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Stabilization Effects of Magnetic Field on a 2D Anisotropic MHD System with Partial Dissipation

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Abstract
To uncover that the magnetic field mechanism can stabilize electrically conducting turbulent fluids, we investigate the stability of a special two dimensional anisotropic MHD system with vertical dissipation in the horizontal velocity component and partial magnetic damping near a background magnetic field. Since the MHD system has only vertical dissipation in the horizontal velocity and vertical magnetic damping, the stability issue and large time behavior problem of the linearized magneto-hydrodynamic system is not trivial. By performing refined energy estimates on the linear system coupled with a careful analysis of the nonlinearities, the stability of a MHD-type system near a background magnetic field is justified for the initial data belonging to $H^3(\mathbf{R}^2)$ space. The authors also build the explicit decay rates of the linearized system.

Keywords MHD equations · Stability · Background magnetic filed · Large-time behavior

Mathematics Subject Classification (2020) 35Q35 · 76D03 · 76E25

1 Introduction

The standard two dimensional incompressible magnetohydrodynamic(MHD) equations obey

$$\begin{cases} \partial_t U + U\cdot\nabla U - \nu\Delta U + \nabla\Pi = B\cdot\nabla B, \\ \partial_t B + U\cdot\nabla B - \mu\Delta B = B\cdot\nabla U, \\ \nabla\cdot U = \nabla\cdot B = 0, \\ U(x,0) = U_0(x), \quad B(x,0) = B_0(x), \end{cases} \tag{1.1}$$

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where $U(t,x)=(U_1(t,x),U_2(t,x))$, $B(t,x)=(B_1(t,x),B_2(t,x))$ are the velocity field and the magnetic field, respectively, Π is scalar pressure. $\nu\ge 0,\mu\ge 0$ denote the kinematic viscosity and the magnetic diffusivity. The system (1.1) comes from the geophysics, astrophysics, cosmology and has been applied in engineering (see [10, 19]). Many mathematicians and physicians have devoted efforts to addressing some fundamental issues on the MHD equations such as well-posedness, stability and large time behavior problems. In 1972, Duvaut and Lions [12] built the local well-posedness in Sobolev spaces and established the global existence with small initial data. Later on, Sermange and Temam [23] established the global well-posedness of the MHD equations (1.1). Since then, there is a large amount of literature on the global well-posedness and regularity issue of the MHD equations with various partial or fractional dissipation (see [6–9]).

In this paper, we will investigate the incompressible MHD fluid system equations:

$$\begin{cases}\partial_t U+U\cdot\nabla U-\nu\begin{pmatrix}\partial_2^2U_1\\0\end{pmatrix}+\nabla\Pi=B\cdot\nabla B,\\ \partial_t B+U\cdot\nabla B+\eta\begin{pmatrix}0\\B_2\end{pmatrix}=B\cdot\nabla U,\\ \nabla\cdot U=\nabla\cdot B=0,\\ U|_{t=0}=U_0,\ B|_{t=0}=B_0.\end{cases}\tag{1.2}$$

Let

$$U^{(0)}=0,\qquad B^{(0)}=e_1=(1,0),$$

which is a steady solution of (1.2). The perturbation (u,b) with

$$u=U-U^{(0)},\qquad b=B-B^{(0)}.$$

satisfying

$$\begin{cases}\partial_t u+u\cdot\nabla u-\nu\begin{pmatrix}\partial_2^2u_1\\0\end{pmatrix}+\nabla\pi=b\cdot\nabla b+\partial_1 b,\\ \partial_t b+u\cdot\nabla b+\eta\begin{pmatrix}0\\b_2\end{pmatrix}=b\cdot\nabla u+\partial_1 u,\\ \nabla\cdot u=\nabla\cdot b=0,\\ u|_{t=0}=u_0,\ b|_{t=0}=b_0.\end{cases}\tag{1.3}$$

During the past thirty years, more and more attention has been paid to the stability and large time behavior problems to the MHD equations with partial dissipations near a background magnetic field. Some studies which stated above are concerned with the fundamental nonlinear phenomena associated with electrically conducting fluids. Many physicians found that the magnetic field can stabilize electrically conducting fluids (see [1, 2, 13, 14]). It is very interesting to understand the stability problem concerning the partially dissipative MHD equations near the background magnetic field. There are substantial developments on the stability problem of the MHD equations with various dissipation near a background magnetic field (see [3, 5, 11, 15, 17, 20–22, 24–26]). The purpose of this paper is to investigate the smoothing and stabilizing effect of the magnetic field on the fluid motion.

Recently, Lai, Wu and Zhang [16] consider the following MHD system

$$\begin{cases} \partial_t u + u\cdot\nabla u - \nu\partial_2^2 u + \nabla\pi = b\cdot\nabla b + \partial_1 b, \\ \partial_t b + u\cdot\nabla b + \eta\begin{pmatrix} 0 \\ b_2 \end{pmatrix} = b\cdot\nabla u + \partial_1 u, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u|_{t=0} = u_0,\ b|_{t=0} = b_0. \end{cases} \tag{1.4}$$

They established the stability of the system (1.4) near a magnetic background. Later on, Lai informed us that they have obtained the stability issue of the system (1.3) with $\partial_2^2 u_1$ replaced by the damping term u_1 in a coming paper.

For convenience, we assume that $\nu = \eta = 1$ throughout this paper. Applying the Helmholtz-Leray projection $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to the equation (1.3) and using the fact that $\nabla\cdot u = \nabla\cdot b = 0$, one has

$$\mathbb{P}\begin{pmatrix} \partial_2^2 u_1 \\ 0 \end{pmatrix} = \Delta^{-1}\partial_2^4 u \triangleq \mathcal{T}_2^2 u,$$

$$\mathbb{P}\begin{pmatrix} 0 \\ b_2 \end{pmatrix} = \Delta^{-1}\partial_1^2 b \triangleq -\mathcal{R}_1^2 b.$$

Then the system (1.3) converts into

$$\begin{cases} \partial_t u = \mathcal{T}_2^2 u + \partial_1 b + \mathbb{P}(b\cdot\nabla b - u\cdot\nabla u), \\ \partial_t b = \mathcal{R}_1^2 b + \partial_1 u + \mathbb{P}(b\cdot\nabla u - u\cdot\nabla b), \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u|_{t=0} = u_0,\ b|_{t=0} = b_0. \end{cases} \tag{1.5}$$

Differentiating (1.5) in t and making several substitutions, we can convert (1.3) into the following new system

$$\begin{cases} \partial_{tt} u - (\mathcal{R}_1^2 + \mathcal{T}_2^2)\partial_t u - \partial_1^2 u + \mathcal{R}_1^2\mathcal{T}_2^2 u = N_1, \\ \partial_{tt} b - (\mathcal{R}_1^2 + \mathcal{T}_2^2)\partial_t b - \partial_1^2 b + \mathcal{R}_1^2\mathcal{T}_2^2 b = N_2, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u|_{t=0} = u_0,\ b|_{t=0} = b_0, \end{cases} \tag{1.6}$$

where N_1 and N_2 are the nonlinear terms,

$$N_1 = (\partial_t - \mathcal{R}_1^2)\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u) + \partial_1\mathbb{P}(b\cdot\nabla u - u\cdot\nabla b),$$

$$N_2 = (\partial_t - \mathcal{T}_2^2)\mathbb{P}(b\cdot\nabla u - u\cdot\nabla b) + \partial_1\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u).$$

Our stability result can be stated as follows.

Theorem 1.1 *Assume* $(u_0, b_0) \in H^3(\mathbf{R}^2)$ *with* $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. *Then there exists a sufficient small* $\varepsilon > 0$ *such that, if*

$$\|u_0\|_{H^3} + \|b_0\|_{H^3} \le \varepsilon. \tag{1.7}$$

Then (1.3) *has a unique global solution that remains uniformly bounded for all time*,

$$\|u\|_{H^3}^2 + \|b\|_{H^3}^2 + \int_0^t \|\partial_2 u_1(\tau)\|_{H^3}^2 d\tau + \int_0^t \|b_2(\tau)\|_{H^3}^2 d\tau + \int_0^t \|\partial_1 u(\tau)\|_{H^2}^2 d\tau \le C\varepsilon^2,$$

where $C > 0$ *is a pure constant.*

Remark 1.2 Due to the lack of the vertical dissipation $\partial_2^2 u_2$, the system (1.3) has less dissipation than that of the system (1.4). We need to establish the H^3-stability and build more subtle energy estimates than that in [16], which obtained the stability in H^2 space.

Remark 1.3 Now we explain why we need the H^3 regularity imposed on the initial data. The most difficult term encountering in the proof is $\int_{\mathbf{R}^2} \partial_1^2 u_2 \partial_2 b_1 \partial_1^2 b_2 dx$. In order to establish the stability, we are obliged to prove the bad term is uniformly integrable over $[0, t]$. Thanks to the Hölder inequality, Sobolev inequality, one has

$$\begin{aligned}
\int_0^t \int_{\mathbf{R}^2} \partial_1^2 u_2 \partial_2 b_1 \partial_1^2 b_2 dx d\tau &\le \int_0^t \|\partial_1^2 u_2\|_{L^2} \|\partial_2 b_1\|_{L^\infty} \|\partial_1^2 b_2\|_{L^2} d\tau \\
&\le C \int_0^t \|\partial_1 u_2\|_{H^2} \|\partial_2 b_1\|_{H^2} \|b_2\|_{H^3} d\tau \\
&\le C \left(\int_0^t \|\partial_1 u_2\|_{H^2}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|b_2\|_{H^3}^2 d\tau \right)^{\frac{1}{2}} \|b_1\|_{L_t^\infty(H^3)}.
\end{aligned}$$

We would like to remark that it is natural to consider the stability problem in anisotropic Sobolev or Besov spaces. To examine this issue, one need to build a subtle energy structure to build an priori estimate. For instance, in the references (see [4, 17]), the authors deal with similar operators (Riesz) in an anisotropic functional setting.

Our second main result explores the large-time behavior of solutions to the linearized system

$$\begin{cases}
\partial_t u - \mathcal{T}_2^2 u - \partial_1 b = 0, \\
\partial_t b - \mathcal{R}_1^2 b - \partial_1 u = 0, \\
\nabla \cdot u = \nabla \cdot b = 0, \\
u|_{t=0} = u_0, \ b|_{t=0} = b_0,
\end{cases} \tag{1.8}$$

which can be converted to the following system of wave equations

$$\begin{cases}
\partial_{tt} u - (\mathcal{R}_1^2 + \mathcal{T}_2^2) \partial_t u - \partial_1^2 u + \mathcal{R}_1^2 \mathcal{T}_2^2 u = 0, \\
\partial_{tt} b - (\mathcal{R}_1^2 + \mathcal{T}_2^2) \partial_t b - \partial_1^2 b + \mathcal{R}_1^2 \mathcal{T}_2^2 b = 0, \\
\nabla \cdot u = \nabla \cdot b = 0, \\
u|_{t=0} = u_0, \ b|_{t=0} = b_0.
\end{cases} \tag{1.9}$$

According to Lemma 2.3, we need to show that $\int_0^t (\|u(\tau)\|_{H^1}^2 + \|b(\tau)\|_{H^1}^2 d\tau$ is finite which requires $(\Lambda_1^{-\sigma}, \Lambda_2^{-2\sigma}) u_0 \in H^{1+\sigma}$, $(\Lambda_1^{-\sigma}, \Lambda_2^{-2\sigma}) b_0 \in H^{1+\sigma}$. We refer the readers to Sect. 4 for details.

Now we give the definition of the partial fractional derivative Λ_i^γ with $i=1,\ 2$ and $\gamma \in \mathbf{R}$ as follows

$$\widehat{\Lambda_i^\gamma f}(\xi) = |\xi_i|^\gamma \hat{f}(\xi).$$

Theorem 1.4 *Let $\sigma > 0$. Assume (u_0, b_0) satisfies*

$$(\Lambda_1^{-\sigma}, \Lambda_2^{-2\sigma})u_0 \in H^{1+\sigma}, \quad (\Lambda_1^{-\sigma}, \Lambda_2^{-2\sigma})b_0 \in H^{1+\sigma}, \quad \nabla \cdot u_0 = \nabla \cdot b_0 = 0.$$

Then the corresponding solution (u, b) of (1.8) satisfies

$$(u, b) \in L^\infty(0, \infty; H^1), \quad (\mathcal{T}_2 u, \mathcal{R}_1 b) \in L^2(0, \infty; H^1).$$

Moreover,

$$\|u(t)\|_{H^1} + \|b(t)\|_{H^1} \le C(1+t)^{-\frac{\sigma}{2}},$$

for any $t > 0$, where $C > 0$ is a positive constant depending on u_0 and b_0,

The next theorem is to evaluate the decay estimates of $\partial_1 u$ and $\partial_1 b$, the extra terms generalized by the perturbation near the background magnetic field $B_0 = (1, 0, 0)$.

Theorem 1.5 *Assume (u_0, b_0) satisfies*

$$\nabla \cdot u_0 = \nabla \cdot b_0 = 0, \quad (u_0, b_0) \in H^1, \quad (\mathcal{T}_2^2 u_0, \mathcal{R}_1^2 b_0) \in L^2,$$
$$(\Lambda\Lambda_1^{-1}\mathcal{T}_2^2 u_0, \Lambda\Lambda_1^{-1}\mathcal{T}_2^2 b_0) \in L^2, \quad (\Lambda_1^{-1}\mathcal{R}_1\mathcal{T}_2 u_0, \Lambda_1^{-1}\mathcal{R}_1\mathcal{T}_2 b_0) \in L^2.$$

Then the corresponding solution (u, b) of (1.9) satisfies

$$\|\partial_t u(t)\|_{L^2} + \|\partial_1 u(t)\|_{L^2} + \|\mathcal{R}_1\mathcal{T}_2 u(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}},$$
$$\|\partial_t b(t)\|_{L^2} + \|\partial_1 b(t)\|_{L^2} + \|\mathcal{R}_1\mathcal{T}_2 b(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}},$$

for any $t > 0$, where $C > 0$ is a constant depending on u_0 and b_0.

A brief outline of the proof. We first present the main ideas in the proof of Theorem 1.1. The method we use to prove Theorem 1.1 is based on a bootstrapping argument. A natural part of the energy functional is

$$E_1(t) = \|u\|_{H^3}^2 + \|b\|_{H^3}^2 + \int_0^t \|\partial_2 u_1(\tau)\|_{H^3}^2 d\tau + \int_0^t \|b_2(\tau)\|_{H^3}^2 d\tau. \tag{1.10}$$

This part is not sufficient to bound the bad term $\int \partial_1 u_1 (\partial_2^3 b_1)^2 dx$ emerged in the H^3 estimate. We have to seek another part of the energy functional

$$E_2(t) = \int_0^t \|\partial_1 u(\tau)\|_{H^2}^2 d\tau, \tag{1.11}$$

which can help us to control those bad terms. We need also verify that $E_2(t)$ can be bounded by a combination of $E_1(t)$ and $E_2(t)$.

Let $E(t) = E_1(t) + \delta E_2(t)$. From Proposition 3.1 and Proposition 3.2, one has

$$E_1(t) \le C(E_1(0) + E_1^{\frac{3}{2}}(0) + E_1^2(0)) + C(E_1^{\frac{3}{2}}(t) + E_2^{\frac{3}{2}}(t)) \\ + C(E_1^2(t) + E_2^2(t)) + C(E_1^{\frac{5}{2}}(t) + E_2^{\frac{5}{2}}(t)),$$

and

$$E_2(t) \le CE_1(0) + CE_1(t) + CE_1^{\frac{3}{2}}(t) + CE_2^{\frac{3}{2}}(t).$$

These two inequalities which together with the definition of $E(t)$ give rise to

$$E(t) \le E(0) + CE^{\frac{3}{2}}(t) + CE^2(t) + CE^{\frac{5}{2}}(t).$$

Thus a bootstrap argument guarantees us the global existence and stability.

