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Stability and Large Time Behavior of the 2D Boussinesq Equations with Mixed Partial Dissipation Near Hydrostatic Equilibrium

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Abstract
This paper establishes the large time behavior of the solution to two dimensional Boussinesq equations with mixed partial dissipation. Our main result is achieved in terms of the global H^2-stability. Finally, we also obtain the decay estimates of linearized Boussinesq equations.

Keywords Boussinesq equations · Large time behavior · H^2-Stability · Partial dissipation

Mathematics Subject Classification (2010) 35Q30 · 76D03 · 76D07

1 Introduction

This note is concerned with the following two dimensional Boussinesq equations with mixed partial dissipation:

$$\begin{cases} \partial_t U_1 + U\cdot\nabla U_1 - \mu\partial_{22}U_1 + \partial_1\overline{\pi} = 0, & (x,t)\in\mathbf{R}^2\times(0,\infty),\\ \partial_t U_2 + U\cdot\nabla U_2 - \kappa\partial_{11}U_2 + \partial_2\overline{\pi} = \Theta, & (x,t)\in\mathbf{R}^2\times(0,\infty),\\ \partial_t\Theta + U\cdot\nabla\Theta - \eta\partial_{11}\Theta = 0, & (x,t)\in\mathbf{R}^2\times(0,\infty),\\ \nabla\cdot U = 0, & (x,t)\in\mathbf{R}^2\times(0,\infty),\\ U|_{t=0} = U_0,\ \Theta|_{t=0} = \Theta_0, & x\in\mathbf{R}^2, \end{cases} \tag{1.1}$$

where the unknown $U = (U_1, U_2)$ denotes the velocity field, $\overline{\pi}$ is the pressure, Θ is the temperature, μ and κ are the velocity viscosity, η is the thermal diffusivity. For the term Θ in the second equation of (1.1) represents the buoyancy forcing generated due to the temperature variation.

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Let

$$u = U - U^0, \theta = \Theta - \Theta^0, \pi = \overline{\pi} - \overline{\pi}^0.$$

Then (u, θ, π) obeys

$$\begin{cases} \partial_t u_1 + u \cdot \nabla u_1 - \mu \partial_{22} u_1 + \partial_1 \pi = 0, & (x,t) \in \mathbf{R}^2 \times (0,\infty), \\ \partial_t u_2 + u \cdot \nabla u_2 - \kappa \partial_{11} u_2 + \partial_2 \pi = \theta, & (x,t) \in \mathbf{R}^2 \times (0,\infty), \\ \partial_t \theta + u \cdot \nabla \theta - \eta \partial_{11} \theta = -u_2, & (x,t) \in \mathbf{R}^2 \times (0,\infty), \\ \nabla \cdot u = 0, & (x,t) \in \mathbf{R}^2 \times (0,\infty), \\ u|_{t=0} = u_0, \theta|_{t=0} = \theta_0, & x \in \mathbf{R}^2, \end{cases} \tag{1.2}$$

where

$$U^0 = 0, \quad \Theta^0 = x_2, \quad \overline{\pi}^0 = \frac{1}{2} x_2^2 \tag{1.3}$$

is the steady solution of (1.1). Many geophysical flows such as atmospheric fronts and ocean circulations can be modeled by the Boussinesq equations. Recently, the stability and large time behavior issues on the Boussinesq equations have gained more and more interests and become the center of mathematic investigation. In the last thirty years, a considerable amount of literature has been published on the stability problem concerning the Boussinesq equations. Some of them focus on the stability of 2D Boussinesq equations with various partial dissipation (see e. g. [1], [2], [4], [5], [8], [9], [10], [11]). In 2019, Ji, Li, Wei and Wu [6] obtained the stability of the 2D Boussinesq eqution (1.2) under the assumption that H^1-norm of initial data is small. However, they didn't give the large time behavior of the system (1.2). Very recently, Lai, Wu and Zhong [7] have established the global existence and stability of 2D Boussinesq equations with partial dissipation and temperature damping in the Sobolev space $H^2(\mathbf{R}^2)$. In addition, the large-time behavior of $\|\nabla u\|_{L^2}$ and $\|\nabla \theta\|_{L^2}$ is also obtained via energy methods. Motivated by [1], [9] and [7], the purpose of this paper is to address large time behavior of the solution to the system (1.2) and decay estimates of linearized equation of system (1.2). Our results are stated as follows.

Theorem 1.1 *Let* $(u_0, \theta_0) \in H^2(\mathbf{R}^2)$ *and* $\nabla \cdot u_0 = 0$. *If*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \le \varepsilon, \tag{1.4}$$

holds for sufficiently small $\varepsilon > 0$, *then, the system* (1.2) *admits a unique global smooth solution satisfying*

$$\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + 2\int_0^t \mu \|\partial_2 u_1(\tau)\|_{H^2}^2 + \kappa \|\partial_1 u_2(\tau)\|_{H^2}^2 + \eta \|\partial_1 \theta(\tau)\|_{H^2}^2 d\tau \le C\varepsilon^2 \tag{1.5}$$

for all $t > 0$ *and* $C = C(\mu, \kappa, \eta)$ *is a positive constant. Moreover,*

$$\|\partial_1 u_2(t)\|_{L^2} \to 0, \quad \|\partial_2 u_1(t)\|_{L^2} \to 0, \quad \|\partial_1 \theta(t)\|_{L^2} \to 0, \quad as \quad t \to \infty. \tag{1.6}$$

Remark 1.2 Compared with Theorem 1.1 in [6], we obtain the stability under the H^2-norm of the initial data (u_0, θ_0) is small because the achievement of large time behavior of the solution (u, θ) to system (1.2) is heavily dependent on the uniform estimate (1.5).

