$

Hence, applying Lemma 3.1, it follows that there exists k^* such that $|\{w>k^*\}|=0$, that is, u_n is bounded as desired.

Now, taking u_n as test function in (3.3), by assumptions (2.2), (2.4) and using Young's inequality, one obtains, if $\|u_n\|_{L^\infty(\Omega)}\leq C$ that u_n is bounded in $H_0^1(\Omega)$. Hence, thanks to Rellich-Kondrachov Theorem, we deduce that up to subsequences,

$$u_n\rightharpoonup u \text{ weakly in } H_0^1(\Omega),$$

$$u_n\to u \text{ strongly in } L^2(\Omega),$$

$$u_n\to u \text{ a.e. in } \Omega.$$

So that, due to the assumptions (2.3) and (2.4), one can pass easily to the limit in (2.1) as n tends to infinity to conclude the proof of Theorem 3.2. □

4 Existence of Renormalized Solutions

The existence result of renormalized solutions for problem (2.1) can be stated as follows

Theorem 4.1 *Assume that (2.2)-(2.5) hold, with $r\in[2,N)$, $m=1$, and that $\gamma+\theta\leq 1$. Suppose that $\beta<\gamma$ and that $b(s)\geq\alpha_0(1+|s|)^q$, $\forall s\in\mathbb{R}$ with $q\in[\beta,\gamma)$. Then there exists at least a renormalized solution u for (2.1) in the sense of Definition 2.2.*

Remark 4.2 In what follows, we will only deal with the case $\theta+\gamma=1$ since in the case $\theta+\gamma<1$, up the change of the unknown $\widetilde{\alpha}(u)$ and by proceeding as in [2, 4] one can deduce that u is a renormalized solution of (2.1) for every $\beta>0$. Indeed, we remark that for every $0<\beta_1<\beta$, we have $\lim\limits_{|s|\to+\infty}\dfrac{\log^\beta(1+|s|)}{|s|^{\beta_1}}=0$, so, by distinguishing the sets where $|s|\leq s_0$ $(s_0>0)$ and where $|s|>s_0$, the assumption (2.4) on Φ could be written as follows

$$|\Phi(x,s)|\leq C'\,c(x)(1+|s|^{\theta'}),$$

where $0<\theta<\theta'<1$ such that $\gamma+\theta'<1$, and C' is a positive constant. Then, we conclude the proof of Theorem 4.1.

Proof We take $T_k(u_n)$ as test function in the approximate problem (3.3); using assumptions (2.2) and (2.4), we obtain

$$\alpha_0\int_\Omega\frac{|\nabla T_k(u_n)|^2}{(1+|u_n|)^\gamma}\leq k^\theta(1+k)^{\frac{\gamma}{2}}\log^\beta(1+k)\int_\Omega c(x)\frac{|\nabla T_k(u_n)|}{(1+|u_n|)^{\frac{\gamma}{2}}}+k\|f\|_{L^1(\Omega)}.$$

By Young's inequality, it follows that

$$\frac{\alpha_0}{2}\int_\Omega \frac{|\nabla T_k(u_n)|^2}{(1+|u_n|)^\gamma} \le Ck^{2\theta}(1+k)^\gamma \log^{2\beta}(1+k)\int_\Omega c^2(x) + k\|f\|_{L^1(\Omega)},$$

so, we get

$$\int_\Omega |\nabla T_k(u_n)|^2 \le Ck^{2\theta}(1+k)^{2\gamma}\log^{2\beta}(1+k)\int_\Omega c^2(x) + k(1+k)^\gamma\|f\|_{L^1(\Omega)},$$

then, we deduce that, for every $k>0$,

$$T_k(u_n) \text{ is bounded in } H_0^1(\Omega). \tag{4.1}$$

Moreover, using $T_k(\widetilde{\alpha}(u_n))$ as test function in the problem (3.3), we deduce that

$$T_k(\widetilde{\alpha}(u_n)) \text{ is bounded in } H_0^1(\Omega). \tag{4.2}$$

The next step is to prove that u_n converges almost everywhere to a measurable function which is almost everywhere finite. To this end, we follow the classical approach of [3, 25]. Let us start by evaluating the measure of the set $\{|\widetilde{\alpha}(u_n)|>k\}$ as $k\to\infty$, we take $\int_0^{\widetilde{\alpha}(u_n)} \frac{dr}{(1+|r|)^2}$ as a test function in (3.3), using assumptions (2.2) and (2.4) lead to

$$\alpha_0\int_\Omega \frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2} \le \int_\Omega c(x)|u_n|^\theta\log^\beta(1+|u_n|)\frac{|\nabla\widetilde{\alpha}(u_n)|}{(1+|\widetilde{\alpha}(u_n)|)^2} + \int_\Omega |f|,$$