We would like to remark that it is very difficult to obtain the large time behavior of the nonlinear problem. The characteristic polynomial associated with (1.9) is given by

$$\lambda^2 + \frac{\xi_1^2 + \xi_2^4}{|\xi|^2}\lambda + \xi_1^2 + \frac{\xi_1^2\xi_2^4}{|\xi|^4} = 0, \tag{1.12}$$

whose eigenvalues are

$$\lambda_1 = \frac{-(\xi_1^2 + \xi_2^4) + \sqrt{(\xi_1^2 - \xi_2^4)^2 - 4\xi_1^2|\xi|^4}}{|\xi|^2}, \quad \lambda_2 = \frac{-(\xi_1^2 + \xi_2^4) - \sqrt{(\xi_1^2 - \xi_2^4)^2 - 4\xi_1^2|\xi|^4}}{|\xi|^2}.$$

It is very easy to verify that $\lambda_1 \sim 0$ as $\xi \to 0$. When the solution is represented by the integral form, the degeneracy of λ_1 makes the decay evaluation of nonlinear terms extremely difficult. It is not easy to estabilish the explicit decay rate of linearized system due to the degenerate of the eigenvalue λ_1.

The rest of this paper is organized as follows. Section 2 presents some crucial Lemmas. The proof of Theorem 1.1 is performed in Sect. 3. Section 4 is devoted to the proof of Theorem 1.4 and that of Theorem 1.5.

2 Preliminaries

In this section, we state some important lemmas which can be found in [18].

Lemma 2.1 *Assume that $f, g, h, \partial_1 f$ and $\partial_2 g$ are all in $L^2(\mathbf{R}^2)$. Then there exists a generic constant $C > 0$ such that*

$$\int_{\mathbf{R}^2} |fgh| dx \le C\|f\|_{L^2(\mathbf{R}^2)}^{\frac{1}{2}} \|\partial_1 f\|_{L^2(\mathbf{R}^2)}^{\frac{1}{2}} \|g\|_{L^2(\mathbf{R}^2)}^{\frac{1}{2}} \|\partial_2 g\|_{L^2(\mathbf{R}^2)}^{\frac{1}{2}} \|h\|_{L^2(\mathbf{R}^2)}.$$

Lemma 2.2 *Assume that $f, \partial_1 f, \partial_2 f$ are bounded in $H^1(\mathbf{R}^2)$, it holds that*

$$\|f\|_{L^\infty(\mathbf{R}^2)} \le C\|f\|_{L^2(\mathbf{R}^2)}^{\frac{1}{4}} \|\partial_1 f\|_{L^2(\mathbf{R}^2)}^{\frac{1}{4}} \|\partial_2 f\|_{L^2(\mathbf{R}^2)}^{\frac{1}{4}} \|\partial_1\partial_2 f\|_{L^2(\mathbf{R}^2)}^{\frac{1}{4}},$$

and

$$\|f\|_{L^\infty(\mathbf{R}^2)} \le C\|f\|_{H^1(\mathbf{R}^2)}^{\frac{1}{2}}\|\partial_1 f\|_{H^1(\mathbf{R}^2)}^{\frac{1}{2}},$$
$$\|f\|_{L^\infty(\mathbf{R}^2)} \le C\|f\|_{H^1(\mathbf{R}^2)}^{\frac{1}{2}}\|\partial_2 f\|_{H^1(\mathbf{R}^2)}^{\frac{1}{2}}.$$

Lemma 2.3 *For given positive constants $C_0, C_1, C_2 > 0$, assume that $f = f(t)$ is a nonnegative function defined on $[0, \infty)$ and satisfies,*

$$\int_0^\infty f(\tau) \le C_0 < \infty, \quad f(t) \le C_1 f(s), \quad 0 \le s < t.$$

Then there exists a positive constant $C_2 = \max\{2C_1 f(0), 4C_0C_1\}$ such that

$$f(t) \le C_2(1+t)^{-1}, \quad \forall t \ge 0.$$

3 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. We first introduce the following energy functional,

$$E(t) = E_1(t) + E_2(t),$$

where

$$E_1(t) = \sup_{0\le\tau\le t} (\|u(\tau)\|_{H^3}^2 + \|b(\tau)\|_{H^3}^2) + \int_0^t \|\partial_2 u_1(\tau)\|_{H^3}^2 + \int_0^t \|b_2(\tau)\|_{H^3}^2, \tag{3.1}$$

$$E_2(t) = \int_0^t \|\partial_1 u(\tau)\|_{H^2}^2. \tag{3.2}$$

3.1 A Priori Estimates

Proposition 3.1 *It holds that*

$$\begin{aligned} E_1(t) \le & C(E_1(0) + E_1^{\frac{3}{2}}(0) + E_1^2(0)) + C(E_1^{\frac{3}{2}}(t) + E_2^{\frac{3}{2}}(t)) \\ & + C(E_1^2(t) + E_2^2(t)) + C(E_1^{\frac{5}{2}}(t) + E_2^{\frac{5}{2}}(t)), \end{aligned}$$

where C is a pure positive constant.

Proof **Step 1. L^2-estimates**

Taking the L^2- inner product to the equations (1.3) with (u, b), one obtains

$$\frac{1}{2}\frac{d}{dt}\|(u,b)\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2 + \|b_2\|_{L^2}^2 = 0. \tag{3.3}$$

Step 2. $\dot{H}^3$-estimates

Applying ∂_i^3 $(i=1,2)$ to (1.3), and then taking the L^2- inner product with $(\partial_i^3 u, \partial_i^3 b)$ to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|(\partial_i^3 u, \partial_i^3 b)\|_{L^2}^2 + \|\partial_2\partial_i^3 u_1\|_{L^2}^2 + \|\partial_i^3 b_2\|_{L^2}^2 =& -\int \partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u dx \\
&+\int \partial_i^3(b\cdot\nabla b)\cdot\partial_i^3 u dx - \int \partial_i^3(u\cdot\nabla b)\cdot\partial_i^3 b dx + \int \partial_i^3(b\cdot\nabla u)\cdot\partial_i^3 b dx \\
&= -\sum_{1\le\alpha\le3} C_3^\alpha \int \partial_i^\alpha u\cdot\nabla\partial_i^{3-\alpha} u\cdot\partial_i^3 u dx + \sum_{1\le\alpha\le3} C_3^\alpha \int \partial_i^\alpha b\cdot\nabla\partial_i^{3-\alpha} b\cdot\partial_i^3 u dx \\
&- \sum_{1\le\alpha\le3} C_3^\alpha \int \partial_i^\alpha u\cdot\nabla\partial_i^{3-\alpha} b\cdot\partial_i^3 b dx + \sum_{1\le\alpha\le3} C_3^\alpha \int \partial_i^\alpha b\cdot\nabla\partial_i^{3-\alpha} u\cdot\partial_i^3 b dx \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where we use the fact that $\nabla\cdot u = \nabla\cdot b = 0$. We write

$$\begin{aligned}
I_1 =& -\sum_{1\le\alpha\le3} C_3^\alpha \int \partial_i^\alpha u\cdot\nabla\partial_i^{3-\alpha} u\cdot\partial_i^3 u dx \\
=& -\sum_{1\le\alpha\le3} C_3^\alpha \int \partial_1^\alpha u\cdot\nabla\partial_1^{3-\alpha} u\cdot\partial_1^3 u dx - \sum_{1\le\alpha\le3} C_3^\alpha \int \partial_2^\alpha u\cdot\nabla\partial_2^{3-\alpha} u\cdot\partial_2^3 u dx \\
=& I_{11} + I_{12}.
\end{aligned}$$

According to the Hölder inequality, one gets

$$\begin{aligned}
I_{11} =& -\sum_{1\le\alpha\le3} C_3^\alpha \int \partial_1^\alpha u\cdot\nabla\partial_1^{3-\alpha} u\cdot\partial_1^3 u dx \\
=& -3\int \partial_1 u\cdot\nabla\partial_1^2 u\cdot\partial_1^3 u dx - 3\int \partial_1^2 u\cdot\nabla\partial_1 u\cdot\partial_1^3 u dx - \int \partial_1^3 u\cdot\nabla u\cdot\partial_1^3 u dx \\
\le& C\|\partial_1^3 u\|_{L^2}(\|\partial_1 u\|_{L^\infty}\|\nabla\partial_1^2 u\|_{L^2} + \|\partial_1^2 u\|_{L^4}\|\nabla\partial_1 u\|_{L^4} + \|\partial_1^3 u\|_{L^2}\|\nabla u\|_{L^\infty}) \\
\le& C\|u\|_{H^3}\|\partial_1 u\|_{H^2}^2.
\end{aligned}$$

To deal with I_{12}, we decompose it into the following three parts

$$\begin{aligned}
I_{12} =& -\sum_{1\le\alpha\le3} C_3^\alpha \int \partial_2^\alpha u\cdot\nabla\partial_2^{3-\alpha} u\cdot\partial_2^3 u dx \\
=& -\sum_{1\le\alpha\le3} C_3^\alpha \int \partial_2^\alpha u_1\partial_1\partial_2^{3-\alpha} u\cdot\partial_2^3 u dx - \sum_{1\le\alpha\le3} C_3^\alpha \int \partial_2^\alpha u_2\partial_2^{4-\alpha} u_1\partial_2^3 u_1 dx \\
&- \sum_{1\le\alpha\le3} C_3^\alpha \int \partial_2^\alpha u_2\partial_2^{4-\alpha} u_2\partial_2^3 u_2 dx \\
=& I_{121} + I_{122} + I_{123}.
\end{aligned}$$

Similar to estimate I_{11}, we find

$$\begin{aligned}
I_{121} =& - \sum_{1\leq\alpha\leq 3} C_3^\alpha \int \partial_2^\alpha u_1 \partial_1 \partial_2^{3-\alpha} u \cdot \partial_2^3 u dx \\
=& - 3\int \partial_2 u_1 \partial_1 \partial_2^2 u \cdot \partial_2^3 u dx - 3\int \partial_2^2 u_1 \partial_1 \partial_2 u \cdot \partial_2^3 u dx - \int \partial_2^3 u_1 \partial_1 u \cdot \partial_2^3 u dx \\
\leq& C\|\partial_2^3 u\|_{L^2} (\|\partial_2 u_1\|_{L^\infty} \|\partial_1 \partial_2^2 u\|_{L^2} + \|\partial_2^2 u_1\|_{L^4} \|\partial_1 \partial_2 u\|_{L^4} + \|\partial_2^3 u_1\|_{L^2} \|\partial_1 u\|_{L^\infty}) \\
\leq& C\|u\|_{H^3} (\|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}$$

Thanks to the Hölder inequality, we have

$$\begin{aligned}
I_{122} =& - \sum_{1\leq\alpha\leq 3} C_3^\alpha \int \partial_2^\alpha u_2 \partial_2^{4-\alpha} u_1 \partial_2^3 u_1 dx \\
=& - 3\int \partial_2 u_2 \partial_2^3 u_1 \partial_2^3 u_1 dx - 3\int \partial_2^2 u_2 \partial_2^2 u_1 \partial_2^3 u_1 dx - \int \partial_2^3 u_2 \partial_2 u_1 \partial_2^3 u_1 dx \\
\leq& C\|\partial_2^3 u_1\|_{L^2} (\|\partial_2 u_2\|_{L^\infty} \|\partial_2^3 u_1\|_{L^2} + \|\partial_2^2 u_2\|_{L^2} \|\partial_2^2 u_1\|_{L^\infty} + \|\partial_2^3 u_2\|_{L^2} \|\partial_2 u_1\|_{L^\infty}) \\
\leq& C\|u\|_{H^3} \|\partial_2 u_1\|_{H^3}^2.
\end{aligned}$$

According to the Hölder inequality and the fact that $\partial_2 u_2 = -\partial_1 u_1$, we get

$$\begin{aligned}
I_{123} =& - \sum_{1\leq\alpha\leq 3} C_3^\alpha \int \partial_2^\alpha u_2 \partial_2^{4-\alpha} u_2 \partial_2^3 u_2 dx \\
=& - 4\int \partial_2 u_2 \partial_2^3 u_2 \partial_2^3 u_2 dx - 3\int \partial_2^2 u_2 \partial_2^2 u_2 \partial_2^3 u_2 dx \\
=& - 4\int \partial_2 u_2 \partial_2^2 \partial_1 u_1 \partial_2^2 \partial_1 u_1 dx - 3\int \partial_2^2 u_2 \partial_2 \partial_1 u_1 \partial_2^2 \partial_1 u_1 dx \\
\leq& C\|\partial_2^2 \partial_1 u_1\|_{L^2} (\|\partial_2^2 \partial_1 u_1\|_{L^2} \|\partial_2 u_2\|_{L^\infty} + \|\partial_2^2 u_2\|_{L^4} \|\partial_2 \partial_1 u_1\|_{L^4}) \\
\leq& C\|u\|_{H^3} \|\partial_2 u_1\|_{H^3}^2.
\end{aligned}$$

Hence, we get

$$I_1 \leq C\|u\|_{H^3} (\|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).$$

To estimate I_2, we split it into five parts

$$\begin{aligned}
I_2 =& \sum_{1\leq\alpha\leq 3} C_3^\alpha \int \partial_i^\alpha b \cdot \nabla \partial_i^{3-\alpha} b \cdot \partial_i^3 u dx \\
=& \sum_{1\leq\alpha\leq 3} C_3^\alpha \int \partial_1^\alpha b \cdot \nabla \partial_1^{3-\alpha} b \cdot \partial_1^3 u dx + \sum_{1\leq\alpha\leq 3} C_3^\alpha \int \partial_2^\alpha b \cdot \nabla \partial_2^{3-\alpha} b \cdot \partial_2^3 u dx \\
=& \sum_{1\leq\alpha\leq 3} C_3^\alpha \int \partial_1^\alpha b \cdot \nabla \partial_1^{3-\alpha} b \cdot \partial_1^3 u dx + \sum_{1\leq\alpha\leq 3} C_3^\alpha \int \partial_2^\alpha b_1 \partial_1 \partial_2^{3-\alpha} b_1 \partial_2^3 u_1 dx
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{1\le\alpha\le3} C_3^\alpha \int \partial_2^\alpha b_1 \partial_1 \partial_2^{3-\alpha} b_2 \partial_2^3 u_2 dx + \sum_{1\le\alpha\le3} C_3^\alpha \int \partial_2^\alpha b_2 \partial_2^{4-\alpha} b_1 \partial_2^3 u_1 dx \\
&+\sum_{1\le\alpha\le3} C_3^\alpha \int \partial_2^\alpha b_2 \partial_2^{4-\alpha} b_2 \partial_2^3 u_2 dx \\
=&I_{21}+I_{22}+\cdots+I_{25}.
\end{aligned}
$$

According to the Hölder inequality and the Sobolev inequality, we conclude

$$
\begin{aligned}
I_{21}=&\sum_{1\le\alpha\le3} C_3^\alpha \int \partial_1^\alpha b\cdot\nabla\partial_1^{3-\alpha} b\cdot\partial_1^3 u dx \\
=&3\int \partial_1 b\cdot\nabla\partial_1^2 b\cdot\partial_1^3 u dx + 3\int \partial_1^2 b\cdot\nabla\partial_1 b\cdot\partial_1^3 u dx + \int \partial_1^3 b\cdot\nabla b\cdot\partial_1^3 u dx \\
\le&C\|\partial_1^3 u\|_{L^2}(\|\partial_1 b\|_{L^\infty}\|\nabla\partial_1^2 b\|_{L^2} + \|\partial_1^2 b\|_{L^4}\|\nabla\partial_1 b\|_{L^4} + \|\partial_1^3 b\|_{L^2}\|\nabla b\|_{L^\infty}) \\
\le&C\|b\|_{H^3}(\|b_2\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2),
\end{aligned}
$$

where we have used the fact that $\|\partial_1 b\|_{H^2}\le C\|b_2\|_{H^3}$. Similarly to the derivation of I_{21}, we have

$$
\begin{aligned}
I_{22}=&\sum_{1\le\alpha\le3} C_3^\alpha \int \partial_2^\alpha b_1 \partial_1 \partial_2^{3-\alpha} b_1 \partial_2^3 u_1 dx \\
=&3\int \partial_2 b_1 \partial_1 \partial_2^2 b_1 \partial_2^3 u_1 dx + 3\int \partial_2^2 b_1 \partial_1 \partial_2 b_1 \partial_2^3 u_1 dx + \int \partial_2^3 b_1 \partial_1 b_1 \partial_2^3 u_1 dx \\
\le&C\|\partial_2^3 u_1\|_{L^2}(\|\partial_2 b_1\|_{L^\infty}\|\partial_1\partial_2^2 b_1\|_{L^2} + \|\partial_2^2 b_1\|_{L^4}\|\partial_1\partial_2 b_1\|_{L^4} + \|\partial_2^3 b_1\|_{L^2}\|\partial_1 b_1\|_{L^\infty}) \\
\le&C\|b\|_{H^3}(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2).
\end{aligned}
$$