Applying the ∂_1 and ∂_2 to $(1.2)_1$ and $(1.2)_2$, respectively, to conclude

$$\pi = \Delta^{-1}\partial_2\theta - \Delta^{-1}\nabla\cdot\nabla\cdot(u\otimes u) + \mu\Delta^{-1}\partial_1\partial_{22}u_1 + \kappa\Delta^{-1}\partial_2\partial_{11}u_2. \tag{1.7}$$

Then, the equation (1.2) can be rewritten as

$$\begin{cases} \partial_t u_1 + u\cdot\nabla u_1 - \mu\partial_{22}u_1 + \partial_1\Delta^{-1}\partial_2\theta - \partial_1\Delta^{-1}\nabla\cdot\nabla\cdot(u\otimes u) \\ \quad + \mu\partial_1\Delta^{-1}\partial_1\partial_{22}u_1 + \kappa\partial_1\Delta^{-1}\partial_2\partial_{11}u_2 = 0, \\ \partial_t u_2 + u\cdot\nabla u_2 - \kappa\partial_{11}u_2 - \partial_1\partial_1\Delta^{-1}\theta - \partial_2\Delta^{-1}\nabla\cdot\nabla\cdot(u\otimes u) \\ \quad + \mu\partial_2\Delta^{-1}\partial_1\partial_{22}u_1 + \kappa\partial_2\Delta^{-1}\partial_2\partial_{11}u_2 = 0, \\ \partial_t\theta + u\cdot\nabla\theta - \eta\partial_{11}\theta = -u_2. \end{cases} \tag{1.8}$$

The linearized equations of (1.8) is

$$\begin{cases} \partial_t u_1 - \Delta^{-1}(\mu\partial_2^4 + \kappa\partial_1^4)u_1 + \partial_1\partial_2\Delta^{-1}\theta = 0, \\ \partial_t u_2 - \Delta^{-1}(\mu\partial_2^4 + \kappa\partial_1^4)u_2 - \partial_1\partial_1\Delta^{-1}\theta = 0, \\ \partial_t\theta - \eta\partial_{11}\theta = -u_2. \end{cases} \tag{1.9}$$

The following theorem gives the explicit decay rates of the solution of (1.9).

Theorem 1.3 *Let (u, θ) be the corresponding solution of (1.9). Then we have the following two conclusions*:

(i) Let $\sigma > 0$. Assume initial data (u_0, θ_0) with $\nabla\cdot u_0 = 0$ satisfying

$$\|\Lambda_1^{-\sigma}u_0\|_{L^2} + \|\Lambda_1^{-\sigma}\theta_0\|_{L^2} + \|\Lambda_1^{-(\sigma+2)}\theta_0\|_{L^2} \le \varepsilon \tag{1.10}$$

for some ε small enough. Then (u, θ) obeys the following decay estimate

$$\|u(t)\|_{L^2} + \|\theta(t)\|_{L^2} \le C\varepsilon t^{-\frac{\sigma}{2}}, \tag{1.11}$$

where $C > 0$ is a constant independent of ε and t.

(ii)Let $m > 0$. Assume initial data (u_0, θ_0) with $\nabla\cdot u_0 = 0$ satisfying

$$\|u_0\|_{L^2} + \|\theta_0\|_{L^2} + \|\Lambda_1^{-2}\theta_0\|_{L^2} \le \varepsilon \tag{1.12}$$

for some ε small enough. Then (u, θ) obeys the following decay estimate

$$\|\partial_1^m u(t)\|_{L^2} + \|\partial_1^m\theta(t)\|_{L^2} \le C\varepsilon t^{-\frac{m}{2}}, \tag{1.13}$$

where $C > 0$ is a constant independent of ε and t.

Remark 1.4 By taking the time derivative on (1.9) and making several substitutions, the system (1.9) turns into the following degenerate wave equations with damping:

$$\begin{cases} \partial_{tt}u_1 + (\mu R_2^2\partial_2^2 + \kappa R_1^2\partial_1^2 - \eta\partial_{11})\partial_t u_1 - (R_1^2 + \mu\eta R_1^2\partial_2^4 + \kappa\eta R_1^2\partial_1^4)u_1 = 0, \\ \partial_{tt}u_2 + (\mu R_2^2\partial_2^2 + \kappa R_1^2\partial_1^2 - \eta\partial_{11})\partial_t u_2 - (R_1^2 + \mu\eta R_1^2\partial_2^4 + \kappa\eta R_1^2\partial_1^4)u_2 = 0, \\ \partial_{tt}\theta + (\mu R_2^2\partial_2^2 + \kappa R_1^2\partial_1^2 - \eta\partial_{11})\partial_t \theta - (R_1^2 + \mu\eta R_1^2\partial_2^4 + \kappa\eta R_1^2\partial_1^4)\theta = 0, \end{cases} \tag{1.14}$$

where $R_i = \partial_i(-\Delta)^{-\frac{1}{2}}$ with $i = 1, 2$ denotes the standard Resiz transform. Compared with the wave equations in [1], this system is more complex. The upper bounds for the kernel function G_1 and G_2, which is presented in Sect. 4, are more sophisticated to handle than that in [1]. These upper bounds play a crucial role in achieving the decay estimate in Theorem 1.3.