and using (3.2), (3.4) we obtain

$$\begin{aligned}\alpha_0\int_\Omega \frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2} &\le \int_{\{|u_n|>k_0\}} c(x)|u_n|^\theta\log^\beta(1+|u_n|)\frac{|\nabla\widetilde{\alpha}(u_n)|}{(1+|\widetilde{\alpha}(u_n)|)^2} \\ &+\int_{\{|u_n|\le k_0\}} c(x)|u_n|^\theta\log^\beta(1+|u_n|)\frac{|\nabla\widetilde{\alpha}(u_n)|}{(1+|\widetilde{\alpha}(u_n)|)^2} + \int_\Omega |f| \\ &\le C\int_\Omega c(x)\log^\beta(1+|\widetilde{\alpha}(u_n)|)\frac{|\nabla\widetilde{\alpha}(u_n)|}{1+|\widetilde{\alpha}(u_n)|} \\ &+C\int_\Omega c(x)\frac{|\nabla\widetilde{\alpha}(u_n)|}{1+|\widetilde{\alpha}(u_n)|} + \int_\Omega |f|.\end{aligned} \tag{4.3}$$

Then, by using Hölder and Young inequalities in the right hand side of (4.3), we obtain

$$\begin{aligned}\frac{\alpha_0}{2}\int_\Omega \frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2} &\le \|c\|_{L^r(\Omega)}\Big(\int_\Omega |\log(1+|\widetilde{\alpha}(u_n)|)|^{2^*}\Big)^{\frac{\beta}{2^*}}\Big(\int_\Omega \frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2}\Big)^{\frac{1}{2}} \\ &+C\int_\Omega c^2(x) + \int_\Omega |f|,\end{aligned}$$

where $r = \frac{2N}{N-\beta(N-2)}$ (note that $\beta < 1$ implies $r < N$). Then, an application of Sobolev inequality leads to

$$\frac{\alpha_0}{2}\int_\Omega \frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2} \leq \|c\|_{L^r(\Omega)}\Big(\int_\Omega \frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2}\Big)^{\frac{\beta+1}{2}}$$

$$+C\int_\Omega c^2(x) + \int_\Omega |f|.$$

Using Young's inequality, we get

$$\int_\Omega |\nabla \log(1+|\widetilde{\alpha}(u_n)|)|^2 \leq C\Big(\Big(\int_\Omega c^r(x)\Big)^{\frac{2}{r(1-\beta)}} + \int_\Omega c^2(x)\Big) + \int_\Omega |f|\Big),$$

so, by Sobolev inequality, we find

$$\Big(\int_\Omega |\log(1+|\widetilde{\alpha}(u_n)|)|^{2^*}\Big)^{\frac{2}{2^*}} \leq C\Big(\Big(\int_\Omega c^r(x)\Big)^{\frac{2}{r(1-\beta)}} + \int_\Omega c^2(x)\Big) + \int_\Omega |f|\Big).$$

Then, for every $k > 0$, the previous estimate implies that

$$|\{|\widetilde{\alpha}(u_n)| > k\}|^{\frac{2}{2^*}} \leq \frac{C}{\log(1+k)^2}\Big(\Big(\int_\Omega c^r(x)\Big)^{\frac{2}{r(1-\beta)}} + \int_\Omega c^2(x)\Big) + \int_\Omega |f|\Big),$$

which yields

$$\lim_{k\to+\infty}\sup_n meas\{|\widetilde{\alpha}(u_n)| > k\} = 0. \tag{4.4}$$

Now, we show that u_n is a Cauchy sequence in measure. For $t, k > 0$, we observe that

$$\{|u_n - u_m| > t\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > t\},$$

which leads to

$$meas(\{|u_n - u_m| > t\}) \leq meas(\{|u_n| > k\})$$

$$+meas(\{|u_m| > k\}) + meas(\{|T_k(u_n) - T_k(u_m)| > t\}).$$

To estimate $meas(\{|T_k(u_n) - T_k(u_m)| > t\})$, by using (4.2) and applying Rellich-Kondrachov theorem, we deduce, up to subsequences, that $T_k(u_n)$ is a Cauchy sequence both in $L^2(\Omega)$ and measure. Then, for any fixed $\varepsilon > 0$, there exists $n_\varepsilon > 0$ such that

$$meas(\{|T_k(u_n) - T_k(u_m)| > t\}) < \frac{\varepsilon}{3},$$

for every $n, m > n_\varepsilon$ and for every $t > 0$.