Thanks to the Hölder inequality again, we have

$$
\begin{aligned}
I_{23}=&\sum_{1\le\alpha\le3} C_3^\alpha \int \partial_2^\alpha b_1 \partial_1 \partial_2^{3-\alpha} b_2 \partial_2^3 u_2 dx \\
=&3\int \partial_2 b_1 \partial_1 \partial_2^2 b_2 \partial_2^3 u_2 dx + 3\int \partial_2^2 b_1 \partial_1 \partial_2 b_2 \partial_2^3 u_2 dx + \int \partial_2^3 b_1 \partial_1 b_2 \partial_2^3 u_2 dx \\
\le&C\|\partial_2^3 u_2\|_{L^2}(\|\partial_2 b_1\|_{L^\infty}\|\partial_1\partial_2^2 b_2\|_{L^2} + \|\partial_2^2 b_1\|_{L^4}\|\partial_1\partial_2 b_2\|_{L^4} + \|\partial_2^3 b_1\|_{L^2}\|\partial_1 b_2\|_{L^\infty}) \\
\le&C\|b\|_{H^3}(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2),
\end{aligned}
$$

$$
\begin{aligned}
I_{24}=&\sum_{1\le\alpha\le3} C_3^\alpha \int \partial_2^\alpha b_2 \partial_2^{4-\alpha} b_1 \partial_2^3 u_1 dx \\
=&3\int \partial_2 b_2 \partial_2^3 b_1 \partial_2^3 u_1 dx + 3\int \partial_2^2 b_2 \partial_2^2 b_1 \partial_2^3 u_1 dx + \int \partial_2^3 b_2 \partial_2 b_1 \partial_2^3 u_1 dx \\
\le&C\|\partial_2^3 u_1\|_{L^2}(\|\partial_2 b_2\|_{L^\infty}\|\partial_2^3 b_1\|_{L^2} + \|\partial_2^2 b_2\|_{L^4}\|\partial_2^2 b_1\|_{L^4} + \|\partial_2^3 b_2\|_{L^2}\|\partial_2 b_1\|_{L^\infty}) \\
\le&C\|b\|_{H^3}(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2),
\end{aligned}
$$

and

$$
\begin{aligned}
I_{25} = &\sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_2^\alpha b_2 \partial_2^{4-\alpha} b_2 \partial_2^3 u_2 dx\\
=&4\int \partial_2 b_2 \partial_2^3 b_2 \partial_2^3 u_2 dx + 3\int \partial_2^2 b_2 \partial_2^2 b_2 \partial_2^3 u_2 dx\\
\le& C\|\partial_2^3 u_2\|_{L^2}(\|\partial_2 b_2\|_{L^\infty}\|\partial_2^3 b_2\|_{L^2} + \|\partial_2^2 b_2\|_{L^4}^2)\\
\le& C\|u\|_{H^3}\|b_2\|_{H^3}^2.
\end{aligned}
$$

Thus we get

$$
I_2 \le C(\|u\|_{H^3} + \|b\|_{H^3})(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
$$

Now we write

$$
\begin{aligned}
I_3 =& -\sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_i^\alpha u\cdot\nabla\partial_i^{3-\alpha} b\cdot\partial_i^3 b dx\\
=& -\sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_1^\alpha u\cdot\nabla\partial_1^{3-\alpha} b\cdot\partial_1^3 b dx - \sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_2^\alpha u\cdot\nabla\partial_2^{3-\alpha} b\cdot\partial_2^3 b dx\\
=& -\sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_1^\alpha u\cdot\nabla\partial_1^{3-\alpha} b\cdot\partial_1^3 b dx - \sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_2^\alpha u_1 \partial_1\partial_2^{3-\alpha} b\cdot\partial_2^3 b dx\\
&- \sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_2^\alpha u_2 \partial_2^{4-\alpha} b\cdot\partial_2^3 b dx\\
=& I_{31} + I_{32} + I_{33}.
\end{aligned}
$$

Thanks to the Hölder inequality and Sobolev embedding, we get

$$
\begin{aligned}
I_{31} =& -\sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_1^\alpha u\cdot\nabla\partial_1^{3-\alpha} b\cdot\partial_1^3 b dx\\
=& -3\int \partial_1 u\cdot\nabla\partial_1^2 b\cdot\partial_1^3 b dx - 3\int \partial_1^2 u\cdot\nabla\partial_1 b\cdot\partial_1^3 b dx - \int \partial_1^3 u\cdot\nabla b\cdot\partial_1^3 b dx\\
\le& C\|\partial_1^3 b\|_{L^2}(\|\partial_1 u\|_{L^\infty}\|\nabla\partial_1^2 b\|_{L^2} + \|\partial_1^2 u\|_{L^4}\|\nabla\partial_1 b\|_{L^4} + \|\partial_1^3 u\|_{L^2}\|\nabla b\|_{L^\infty})\\
\le& C\|b\|_{H^3}(\|b_2\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_{32} =& -\sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_2^\alpha u_1 \partial_1\partial_2^{3-\alpha} b\cdot\partial_2^3 b dx\\
=& -3\int \partial_2 u_1 \partial_1\partial_2^2 b\cdot\partial_2^3 b dx - 3\int \partial_2^2 u_1 \partial_1\partial_2 b\cdot\partial_2^3 b dx - \int \partial_2^3 u_1 \partial_1 b\cdot\partial_2^3 b dx\\
\le& C\|\partial_2^3 b\|_{L^2}(\|\partial_2 u_1\|_{L^\infty}\|\partial_1\partial_2^2 b\|_{L^2} + \|\partial_2^2 u_1\|_{L^4}\|\partial_1\partial_2 b\|_{L^4} + \|\partial_2^3 u_1\|_{L^2}\|\partial_1 b\|_{L^\infty})\\
\le& C\|b\|_{H^3}(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2).
\end{aligned}
$$

To bound I_{33}, we further decompose it into three terms

$$
\begin{aligned}
I_{33} =& -\sum_{1\le\alpha\le3} C_3^\alpha \int \partial_2^\alpha u_2 \partial_2^{4-\alpha} b \cdot \partial_2^3 b dx \\
=& -3\int \partial_2 u_2 \partial_2^3 b \cdot \partial_2^3 b dx - 3\int \partial_2^2 u_2 \partial_2^2 b \cdot \partial_2^3 b dx - \int \partial_2^3 u_2 \partial_2 b \cdot \partial_2^3 b dx \\
=& I_{331} + I_{332} + I_{333}.
\end{aligned}
$$

By the Hölder inequality and the Sobolev inequality, we obtain

$$
\begin{aligned}
I_{331} =& -3\int \partial_2 u_2 \partial_2^3 b \cdot \partial_2^3 b dx \\
=& -3\int \partial_2 u_2 \partial_2^3 b_1 \partial_2^3 b_1 dx - 3\int \partial_2 u_2 \partial_2^3 b_2 \partial_2^3 b_2 dx \\
\le& C\|\partial_2 u_2\|_{L^\infty} \|\partial_2^3 b_2\|_{L^2}^2 + 3\int \partial_1 u_1 (\partial_2^3 b_1)^2 dx \\
\le& C\|u\|_{H^3} \|b_2\|_{H^3}^2 + 3\int \partial_1 u_1 (\partial_2^3 b_1)^2 dx.
\end{aligned}
$$

Integrating by parts and using the fact that $\partial_2 u_2 = -\partial_1 u_1$, we get

$$
\begin{aligned}
I_{332} =& -\frac{3}{2}\int \partial_2^2 u_2 \partial_2 (\partial_2^2 b)^2 dx = \frac{3}{2}\int \partial_2^3 u_2 (\partial_2^2 b)^2 dx \\
=& -\frac{3}{2}\int \partial_2^2 \partial_1 u_1 (\partial_2^2 b)^2 dx = 3\int \partial_2^2 u_1 \partial_2^2 b \cdot \partial_1 \partial_2^2 b dx \\
\le& C\|\partial_2^2 u_1\|_{L^4} \|\partial_2^2 b\|_{L^4} \|\partial_1 \partial_2^2 b\|_{L^2} \\
\le& C\|b\|_{H^3} (\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2).
\end{aligned}
$$

In view of the divergence-free conditions, integration by parts and the Hölder inequality, one can obtain

$$
\begin{aligned}
I_{333} =& \int \partial_2^2 \partial_1 u_1 \partial_2 b \cdot \partial_2^3 b dx \\
=& -\int \partial_2^2 u_1 \partial_2 \partial_1 b \cdot \partial_2^3 b dx - \int \partial_2^2 u_1 \partial_2 b \cdot \partial_2^3 \partial_1 b dx \\
=& -\int \partial_2^2 u_1 \partial_2 \partial_1 b \cdot \partial_2^3 b dx + \int \partial_2^3 u_1 \partial_2 b \cdot \partial_2^2 \partial_1 b dx + \int \partial_2^2 u_1 \partial_2^2 b \cdot \partial_2^2 \partial_1 b dx \\
\le& C\|\partial_2^2 u_1\|_{L^4} \|\partial_2 \partial_1 b\|_{L^4} \|\partial_2^3 b\|_{L^2} + (\|\partial_2^3 u_1\|_{L^2} \|\partial_2 b\|_{L^\infty} + \|\partial_2^2 u_1\|_{L^4} \|\partial_2^2 b\|_{L^4}) \|\partial_2^2 \partial_1 b\|_{L^2} \\
\le& C\|b\|_{H^3} (\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2).
\end{aligned}
$$

Then, it leads to

$$
I_3 \le C(\|u\|_{H^3} + \|b\|_{H^3})(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2) + 3\int \partial_1 u_1 (\partial_2^3 b_1)^2 dx.
$$

In order to bound I_4, we decompose it into the following terms

$$
\begin{aligned}
I_4 = & \sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_i^\alpha b\cdot\nabla\partial_i^{3-\alpha}u\cdot\partial_i^3 b dx \\
= & \sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_1^\alpha b\cdot\nabla\partial_1^{3-\alpha}u\cdot\partial_1^3 b dx + \sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_2^\alpha b\cdot\nabla\partial_2^{3-\alpha}u\cdot\partial_2^3 b dx \\
= & \sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_1^\alpha b\cdot\nabla\partial_1^{3-\alpha}u\cdot\partial_1^3 b dx + \sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_2^\alpha b_1\partial_1\partial_2^{3-\alpha}u_1\partial_2^3 b_1 dx \\
& + \sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_2^\alpha b_1\partial_1\partial_2^{3-\alpha}u_2\partial_2^3 b_2 dx + \sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_2^\alpha b_2\partial_2^{4-\alpha}u_1\partial_2^3 b_1 dx \\
& + \sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_2^\alpha b_2\partial_2^{4-\alpha}u_2\partial_2^3 b_2 dx \\
= & I_{41} + I_{42} + \cdots + I_{45}.
\end{aligned}
$$

Thanks to the Hölder inequality, one has

$$
\begin{aligned}
I_{41} = & \sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_1^\alpha b\cdot\nabla\partial_1^{3-\alpha}u\cdot\partial_1^3 b dx \\
= & 3\int \partial_1 b\cdot\nabla\partial_1^2 u\cdot\partial_1^3 b dx + 3\int \partial_1^2 b\cdot\nabla\partial_1 u\cdot\partial_1^3 b dx + \int \partial_1^3 b\cdot\nabla u\cdot\partial_1^3 b dx \\
\le & C\|\partial_1^3 b\|_{L^2}(\|\partial_1 b\|_{L^\infty}\|\nabla\partial_1^2 u\|_{L^2} + \|\partial_1^2 b\|_{L^4}\|\nabla\partial_1 u\|_{L^4} + \|\partial_1^3 b\|_{L^2}\|\nabla u\|_{L^\infty}) \\
\le & C\|u\|_{H^3}\|b_2\|_{H^3}^2.
\end{aligned}
$$

By using integration by parts several times and the Hölder inequality, we deduce

$$
\begin{aligned}
I_{42} = & \sum_{1\le\alpha\le 3} C_3^\alpha \int \partial_2^\alpha b_1\partial_1\partial_2^{3-\alpha}u_1\partial_2^3 b_1 dx \\
= & 3\int \partial_2 b_1\partial_1\partial_2^2 u_1\partial_2^3 b_1 dx + 3\int \partial_2^2 b_1\partial_1\partial_2 u_1\partial_2^3 b_1 dx + \int \partial_1 u_1(\partial_2^3 b_1)^2 dx \\
= & -3\int \partial_2\partial_1 b_1\partial_2^2 u_1\partial_2^3 b_1 dx - 3\int \partial_2 b_1\partial_2^2 u_1\partial_2^3\partial_1 b_1 dx + \frac{3}{2}\int \partial_1\partial_2 u_1\partial_2(\partial_2^2 b_1)^2 dx \\
& + \int \partial_1 u_1(\partial_2^3 b_1)^2 dx \\
= & -3\int \partial_2\partial_1 b_1\partial_2^2 u_1\partial_2^3 b_1 dx + 3\int \partial_2^2 b_1\partial_2^2 u_1\partial_2^2\partial_1 b_1 dx \\
& + 3\int \partial_2 b_1\partial_2^3 u_1\partial_2^2\partial_1 b_1 dx - \frac{3}{2}\int \partial_1\partial_2^2 u_1(\partial_2^2 b_1)^2 dx + \int \partial_1 u_1(\partial_2^3 b_1)^2 dx \\
= & -3\int \partial_2\partial_1 b_1\partial_2^2 u_1\partial_2^3 b_1 dx + 3\int \partial_2^2 b_1\partial_2^2 u_1\partial_2^2\partial_1 b_1 dx \\
& + 3\int \partial_2 b_1\partial_2^3 u_1\partial_2^2\partial_1 b_1 dx + 3\int \partial_2^2 u_1\partial_2^2 b_1\partial_1\partial_2^2 b_1 dx + \int \partial_1 u_1(\partial_2^3 b_1)^2 dx
\end{aligned}
$$

$$
\begin{aligned}
\leq & C\|\partial_2\partial_1 b_1\|_{L^4}\|\partial_2^2 u_1\|_{L^4}\|\partial_2^3 b_1\|_{L^2} + C\|\partial_2^2 b_1\|_{L^4}\|\partial_2^2 u_1\|_{L^4}\|\partial_2^2\partial_1 b_1\|_{L^2} \\
& + C\|\partial_2 b_1\|_{L^\infty}\|\partial_2^3 u_1\|_{L^2}\|\partial_2^2\partial_1 b_1\|_{L^2} + C\|\partial_2^2 u_1\|_{L^4}\|\partial_2^2 b_1\|_{L^4}\|\partial_1\partial_2^2 b_1\|_{L^2} \\
& + \int \partial_1 u_1(\partial_2^3 b_1)^2 dx \\
\leq & C\|b\|_{H^3}(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2) + \int \partial_1 u_1(\partial_2^3 b_1)^2 dx.
\end{aligned}
$$

According to the Hölder inequality and Sobolev inequality again, one arrives at

$$
\begin{aligned}
I_{43} = & \sum_{1\leq\alpha\leq 3} C_3^\alpha \int \partial_2^\alpha b_1 \partial_1 \partial_2^{3-\alpha} u_2 \partial_2^3 b_2 dx \\
= & 3\int \partial_2 b_1 \partial_1 \partial_2^2 u_2 \partial_2^3 b_2 dx + 3\int \partial_2^2 b_1 \partial_1 \partial_2 u_2 \partial_2^3 b_2 dx + \int \partial_2^3 b_1 \partial_1 u_2 \partial_2^3 b_2 dx \\
\leq & C\|\partial_2^3 b_2\|_{L^2}(\|\partial_2 b_1\|_{L^\infty}\|\partial_1\partial_2^2 u_2\|_{L^2} + \|\partial_2^2 b_1\|_{L^4}\|\partial_1\partial_2 u_2\|_{L^4} + \|\partial_2^3 b_1\|_{L^2}\|\partial_1 u_2\|_{L^\infty}) \\
\leq & C\|b\|_{H^3}(\|b_2\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}
$$