Remark 1.5 Now we explain why we cannot obtain the decay rate of the system (1.8). The methods of proving Theorem 1.3 heavily depends on the spectral analysis of the wave equations (1.14). Unfortunately, it is very difficult for us to decouple the system (1.8). Consequently, we cannot build the decay estimates of system (1.8) via spectral methods. It is of great interest to address this problem.

The rest of this paper is organized as follows. Some crucial lemmas are presented in Sect. 2. We first build a priori estimates and exploy the bootstrap argument to establish H^2-stability in Sect. 3. The large time behavior of the solution to system (1.1) is also obtained in Sect. 3. The proof of Theorem 1.3 can be found in Sect. 4.

Notation We recall the definition of the fractional Laplacian, $\widehat{\Lambda_i^\beta f}(\xi) = |\xi_i|^\beta \hat{f}(\xi)$, for any real number β and $i = 1, 2$, $\xi = (\xi_1, \xi_2)$.

2 Several Useful Lemmas

For the convenience, we first recall the following version of the two dimensional anisotropic inequalities in the whole space $\mathbf{R}^2$. Lemma 2.1 is due to Cao and Wu [3].

Lemma 2.1 *Assume $f, g, h, \partial_1 g$ and $\partial_2 h$ are in $L^2(\mathbf{R}^2)$, then, for a constant C,*

$$\int_{\mathbf{R}^2} |fgh|dx \le C\|f\|_{L^2(\mathbf{R}^2)}\|g\|_{L^2(\mathbf{R}^2)}^{\frac{1}{2}}\|\partial_1 g\|_{L^2(\mathbf{R}^2)}^{\frac{1}{2}}\|h\|_{L^2(\mathbf{R}^2)}^{\frac{1}{2}}\|\partial_2 h\|_{L^2(\mathbf{R}^2)}^{\frac{1}{2}}. \tag{2.1}$$

Lemma 2.2 *Let $f = f(t)$, with $t \in [0, \infty)$ be nonnegative continuous function. Assume f is integrable on $[0, \infty)$,*

$$\int_0^\infty f(t)dt < \infty.$$

Assume that for any $\delta > 0$, there is $\rho > 0$ such that, for any $0 \le t_1 < t_2$ with $t_2 - t_1 \le \rho$, either $f(t_2) \le f(t_1)$ or $f(t_2) \ge f(t_1)$ and $f(t_2) - f(t_1) \le \delta$. Then

$$f(t) \to 0, \quad as \quad t \to \infty. \tag{2.2}$$

This Lemma can be found in [7].

3 Proofs of Theorem 1.1

3.1 H^2-Stability

For the sake of conciseness, we construct a suitable energy functional:

$$E(t) = \sup_{0\le\tau\le t} (\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2) + 2\int_0^t \mu\|\partial_2 u_1(\tau)\|_{H^2}^2 + \kappa\|\partial_1 u_2(\tau)\|_{H^2}^2 + \eta\|\partial_1\theta(\tau)\|_{H^2}^2 d\tau. \tag{3.1}$$

Step 1 L^2-energy estimate. A standard energy method yields

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\int_0^t \mu\|\partial_2 u_1(\tau)\|_{L^2}^2 + \kappa\|\partial_1 u_2(\tau)\|_{L^2}^2 + \eta\|\partial_1\theta(\tau)\|_{L^2}^2 d\tau \le \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2. \tag{3.2}$$

Step 2 $\dot{H}^2$-energy estimate. Applying Δ to both sides of the first, the second and the third equation of (1.2), respectively, then taking the L^2-inner product with $(\Delta u_1, \Delta u_2, \Delta\theta)$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Delta u(t)\|_{L^2}^2 + \|\Delta\theta(t)\|_{L^2}^2) + \mu\|\partial_2\Delta u_1\|_{L^2}^2 + \kappa\|\partial_1\Delta u_2\|_{L^2}^2 + \eta\|\partial_1\Delta\theta\|_{L^2}^2 \\
&= -\int_{\mathbf{R}^2} \Delta(u\cdot\nabla u_1)\Delta u_1 + \Delta(u\cdot\nabla u_2)\Delta u_2 dx + \int_{\mathbf{R}^2}(\Delta\theta\Delta u_2 - \Delta u_2\Delta\theta)dx \\
&\quad - \int_{\mathbf{R}^2}\Delta(u\cdot\nabla\theta)\Delta\theta dx := I_1 + I_2 + I_3.
\end{aligned} \tag{3.3}$$