We remark that, due to the proprety of $\widetilde{\alpha}$ ($\widetilde{\alpha}$ is C^1 increasing), we have

$$\{|u_n| > k\} = \{\widetilde{\alpha}(u_n) > \widetilde{\alpha}(k)\} \cup \{\widetilde{\alpha}(u_n) < \widetilde{\alpha}(-k)\},$$

so that

$$meas(\{|u_n| > k\}) = meas(\{\widetilde{\alpha}(u_n) > \widetilde{\alpha}(k)\}) + meas(\{\widetilde{\alpha}(u_n) < \widetilde{\alpha}(-k)\}).$$

Then, using (4.4) and the fact that $\widetilde{\alpha}(\pm\infty) = \pm\infty$, there exists $k_0 > 0$ such that for any fixed $\varepsilon > 0$, we have

$$meas(\{|u_n| > k\}) + meas(\{|u_m| > k\}) \le \frac{2\varepsilon}{3},$$

for every $n, m \in \mathbb{N}$ and for every $k > k_0$.

Hence, for every $\varepsilon > 0$, we obtain

$$meas(\{|u_n - u_m| > t\}) < \frac{\varepsilon}{3},$$

for every $n, m > n_\varepsilon$.

Hence, we deduce that u_n is a Cauchy sequence in measure which means that there exists a measurable function u which is finite almost everywhere in Ω such that up to a subsequence still indexed by n

$$u_n \to u \text{ a.e. in } \Omega, \tag{4.5}$$

$$T_k(u_n) \to T_k(u) \text{ weakly in } H_0^1(\Omega). \tag{4.6}$$

Next, we prove that

$$\frac{1}{m} \int_{\{|\widetilde{\alpha}(u_n)| \le m\}} \alpha(u_n) a_n(x, u_n) |\nabla u_n|^2 = \omega(n, m), \tag{4.7}$$

where $\omega(n, m)$ denotes any quantity that vanishes as the arguments goes to its natural limit (that is $n \to +\infty$, $m \to +\infty$).

We use $\displaystyle\frac{1}{m} \int_{T_k(\widetilde{\alpha}(u_n))}^{T_m(\widetilde{\alpha}(u_n))} \frac{ds}{(1+|s|)^q}$ as test function in (3.3) with $m > k \ge k_0$, using (2.2), (3.2) and (3.4), we obtain

$$\frac{\alpha_0}{m} \int_{\{k < |\widetilde{\alpha}(u_n)| \le m\}} \left(\frac{1 + |T_n(s)|}{1 + |\widetilde{\alpha}(u_n)|}\right)^q |\nabla \widetilde{\alpha}(u_n)|^2$$

$$\le \frac{C}{m} \int_{\{k < |\widetilde{\alpha}(u_n)| \le m\}} c(x)(1 + |\widetilde{\alpha}(u_n))|)^{1-q+\beta} |\nabla T_m(\widetilde{\alpha}(u_n))|$$

$$+ \frac{1}{m} \int_\Omega f_n \int_{T_k(\widetilde{\alpha}(u_n))}^{T_m(\widetilde{\alpha}(u_n))} \frac{ds}{(1+|s|)^q}.$$

Since $|\widetilde{\alpha}(u_n)| \le m$ is equivalent to $|u_n| \le m_1 = \max\{\widetilde{\alpha}^{-1}(m), -\widetilde{\alpha}^{-1}(-m)\}$, for $n > m_1$, we obtain

$$\frac{\alpha_0}{m} \int_{\{k < |\widetilde{\alpha}(u_n)| \le m\}} \left(\frac{1 + |s|}{1 + |\widetilde{\alpha}(u_n)|}\right)^q |\nabla \widetilde{\alpha}(u_n)|^2 \tag{4.8}$$

$$\le \frac{1}{m} \int_{\{k < |\widetilde{\alpha}(u_n)| \le m\}} c(x)(1 + |\widetilde{\alpha}(u_n))|)^{1-q+\beta} |\nabla T_m(\widetilde{\alpha}(u_n))|$$

$$+ \frac{1}{m} \int_\Omega f_n \int_{T_k(\widetilde{\alpha}(u_n))}^{T_m(\widetilde{\alpha}(u_n))} \frac{ds}{(1+|s|)^q}.$$

Remark that $\lim_{|s|\to+\infty}\dfrac{1+|s|}{1+|\widetilde{\alpha}(s)|}=+\infty$, then, for some $C_1>0$, using the assumption (2.2) in the left hand side of (4.8) and if $k\geq k^*$, we have

$$\frac{\alpha_0}{m}\int_{\{k<|\widetilde{\alpha}(u_n)|\leq m\}}\Big(\frac{1+|s|}{1+|\widetilde{\alpha}(u_n)|}\Big)^q|\nabla\widetilde{\alpha}(u_n)|^2\geq\frac{C_1\alpha_0}{m}\int_{\{|\widetilde{\alpha}(u_n)|\leq m\}}|\nabla\widetilde{\alpha}(u_n)|^2.$$