Similarly, it leads to

$$
\begin{aligned}
I_{44} = & \sum_{1\leq\alpha\leq 3} C_3^\alpha \int \partial_2^\alpha b_2 \partial_2^{4-\alpha} u_1 \partial_2^3 b_1 dx \\
= & 3\int \partial_2 b_2 \partial_2^3 u_1 \partial_2^3 b_1 dx + 3\int \partial_2^2 b_2 \partial_2^2 u_1 \partial_2^3 b_1 dx + \int \partial_2^3 b_2 \partial_2 u_1 \partial_2^3 b_1 dx \\
\leq & C\|\partial_2^3 b_1\|_{L^2}(\|\partial_2 b_2\|_{L^\infty}\|\partial_2^3 u_1\|_{L^2} + \|\partial_2^2 b_2\|_{L^4}\|\partial_2^2 u_1\|_{L^4} + \|\partial_2^3 b_2\|_{L^2}\|\partial_2 u_1\|_{L^\infty}) \\
\leq & C\|b\|_{H^3}(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2).
\end{aligned}
$$

and

$$
\begin{aligned}
I_{45} = & \sum_{1\leq\alpha\leq 3} C_3^\alpha \int \partial_2^\alpha b_2 \partial_2^{4-\alpha} u_2 \partial_2^3 b_2 dx \\
= & 3\int \partial_2 b_2 \partial_2^3 u_2 \partial_2^3 b_2 dx + 3\int \partial_2^2 b_2 \partial_2^2 u_2 \partial_2^3 b_2 dx + \int \partial_2^3 b_2 \partial_2 u_2 \partial_2^3 b_2 dx \\
\leq & C\|\partial_2^3 b_2\|_{L^2}(\|\partial_2 b_2\|_{L^\infty}\|\partial_2^3 u_2\|_{L^2} + \|\partial_2^2 b_2\|_{L^4}\|\partial_2^2 u_2\|_{L^4} + \|\partial_2^3 b_2\|_{L^2}\|\partial_2 u_2\|_{L^\infty}) \\
\leq & C\|u\|_{H^3}\|b_2\|_{H^3}^2.
\end{aligned}
$$

Collecting the estimate from I_{41} to I_{45} to give

$$
I_4 \leq C(\|u\|_{H^3} + \|b\|_{H^3})(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2) + \int \partial_1 u_1(\partial_2^3 b_1)^2 dx.
$$

Then one has

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|(u,b)\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|b_2\|_{H^3}^2 \\
\leq & C\|(u,b)\|_{H^3}(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2) + 4\int \partial_1 u_1(\partial_2^3 b_1)^2 dx.
\end{aligned} \tag{3.4}
$$

To handle the worst term $\int \partial_1 u_1 (\partial_2^3 b_1)^2 dx$, thanks to the following equation

$$\partial_1 u_1 = \partial_t b_1 + u \cdot \nabla b_1 - b \cdot \nabla u_1, \tag{3.5}$$

one can verify

$$\begin{aligned}
\int \partial_1 u_1 (\partial_2^3 b_1)^2 dx &= \int (\partial_t b_1 + u \cdot \nabla b_1 - b \cdot \nabla u_1)(\partial_2^3 b_1)^2 dx \\
&= \frac{d}{dt} \int b_1 (\partial_2^3 b_1)^2 dx - 2 \int b_1 \partial_2^3 b_1 \partial_2^3 \partial_t b_1 + \int (u \cdot \nabla b_1)(\partial_2^3 b_1)^2 dx \\
&\quad - \int (b \cdot \nabla u_1)(\partial_2^3 b_1)^2 dx \\
&= \frac{d}{dt} \int b_1 (\partial_2^3 b_1)^2 dx + 2 \int b_1 \partial_2^3 b_1 \partial_2^3 (u \cdot \nabla b_1) dx - 2 \int b_1 \partial_2^3 b_1 \partial_2^3 (b \cdot \nabla u_1) dx \\
&\quad - 2 \int b_1 \partial_2^3 b_1 \partial_2^3 \partial_1 u_1 dx + \int (u \cdot \nabla b_1)(\partial_2^3 b_1)^2 dx - \int (b \cdot \nabla u_1)(\partial_2^3 b_1)^2 dx \\
&= \frac{d}{dt} \int b_1 (\partial_2^3 b_1)^2 dx + 2 \sum_{1 \le \alpha \le 3} C_3^\alpha \int b_1 \partial_2^3 b_1 \partial_2^\alpha u \cdot \nabla \partial_2^{3-\alpha} b_1 dx + 2 \int b_1 \partial_2^3 b_1 u \cdot \nabla \partial_2^3 b_1 dx \\
&\quad - 2 \int b_1 \partial_2^3 b_1 \partial_2^3 (b \cdot \nabla u_1) dx - 2 \int b_1 \partial_2^3 b_1 \partial_2^3 \partial_1 u_1 dx \\
&\quad + \int (u \cdot \nabla b_1)(\partial_2^3 b_1)^2 dx - \int (b \cdot \nabla u_1)(\partial_2^3 b_1)^2 dx \\
&= J_1 + J_2 + \cdots + J_7.
\end{aligned}$$

By $\nabla \cdot u = 0$, we can easily verify

$$\begin{aligned}
J_3 + J_6 &= 2 \int b_1 \partial_2^3 b_1 u \cdot \nabla \partial_2^3 b_1 dx + \int (u \cdot \nabla b_1)(\partial_2^3 b_1)^2 dx \\
&= \int b_1 u \cdot \nabla (\partial_2^3 b_1)^2 dx + \int (u \cdot \nabla b_1)(\partial_2^3 b_1)^2 dx \\
&= \int u \cdot \nabla (b_1 (\partial_2^3 b_1)^2) dx \\
&= 0.
\end{aligned}$$

To deal with J_2, we first split it into the following four terms,

$$\begin{aligned}
J_2 &= 2 \sum_{1 \le \alpha \le 3} C_3^\alpha \int b_1 \partial_2^3 b_1 \partial_2^\alpha u \cdot \nabla \partial_2^{3-\alpha} b_1 dx \\
&= 2 \sum_{1 \le \alpha \le 3} C_3^\alpha \int b_1 \partial_2^3 b_1 \partial_2^\alpha u_1 \partial_1 \partial_2^{3-\alpha} b_1 dx + 2 \sum_{1 \le \alpha \le 3} C_3^\alpha \int b_1 \partial_2^3 b_1 \partial_2^\alpha u_2 \partial_2^{4-\alpha} b_1 dx \\
&= 2 \sum_{1 \le \alpha \le 3} C_3^\alpha \int b_1 \partial_2^3 b_1 \partial_2^\alpha u_1 \partial_1 \partial_2^{3-\alpha} b_1 dx + 6 \int b_1 \partial_2^3 b_1 \partial_2 u_2 \partial_2^3 b_1 dx
\end{aligned}$$

$$
\begin{aligned}
&+6\int b_1\partial_2^3 b_1\partial_2^2 u_2\partial_2^2 b_1 dx+2\int b_1\partial_2^3 b_1\partial_2^3 u_2\partial_2 b_1 dx\\
=&J_{21}+J_{22}+J_{23}+J_{24}.
\end{aligned}
$$

For the first term, the Hölder inequality guarantees that

$$
\begin{aligned}
J_{21}=&2\sum_{1\le\alpha\le3}C_3^\alpha\int b_1\partial_2^3 b_1\partial_2^\alpha u_1\partial_1\partial_2^{3-\alpha}b_1 dx\\
=&6\int b_1\partial_2^3 b_1\partial_2 u_1\partial_1\partial_2^2 b_1 dx+6\int b_1\partial_2^3 b_1\partial_2^2 u_1\partial_1\partial_2 b_1 dx+2\int b_1\partial_2^3 b_1\partial_2^3 u_1\partial_1 b_1 dx\\
\le&C\|b_1\|_{L^\infty}\|\partial_2^3 b_1\|_{L^2}(\|\partial_2 u_1\|_{L^\infty}\|\partial_1\partial_2^2 b_1\|_{L^2}+\|\partial_2^2 u_1\|_{L^4}\|\partial_1\partial_2 b_1\|_{L^4}\\
&+\|\partial_2^3 u_1\|_{L^2}\|\partial_1 b_1\|_{L^\infty})\\
\le&C\|b\|_{H^3}^2(\|b_2\|_{H^3}^2+\|\partial_2 u_1\|_{H^3}^2).
\end{aligned}
$$

In term of (3.5), we get

$$
\begin{aligned}
J_{22}=&6\int b_1\partial_2^3 b_1\partial_2 u_2\partial_2^3 b_1 dx=-6\int b_1\partial_1 u_1(\partial_2^3 b_1)^2 dx\\
=&-6\int b_1(\partial_t b_1+u\cdot\nabla b_1-b\cdot\nabla u_1)(\partial_2^3 b_1)^2 dx\\
=&-3\int \partial_t(b_1)^2(\partial_2^3 b_1)^2 dx-6\int b_1(u\cdot\nabla b_1)(\partial_2^3 b_1)^2 dx+6\int b_1(b\cdot\nabla u_1)(\partial_2^3 b_1)^2 dx\\
=&-3\frac{d}{dt}\int (b_1)^2(\partial_2^3 b_1)^2 dx+6\int (b_1)^2\partial_2^3 b_1\partial_2^3\partial_t b_1 dx\\
&-6\int b_1(u\cdot\nabla b_1)(\partial_2^3 b_1)^2 dx+6\int b_1(b\cdot\nabla u_1)(\partial_2^3 b_1)^2 dx\\
=&-3\frac{d}{dt}\int (b_1)^2(\partial_2^3 b_1)^2 dx-6\int (b_1)^2\partial_2^3 b_1\partial_2^3(u\cdot\nabla b_1) dx\\
&+6\int (b_1)^2\partial_2^3 b_1\partial_2^3(b\cdot\nabla u_1) dx\\
&+6\int (b_1)^2\partial_2^3 b_1\partial_2^3\partial_1 u_1 dx-6\int b_1(u\cdot\nabla b_1)(\partial_2^3 b_1)^2 dx+6\int b_1(b\cdot\nabla u_1)(\partial_2^3 b_1)^2 dx\\
=&-3\frac{d}{dt}\int (b_1)^2(\partial_2^3 b_1)^2 dx-6\sum_{1\le\alpha\le3}C_3^\alpha\int (b_1)^2\partial_2^3 b_1\partial_2^\alpha u\cdot\nabla\partial_2^{3-\alpha}b_1 dx\\
&-6\int (b_1)^2\partial_2^3 b_1 u\cdot\nabla\partial_2^3 b_1 dx+6\int (b_1)^2\partial_2^3 b_1\partial_2^3(b\cdot\nabla u_1) dx\\
&+6\int (b_1)^2\partial_2^3 b_1\partial_2^3\partial_1 u_1 dx-6\int b_1(u\cdot\nabla b_1)(\partial_2^3 b_1)^2 dx+6\int b_1(b\cdot\nabla u_1)(\partial_2^3 b_1)^2 dx\\
=&N_1+N_2+\cdots+N_7.
\end{aligned}
$$

According to $\nabla \cdot u = 0$, we obtain

$$\begin{aligned}
N_3 + N_6 =& - 6 \int (b_1)^2 \partial_2^3 b_1 u \cdot \nabla \partial_2^3 b_1 dx - 6 \int b_1 (u \cdot \nabla b_1)(\partial_2^3 b_1)^2 dx \\
=& - 3 \int u \cdot \nabla (\partial_2^3 b_1)^2 (b_1)^2 dx - 3 \int u \cdot \nabla (b_1)^2 (\partial_2^3 b_1)^2 dx \\
=& - 3 \int u \cdot \nabla ((\partial_2^3 b_1)^2 (b_1)^2) dx \\
=& 0.
\end{aligned}$$

Thanks to the Hölder inequality and Lemma 2.2, we have

$$\begin{aligned}
N_2 =& - 6 \sum_{1 \le \alpha \le 3} C_3^\alpha \int (b_1)^2 \partial_2^3 b_1 \partial_2^\alpha u \cdot \nabla \partial_2^{3-\alpha} b_1 dx \\
=& - 6 \sum_{1 \le \alpha \le 3} C_3^\alpha \int (b_1)^2 \partial_2^3 b_1 \partial_2^\alpha u_1 \partial_1 \partial_2^{3-\alpha} b_1 dx \\
& - 6 \sum_{1 \le \alpha \le 3} C_3^\alpha \int (b_1)^2 \partial_2^3 b_1 \partial_2^\alpha u_2 \partial_2^{4-\alpha} b_1 dx \\
=& - 18 \int (b_1)^2 \partial_2^3 b_1 \partial_2 u_1 \partial_1 \partial_2^2 b_1 dx - 18 \int (b_1)^2 \partial_2^3 b_1 \partial_2^2 u_1 \partial_1 \partial_2 b_1 dx \\
& - 6 \int (b_1)^2 \partial_2^3 b_1 \partial_2^3 u_1 \partial_1 b_1 dx \\
& - 18 \int (b_1)^2 \partial_2^3 b_1 \partial_2 u_2 \partial_2^3 b_1 dx - 18 \int (b_1)^2 \partial_2^3 b_1 \partial_2^2 u_2 \partial_2^2 b_1 dx \\
& - 6 \int (b_1)^2 \partial_2^3 b_1 \partial_2^3 u_2 \partial_2 b_1 dx \\
\le& C \|b_1\|_{L^\infty}^2 \|\partial_2^3 b_1\|_{L^2} (\|\partial_2 u_1\|_{L^\infty} \|\partial_1 \partial_2^2 b_1\|_{L^2} + \|\partial_2^2 u_1\|_{L^\infty} \|\partial_1 \partial_2 b_1\|_{L^2} \\
& + \|\partial_2^3 u_1\|_{L^2} \|\partial_1 b_1\|_{L^\infty} + \|\partial_2 u_2\|_{L^\infty} \|\partial_2^3 b_1\|_{L^2} + \|\partial_2^2 u_2\|_{L^\infty} \|\partial_2^2 b_1\|_{L^2} \\
& + \|\partial_2^3 u_2\|_{L^2} \|\partial_2 b_1\|_{L^\infty}) \\
\le& C \|b_1\|_{H^1} \|\partial_1 b_1\|_{H^1} \|b\|_{H^3}^2 (\|\partial_2 u_1\|_{H^3} + \|\partial_1 u\|_{H^2}) \\
\le& C \|b\|_{H^3}^3 (\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}$$

By the Leibniz formula and Lemma 2.2, we infer that

$$\begin{aligned}
N_4 =& 6 \int (b_1)^2 \partial_2^3 b_1 \partial_2^3 (b \cdot \nabla u_1) dx \\
=& 6 \sum_{0 \le \alpha \le 3} C_3^\alpha \int (b_1)^2 \partial_2^3 b_1 \partial_2^\alpha b \cdot \nabla \partial_2^{3-\alpha} u_1 dx \\
=& 6 \int (b_1)^2 \partial_2^3 b_1 b \cdot \nabla \partial_2^3 u_1 dx + 18 \int (b_1)^2 \partial_2^3 b_1 \partial_2 b \cdot \nabla \partial_2^2 u_1 dx
\end{aligned}$$

$$
\begin{aligned}
&+18\int (b_1)^2\partial_2^3 b_1\partial_2^2 b\cdot\nabla\partial_2 u_1 dx+6\int (b_1)^2\partial_2^3 b_1\partial_2^3 b\cdot\nabla u_1 dx\\
=&6\int (b_1)^2\partial_2^3 b_1 b\cdot\nabla\partial_2^3 u_1 dx+18\int (b_1)^2\partial_2^3 b_1\partial_2 b\cdot\nabla\partial_2^2 u_1 dx\\
&+18\int (b_1)^2\partial_2^3 b_1\partial_2^2 b\cdot\nabla\partial_2 u_1 dx+6\int (b_1)^2\partial_2^3 b_1\partial_2^3 b_1\partial_1 u_1 dx\\
&+6\int (b_1)^2\partial_2^3 b_1\partial_2^3 b_2\partial_2 u_1 dx\\
\le& C\|b_1\|_{L^\infty}^2\|\partial_2^3 b_1\|_{L^2}(\|b\|_{L^\infty}\|\nabla\partial_2^3 u_1\|_{L^2}+\|\partial_2 b\|_{L^4}\|\nabla\partial_2^2 u_1\|_{L^4}\\
&+\|\partial_2^2 b\|_{L^4}\|\nabla\partial_2 u_1\|_{L^4}+\|\partial_2^3 b_1\|_{L^2}\|\partial_1 u_1\|_{L^\infty}+\|\partial_2^3 b_2\|_{L^2}\|\partial_2 u_1\|_{L^\infty})\\
\le& C\|b_1\|_{H^1}\|\partial_1 b_1\|_{H^1}\|b\|_{H^3}^2(\|\partial_2 u_1\|_{H^3}^2+\|\partial_1 u\|_{H^2}^2)\\
\le& C\|b\|_{H^3}^3(\|b_2\|_{H^3}^2+\|\partial_2 u_1\|_{H^3}^2+\|\partial_1 u\|_{H^2}^2).
\end{aligned}
$$