It is not difficult to check that $I_2 = 0$. To estimate I_1, we decompose I_1 into the following form:

$$\begin{aligned}
I_1 = &-\int_{\mathbf{R}^2} \partial_{11}u\cdot\nabla u_1\partial_{11}u_1 + 2\partial_1 u\cdot\nabla\partial_1 u_1\partial_{11}u_1 dx \\
&-\int_{\mathbf{R}^2} \partial_{22}u\cdot\nabla u_1\partial_{22}u_1 + 2\partial_2 u\cdot\nabla\partial_2 u_1\partial_{22}u_1 dx \\
&-\int_{\mathbf{R}^2} \partial_{11}u\cdot\nabla u_2\partial_{11}u_2 + 2\partial_1 u\cdot\nabla\partial_1 u_2\partial_{11}u_2 dx \\
&-\int_{\mathbf{R}^2} \partial_{22}u\cdot\nabla u_2\partial_{22}u_2 + 2\partial_2 u\cdot\nabla\partial_2 u_2\partial_{22}u_2 dx \\
:= &\ I_{11} + I_{12} + I_{13} + I_{14}.
\end{aligned} \tag{3.4}$$

Thanks to the fact that $\nabla \cdot u = 0$ and the Sobolev embedding, one gets

$$\begin{aligned} I_{11} =& -3\int_{\mathbf{R}^2} (\partial_{11}u_1)^2\partial_1 u_1 dx - \int_{\mathbf{R}^2} \partial_{11}u_2\partial_2 u_1\partial_{11}u_1 dx \\ & -2\int_{\mathbf{R}^2} \partial_1 u_2\partial_2\partial_1 u_1\partial_{11}u_1 dx \\ \leq & C\|\partial_1 u_1\|_{L^2}\|\partial_{11}u_1\|_{L^4}^2 + C\|\partial_2 u_1\|_{L^\infty}\|\partial_{11}u_2\|_{L^2}\|\partial_{11}u_1\|_{L^2} \\ & + C\|\partial_1 u_2\|_{L^\infty}\|\partial_2\partial_1 u_1\|_{L^2}\|\partial_{11}u_1\|_{L^2} \\ \leq & C\|u\|_{H^2}(\|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 u_2\|_{H^2}^2). \end{aligned} \tag{3.5}$$

Similarly,

$$I_{12}, I_{13}, I_{14} \leq C\|u\|_{H^2}(\|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 u_2\|_{H^2}^2). \tag{3.6}$$

Next, we split I_3 into the following two parts:

$$\begin{aligned} I_3 =& -\int_{\mathbf{R}^2} \partial_{11}(u\cdot\nabla\theta)\partial_{11}\theta + \partial_{22}(u\cdot\nabla\theta)\partial_{22}\theta dx \\ =& -\int_{\mathbf{R}^2} \partial_{11}u\cdot\nabla\theta\partial_{11}\theta + 2\partial_1 u\cdot\nabla\partial_1\theta\partial_{11}\theta dx \\ & -\int_{\mathbf{R}^2} \partial_{22}u\cdot\nabla\theta\partial_{22}\theta + 2\partial_2 u\cdot\nabla\partial_2\theta\partial_{22}\theta dx \\ :=& I_{31} + I_{32}. \end{aligned}$$

We can infer from the Hölder inequality and the Sobolev inequality

$$\begin{aligned} I_{31} =& -\int_{\mathbf{R}^2} \partial_{11}u_1\partial_1\theta\partial_{11}\theta dx - \int_{\mathbf{R}^2} \partial_{11}u_2\partial_2\theta\partial_{11}\theta dx - 2\int_{\mathbf{R}^2} \partial_1 u\cdot\nabla\partial_1\theta\partial_{11}\theta dx \\ \leq & C\|\partial_{11}u_1\|_{L^2}\|\partial_1\theta\|_{L^4}\|\partial_{11}\theta\|_{L^4} + C\|\partial_{11}u_2\|_{L^4}\|\partial_2\theta\|_{L^2}\|\partial_{11}\theta\|_{L^4} \\ & + C\|\partial_1 u\|_{L^2}\|\nabla\partial_1\theta\|_{L^4}\|\partial_{11}\theta\|_{L^4} \\ \leq & C\|u\|_{H^2}\|\partial_1\theta\|_{H^2}^2 + C\|\theta\|_{H^2}(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2) \\ \leq & C(\|u\|_{H^2} + \|\theta\|_{H^2})(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2). \end{aligned} \tag{3.7}$$

To handle I_{32}, we write

$$\begin{aligned} I_{32} \leq & C\left|\int_{\mathbf{R}^2} \partial_{22}u_1\partial_1\theta\partial_{22}\theta + \partial_2 u_1\partial_1\partial_2\theta\partial_{22}\theta dx\right| \\ & + C\left|\int_{\mathbf{R}^2} \partial_{22}u_2\partial_2\theta\partial_{22}\theta dx\right| + C\left|\int_{\mathbf{R}^2} \partial_2 u_2(\partial_{22}\theta)^2 dx\right| \\ :=& I_{321} + I_{322} + I_{323}. \end{aligned}$$

According to the Hölder inequality, it deduces

$$\begin{aligned} I_{321} \leq & C(\|\partial_{22}u_1\|_{L^4}\|\partial_1\theta\|_{L^4} + \|\partial_2 u_1\|_{L^4}\|\partial_1\partial_2\theta\|_{L^4})\|\partial_{22}\theta\|_{L^2} \\ \leq & C\|\theta\|_{H^2}(\|\partial_2 u_1\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2). \end{aligned} \tag{3.8}$$