Next, we estimate the first term in the right hand side of (4.8), using Hölder inequality with

$$\frac{1-q+\beta}{2^*}+\frac{1}{2}+\frac{N-(1-q+\beta)(N-2)}{2N}=1,$$

we obtain

$$\frac{1}{m}\int_{\{k<|\widetilde{\alpha}(u_n)|\leq m\}}c(x)(1+|\widetilde{\alpha}(u_n))|)^{1-q+\beta}|\nabla T_m(\widetilde{\alpha}(u_n))|$$

$$\leq\frac{C}{m}\|c\|_{L^{\frac{2N}{N-(1-q+\beta)(N-2)}}(\Omega)}\Big(meas(\Omega)^{2^*}+\int_\Omega|T_m(\widetilde{\alpha}(u_n))|^{2^*}\Big)^{\frac{1-q+\beta}{2^*}}\Big(\int_\Omega|\nabla T_m(\widetilde{\alpha}(u_n))|^2\Big)^{\frac{1}{2}}$$

$$\leq\frac{C}{m}\|c\|_{L^{\frac{2N}{N-(1-q+\beta)(N-2)}}(\Omega)}\Big(\int_\Omega|\nabla T_m(\widetilde{\alpha}(u_n))|^2\Big)^{\frac{1}{2}}$$

$$+\frac{C}{m}\|c\|_{L^{\frac{2N}{N-(1-q+\beta)(N-2)}}(\Omega)}\Big(\int_\Omega|T_m(\widetilde{\alpha}(u_n))|^{2^*}\Big)^{\frac{1-q+\beta}{2^*}}\Big(\int_\Omega|\nabla T_m(\widetilde{\alpha}(u_n))|^2\Big)^{\frac{1}{2}}.$$

Using Sobolev together with Young inequalities, lead to

$$\frac{1}{m}\int_{\{k<|\widetilde{\alpha}(u_n)|\leq m\}}c(x)(1+|\widetilde{\alpha}(u_n))|)^{1-q+\beta}|\nabla T_m(\widetilde{\alpha}(u_n))|$$

$$\leq\frac{C}{m}\|c\|_{L^{\frac{2N}{N-(1-q+\beta)(N-2)}}(\Omega)}\Big(\int_\Omega|\nabla T_m(\widetilde{\alpha}(u_n))|^2\Big)^{\frac{1}{2}}$$

$$+\frac{C}{m}\|c\|_{L^{\frac{2N}{N-(1-q+\beta)(N-2)}}(\Omega)}\Big(\int_\Omega|\nabla T_m(\widetilde{\alpha}(u_n))|^2\Big)^{\frac{2-q+\beta}{2}}$$

$$\leq\frac{C}{m}\Big(\|c\|^2_{L^{\frac{2N}{N-(1-q+\beta)(N-2)}}(\Omega)}+\|c\|^{\frac{2}{q-\beta}}_{L^{\frac{2N}{N-(1-q+\beta)(N-2)}}(\Omega)}\Big)+\frac{C_1\alpha_0}{2m}\int_\Omega|\nabla T_m(\widetilde{\alpha}(u_n))|^2.$$

Hence, from the previous result, one can deduce that

$$\frac{1}{m}\int_{\{|\widetilde{\alpha}(u_n)|\leq m\}}|\nabla\widetilde{\alpha}(u_n)|^2 \tag{4.9}$$

$$\leq\frac{C}{m}\Big(\|c\|^2_{L^{\frac{2N}{N-(1-q+\beta)(N-2)}}(\Omega)}+\|c\|^{\frac{2}{q-\beta}}_{L^{\frac{2N}{N-(1-q+\beta)(N-2)}}(\Omega)}\Big)$$

$$+\frac{1}{m}\int_\Omega|\nabla T_k(\widetilde{\alpha}(u_n))|^2+\frac{1}{m}\int_\Omega f_n\int_{T_k(\widetilde{\alpha}(u_n))}^{T_m(\widetilde{\alpha}(u_n))}\frac{ds}{(1+|s|)^q}.$$

Now, we pass to the limit as n goes to infinity and then as m tends to infinity in the right hand side of (4.9). By virtue of (4.2), one has

$$\frac{1}{m}\int_{\Omega}|\nabla T_k(\widetilde{\alpha}(u_n))|^2=\omega(n,m).$$

As regards the last term in right hand side of (4.9), the fact that f_n is bounded in $L^1(\Omega)$ gives that

$$\Big|\frac{1}{m}\int_{\Omega}f_n\int_{T_k(\widetilde{\alpha}(u_n))}^{T_m(\widetilde{\alpha}(u_n))}\frac{ds}{(1+|s|)^q}\Big|$$

$$\leq \|f\|_{L^1(\Omega)}\frac{1}{m}\int_0^m\frac{ds}{(1+|s|)^q}+\|f\|_{L^1(\Omega)}\frac{1}{m}\int_0^k\frac{ds}{(1+|s|)^q},$$

since $q\leq 1$, by Hôspital's rule, one has $\lim_{m\to\infty}\frac{1}{m}\int_0^m\frac{ds}{(1+|s|)^q}=0$, so that

$$\frac{1}{m}\int_{\Omega}f_n\int_0^{T_m(\widetilde{\alpha}(u_n))}\frac{1}{(1+|s|)^q}\,ds=\omega(n,m).$$

Therefore, we conclude the proof of (4.7).