We infer from the Hölder inequality, Lemma 2.2 and the Sobolev inequality

$$
\begin{aligned}
N_5=&6\int (b_1)^2\partial_2^3 b_1\partial_2^3\partial_1 u_1 dx\\
\le& C\|b_1\|_{L^\infty}^2\|\partial_2^3 b_1\|_{L^2}\|\partial_2^3\partial_1 u_1\|_{L^2}\\
\le& C\|b_1\|_{H^1}\|\partial_1 b_1\|_{H^1}\|\partial_2^3 b_1\|_{L^2}\|\partial_2^3\partial_1 u_1\|_{L^2}\\
\le& C\|b\|_{H^3}^2(\|b_2\|_{H^3}^2+\|\partial_2 u_1\|_{H^3}^2).
\end{aligned}
$$

Reasoning with the same method yields

$$
\begin{aligned}
N_7=&6\int b_1(b\cdot\nabla u_1)(\partial_2^3 b_1)^2 dx\\
=&6\int b_1 b_1\partial_1 u_1(\partial_2^3 b_1)^2 dx+6\int b_1 b_2\partial_2 u_1(\partial_2^3 b_1)^2 dx\\
\le& C\|b_1\|_{L^\infty}\|\partial_2^3 b_1\|_{L^2}^2(\|b_1\|_{L^\infty}\|\partial_1 u_1\|_{L^\infty}+\|b_2\|_{L^\infty}\|\partial_2 u_1\|_{L^\infty})\\
\le& C\|b_1\|_{H^1}^{\frac12}\|\partial_1 b_1\|_{H^1}^{\frac12}\|b\|_{H^3}^2(\|b_1\|_{H^1}^{\frac12}\|\partial_1 b_1\|_{H^1}^{\frac12}\|\partial_1 u\|_{H^2}+\|b_2\|_{H^2}\|\partial_2 u_1\|_{H^2})\\
\le& C\|b\|_{H^3}^3(\|b_2\|_{H^3}^2+\|\partial_2 u_1\|_{H^3}^2+\|\partial_1 u\|_{H^2}^2).
\end{aligned}
$$

Collecting all the estimate from N_1 to N_7 yields

$$
J_{22}\le -3\frac{d}{dt}\int (b_1)^2(\partial_2^3 b_1)^2+C(\|b\|_{H^3}^2+\|b\|_{H^3}^3)(\|b_2\|_{H^3}^2+\|\partial_2 u_1\|_{H^3}^2+\|\partial_1 u\|_{H^2}^2). \quad (3.6)
$$

With the help of $\nabla\cdot u=0$ and integration by parts, we infer that

$$
\begin{aligned}
J_{23}=&6\int b_1\partial_2^3 b_1\partial_2^2 u_2\partial_2^2 b_1 dx\\
=&3\int b_1\partial_2^2 u_2\partial_2(\partial_2^2 b_1)^2 dx=-3\int b_1\partial_2\partial_1 u_1\partial_2(\partial_2^2 b_1)^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&=3\int \partial_2 b_1\partial_2\partial_1 u_1(\partial_2^2 b_1)^2dx+3\int b_1\partial_2^2\partial_1 u_1(\partial_2^2 b_1)^2dx\\
&=-3\int \partial_2\partial_1 b_1\partial_2 u_1(\partial_2^2 b_1)^2dx-6\int \partial_2 b_1\partial_2 u_1\partial_2^2 b_1\partial_1\partial_2^2 b_1dx\\
&\quad-3\int \partial_1 b_1\partial_2^2 u_1(\partial_2^2 b_1)^2dx-6\int b_1\partial_2^2 u_1\partial_2^2 b_1\partial_1\partial_2^2 b_1dx\\
&\le C\|\partial_2\partial_1 b_1\|_{L^2}\|\partial_2 u_1\|_{L^\infty}\|\partial_2^2 b_1\|_{L^4}^2+C\|\partial_2 b_1\|_{L^4}\|\partial_2 u_1\|_{L^\infty}\|\partial_2^2 b_1\|_{L^4}\|\partial_1\partial_2^2 b_1\|_{L^2}\\
&\quad+C\|\partial_1 b_1\|_{L^\infty}\|\partial_2^2 u_1\|_{L^2}\|\partial_2^2 b_1\|_{L^4}^2+C\|b_1\|_{L^\infty}\|\partial_2^2 u_1\|_{L^4}\|\partial_2^2 b_1\|_{L^4}\|\partial_1\partial_2^2 b_1\|_{L^2}\\
&\le C\|b\|_{H^3}^2(\|b_2\|_{H^3}^2+\|\partial_2 u_1\|_{H^3}^2).
\end{aligned}
$$

Making use of the fact that $\partial_2 u_2=-\partial_1 u_1$ and integrating by parts, one gets

$$
\begin{aligned}
J_{24}=&2\int b_1\partial_2^3 b_1\partial_2^3 u_2\partial_2 b_1dx=-2\int b_1\partial_2^3 b_1\partial_2^2\partial_1 u_1\partial_2 b_1dx\\
=&2\int \partial_1 b_1\partial_2^3 b_1\partial_2^2 u_1\partial_2 b_1dx+2\int b_1\partial_2^3 b_1\partial_2^2 u_1\partial_2\partial_1 b_1dx+2\int b_1\partial_2^3\partial_1 b_1\partial_2^2 u_1\partial_2 b_1dx\\
=&2\int \partial_1 b_1\partial_2^3 b_1\partial_2^2 u_1\partial_2 b_1dx+2\int b_1\partial_2^3 b_1\partial_2^2 u_1\partial_2\partial_1 b_1dx\\
&-2\int \partial_2 b_1\partial_2^2\partial_1 b_1\partial_2^2 u_1\partial_2 b_1dx-2\int b_1\partial_2^2\partial_1 b_1\partial_2^3 u_1\partial_2 b_1dx\\
&-2\int b_1\partial_2^2\partial_1 b_1\partial_2^2 u_1\partial_2^2 b_1dx\\
\le& C\|\partial_2^3 b_1\|_{L^2}\|\partial_2^2 u_1\|_{L^4}(\|\partial_1 b_1\|_{L^\infty}\|\partial_2 b_1\|_{L^4}+\|b_1\|_{L^\infty}\|\partial_2\partial_1 b_1\|_{L^4})\\
&+C\|\partial_2^2\partial_1 b_1\|_{L^2}(\|\partial_2 b_1\|_{L^4}^2\|\partial_2^2 u_1\|_{L^\infty}+\|b_1\|_{L^\infty}\|\partial_2^3 u_1\|_{L^2}\|\partial_2 b_1\|_{L^\infty}\\
&+\|b_1\|_{L^\infty}\|\partial_2^2 u_1\|_{L^\infty}\|\partial_2^2 b_1\|_{L^2})\\
\le& C\|b\|_{H^3}^2(\|b_2\|_{H^3}^2+\|\partial_2 u_1\|_{H^3}^2).
\end{aligned}
$$

Thus, it reaches

$$
J_2\le-3\frac{d}{dt}\int(b_1)^2(\partial_2^3 b_1)^2+C(\|b\|_{H^3}^2+\|b\|_{H^3}^3)(\|b_2\|_{H^3}^2+\|\partial_2 u_1\|_{H^3}^2+\|\partial_1 u\|_{H^2}^2).
$$

To deal with J_4. We split it into the following terms

$$
\begin{aligned}
J_4=&-2\int b_1\partial_2^3 b_1\partial_2^3(b\cdot\nabla u_1)dx\\
=&-2\sum_{0\le\alpha\le3}C_3^\alpha\int b_1\partial_2^3 b_1\partial_2^\alpha b\cdot\nabla\partial_2^{3-\alpha}u_1dx\\
=&-2\int b_1\partial_2^3 b_1 b\cdot\nabla\partial_2^3 u_1dx-6\int b_1\partial_2^3 b_1\partial_2 b\cdot\nabla\partial_2^2 u_1dx\\
&-6\int b_1\partial_2^3 b_1\partial_2^2 b\cdot\nabla\partial_2 u_1dx-2\int b_1\partial_2^3 b_1\partial_2^3 b\cdot\nabla u_1dx
\end{aligned}
$$

$$=J_{41}+J_{42}+J_{43}+J_{44}.$$

Thanks to the Hölder inequality and Lemma 2.2, we obtain

$$\begin{aligned}
J_{41}=&-2\int b_1\partial_2^3 b_1 b\cdot\nabla\partial_2^3 u_1 dx\\
\le& C\|b_1\|_{L^\infty}\|\partial_2^3 b_1\|_{L^2}\|b\|_{L^\infty}\|\nabla\partial_2^3 u_1\|_{L^2}\\
\le& C\|b_1\|_{H^1}^{\frac{1}{2}}\|\partial_1 b_1\|_{H^1}^{\frac{1}{2}}\|b\|_{H^1}^{\frac{1}{2}}\|\partial_1 b\|_{H^1}^{\frac{1}{2}}\|\partial_2^3 b_1\|_{L^2}\|\nabla\partial_2^3 u_1\|_{L^2}\\
\le& C\|b\|_{H^3}^2(\|b_2\|_{H^3}^2+\|\partial_2 u_1\|_{H^3}^2).
\end{aligned}$$

For J_{42}, we deduce from integration by parts, Hölder's inequality and Sobolev's inequality

$$\begin{aligned}
J_{42}=&-6\int b_1\partial_2^3 b_1\partial_2 b\cdot\nabla\partial_2^2 u_1 dx\\
=&-6\int b_1\partial_2^3 b_1\partial_2 b_1\partial_1\partial_2^2 u_1 dx-6\int b_1\partial_2^3 b_1\partial_2 b_2\partial_2^3 u_1 dx\\
=&6\int \partial_1 b_1\partial_2^3 b_1\partial_2 b_1\partial_2^2 u_1 dx+6\int b_1\partial_2^3\partial_1 b_1\partial_2 b_1\partial_2^2 u_1 dx\\
&+6\int b_1\partial_2^3 b_1\partial_2\partial_1 b_1\partial_2^2 u_1 dx-6\int b_1\partial_2^3 b_1\partial_2 b_2\partial_2^3 u_1 dx\\
=&6\int \partial_1 b_1\partial_2^3 b_1\partial_2 b_1\partial_2^2 u_1 dx-6\int \partial_2 b_1\partial_2^2\partial_1 b_1\partial_2 b_1\partial_2^2 u_1 dx-6\int b_1\partial_2^2\partial_1 b_1\partial_2^2 b_1\partial_2^2 u_1 dx\\
&-6\int b_1\partial_2^2\partial_1 b_1\partial_2 b_1\partial_2^3 u_1 dx+6\int b_1\partial_2^3 b_1\partial_2\partial_1 b_1\partial_2^2 u_1 dx-6\int b_1\partial_2^3 b_1\partial_2 b_2\partial_2^3 u_1 dx\\
\le& C\|\partial_1 b_1\|_{L^\infty}\|\partial_2^3 b_1\|_{L^2}\|\partial_2 b_1\|_{L^4}\|\partial_2^2 u_1\|_{L^4}+C\|\partial_2 b_1\|_{L^4}^2\|\partial_2^2\partial_1 b_1\|_{L^2}\|\partial_2^2 u_1\|_{L^\infty}\\
&+C\|b_1\|_{L^\infty}\|\partial_2^2\partial_1 b_1\|_{L^2}(\|\partial_2^2 b_1\|_{L^4}\|\partial_2^2 u_1\|_{L^4}+\|\partial_2 b_1\|_{L^\infty}\|\partial_2^3 u_1\|_{L^2})\\
&+C\|b_1\|_{L^\infty}\|\partial_2^3 b_1\|_{L^2}(\|\partial_2\partial_1 b_1\|_{L^4}\|\partial_2^2 u_1\|_{L^4}+\|\partial_2 b_2\|_{L^\infty}\|\partial_2^3 u_1\|_{L^2})\\
\le& C\|b\|_{H^3}^2(\|b_2\|_{H^3}^2+\|\partial_2 u_1\|_{H^3}^2).
\end{aligned}$$

With the help of integration by parts and Hölder's inequality, it reaches

$$\begin{aligned}
J_{43}=&-6\int b_1\partial_2^3 b_1\partial_2^2 b\cdot\nabla\partial_2 u_1 dx\\
=&-6\int b_1\partial_2^3 b_1\partial_2^2 b_1\partial_1\partial_2 u_1 dx-6\int b_1\partial_2^3 b_1\partial_2^2 b_2\partial_2^2 u_1 dx\\
=&-3\int b_1\partial_1\partial_2 u_1\partial_2(\partial_2^2 b_1)^2 dx-6\int b_1\partial_2^3 b_1\partial_2^2 b_2\partial_2^2 u_1 dx\\
=&3\int \partial_2 b_1\partial_1\partial_2 u_1(\partial_2^2 b_1)^2 dx+3\int b_1\partial_1\partial_2^2 u_1(\partial_2^2 b_1)^2 dx-6\int b_1\partial_2^3 b_1\partial_2^2 b_2\partial_2^2 u_1 dx\\
=&-3\int \partial_2\partial_1 b_1\partial_2 u_1(\partial_2^2 b_1)^2 dx-6\int \partial_2 b_1\partial_2 u_1\partial_2^2 b_1\partial_1\partial_2^2 b_1 dx
\end{aligned}$$

$$
\begin{aligned}
& -3\int \partial_1 b_1 \partial_2^2 u_1 (\partial_2^2 b_1)^2 dx - 6\int b_1 \partial_2^2 u_1 \partial_2^2 b_1 \partial_1 \partial_2^2 b_1 dx - 6\int b_1 \partial_2^3 b_1 \partial_2^2 b_2 \partial_2^2 u_1 dx \\
\le & C\|\partial_2 u_1\|_{L^\infty} \|\partial_2^2 b_1\|_{L^4} (\|\partial_2^2 b_1\|_{L^4} \|\partial_2 \partial_1 b_1\|_{L^2} + \|\partial_2 b_1\|_{L^4} \|\partial_2^2 \partial_1 b_1\|_{L^2}) \\
& + C\|\partial_1 b_1\|_{L^\infty} \|\partial_2^2 u_1\|_{L^2} \|\partial_2^2 b_1\|_{L^4}^2 + C\|b_1\|_{L^\infty} \|\partial_2^2 u_1\|_{L^4} \|\partial_2^2 b_1\|_{L^4} \|\partial_1 \partial_2^2 b_1\|_{L^2} \\
& + C\|b_1\|_{L^\infty} \|\partial_2^3 b_1\|_{L^2} \|\partial_2^2 b_2\|_{L^4} \|\partial_2^2 u_1\|_{L^4} \\
\le & C\|b\|_{H^3}^2 (\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2).
\end{aligned}
$$