Integrating by parts and the Hölder inequality give rise to

$$
\begin{aligned}
I_{322} \leq & C\left|\int_{\mathbf{R}^2} \partial_1 \partial_2 u_1 \partial_2 \theta \partial_{22} \theta d x\right| \\
\leq & C\left|\int_{\mathbf{R}^2} \partial_2 u_1\left(\partial_1 \partial_2 \theta \partial_{22} \theta+\partial_2 \theta \partial_{22} \partial_1 \theta\right) d x\right| \\
\leq & C\left\|\partial_2 u_1\right\|_{L^4}\left(\left\|\partial_1 \partial_2 \theta\right\|_{L^4}\left\|\partial_{22} \theta\right\|_{L^2}+\left\|\partial_2 \theta\right\|_{L^4}\left\|\partial_1 \partial_{22} \theta\right\|_{L^2}\right) \\
\leq & C\|\theta\|_{H^2}\left(\left\|\partial_2 u_1\right\|_{H^2}^2+\left\|\partial_1 \theta\right\|_{H^2}^2\right).
\end{aligned} \tag{3.9}
$$

Form Lemma 2.1 and the Young inequality, one can follow that

$$
\begin{aligned}
I_{323} \leq & C\left|\int_{\mathbf{R}^2} \partial_1 u_1\left(\partial_{22} \theta\right)^2 d x\right| \\
\leq & C\left|\int_{\mathbf{R}^2} u_1 \partial_1 \partial_{22} \theta \partial_{22} \theta d x\right| \\
\leq & C\left\|\partial_1 \partial_{22} \theta\right\|_{L^2}\left\|\partial_{22} \theta\right\|_{L^2}^{\frac{1}{2}}\left\|\partial_1 \partial_{22} \theta\right\|_{L^2}^{\frac{1}{2}}\left\|u_1\right\|_{L^2}^{\frac{1}{2}}\left\|\partial_2 u_1\right\|_{L^2}^{\frac{1}{2}} \\
\leq & C\left(\|u\|_{H^2}+\|\theta\|_{H^2}\right)\left(\left\|\partial_2 u_1\right\|_{H^2}^2+\left\|\partial_1 \theta\right\|_{H^2}^2\right).
\end{aligned} \tag{3.10}
$$

Combining the estimates from (3.5) to (3.10) and integrating over $[0, t]$, we can obtain

$$
\|\Delta u(t)\|_{L^2}^2+\|\Delta \theta(t)\|_{L^2}^2+2 \int_0^t \mu\left\|\partial_2 \Delta u_1\right\|_{L^2}^2+\kappa\left\|\partial_1 \Delta u_2\right\|_{L^2}^2+\eta\left\|\partial_1 \Delta \theta\right\|_{L^2}^2 d \tau \tag{3.11}
$$

$$
\leq\left\|\Delta u_0\right\|_{L^2}^2+\left\|\Delta \theta_0\right\|_{L^2}^2+C \int_0^t\left(\|u\|_{H^2}+\|\theta\|_{H^2}\right)\left(\left\|\partial_2 u_1\right\|_{H^2}^2+\left\|\partial_1 u_2\right\|_{H^2}^2+\left\|\partial_1 \theta\right\|_{H^2}^2\right) d \tau .
$$

Adding (3.2) and (3.11) leads to

$$
\begin{aligned}
& \|u(t)\|_{H^2}^2+\|\theta(t)\|_{H^2}^2+2 \int_0^t \mu\left\|\partial_2 u_1\right\|_{H^2}^2+\kappa\left\|\partial_1 u_2\right\|_{H^2}^2+\eta\left\|\partial_1 \theta\right\|_{H^2}^2 d \tau \\
& \quad \leq C_0\left(\left\|u_0\right\|_{H^2}^2+\left\|\theta_0\right\|_{H^2}^2\right)+C_1 \sup _{0 \leq \tau \leq t}\left(\|u(\tau)\|_{H^2}+\|\theta(\tau)\|_{H^2}\right) \\
& \quad \times \int_0^t\left(\left\|\partial_2 u_1\right\|_{H^2}^2+\left\|\partial_1 u_2\right\|_{H^2}^2+\left\|\partial_1 \theta\right\|_{H^2}^2\right) d \tau,
\end{aligned} \tag{3.12}
$$

which along with the definition of $E(t)$ ensures

$$
E(t) \leq C_0 E(0)+C_1 E^{\frac{3}{2}}(t). \tag{3.13}
$$

To apply the bootstrapping argument, we make the ansatz

$$
E(t) \leq \frac{1}{4 C_1^2}. \tag{3.14}
$$

We choose ε suitable small such that the initial H^2-norm $E(0)$ sufficiently small, namely,

$$
E(0):=\left\|u_0\right\|_{H^2}^2+\left\|\theta_0\right\|_{H^2}^2 \leq \varepsilon^2:=\frac{1}{4 C_1^2 C_0}. \tag{3.15}
$$

In fact, when (3.14) and (3.15) holds, (3.13) implies

$$E(t) \le \frac{1}{4C_1^2} + \frac{1}{2}E(t).$$

Therefore, the bootstrapping argument then concludes that, for all $t > 0$

$$E(t) \le \frac{1}{8C_1^2} \le \frac{C_0}{2}\varepsilon^2,$$

which gives the desired inequality (1.5).