Now we prove that for any $k>0$,

$$T_k(u_n)\to T_k(u)\ \text{ strongly in }\ H_0^1(\Omega). \tag{4.10}$$

We follow the method of [20]. Let $h>k$ and take the test function $\varphi_{h,k}(u_n)=T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))$ in (3.3), we have

$$\int_{\Omega}a_n(x,u_n)\nabla u_n\nabla\varphi_{h,k}(u_n)+\int_{\Omega}\Phi_n(x,u_n)\nabla\varphi_{h,k}(u_n) \tag{4.11}$$

$$=\int_{\Omega}f_n\varphi_{h,k}(u_n).$$

In what follows, we study the behavior of each term of (4.11) as $n\to+\infty$ and $h\to+\infty$. By (4.5), we have $\varphi_{h,k}(u_n)$ converges to $T_{2k}(u-T_h(u))$ almost everywhere in Ω as $n\to+\infty$ and that $T_{2k}(u-T_h(u))$ goes to zero as h tends to $+\infty$, so, by the Lebesgue's convergence theorem, we obtain

$$\int_{\Omega}f_n\varphi_{h,k}(u_n)=\omega(n,h). \tag{4.12}$$

Let $M=4k+h$, for $n>M$, one can write,

$$\int_{\Omega}a_n(x,u_n)\nabla u_n\nabla\varphi_{h,k}(u_n) \tag{4.13}$$

$$=\int_{\Omega}a(x,T_k(u_n))\nabla T_k(u_n)\nabla(T_k(u_n)-T_k(u))$$

$$+\int_{\{|u_n|>k\}} a(x, T_M(u_n))\nabla u_n \nabla(u_n - T_h(u_n))$$
$$-\int_{\{|u_n|>k\}} a(x, T_M(u_n))\nabla T_M(u_n)\nabla T_k(u).$$

Using (2.3), (4.5) and (4.6) yield that $a(x, T_M(u_n))\nabla T_M(u_n)$ converges weakly in $(L^2(\Omega))^N$ to $a(x, T_M(u))\nabla T_M(u)$ and that $\nabla T_k(u)\chi_{\{|u_n|>k\}}$ converges strongly to zero in $(L^2(\Omega))^N$. Moreover, since the second term on the right hand side of (4.13) is positive, we deduce that

$$\int_\Omega a_n(x, u_n)\nabla u_n \nabla\varphi_{h,k}(u_n) \tag{4.14}$$
$$\geq \int_\Omega a(x, T_k(u_n))\nabla T_k(u_n)\nabla(T_k(u_n) - T_k(u)) + \omega(n).$$

Now, we deal with the second term in the left hand side of (4.11), we have for $n > M$

$$\int_\Omega \Phi_n(x, u_n)\nabla\varphi_{h,k}(u_n) \tag{4.15}$$
$$= \int_\Omega \Phi(x, T_k(u_n))\nabla(T_k(u_n) - T_k(u))$$
$$+\int_{\{|u_n|>k\}} \Phi(x, T_M(u_n))\nabla(u_n - T_h(u_n))\chi_{\{|u_n|\leq M\}}$$
$$-\int_{\{|u_n|>k\}} \Phi(x, T_M(u_n))\nabla T_k(u).$$

Due to the assumption (2.4), one has $|\Phi(x, T_k(u_n))| \leq Cc(x) \in L^2(\Omega)$ where C is a constant depending on k. On the other hand, by (4.5) we have

$$\Phi(x, T_k(u_n)) \to \Phi(x, T_k(u)) \text{ a.e. in } \Omega.$$

Then, by Lebesgue's convergence theorem, we deduce that

$$\Phi(x, T_k(u_n)) \to \Phi(x, T_k(u)) \text{ strongly in } (L^2(\Omega))^N.$$

Moreoever, using (4.6) and the fact that u is almost everywhere finite, we obtain

$$\int_\Omega \Phi(x, T_k(u_n))\nabla(T_k(u_n) - T_k(u)) = \omega(n), \tag{4.16}$$

$$\int_{\{|u_n|>k\}} \Phi(x, T_M(u_n))\nabla T_k(u) = \omega(n), \tag{4.17}$$

and

$$\int_{\{|u_n|>k\}} \Phi(x, T_M(u_n))\nabla T_M(u_n)\chi_{\{u_n>h\}} \tag{4.18}$$
$$= \int_{\{|u|>k\}} \Phi(x, T_M(u))\nabla T_M(u)\chi_{\{u>h\}} + \omega(n) = \omega(n, h).$$