From the Hölder's inequality and the estimate of J_{22} in (3.6), it follows that

$$
\begin{aligned}
J_{44} =& -2\int b_1 \partial_2^3 b_1 \partial_2^3 b \cdot \nabla u_1 dx \\
=& -2\int b_1 \partial_2^3 b_1 \partial_2^3 b_1 \partial_1 u_1 dx - 2\int b_1 \partial_2^3 b_1 \partial_2^3 b_2 \partial_2 u_1 dx \\
\le & -2\int b_1 \partial_2^3 b_1 \partial_2^3 b_1 \partial_1 u_1 dx + C\|b_1\|_{L^\infty} \|\partial_2^3 b_1\|_{L^2} \|\partial_2^3 b_2\|_{L^2} \|\partial_2 u_1\|_{L^\infty} \\
\le & C\|b\|_{H^3}^2 (\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2) + \frac{1}{3} J_{22} \\
\le & -\frac{d}{dt}\int (b_1)^2 (\partial_2^3 b_1)^2 + C(\|b\|_{H^3}^2 + \|b\|_{H^3}^3)(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}
$$

So,

$$
J_4 \le -\frac{d}{dt}\int (b_1)^2 (\partial_2^3 b_1)^2 + C(\|b\|_{H^3}^2 + \|b\|_{H^3}^3)(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
$$

Integration by parts and the Hölder inequality lead to

$$
\begin{aligned}
J_5 =& -2\int b_1 \partial_2^3 b_1 \partial_2^3 \partial_1 u_1 dx \\
=& 2\int \partial_1 b_1 \partial_2^3 b_1 \partial_2^3 u_1 dx + 2\int b_1 \partial_2^3 \partial_1 b_1 \partial_2^3 u_1 dx \\
=& 2\int \partial_1 b_1 \partial_2^3 b_1 \partial_2^3 u_1 dx - 2\int \partial_2 b_1 \partial_2^2 \partial_1 b_1 \partial_2^3 u_1 dx - 2\int b_1 \partial_2^2 \partial_1 b_1 \partial_2^4 u_1 dx \\
\le & C\|\partial_2^3 u_1\|_{L^2} (\|\partial_1 b_1\|_{L^\infty} \|\partial_2^3 b_1\|_{L^2} + \|\partial_2 b_1\|_{L^\infty} \|\partial_2^2 \partial_1 b_1\|_{L^2}) \\
& + C\|b_1\|_{L^\infty} \|\partial_2^2 \partial_1 b_1\|_{L^2} \|\partial_2^4 u_1\|_{L^2} \\
\le & C\|b\|_{H^3} (\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2).
\end{aligned}
$$

Thanks to the Hölder inequality and (3.6), we find

$$
\begin{aligned}
J_7 =& -\int (b \cdot \nabla u_1)(\partial_2^3 b_1)^2 dx \\
=& -\int b_1 \partial_1 u_1 (\partial_2^3 b_1)^2 dx - \int b_2 \partial_2 u_1 (\partial_2^3 b_1)^2 dx
\end{aligned}
$$

$$\begin{aligned}
&\leq \|b_2\|_{L^\infty}\|\partial_2 u_1\|_{L^\infty}\|\partial_2^3 b_1\|_{L^2}^2 - \int b_1\partial_1 u_1(\partial_2^3 b_1)^2 dx\\
&\leq C\|b\|_{H^3}^2(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2) + \frac{1}{6}J_{22}\\
&\leq -\frac{1}{2}\frac{d}{dt}\int (b_1)^2(\partial_2^3 b_1)^2 dx + C(\|b\|_{H^3}^2 + \|b\|_{H^3}^3)(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}$$

Collecting all the estimates from J_1 to J_7, we get

$$\begin{aligned}
\int \partial_1 u_1(\partial_2^3 b_1)^2 dx \leq &\frac{d}{dt}\int b_1(\partial_2^3 b_1)^2 dx - \frac{9}{2}\frac{d}{dt}\int (b_1)^2(\partial_2^3 b_1)^2 dx\\
&+ C(\|b\|_{H^3} + \|b\|_{H^3}^2 + \|b\|_{H^3}^3)\\
&\times (\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.4), it shows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u,b)\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|b_2\|_{H^3}^2\\
&\leq 4\frac{d}{dt}\int b_1(\partial_2^3 b_1)^2 dx - 18\frac{d}{dt}\int (b_1)^2(\partial_2^3 b_1)^2 dx\\
&\quad + C(\|b\|_{H^3} + \|b\|_{H^3}^2 + \|b\|_{H^3}^3)(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2),
\end{aligned}$$

which implies

$$\begin{aligned}
&\|(u,b)\|_{H^3}^2 + 2\int_0^t (\|\partial_2 u_1\|_{H^3}^2 + \|b_2\|_{H^3}^2)\\
&\leq \|(u_0,b_0)\|_{H^3}^2 + 8\int b_1(\partial_2^3 b_1)^2 dx - 8\int b_1(x,0)(\partial_2^3 b_1)^2(x,0)dx\\
&\quad - 36\int (b_1)^2(\partial_2^3 b_1)^2 dx + 36\int (b_1)^2(x,0)(\partial_2^3 b_1)^2(x,0)dx\\
&\quad + C\int_0^t (\|b\|_{H^3} + \|b\|_{H^3}^2 + \|b\|_{H^3}^3)(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2)\\
&\leq \|(u_0,b_0)\|_{H^3}^2 + C(\|b_0\|_{L^\infty} + \|b_0\|_{L^\infty}^2)\|\partial_2^3 b_0\|_{L^2}^2\\
&\quad + C(\|b_1\|_{L^\infty} + \|b_1\|_{L^\infty}^2)\|\partial_2^3 b_1\|_{L^2}^2\\
&\quad + C\sup_{0\leq\tau\leq t}(\|b\|_{H^3} + \|b\|_{H^3}^2 + \|b\|_{H^3}^3)\int_0^t (\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
E_1(t) \leq &C(E_1(0) + E_1^{\frac{3}{2}}(0) + E_1^2(0)) + C(E_1^{\frac{3}{2}}(t) + E_2^{\frac{3}{2}}(t))\\
&+ C(E_1^2(t) + E_2^2(t)) + C(E_1^{\frac{5}{2}}(t) + E_2^{\frac{5}{2}}(t)).
\end{aligned} \tag{3.8}$$

Hence we finish the proof of Proposition 3.1. □

In order to close the energy estimate, we need the following result.

Proposition 3.2 *Suppose that $E_1(t)$ and $E_2(t)$ are defined as in (3.1) and (3.2). Then exists a positive constant $C > 0$, such that*

$$E_2(t) \le CE_1(0) + CE_1(t) + CE_1^{\frac{3}{2}}(t) + CE_2^{\frac{3}{2}}(t).$$

Proof We first rewrite the second equation of the system (1.3) as

$$\partial_1 u = \partial_t b + u \cdot \nabla b + \begin{pmatrix} 0 \\ b_2 \end{pmatrix} - b \cdot \nabla u. \tag{3.9}$$

Taking the L^2-inner product to (3.9) with $\partial_1 u$ yields

$$\begin{aligned}
\|\partial_1 u\|_{L^2}^2 =& \langle \partial_t b, \partial_1 u\rangle + \langle u \cdot \nabla b, \partial_1 u\rangle + \langle b_2, \partial_1 u_2\rangle - \langle b \cdot \nabla u, \partial_1 u\rangle \\
=& \frac{d}{dt}\langle b, \partial_1 u\rangle - \langle b, \partial_1(-u \cdot \nabla u - \nabla \pi + \begin{pmatrix} \partial_2^2 u_1 \\ 0 \end{pmatrix} + b \cdot \nabla b + \partial_1 b)\rangle \\
&+ \langle u \cdot \nabla b, \partial_1 u\rangle + \langle b_2, \partial_1 u_2\rangle - \langle b \cdot \nabla u, \partial_1 u\rangle \\
=& \frac{d}{dt}\langle b, \partial_1 u\rangle + \|\partial_1 b\|_{L^2}^2 + \langle b, \partial_1(u \cdot \nabla u)\rangle - \langle b_1, \partial_1 \partial_2^2 u_1\rangle \\
&- \langle b, \partial_1(b \cdot \nabla b)\rangle + \langle u \cdot \nabla b, \partial_1 u\rangle + \langle b_2, \partial_1 u_2\rangle - \langle b \cdot \nabla u, \partial_1 u\rangle \\
=& \frac{d}{dt}\langle b, \partial_1 u\rangle + \|\partial_1 b\|_{L^2}^2 + M_1 + M_2 + \cdots + M_6,
\end{aligned}$$

where we use the equation

$$\partial_t u = -u \cdot \nabla u - \nabla \pi + \begin{pmatrix} \partial_2^2 u_1 \\ 0 \end{pmatrix} + b \cdot \nabla b + \partial_1 b.$$

Taking advantage of integration by parts and the Hölder inequality, we see that

$$\begin{aligned}
M_1 =& \int b \cdot \partial_1(u \cdot \nabla u)dx = -\int \partial_1 b \cdot (u \cdot \nabla u)dx \\
=& -\int \partial_1 b \cdot u_1 \partial_1 u dx - \int \partial_1 b_1 u_2 \partial_2 u_1 dx - \int \partial_1 b_2 u_2 \partial_2 u_2 dx \\
\le& \|\partial_1 b\|_{L^2}\|u_1\|_{L^\infty}\|\partial_1 u\|_{L^2} + \|\partial_1 b_1\|_{L^2}\|u_2\|_{L^\infty}\|\partial_2 u_1\|_{L^2} + \|\partial_1 b_2\|_{L^2}\|u_2\|_{L^\infty}\|\partial_2 u_2\|_{L^2} \\
\le& C\|u\|_{H^2}(\|b_2\|_{H^1}^2 + \|\partial_2 u_1\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2).
\end{aligned}$$

It is easy to see

$$\begin{aligned}
M_2 =& \int \partial_1 b_1 \partial_2^2 u_1 dx \\
\le& \|\partial_1 b_1\|_{L^2}\|\partial_2^2 u_1\|_{L^2} \\
\le& C(\|b_2\|_{H^1}^2 + \|\partial_2 u_1\|_{H^1}^2).
\end{aligned}$$

From integration by parts and Hölder's inequality, it follows

$$\begin{aligned} M_3 = & -\int b\cdot\partial_1(b\cdot\nabla b)dx = \int \partial_1 b\cdot(b\cdot\nabla b)dx \\ = & \int \partial_1 b\cdot b_1\partial_1 b dx + \int \partial_1 b\cdot b_2\partial_2 b dx \\ \le & \|b_1\|_{L^\infty}\|\partial_1 b\|_{L^2}^2 + \|\partial_1 b\|_{L^2}\|b_2\|_{L^4}\|\partial_2 b\|_{L^4} \\ \le & C\|b\|_{H^2}\|b_2\|_{H^1}^2. \end{aligned}$$

From Lemma 2.1, we obviously deduce

$$\begin{aligned} M_4 = & \int u\cdot\nabla b\cdot\partial_1 u dx \\ = & \int u_1\partial_1 b\cdot\partial_1 u dx + \int u_2\partial_2 b\cdot\partial_1 u dx \\ \le & \|u_1\|_{L^\infty}\|\partial_1 b\|_{L^2}\|\partial_1 u\|_{L^2} + \|u_2\|_{L^2}^{\frac12}\|\partial_2 u_2\|_{L^2}^{\frac12}\|\partial_2 b\|_{L^2}^{\frac12}\|\partial_2\partial_1 b\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2} \\ \le & C(\|u\|_{H^2}+\|b\|_{H^2})(\|b_2\|_{H^2}^2+\|\partial_1 u\|_{L^2}^2). \end{aligned}$$

Young's inequality ensures

$$M_5 \le C\|b_2\|_{L^2}^2 + \frac{1}{2}\|\partial_1 u_2\|_{L^2}^2.$$

Thanks to the Hölder inequality, we find

$$\begin{aligned} M_6 = & -\int b\cdot\nabla u\cdot\partial_1 u dx \\ = & -\int b_1\partial_1 u\cdot\partial_1 u dx - \int b_2\partial_2 u\cdot\partial_1 u dx \\ \le & \|b_1\|_{L^\infty}\|\partial_1 u\|_{L^2}^2 + \|b_2\|_{L^\infty}\|\partial_2 u\|_{L^2}\|\partial_1 u\|_{L^2} \\ \le & C(\|u\|_{H^1}+\|b\|_{H^2})(\|b_2\|_{H^2}^2+\|\partial_1 u\|_{L^2}^2). \end{aligned}$$

Collecting the bounds for M_1 through M_6, one has

$$\begin{aligned} \frac{1}{2}\|\partial_1 u\|_{L^2}^2 \le & \frac{d}{dt}\langle b,\partial_1 u\rangle + C(\|b_2\|_{H^1}^2 + \|\partial_2 u_1\|_{H^1}^2) \\ & + C(\|u\|_{H^2}+\|b\|_{H^2})(\|b_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^1}^2 + \|\partial_1 u\|_{L^2}^2). \end{aligned} \tag{3.10}$$

Now we turn to establish $\dot{H}^2$-norm of $\partial_1 u$. Applying ∂_i^2 to (3.9) and then taking L^2-inner product to (3.9) with $\partial_i^2\partial_1 u$ to conclude

$$\begin{aligned} \|\partial_1\partial_i^2 u\|_{L^2}^2 = & \langle\partial_t\partial_i^2 b,\partial_1\partial_i^2 u\rangle + \langle\partial_i^2(u\cdot\nabla b),\partial_1\partial_i^2 u\rangle + \langle\partial_i^2 b_2,\partial_1\partial_i^2 u_2\rangle - \langle\partial_i^2(b\cdot\nabla u),\partial_1\partial_i^2 u\rangle \\ = & \frac{d}{dt}\langle\partial_i^2 b,\partial_1\partial_i^2 u\rangle - \langle\partial_i^2 b,\partial_1\partial_i^2(-u\cdot\nabla u - \nabla\pi + \begin{pmatrix}\partial_2^2 u_1\\ 0\end{pmatrix} + b\cdot\nabla b + \partial_1 b)\rangle \\ & + \langle\partial_i^2(u\cdot\nabla b),\partial_1\partial_i^2 u\rangle + \langle\partial_i^2 b_2,\partial_1\partial_i^2 u_2\rangle - \langle\partial_i^2(b\cdot\nabla u),\partial_1\partial_i^2 u\rangle \end{aligned}$$

$$
\begin{aligned}
=&\frac{d}{dt}\langle \partial_i^2 b, \partial_1\partial_i^2 u\rangle + \|\partial_1\partial_i^2 b\|_{L^2}^2 + \langle \partial_i^2 b, \partial_1\partial_i^2(u\cdot\nabla u)\rangle - \langle \partial_i^2 b_1, \partial_1\partial_i^2\partial_2^2 u_1\rangle \\
&- \langle \partial_i^2 b, \partial_1\partial_i^2(b\cdot\nabla b)\rangle + \langle \partial_i^2(u\cdot\nabla b), \partial_1\partial_i^2 u\rangle + \langle \partial_i^2 b_2, \partial_1\partial_i^2 u_2\rangle \\
&- \langle \partial_i^2(b\cdot\nabla u), \partial_1\partial_i^2 u\rangle \\
=&\frac{d}{dt}\langle \partial_i^2 b, \partial_1\partial_i^2 u\rangle + \|\partial_1\partial_i^2 b\|_{L^2}^2 + K_1 + K_2 + \cdots + K_6.
\end{aligned}
$$

According to integration by parts and Leibniz's formula, K_1 can be divided into the following terms

$$
\begin{aligned}
K_1 =& \int \partial_i^2 b\cdot\partial_1\partial_i^2(u\cdot\nabla u)dx = -\int \partial_i^2\partial_1 b\cdot\partial_i^2(u\cdot\nabla u)dx \\
=&-\sum_{0\le\alpha\le 2} C_2^\alpha \int \partial_i^2\partial_1 b\cdot\partial_i^\alpha u\cdot\nabla\partial_i^{2-\alpha} u dx \\
=&-\sum_{0\le\alpha\le 2} C_2^\alpha \int \partial_i^2\partial_1 b\cdot\partial_i^\alpha u_1\partial_1\partial_i^{2-\alpha} u dx - \sum_{0\le\alpha\le 2} C_2^\alpha \int \partial_i^2\partial_1 b_1\partial_i^\alpha u_2\partial_2\partial_i^{2-\alpha} u_1 dx \\
&-\sum_{0\le\alpha\le 2} C_2^\alpha \int \partial_i^2\partial_1 b_2\partial_i^\alpha u_2\partial_2\partial_i^{2-\alpha} u_2 dx \\
=&K_{11} + K_{12} + K_{13}.
\end{aligned}
$$