3.2 Large Time Behavior of the Boussinesq equation (1.2)

Now we pay our attention to show the inequality (1.6). Applying ∂_2 to $(1.8)_1$ and ∂_1 to $(1.8)_2$, then taking the L^2-inner product with $\partial_2 u_1$ and $\partial_1 u_2$, respectively. After performing L^2-inner product on both side of $(1.8)_3$ with $\partial_1\theta$, we add them to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial_2 u_1(t)\|_{L^2}^2 + \|\partial_1 u_2(t)\|_{L^2}^2 + \|\partial_1\theta(t)\|_{L^2}^2) + \mu\|\partial_{22}u_1\|_{L^2}^2 + \kappa\|\partial_{11}u_2\|_{L^2}^2 + \eta\|\partial_{11}\theta\|_{L^2}^2\\
&= -\int_{\mathbf{R}^2} \partial_2 u\cdot\nabla u_1\partial_2 u_1 + \partial_1 u\cdot\nabla u_2\partial_1 u_2 dx\\
&\quad -\int_{\mathbf{R}^2} \partial_2\partial_1\partial_2\Delta^{-1}\theta\partial_2 u_1 - \partial_1\partial_1\partial_1\Delta^{-1}\theta\partial_1 u_2 dx\\
&\quad +\int_{\mathbf{R}^2} \partial_2\partial_1\Delta^{-1}\nabla\cdot\nabla\cdot(u\otimes u)\partial_2 u_1 dx + \int_{\mathbf{R}^2} \partial_1\partial_2\Delta^{-1}\nabla\cdot\nabla\cdot(u\otimes u)\partial_1 u_2 dx\\
&\quad -\int_{\mathbf{R}^2} \partial_1 u\cdot\nabla\theta\partial_1\theta dx - \int_{\mathbf{R}^2} \partial_1 u_2\partial_1\theta dx\\
&\quad +\int_{\mathbf{R}^2} (\mu\partial_1\Delta^{-1}\partial_1\partial_{22}u_1 + \kappa\partial_1\Delta^{-1}\partial_2\partial_{11}u_2)\partial_2 u_1 dx\\
&\quad +\int_{\mathbf{R}^2} (\mu\partial_2\Delta^{-1}\partial_1\partial_{22}u_1 + \kappa\partial_2\Delta^{-1}\partial_2\partial_{11}u_2)\partial_1 u_2 dx\\
&:= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + +J_7 + J_8.
\end{aligned}\tag{3.16}$$

Thanks to the fact $\nabla\cdot u = 0$, it's not hard to see that

$$\begin{aligned}
J_1 &= -\int_{\mathbf{R}^2} (\partial_2 u_1)^2\partial_1 u_1 + \partial_2 u_2(\partial_2 u_1)^2 + \partial_1 u_1(\partial_1 u_2)^2 + (\partial_1 u_2)^2\partial_2 u_2 dx\\
&= 0.
\end{aligned}$$

By L^p- boundedness of the Riesz transform and the Hölder inequality, we get

$$\begin{aligned}
J_2 &\le C\|R_{22}\partial_1\theta\|_{L^2}\|\partial_2 u_1\|_{L^2} + C\|R_{11}\partial_1\theta\|_{L^2}\|\partial_1 u_2\|_{L^2}\\
&\le C\|\partial_1\theta\|_{L^2}(\|\partial_2 u_1\|_{L^2} + \|\partial_1 u_2\|_{L^2})\\
&\le C\|u\|_{H^2}\|\theta\|_{H^2}.
\end{aligned}$$

Thanks to L^p- boundedness of the Riesz transform and Sobolev's embedding, one arrives at

$$\begin{aligned}J_3 \leq& C\|R_2R_1(\partial_1(u\cdot\nabla u_1)+\partial_2(u\cdot\nabla u_2))\|_{L^2}\|\partial_2 u_1\|_{L^2}\\ \leq& C\|\partial_1(u\cdot\nabla u_1)+\partial_2(u\cdot\nabla u_2)\|_{L^2}\|\partial_2 u_1\|_{L^2}\\ \leq& C\|\partial_1 u\cdot\nabla u_1+u\cdot\nabla\partial_1 u_1+\partial_2 u\cdot\nabla u_2+u\cdot\nabla\partial_2 u_2\|_{L^2}\|\partial_2 u_1\|_{L^2}\\ \leq& C(\|\partial_1 u\|_{L^4}\|\nabla u_1\|_{L^4}+\|u\|_{L^\infty}\|\nabla\partial_1 u_1\|_{L^2}+C\|\partial_2 u\|_{L^4}\|\nabla u_2\|_{L^4})\|\partial_2 u_1\|_{L^2}\\ \leq& C\|u\|_{H^2}^3.\end{aligned}$$

Similarly,

$$J_4 \leq C\|u\|_{H^2}^3.$$

Applying the Hölder inequality to get

$$J_5 \leq C\|\partial_1 u\|_{L^4}\|\nabla\theta\|_{L^4}\|\partial_1\theta\|_{L^2} \leq C\|u\|_{H^2}\|\theta\|_{H^2}^2,$$

and

$$J_6 \leq C\|\partial_1 u_2\|_{L^2}\|\partial_1\theta\|_{L^2} \leq C\|u\|_{H^2}\|\theta\|_{H^2}.$$