Putting together (4.12), (4.14), (4.17) and (4.18), from (4.11) it follows that

$$\int_\Omega a(x, T_k(u_n))\nabla T_k(u_n)\nabla(T_k(u_n) - T_k(u)) \le \omega(n, h).$$

Moreover, writing

$$\int_\Omega a(x, T_k(u_n))|\nabla(T_k(u_n) - T_k(u))|^2$$

$$= \int_\Omega a(x, T_k(u_n))\nabla T_k(u_n)\nabla(T_k(u_n) - T_k(u))$$

$$- \int_\Omega a(x, T_k(u_n))\nabla T_k(u)\nabla(T_k(u_n) - T_k(u)),$$

so, by (4.6), letting n tends to infinity, we obtain

$$\int_\Omega a(x, T_k(u_n))|\nabla(T_k(u_n) - T_k(u))|^2 = \omega(n).$$

Moreover, using (2.2), we conclude that (4.10) holds.

Now we pass to the limit in the approximated problem (3.3). Let S be a function in $W^{1,\infty}(\mathbb{R})$ with compact support, contained in $[-k, k]$, $k > 0$ and let $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Using $S(u_n)\varphi$ as test function in (3.3) we have

$$\int_\Omega S'(u_n)\alpha(u_n)a_n(x, u_n)|\nabla u_n|^2\varphi + \int_\Omega S(u_n)a_n(x, u_n)\nabla u_n\nabla\varphi \tag{4.19}$$

$$+ \int_\Omega S(u_n)\Phi_n(x, u_n)\nabla\varphi + \int_\Omega S'(u_n)\alpha(u_n)\Phi_n(x, u_n)\nabla u_n\varphi$$

$$= \int_\Omega f_n S(u_n)\varphi.$$

Since S has a compact support contained in $[-k, k]$, the strong convergence of f_n to f in $L^1(\Omega)$ together with (4.5) imply that

$$\int_\Omega f_n S(u_n)\varphi = \int_\Omega f_n S(u)\varphi + \omega(n).$$

For $n > k$, using assumption (2.4), the pointwise convergence of u_n to u together with the Lebesgue's convergence theorem yield that

$$\int_\Omega S(u_n)\Phi_n(x, u_n)\nabla\varphi = \int_\Omega S(u)\Phi(x, u)\nabla\varphi + \omega(n).$$

Similarly by (4.6) we obtain

$$\int_\Omega S'(u_n)\Phi_n(x, u_n)\nabla u_n\varphi = \int_\Omega S'(u)\Phi(x, u)\nabla u\varphi + \omega(n).$$

In view of (2.3) and (4.6) we obtain

$$\int_\Omega S(u_n)a_n(x,u_n)\nabla u_n\nabla\varphi = \int_\Omega S(u_n)a(x,T_k(u_n))\nabla T_k(u_n)\nabla\varphi$$
$$= \int_\Omega S(u)a(x,u)\nabla u\nabla\varphi + \omega(n).$$

Finally, thanks to (4.10) we get

$$\int_\Omega S'(u_n)a_n(x,u_n)|\nabla u_n|^2\varphi$$
$$= \int_\Omega S'(u_n)\alpha(u_n)a(x,T_k(u_n))|\nabla T_k(u_n)|^2\varphi$$
$$= \int_\Omega S'(u_n)a(x,u)|\nabla u|^2\varphi + \omega(n).$$

Gathering all the previous results, we deduce that the condition (2.9) in the definition of renormalized solution holds. The condition (2.8) follows from (4.7) and (4.10). Since u is finite almost everywhere in Ω and since $T_k(u) \in H_0^1(\Omega)$ for every $k > 0$, we deduce that u is a renormalized solution of problem (2.1) and the proof of Theorem 4.1 is completed. □

5 Non Coercive Operator with a Lower Order Term

In this section, we consider the following problem similar to (2.1) of the form

$$\begin{cases} -\operatorname{div}\Big(a(x,u)\nabla u + \Phi(x,u)\Big) + g(u) = f \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{5.1}$$

where g is a continuous function in $\mathbb{R}$ such that:

$$g(s)s \geq 0, \ \forall s \in \mathbb{R}, \tag{5.2}$$

$$\lim_{s\to\pm\infty} |g(s)| = +\infty. \tag{5.3}$$

We assume that there exist $\delta_1,\ \delta_2 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{|s|}{(1+|\widetilde{\alpha}(s)|)^{\delta_2}(1+|g(s)|)^{\delta_1}} = \ell' \in \mathbb{R}^+,$$

which means the existence of a real number $k_1 > 0$ and a constant $C > 0$ such that for every $|s| > k_1$, one has

$$|s| \leq C(1+|\widetilde{\alpha}(s)|)^{\delta_2}(1+|g(s)|)^{\delta_1}. \tag{5.4}$$

As we said in the introduction, the presence of the lower order term g is crucial in the sense that it guarantees to existence of renormalized solutions when $\theta = 1$ and $\beta = 0$.