Thanks to the Hölder inequality and the Sobolev inequality, we can get

$$
\begin{aligned}
K_{11} =&-\sum_{0\le\alpha\le 2} C_2^\alpha \int \partial_i^2\partial_1 b\cdot\partial_i^\alpha u_1\partial_1\partial_i^{2-\alpha} u dx \\
=&-\int \partial_i^2\partial_1 b\cdot u_1\partial_1\partial_i^2 u dx - 2\int \partial_i^2\partial_1 b\cdot\partial_i u_1\partial_1\partial_i u dx - \int \partial_i^2\partial_1 b\cdot\partial_i^2 u_1\partial_1 u dx \\
\le&\|\partial_i^2\partial_1 b\|_{L^2}(\|u_1\|_{L^\infty}\|\partial_1\partial_i^2 u\|_{L^2} + 2\|\partial_i u_1\|_{L^4}\|\partial_1\partial_i u\|_{L^4} + \|\partial_i^2 u_1\|_{L^4}\|\partial_1 u\|_{L^4}) \\
\le&C\|u\|_{H^3}(\|b_2\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}
$$

Reasoning in the same way gives

$$
\begin{aligned}
K_{12} =&-\sum_{0\le\alpha\le 2} C_2^\alpha \int \partial_i^2\partial_1 b_1\partial_i^\alpha u_2\partial_2\partial_i^{2-\alpha} u_1 dx \\
\le&C\sum_{0\le\alpha\le 2} \|\partial_i^2\partial_1 b_1\|_{L^2}\|\partial_i^\alpha u_2\|_{L^4}\|\partial_2\partial_i^{2-\alpha} u_1\|_{L^4} \\
\le&C\|u\|_{H^3}(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2),
\end{aligned}
$$

and

$$
\begin{aligned}
K_{13} =&-\sum_{0\le\alpha\le 2} C_2^\alpha \int \partial_i^2\partial_1 b_2\partial_i^\alpha u_2\partial_2\partial_i^{2-\alpha} u_2 dx \\
=&-\int \partial_i^2\partial_1 b_2 u_2\partial_2\partial_i^2 u_2 dx - 2\int \partial_i^2\partial_1 b_2\partial_i u_2\partial_2\partial_i u_2 dx - \int \partial_i^2\partial_1 b_2\partial_i^2 u_2\partial_2 u_2 dx
\end{aligned}
$$

$$\leq C\|\partial_i^2\partial_1 b_2\|_{L^2}(\|u_2\|_{L^\infty}\|\partial_2\partial_i^2 u_2\|_{L^2}+\|\partial_i u_2\|_{L^4}\|\partial_2\partial_i u_2\|_{L^4}+\|\partial_i^2 u_2\|_{L^4}\|\partial_2 u_2\|_{L^4})$$
$$\leq C\|u\|_{H^3}(\|b_2\|_{H^3}^2+\|\partial_1 u\|_{H^2}^2).$$

So, we have

$$K_1\leq C\|u\|_{H^3}(\|b_2\|_{H^3}^2+\|\partial_2 u_1\|_{H^3}^2+\|\partial_1 u\|_{H^2}^2).$$

Integrating by parts and using the Young inequality, we get

$$K_2=-\int\partial_i^2 b_1\partial_1\partial_i^2\partial_2^2 u_1 dx=\int\partial_1\partial_i^2 b_1\partial_i^2\partial_2^2 u_1 dx$$
$$\leq C(\|b_2\|_{H^3}^2+\|\partial_2 u_1\|_{H^3}^2).$$

Thanks to integration by parts and the Leibniz law, we obtain

$$\begin{aligned}K_3=&-\int\partial_i^2 b\cdot\partial_1\partial_i^2(b\cdot\nabla b)dx\\
=&\int\partial_1\partial_i^2 b\cdot\partial_i^2(b\cdot\nabla b)dx=\sum_{0\leq\alpha\leq 2}C_2^\alpha\int\partial_1\partial_i^2 b\cdot\partial_i^\alpha b\cdot\nabla\partial_i^{2-\alpha}bdx\\
=&\sum_{0\leq\alpha\leq 2}C_2^\alpha\int\partial_1\partial_i^2 b\cdot\partial_i^\alpha b_1\partial_1\partial_i^{2-\alpha}bdx+\sum_{0\leq\alpha\leq 2}C_2^\alpha\int\partial_1\partial_i^2 b\cdot\partial_i^\alpha b_2\partial_2\partial_i^{2-\alpha}bdx\\
=&\int\partial_1\partial_i^2 b\cdot b_1\partial_1\partial_i^2 bdx+2\int\partial_1\partial_i^2 b\cdot\partial_i b_1\partial_1\partial_i bdx+\int\partial_1\partial_i^2 b\cdot\partial_i^2 b_1\partial_1 bdx\\
&+\int\partial_1\partial_i^2 b\cdot b_2\partial_2\partial_i^2 bdx+2\int\partial_1\partial_i^2 b\cdot\partial_i b_2\partial_2\partial_i bdx+\int\partial_1\partial_i^2 b\cdot\partial_i^2 b_2\partial_2 bdx\\
\leq&C\|\partial_1\partial_i^2 b\|_{L^2}(\|b_1\|_{L^\infty}\|\partial_1\partial_i^2 b\|_{L^2}+\|\partial_i b_1\|_{L^4}\|\partial_1\partial_i b\|_{L^4}+\|\partial_i^2 b_1\|_{L^4}\|\partial_1 b\|_{L^4}\\
&+\|b_2\|_{L^\infty}\|\partial_2\partial_i^2 b\|_{L^2}+\|\partial_i b_2\|_{L^4}\|\partial_2\partial_i b\|_{L^4}+\|\partial_i^2 b_2\|_{L^4}\|\partial_2 b\|_{L^4})\\
\leq&C\|b\|_{H^3}\|b_2\|_{H^3}^2.\end{aligned}$$

Using the Leibniz law again, we decompose K_4 into the following form:

$$\begin{aligned}K_4=&\int\partial_i^2(u\cdot\nabla b)\cdot\partial_1\partial_i^2 udx\\
=&\sum_{0\leq\alpha\leq 2}C_2^\alpha\int\partial_i^\alpha u\cdot\nabla\partial_i^{2-\alpha}b\cdot\partial_1\partial_i^2 udx\\
=&\sum_{0\leq\alpha\leq 2}C_2^\alpha\int\partial_i^\alpha u_1\partial_1\partial_i^{2-\alpha}b\cdot\partial_1\partial_i^2 udx+\sum_{0\leq\alpha\leq 2}C_2^\alpha\int\partial_i^\alpha u_2\partial_2\partial_i^{2-\alpha}b\cdot\partial_1\partial_i^2 udx\\
=&K_{41}+K_{42}.\end{aligned}$$

By the Hölder inequality and Sobolev inequality, we thus get

$$K_{41}=\sum_{0\leq\alpha\leq 2}C_2^\alpha\int\partial_i^\alpha u_1\partial_1\partial_i^{2-\alpha}b\cdot\partial_1\partial_i^2 udx$$

$$
\begin{aligned}
&= \int u_1\partial_1\partial_i^2 b\cdot\partial_1\partial_i^2 u dx + 2\int \partial_i u_1\partial_1\partial_i b\cdot\partial_1\partial_i^2 u dx + \int \partial_i^2 u_1\partial_1 b\cdot\partial_1\partial_i^2 u dx\\
&\leq C\|\partial_1\partial_i^2 u\|_{L^2}(\|u_1\|_{L^\infty}\|\partial_1\partial_i^2 b\|_{L^2} + \|\partial_i u_1\|_{L^4}\|\partial_1\partial_i b\|_{L^4} + \|\partial_i^2 u_1\|_{L^4}\|\partial_1 b\|_{L^4})\\
&\leq C\|u\|_{H^3}(\|b_2\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}
$$

We first split K_{42} into the following terms

$$
\begin{aligned}
K_{42} =& \sum_{0\leq\alpha\leq 2} C_2^\alpha \int \partial_i^\alpha u_2\partial_2\partial_i^{2-\alpha} b\cdot\partial_1\partial_i^2 u dx\\
=& \int u_2\partial_2\partial_i^2 b\cdot\partial_1\partial_i^2 u dx + 2\int \partial_i u_2\partial_2\partial_i b\cdot\partial_1\partial_i^2 u dx + \int \partial_i^2 u_2\partial_2 b\cdot\partial_1\partial_i^2 u dx\\
=& K_{421} + K_{422} + K_{423}.
\end{aligned}
$$

Using integration by parts and the Hölder inequality, we can obtain

$$
\begin{aligned}
K_{421} =& \int u_2\partial_2\partial_i^2 b\cdot\partial_1\partial_i^2 u dx\\
=& \int u_2\partial_2\partial_i^2 b_1\partial_1\partial_i^2 u_1 dx + \int u_2\partial_2\partial_i^2 b_2\partial_1\partial_i^2 u_2 dx\\
=& -\int \partial_1 u_2\partial_2\partial_i^2 b_1\partial_i^2 u_1 dx - \int u_2\partial_2\partial_i^2\partial_1 b_1\partial_i^2 u_1 dx + \int u_2\partial_2\partial_i^2 b_2\partial_1\partial_i^2 u_2 dx\\
=& -\int \partial_1 u_2\partial_2\partial_1^2 b_1\partial_1^2 u_1 dx - \int \partial_1 u_2\partial_2^3 b_1\partial_2^2 u_1 dx\\
&+ \int \partial_2 u_2\partial_i^2\partial_1 b_1\partial_i^2 u_1 dx + \int u_2\partial_i^2\partial_1 b_1\partial_i^2\partial_2 u_1 dx - \int u_2\partial_i^2\partial_1 b_1\partial_1\partial_i^2 u_2 dx\\
\leq& \|\partial_1 u_2\|_{L^\infty}(\|\partial_2\partial_1^2 b_1\|_{L^2}\|\partial_1^2 u_1\|_{L^2} + \|\partial_2^3 b_1\|_{L^2}\|\partial_2^2 u_1\|_{L^2})\\
&+ \|\partial_i^2\partial_1 b_1\|_{L^2}(\|\partial_2 u_2\|_{L^\infty}\|\partial_i^2 u_1\|_{L^2} + \|u_2\|_{L^\infty}\|\partial_i^2\partial_2 u_1\|_{L^2} + \|u_2\|_{L^\infty}\|\partial_1\partial_i^2 u_2\|_{L^2})\\
\leq& C(\|u\|_{H^3} + \|b\|_{H^3})(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}
$$

Due to Lemma 2.1, the last two terms in K_{42}, one can get the following upper bounds

$$
\begin{aligned}
K_{422} + K_{423} =& 2\int \partial_i u_2\partial_2\partial_i b\cdot\partial_1\partial_i^2 u dx + \int \partial_i^2 u_2\partial_2 b\cdot\partial_1\partial_i^2 u dx\\
\leq& C\|\partial_i u_2\|_{L^2}^{\frac{1}{2}}\|\partial_i\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_i b\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_i\partial_1 b\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_i^2 u\|_{L^2}\\
&+ C\|\partial_i^2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_i^2\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2 b\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 b\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_i^2 u\|_{L^2}\\
\leq& C(\|u\|_{H^3} + \|b\|_{H^3})(\|b_2\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}
$$

So,

$$
K_4 \leq C(\|u\|_{H^3} + \|b\|_{H^3})(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
$$

By the Hölder inequality and Sobolev inequality, we deduce

$$\begin{aligned}K_5 =& \int \partial_i^2 b_2 \partial_1 \partial_i^2 u_2 dx\\ \leq& \|\partial_i^2 b_2\|_{L^2} \|\partial_1 \partial_i^2 u_2\|_{L^2}\\ \leq& C\|b_2\|_{H^3}^2 + \frac{1}{2}\|\partial_1 \partial_i^2 u\|_{L^2}^2.\end{aligned}$$

Thanks to the Leibniz formula and Hölder inequality, we have

$$\begin{aligned}K_6 =& -\int \partial_i^2 (b \cdot \nabla u) \cdot \partial_1 \partial_i^2 u dx\\ =& -\sum_{0\leq\alpha\leq 2} C_2^\alpha \int \partial_i^\alpha b \cdot \nabla \partial_i^{2-\alpha} u \cdot \partial_1 \partial_i^2 u dx\\ =& -\sum_{0\leq\alpha\leq 2} C_2^\alpha \int \partial_i^\alpha b_1 \partial_1 \partial_i^{2-\alpha} u \cdot \partial_1 \partial_i^2 u dx - \sum_{0\leq\alpha\leq 2} C_2^\alpha \int \partial_i^\alpha b_2 \partial_2 \partial_i^{2-\alpha} u \cdot \partial_1 \partial_i^2 u dx\\ =& -\int b_1 \partial_1 \partial_i^2 u \cdot \partial_1 \partial_i^2 u dx - 2\int \partial_i b_1 \partial_1 \partial_i u \cdot \partial_1 \partial_i^2 u dx - \int \partial_i^2 b_1 \partial_1 u \cdot \partial_1 \partial_i^2 u dx\\ &- \int b_2 \partial_2 \partial_i^2 u \cdot \partial_1 \partial_i^2 u dx - 2\int \partial_i b_2 \partial_2 \partial_i u \cdot \partial_1 \partial_i^2 u dx - \int \partial_i^2 b_2 \partial_2 u \cdot \partial_1 \partial_i^2 u dx\\ \leq& C\|\partial_1 \partial_i^2 u\|_{L^2} (\|b_1\|_{L^\infty} \|\partial_1 \partial_i^2 u\|_{L^2} + \|\partial_i b_1\|_{L^4} \|\partial_1 \partial_i u\|_{L^4} + \|\partial_i^2 b_1\|_{L^4} \|\partial_1 u\|_{L^4}\\ &+ \|b_2\|_{L^\infty} \|\partial_2 \partial_i^2 u\|_{L^2} + \|\partial_i b_2\|_{L^4} \|\partial_2 \partial_i u\|_{L^4} + \|\partial_i^2 b_2\|_{L^4} \|\partial_2 u\|_{L^4})\\ \leq& C(\|u\|_{H^3} + \|b\|_{H^3})(\|b_2\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).\end{aligned}$$

Combining all the estimates from K_1 to K_6, we find

$$\begin{aligned}\frac{1}{2}\|\partial_1 \partial_i^2 u\|_{L^2}^2 \leq& \frac{d}{dt}\langle \partial_i^2 b, \partial_i^2 \partial_1 u\rangle + C(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2)\\ &+ C(\|u\|_{H^3} + \|b\|_{H^3})(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).\end{aligned} \tag{3.11}$$

Putting (3.10) and (3.11) together gives

$$\begin{aligned}\frac{1}{2}\|\partial_1 u\|_{H^2}^2 \leq& \frac{d}{dt}\langle b, \partial_1 u\rangle + \frac{d}{dt}\langle \partial_i^2 b, \partial_i^2 \partial_1 u\rangle + C(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2)\\ &+ C(\|u\|_{H^3} + \|b\|_{H^3})(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).\end{aligned} \tag{3.12}$$

Integrating (3.12) over $[0, t]$ leads to

$$\begin{aligned}\int_0^t \|\partial_1 u\|_{H^2}^2 \leq& 2\int b \cdot \partial_1 u dx - 2\int b(x,0) \cdot \partial_1 u(x,0) dx + 2\int \partial_i^2 b \cdot \partial_i^2 \partial_1 u dx\\ &- 2\int \partial_i^2 b(x,0) \cdot \partial_i^2 \partial_1 u(x,0) dx + C\int_0^t (\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2)\\ &+ C\int_0^t (\|u\|_{H^3} + \|b\|_{H^3})(\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2)\end{aligned}$$

$$
\begin{aligned}
&\leq C(\|b_0\|_{L^2}\|\partial_1 u_0\|_{L^2} + \|\partial_i^2 b_0\|_{L^2}\|\partial_i^2\partial_1 u_0\|_{L^2}) \\
&\quad + C(\|b\|_{L^2}\|\partial_1 u\|_{L^2} + \|\partial_i^2 b\|_{L^2}\|\partial_i^2\partial_1 u\|_{L^2}) + C\int_0^t (\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2) \\
&\quad + C\sup_{0\leq\tau\leq t}(\|u\|_{H^3} + \|b\|_{H^3})\int_0^t (\|b_2\|_{H^3}^2 + \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2),
\end{aligned}
$$

which implies that

$$
E_2(t) \leq CE_1(0) + CE_1(t) + CE_1^{\frac{3}{2}}(t) + CE_2^{\frac{3}{2}}(t). \tag{3.13}
$$