Integrating by parts and the L^p- boundedness of the Riesz transform give rise to

$$J_7+J_8 \leq C\|u\|_{H^2}^2.$$

Inserting the estimate from J_1 to J_6 into (3.16) and integrating over $[s,t]$ with $0<s<t<\infty$ to obtain

$$\begin{aligned}&(\|\partial_2 u_1(t)\|_{L^2}^2+\|\partial_1 u_2(t)\|_{L^2}^2+\|\partial_1\theta(t)\|_{L^2}^2)\\ &\quad-(\|\partial_2 u_1(s)\|_{L^2}^2+\|\partial_1 u_2(s)\|_{L^2}^2+\|\partial_1\theta(s)\|_{L^2}^2) \leq C(\varepsilon^2+\varepsilon^3)(t-s).\end{aligned} \tag{3.17}$$

Thanks to (1.5), one has

$$\int_0^\infty \|\partial_2 u_1(\tau)\|_{L^2}^2+\|\partial_1 u_2(\tau)\|_{L^2}^2+\|\partial_1\theta(\tau)\|_{L^2}^2 d\tau \leq C\varepsilon^2.$$

Therefore, as a result of Lemma 2.2, we conclude that

$$\|\partial_1 u_2(t)\|_{L^2}\to 0,\quad \|\partial_2 u_1(t)\|_{L^2}\to 0,\quad \|\partial_1\theta(t)\|_{L^2}\to 0,\quad \text{as}\quad t\to\infty.$$

This helps us to complete the proof of Theorem 1.1.

4 Proofs of Theorem 1.3

Lemma 4.1 *Assume that ϕ satisfies the follow equation in $\mathbf{R}^2$,*

$$\partial_{tt}\phi+(\mu R_2^2\partial_2^2+\kappa R_1^2\partial_1^2-\eta\partial_{11})\partial_t\phi-(R_1^2+\mu\eta R_1^2\partial_2^4+\kappa\eta R_1^2\partial_1^4)\phi=0, \tag{4.1}$$

with the initial conditions

$$\phi(x,0)=\phi_0(x), \quad \partial_t\phi(x,0)=\phi_1(x).$$

Then the solution ϕ *to* (4.1) *can be explicitly represented as*

$$\phi(x,t)=G_1\left(\phi_1-\frac{1}{2}(\Delta^{-1}(\mu\partial_2^4+\kappa\partial_1^4)+\eta\partial_{11})\phi_0\right)+G_2\phi_0, \tag{4.2}$$

where G_1 *and* G_2 *are given as follows,*

$$\widehat{G}_1(\xi,t)=\frac{e^{\lambda_2 t}-e^{\lambda_1 t}}{\lambda_2-\lambda_1}, \quad \widehat{G}_2(\xi,t)=\frac{1}{2}(e^{\lambda_1 t}+e^{\lambda_2 t}), \tag{4.3}$$

with λ_1 *and* λ_2 *being the roots of the characteristic equation*

$$\lambda^2+\left(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\right)\lambda+\frac{\xi_1^2+\eta\xi_1^2(\mu\xi_2^4+\kappa\xi_1^4)}{|\xi|^2}=0,$$

or

$$\lambda_1=-\frac{1}{2}\left(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\right)-\frac{1}{2}\sqrt{\Gamma}, \tag{4.4}$$

$$\lambda_2=-\frac{1}{2}\left(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\right)+\frac{1}{2}\sqrt{\Gamma}, \tag{4.5}$$

here

$$\Gamma=\left(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\right)^2-\frac{4\xi_1^2+4\eta\xi_1^2(\mu\xi_2^4+\kappa\xi_1^4)}{|\xi|^2}. \tag{4.6}$$

Proof Applying the Fourier transform on the space variable x to both sides of (4.1), we obtain

$$\partial_{tt}\widehat{\phi}+\left(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\right)\partial_t\widehat{\phi}+\frac{\xi_1^2+\eta\xi_1^2(\mu\xi_2^4+\kappa\xi_1^4)}{|\xi|^2}\widehat{\phi}=0,$$

namely,

$$(\partial_t-\lambda_2)(\partial_t-\lambda_1)\widehat{\phi}=0 \quad \text{or} \quad (\partial_t-\lambda_1)(\partial_t-\lambda_2)\widehat{\phi}=0.$$

It is not difficult to rewrite the wave equation into two different systems,

$$(\partial_t-\lambda_2)\widehat{\phi}=\widehat{f}, \tag{4.7}$$

$$(\partial_t-\lambda_1)\widehat{f}=0, \tag{4.8}$$

or

$$(\partial_t-\lambda_1)\widehat{\phi}=\widehat{g}, \tag{4.9}$$

$$(\partial_t-\lambda_2)\widehat{g}=0. \tag{4.10}$$

By taking the difference of (4.9) and (4.7), it deduces

$$\widehat{\phi}(\xi,t)=(\lambda_2-\lambda_1)^{-1}(\widehat{g}-\widehat{f}). \tag{4.11}$$