Theorem 5.1 *Assume that (2.2)-(2.4) and (5.2)-(5.3) hold with $r > N$ and $m > \frac{N}{2}$. Furthermore, we suppose that $\theta = 1$, $\beta = 0$. If $\delta_1 < \frac{1}{N} - \frac{1}{2m}$ and $\delta_2 = 1$, then there exists a weak solution u for (2.1) in the sense of Definition 2.1.*

Proof Let us consider the following approximate problem similar to (3.3) admitting a solution $u_n \in H_0^1(\Omega)$ by Schauder's fixed point theorem.

$$\begin{cases} -\operatorname{div}\Big(a_n(x,u_n)\nabla u_n\Big) - \operatorname{div}\Big(\Phi_n(x,u_n)\Big) + g(u_n) = f_n \ \text{ in } \Omega, \\ u_n = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{5.5}$$

By taking $T_k(u_n)$ as test function in (5.5), using (5.2), it's easy to check that

$$g(u_n) \text{ is bounded in } L^1(\Omega). \tag{5.6}$$

Now, let $j > 0$, by (5.3), there exists $j_0 > 0$ such that $|g(s)| \geq j$ for every $j \geq j_0$. Then, using (5.6), we obtain

$$meas(\{|u_n| > j\}) \leq \frac{1}{j}\|g(u_n)\|_{L^1(\Omega)} \leq \frac{C}{j},$$

which leads to

$$\lim_{j\to+\infty} \sup_n meas\{|u_n| \geq j\} = 0. \tag{5.7}$$

Thus, (4.6) and Fatou's lemma yield that u is almost everywhere finite in Ω.

As in the proof of Theorem 3.2, we use $\Psi(\widetilde{\alpha}(u_n))$ as test function in (5.5), dropping the positive term, using assumptions (2.2), (2.4), condition (5.4) and for $k \geq k_1$, we obtain

$$\alpha_0 \int_{\{|\widetilde{\alpha}(u_n)|>k\}} \frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2} \leq C\int_{\{|\widetilde{\alpha}(u_n)|>k\}} c(x)(1+|g(u_n)|)^{\delta_1}\frac{|\nabla\widetilde{\alpha}(u_n)|}{1+|\widetilde{\alpha}(u_n)|} + \int_{\{|\widetilde{\alpha}(u_n)|>k\}} |f|,$$

by Young inequality, we obtain

$$\frac{\alpha_0}{2}\int_{\{|\widetilde{\alpha}(u_n)|>k\}} \frac{|\nabla\widetilde{\alpha}(u_n)|^2}{(1+|\widetilde{\alpha}(u_n)|)^2} \leq C\int_{\{|\widetilde{\alpha}(u_n)|>k\}} c^2(x)(1+|g(u_n)|)^{2\delta_1} + \int_{\{|\widetilde{\alpha}(u_n)|>k\}} |f|,$$

and applying Hölder inequality with $\dfrac{1}{m} + 2\delta_1 + \dfrac{m-1-2\delta_1 m}{m} = 1$, it results

$$\int_{A_k} |\nabla \log(1+|\widetilde{\alpha}(u_n)|)|^2 \leq C\|c\|_{L^{2m}(A_k)}\Big(\int_{A_k}(1+|g(u_n)|)\Big)^{2\delta_1}|A_k|^{\frac{m-1-2\delta_1 m}{m}} + \|f\|_{L^m(A_k)}|A_k|^{\frac{m-1-2\delta_1 m}{m}}|\Omega|^{2\delta_1},$$

where $A_k = \{|\widetilde{\alpha}(u_n)| > k\}$. Thus, thanks to (5.6) and the proof of Therem 3.2, it follows that

$$|A_l| \leq \frac{C}{(l-h)^{2^*}}\Big(\|c\|_{L^{2m}(\Omega)} + \|f\|_{L^m(\Omega)}\Big)^{\frac{2^*}{2}}|A_h|^{\frac{2^*}{2}(\frac{m-1-2\delta_1 m}{m})} \ \forall l > h.$$

Since $m > \frac{N}{2}$ and $\delta_1 < \frac{1}{N} - \frac{1}{2m}$ imply that $\frac{2^*}{2}(\frac{m-1-2\delta_1 m}{m}) > 1$. Then, applying Lemma 3.1, there exists k^* such that $|\{w > k^*\}| = 0$, that is, u_n is bounded. □

Theorem 5.2 *Assume that* (2.2)-(2.5), *with* $r = N$, $m = 1$, $\beta = 0$ *and* $\theta = 1$. *Assume that* (5.4) *holds with* $\delta_2 \in (0, 1)$ *and* $\delta_1 = \frac{1-\delta_2}{2^*}$. *Then there exists at least a renormalized solution* u *for* (2.1) *in the sense of Definition* 2.2.