Hence we finish the proof of Proposition 3.2. □

3.2. Proof of Theorem 1.1

Proof Multiplying (3.13) by $\frac{1}{2C}$, and then adding the resulting inequality to (3.8), one has

$$
\begin{aligned}
E_1(t) + \frac{1}{2C}E_2(t) \leq& C(E_1(0) + E_1^{\frac{3}{2}}(0) + E_1^2(0)) + C(E_1^{\frac{3}{2}}(t) + E_2^{\frac{3}{2}}(t)) \\
&+ \frac{1}{2}E_1(t) + C(E_1^2(t) + E_2^2(t)) + C(E_1^{\frac{5}{2}}(t) + E_2^{\frac{5}{2}}(t)),
\end{aligned}
$$

which together with the definition of $E(t)$ implies

$$
E(t) \leq C_1(E(0) + E^{\frac{3}{2}}(0) + E^2(0)) + C_2 E^{\frac{3}{2}}(t) + C_3 E^2(t) + C_4 E^{\frac{5}{2}}(t).
$$

We take $\|(u_0, b_0)\|_{H^3}$ to be sufficiently small,

$$
C_1(E(0) + E^{\frac{3}{2}}(0) + E^2(0)) \leq \frac{1}{4}\min\left\{\frac{1}{36C_2^2}, \frac{1}{6C_3}, (\frac{1}{6C_4})^{\frac{2}{3}}\right\}.
$$

The bootstrapping argument starts with the ansatz that

$$
E(t) \leq \min\left\{\frac{1}{36C_2^2}, \frac{1}{6C_3}, (\frac{1}{6C_4})^{\frac{2}{3}}\right\}.
$$

Then we can infer that

$$
E(t) \leq C_1(E(0) + E^{\frac{3}{2}}(0) + E^2(0)) + \frac{1}{2}E(t),
$$

which gives

$$
\begin{aligned}
E(t) \leq& 2C_1(E(0) + E^{\frac{3}{2}}(0) + E^2(0)) \\
\leq& \frac{1}{2}\min\left\{\frac{1}{36C_2^2}, \frac{1}{6C_3}, (\frac{1}{6C_4})^{\frac{2}{3}}\right\},
\end{aligned}
$$

This inequality implies the desired estimate. Theorem 1.1 is completed. □

4 Proof of Theorem 1.4 and Theorem 1.5

This section proves Theorem 1.4 and Theorem 1.5. We are now ready to prove Theorem 1.4.

Proof From (1.8), we find

$$\frac{1}{2}\frac{d}{dt}\|(u,b)\|_{L^2}^2+\|\mathcal{T}_2 u\|_{L^2}^2+\|\mathcal{R}_1 b\|_{L^2}^2=0, \tag{4.1}$$

and

$$\frac{1}{2}\frac{d}{dt}\|(\nabla u,\nabla b)\|_{L^2}^2+\|\mathcal{T}_2 \nabla u\|_{L^2}^2+\|\mathcal{R}_1 \nabla b\|_{L^2}^2=0. \tag{4.2}$$

From (4.1) and (4.2), we find

$$\frac{d}{dt}A(t)+B(t)=0, \tag{4.3}$$

where

$$\begin{aligned}A(t)=&\|u\|_{L^2}^2+\|b\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2,\\ B(t)=&2\|\mathcal{T}_2 u\|_{L^2}^2+2\|\mathcal{R}_1 b\|_{L^2}^2+2\|\mathcal{T}_2\nabla u\|_{L^2}^2+2\|\mathcal{R}_1\nabla b\|_{L^2}^2.\end{aligned}$$

Applying $\Lambda_1^{-\sigma},\ \Lambda_2^{-2\sigma}$ to (1.8) and dotting them with $(\Lambda_1^{-\sigma}u,\ \Lambda_1^{-\sigma}b)$ and $(\Lambda_2^{-2\sigma}u,\ \Lambda_2^{-2\sigma}b)$ in $H^{1+\sigma}$, respectively, we obtain

$$\begin{aligned}&\frac{d}{dt}L(t)+2\|\mathcal{T}_2(\Lambda_1^{-\sigma},\Lambda_2^{-2\sigma})u\|_{L^2}^2+2\|\mathcal{R}_1(\Lambda_1^{-\sigma},\Lambda_2^{-2\sigma})b\|_{L^2}^2\\ &\quad+2\|\mathcal{T}_2\Lambda^{1+\sigma}(\Lambda_1^{-\sigma},\Lambda_2^{-2\sigma})u\|_{L^2}^2+2\|\mathcal{R}_1\Lambda^{1+\sigma}(\Lambda_1^{-\sigma},\Lambda_2^{-2\sigma})b\|_{L^2}^2=0,\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}L(t)=&\|(\Lambda_1^{-\sigma},\Lambda_2^{-2\sigma})u\|_{L^2}^2+\|(\Lambda_1^{-\sigma},\Lambda_2^{-2\sigma})b\|_{L^2}^2\\ &+\|\Lambda^{1+\sigma}(\Lambda_1^{-\sigma},\Lambda_2^{-2\sigma})u\|_{L^2}^2+\|\Lambda^{1+\sigma}(\Lambda_1^{-\sigma},\Lambda_2^{-2\sigma})b\|_{L^2}^2.\end{aligned}$$

It's easy to get from (4.3)

$$L(t)\le L(0). \tag{4.5}$$

The next job is to prove that

$$A(t)\le CB^{\frac{\sigma}{1+\sigma}}(t)L^{\frac{1}{1+\sigma}}(t). \tag{4.6}$$

Applying the Plancherel theorem and the Hölder inequality to obtain

$$\begin{aligned}\|u\|_{L^2}^2=&\int|\hat{u}(\xi,t)|^2d\xi\\ =&\int(|\xi_2|^4|\hat{u}(\xi,t)|^2)^{\frac{\sigma}{1+\sigma}}(|\xi_2|^{-4\sigma}|\hat{u}(\xi,t)|^2)^{\frac{1}{1+\sigma}}d\xi\end{aligned}$$

$$
\begin{aligned}
&\leq \Big(\int |\xi|^{-2}|\xi_2|^4|\xi|^2|\hat{u}(\xi,t)|^2 d\xi\Big)^{\frac{\sigma}{1+\sigma}} \Big(\int |\xi_2|^{-4\sigma}|\hat{u}(\xi,t)|^2 d\xi\Big)^{\frac{1}{1+\sigma}} \\
&\leq \|\mathcal{T}_2\nabla u\|_{L^2}^{\frac{2\sigma}{1+\sigma}} \|\Lambda_2^{-2\sigma} u\|_{L^2}^{\frac{2}{1+\sigma}}.
\end{aligned}
$$

and

$$
\begin{aligned}
\|\nabla u\|_{L^2}^2 &= \int |\xi|^2|\hat{u}(\xi,t)|^2 d\xi \\
&= \int (|\xi_2|^4|\hat{u}(\xi,t)|^2)^{\frac{\sigma}{1+\sigma}} (|\xi_2|^{-4\sigma}|\xi|^{2(1+\sigma)}|\hat{u}(\xi,t)|^2)^{\frac{1}{1+\sigma}} d\xi \\
&\leq \Big(\int |\xi|^{-2}|\xi_2|^4|\xi|^2|\hat{u}(\xi,t)|^2 d\xi\Big)^{\frac{\sigma}{1+\sigma}} \Big(\int |\xi_2|^{-4\sigma}|\xi|^{2(1+\sigma)}|\hat{u}(\xi,t)|^2 d\xi\Big)^{\frac{1}{1+\sigma}} \\
&\leq \|\mathcal{T}_2\nabla u\|_{L^2}^{\frac{2\sigma}{1+\sigma}} \|\Lambda^{1+\sigma}\Lambda_2^{-2\sigma} u\|_{L^2}^{\frac{2}{1+\sigma}}.
\end{aligned}
$$

Repeating the same argument gives

$$
\begin{aligned}
\|b\|_{L^2}^2 &= \int |\hat{b}(\xi,t)|^2 d\xi \\
&= \int (|\xi_1|^2|\hat{b}(\xi,t)|^2)^{\frac{\sigma}{1+\sigma}} (|\xi_1|^{-2\sigma}|\hat{b}(\xi,t)|^2)^{\frac{1}{1+\sigma}} d\xi \\
&\leq \Big(\int |\xi|^{-2}|\xi_1|^2|\xi|^2|\hat{b}(\xi,t)|^2 d\xi\Big)^{\frac{\sigma}{1+\sigma}} \Big(\int |\xi_1|^{-2\sigma}|\hat{b}(\xi,t)|^2 d\xi\Big)^{\frac{1}{1+\sigma}} \\
&\leq \|\mathcal{R}_1\nabla b\|_{L^2}^{\frac{2\sigma}{1+\sigma}} \|\Lambda_1^{-\sigma} b\|_{L^2}^{\frac{2}{1+\sigma}}.
\end{aligned}
$$

and

$$
\begin{aligned}
\|\nabla b\|_{L^2}^2 &= \int |\xi|^2|\hat{b}(\xi,t)|^2 d\xi \\
&= \int (|\xi_1|^2|\hat{b}(\xi,t)|^2)^{\frac{\sigma}{1+\sigma}} (|\xi_1|^{-2\sigma}|\xi|^{2(1+\sigma)}|\hat{b}(\xi,t)|^2)^{\frac{1}{1+\sigma}} d\xi \\
&\leq \Big(\int |\xi|^{-2}|\xi_1|^2|\xi|^2|\hat{b}(\xi,t)|^2 d\xi\Big)^{\frac{\sigma}{1+\sigma}} \Big(\int |\xi_1|^{-2\sigma}|\xi|^{2(1+\sigma)}|\hat{b}(\xi,t)|^2 d\xi\Big)^{\frac{1}{1+\sigma}} \\
&\leq \|\mathcal{R}_1\nabla b\|_{L^2}^{\frac{2\sigma}{1+\sigma}} \|\Lambda^{1+\sigma}\Lambda_1^{-\sigma} b\|_{L^2}^{\frac{2}{1+\sigma}}.
\end{aligned}
$$

Combining the above four inequalities can obtain (4.6). Combining (4.5) and (4.6), we immediately get

$$
B(t) \geq C L^{-\frac{1}{\sigma}}(0) A^{1+\frac{1}{\sigma}}(t),
$$

which together with (4.3) yields

$$
\frac{d}{dt}A(t) + C L^{-\frac{1}{\sigma}}(0) A^{1+\frac{1}{\sigma}}(t) \leq 0.
$$

This inequality gives rise to

$$A(t) \le \left(A^{-\frac{1}{\sigma}}(0) + \frac{C}{\sigma} L^{-\frac{1}{\sigma}}(0) t \right)^{-\sigma},$$

which completes the proof of Theorem 1.4. □

We proceed to prove Theorem 1.5.

Proof Taking the L^2-inner product to the equation $(1.9)_1$ with $\partial_t u$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\partial_t u\|_{L^2}^2 + \|\mathcal{R}_1\mathcal{T}_2 u\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2) \\ &\quad + \|\partial_t \mathcal{R}_1 u\|_{L^2}^2 + \|\partial_t \mathcal{T}_2 u\|_{L^2}^2 = 0. \end{aligned} \tag{4.7}$$

Similarly,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\partial_t \Lambda\Lambda_1^{-1} u\|_{L^2}^2 + \|\mathcal{T}_2 u\|_{L^2}^2 + \|\Lambda u\|_{L^2}^2) \\ &\quad + \|\partial_t u\|_{L^2}^2 + \|\partial_t \mathcal{T}_2 \Lambda\Lambda_1^{-1} u\|_{L^2}^2 = 0. \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\partial_t \Lambda_1^{-1} u\|_{L^2}^2 + \|\mathcal{R}_1\mathcal{T}_2\Lambda_1^{-1} u\|_{L^2}^2 + \|u\|_{L^2}^2) \\ &\quad + \|\partial_t \mathcal{R}_1 \Lambda_1^{-1} u\|_{L^2}^2 + \|\partial_t \mathcal{T}_2 \Lambda_1^{-1} u\|_{L^2}^2 = 0. \end{aligned} \tag{4.9}$$

Taking L^2-inner product to $(1.9)_1$ with u, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\mathcal{R}_1 u\|_{L^2}^2 + \|\mathcal{T}_2 u\|_{L^2}^2 + 2\langle \partial_t u, u\rangle) \\ &\quad + \|\mathcal{R}_1\mathcal{T}_2 u\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2 - \|\partial_t u\|_{L^2}^2 = 0. \end{aligned} \tag{4.10}$$

Combining (4.7), (4.8), (4.9) and (4.10) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\partial_t u\|_{L^2}^2 + \|\mathcal{R}_1\mathcal{T}_2 u\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2 + \|\partial_t \Lambda\Lambda_1^{-1} u\|_{L^2}^2 + \|\mathcal{T}_2 u\|_{L^2}^2 \\ &\quad + \|\Lambda u\|_{L^2}^2 + \|\partial_t \Lambda_1^{-1} u\|_{L^2}^2 + \|\mathcal{R}_1\mathcal{T}_2\Lambda_1^{-1} u\|_{L^2}^2 + \|u\|_{L^2}^2 + \frac{1}{2}\|\mathcal{R}_1 u\|_{L^2}^2 \\ &\quad + \frac{1}{2}\|\mathcal{T}_2 u\|_{L^2}^2 + \langle \partial_t u, u\rangle) + \|\partial_t \mathcal{R}_1 u\|_{L^2}^2 + \|\partial_t \mathcal{T}_2 u\|_{L^2}^2 + \|\partial_t u\|_{L^2}^2 \\ &\quad + \|\partial_t \mathcal{T}_2 \Lambda\Lambda_1^{-1} u\|_{L^2}^2 + \|\partial_t \mathcal{R}_1 \Lambda_1^{-1} u\|_{L^2}^2 + \|\partial_t \mathcal{T}_2 \Lambda_1^{-1} u\|_{L^2}^2 \\ &\quad + \frac{1}{2}\|\mathcal{R}_1\mathcal{T}_2 u\|_{L^2}^2 + \frac{1}{2}\|\partial_1 u\|_{L^2}^2 - \frac{1}{2}\|\partial_t u\|_{L^2}^2 = 0. \end{aligned} \tag{4.11}$$

We can easily verify that

$$\frac{d}{dt}\|\partial_t u\|_{L^2}^2 + \|u\|_{L^2}^2 + \langle \partial_t u, u\rangle \ge \frac{1}{2}\|\partial_t u\|_{L^2}^2 + \|u\|_{L^2}^2.$$

Thus, by integrating both sides of (4.11) over $(0, t)$, we derive

$$\int_0^t (\|\partial_t u\|_{L^2}^2 + \|\mathcal{R}_1\mathcal{T}_2 u\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2) d\tau$$
$$\leq C(\|u_0\|_{H^1}^2 + \|\partial_t u_0\|_{L^2}^2 + \|\Lambda\Lambda_1^{-1}\partial_t u_0\|_{L^2}^2 + \|\Lambda_1^{-1}\partial_t u_0\|_{L^2}^2 + \|\mathcal{R}_1\mathcal{T}_2\Lambda_1^{-1} u_0\|_{L^2}^2).$$

It follows from (4.7) that

$$\frac{d}{dt}(\|\partial_t u\|_{L^2}^2 + \|\mathcal{R}_1\mathcal{T}_2 u\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2) \leq 0,$$

which together with Lemma 2.3 implies

$$\|\partial_t u\|_{L^2}^2 + \|\mathcal{R}_1\mathcal{T}_2 u\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2 \leq C(1+t)^{-1}.$$

We can prove the same result is true for b along the same method. The proof of Theorem 1.5 is thus completed. □

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Declarations

Competing Interests The authors declare that they have no conflict of interest.

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