Proof Due to (5.6) and (5.7), the proof of Theorem 5.2 is similar to one of Theorem 4.1, the only difference is the convergence result (4.7). In order to prove it, we use $\frac{1}{m}T_m(\widetilde{\alpha}(u_n))$ as test function in (5.5), dropping the positive term and using (5.4) with $\delta_1 = \frac{1-\delta_2}{2^*}$ give

$$\begin{aligned} &\frac{1}{m}\int_{\{|\widetilde{\alpha}(u_n)|\le m\}} |\nabla\widetilde{\alpha}(u_n)|^2 \\ &\le \frac{1}{m}\int_{\{|\widetilde{\alpha}(u_n)|\le m\}} c(x)(1+|\widetilde{\alpha}(u_n)|)^{\delta_2}(1+|g(u_n)|)^{\frac{1-\delta_2}{2^*}}|\nabla T_m(\widetilde{\alpha}(u_n))| \\ &\quad + \frac{1}{m}\int_\Omega f_n T_m(\widetilde{\alpha}(u_n)). \end{aligned} \tag{5.8}$$

Now we estimate the first term in the right hand side of (5.8), using Hölder inequality with $\frac{1}{N} + \frac{\delta_2}{2^*} + \frac{1-\delta_2}{2^*} + \frac{1}{2} = 1$ and by (5.6), we obtain

$$\begin{aligned} &\frac{1}{m}\int_{\{|\widetilde{\alpha}(u_n)|\le m\}} c(x)(1+|\widetilde{\alpha}(u_n)|)^{\delta_2}(1+|g(u_n)|)^{\frac{1-\delta_2}{2^*}}|\nabla T_m(\widetilde{\alpha}(u_n))| \\ &\le \frac{C}{m}\|c\|_{L^N(\Omega)}\Big(\int_\Omega |T_m(\widetilde{\alpha}(u_n))|^{2^*}\Big)^{\frac{\delta_2}{2^*}}\Big(\int_\Omega (1+|g(u_n)|)\Big)^{\frac{1-\delta_2}{2^*}}\Big(\int_\Omega |\nabla T_m(\widetilde{\alpha}(u_n))|^2\Big)^{\frac{1}{2}} \\ &\quad + \frac{C}{m}\|c\|_{L^N(\Omega)}\Big(\int_\Omega (1+|g(u_n)|)\Big)^{\frac{1-\delta_2}{2^*}}\Big(\int_\Omega |\nabla T_m(\widetilde{\alpha}(u_n))|^2\Big)^{\frac{1}{2}} \\ &\le \frac{C}{m}\|c\|_{L^N(\Omega)}\Big(\int_\Omega |T_m(\widetilde{\alpha}(u_n))|^{2^*}\Big)^{\frac{\delta_2}{2^*}}\Big(\int_\Omega |\nabla T_m(\widetilde{\alpha}(u_n))|^2\Big)^{\frac{1}{2}} \\ &\quad + \frac{C}{m}\|c\|_{L^N(\Omega)}\Big(\int_\Omega |\nabla T_m(\widetilde{\alpha}(u_n))|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Using Sobolev and Young inequalities, it yields that

$$\begin{aligned} &\frac{1}{m}\int_{\{|\widetilde{\alpha}(u_n)|\le m\}} |\nabla\widetilde{\alpha}(u_n)|^2 \\ &\le \frac{C}{m}\Big(\|c\|_{L^N(\Omega)}^{\frac{2}{2-\delta_2}} + \|c\|_{L^N(\Omega)}^{\frac{2}{1-\delta_2}}\Big) + \frac{1}{m}\int_\Omega f_n T_m(\widetilde{\alpha}(u_n)). \end{aligned} \tag{5.9}$$

We pass to the limit in each term in the right hand side of (5.9) as n and m tends to infinity respectively. Since the first term in the right hand side easily goes to zero as $m \to +\infty$, using Lebesgue's convergence theorem and the fact that u is finite almost everywhere in Ω, we deduce that

$$\frac{1}{m}\int_{\Omega} f_n T_m(\widetilde{\alpha}(u_n)) = \omega(n,m).$$

Thus, (4.7) holds true. At last, repeating the proof of Theorem 4.1, we conclude that u is a renormalized solution of (5.1). Therefore, the proof Theorem 5.1 is completely proved. □

Data Availability Data sharing is not applicable to this article as no new data were generated or analysed during the current study.

Declarations

Competing Interests The authors declare no conflict of interest.

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