Then, (4.8) and (4.10) yield

$$\widehat{f}(\xi,t)=e^{\lambda_1 t}\widehat{f}(\xi,0)=e^{\lambda_1 t}(\widehat{\phi}_1-\lambda_2\widehat{\phi}_0), \tag{4.12}$$

$$\widehat{g}(\xi,t)=e^{\lambda_2 t}\widehat{g}(\xi,0)=e^{\lambda_2 t}(\widehat{\phi}_1-\lambda_1\widehat{\phi}_0). \tag{4.13}$$

Inserting (4.12) into (4.11) leads to

$$\begin{aligned}\widehat{\phi}(\xi,t)&=(\lambda_2-\lambda_1)^{-1}\Big((e^{\lambda_2 t}-e^{\lambda_1 t})\widehat{\phi}_1+(\lambda_2 e^{\lambda_1 t}-\lambda_1 e^{\lambda_2 t})\widehat{\phi}_0\Big)\\ &=\frac{e^{\lambda_2 t}-e^{\lambda_1 t}}{\lambda_2-\lambda_1}(\widehat{\phi}_1-\lambda_2\widehat{\phi}_0)+e^{\lambda_2 t}\widehat{\phi}_0\\ &=\frac{e^{\lambda_2 t}-e^{\lambda_1 t}}{\lambda_2-\lambda_1}\Big(\widehat{\phi}_1+\frac12(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)\widehat{\phi}_0\Big)+\frac12(e^{\lambda_1 t}+e^{\lambda_2 t})\widehat{\phi}_0\\ &=\widehat{G}_1\Big(\widehat{\phi}_1+\frac12(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)\widehat{\phi}_0\Big)+\widehat{G}_2\widehat{\phi}_0,\end{aligned} \tag{4.14}$$

where we used the definition of λ_2 in the third inequality. This completes the proof of Lemma 4.1. □

Due to the fact that $\widehat{G}_1(\xi,t)$ and $\widehat{G}_2(\xi,t)$ have a strong dependence on frequency, we need to be divided frequency space into several subdomains to obtain the optimal upper bound of $\widehat{G}_1(\xi,t)$ and $\widehat{G}_2(\xi,t)$.

Lemma 4.2 *Let* $\mathbf{R}^2=S_1\cup S_2$. *Here*

$$S_1=\Big\{\xi\in\mathbf{R}^2:\Gamma=\Big(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\Big)^2-\frac{4\xi_1^2+4\eta\xi_1^2(\mu\xi_2^4+\kappa\xi_1^4)}{|\xi|^2}\le\frac14\Big(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\Big)^2\Big\},$$

$S_2=\mathbf{R}^2\backslash S_1$.

Then $\widehat{G}_1(\xi,t)$ *and* $\widehat{G}_2(\xi,t)$ *satisfy the following estimates*:

(a) $\forall\xi\in S_1$,

$$Re\lambda_1\le-\frac12\Big(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\Big),\quad Re\lambda_2\le-\frac14\Big(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\Big),$$

$$\begin{aligned}|\widehat{G}_1(\xi,t)|&\le te^{-\frac14(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)t},\\ |\widehat{G}_2(\xi,t)|&\le Ce^{-\frac{\eta}{4}\xi_1^2 t}.\end{aligned} \tag{4.15}$$

(b) $\forall\xi\in S_2$,

$$\lambda_1\le-\frac34\Big(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\Big),\quad \lambda_2\le-c_0\xi_1^2,$$

$$
\begin{aligned}
&|\widehat{G}_1(\xi,t)| \le \frac{C}{\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2}\left(e^{-\frac{3}{4}(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)t}+e^{-c_0\xi_1^2 t}\right),\\
&|\widehat{G}_2(\xi,t)| \le Ce^{-c\xi_1^2 t}.
\end{aligned}
\tag{4.16}
$$

Proof (a) For $\xi \in S_1$, we divide S_1 into the following two regions:

$$
\begin{aligned}
&S_{11} = \left\{\xi \in S_1 : 0 \le \Gamma \le \frac{1}{4}\left(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\right)^2\right\},\\
&S_{12} = \{\xi \in S_1 : \Gamma < 0\}.
\end{aligned}
$$

For any $\xi \in S_{11}$, according to the definition of λ_1 and λ_2 in (4.4) and (4.5), λ_1 and λ_2 are real roots and satisfy

$$
\begin{aligned}
&\lambda_1 \le -\frac{1}{2}\left(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\right),\\
&\lambda_2 \le -\frac{1}{4}\left(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\right).
\end{aligned}
$$

By the mean-value theorem, we know

$$
\begin{aligned}
&|\widehat{G}_1(\xi,t)| \le te^{\lambda_2 t} \le te^{-\frac{1}{4}(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)t},\\
&|\widehat{G}_2(\xi,t)| \le Ce^{-\frac{1}{4}(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)t}.
\end{aligned}
$$

For any $\xi \in S_{12}$, λ_1 and λ_2 are a pair of complex conjugate roots, then one has

$$
\begin{aligned}
\widehat{G}_1(\xi,t) &= e^{-\frac{1}{2}(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)t}\,\frac{e^{\frac{i\sqrt{-\Gamma}}{2}t}-e^{-\frac{i\sqrt{-\Gamma}}{2}t}}{i\sqrt{-\Gamma}},\\
&= e^{-\frac{1}{2}(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)t}\,\frac{2\sin(\frac{\sqrt{-\Gamma}}{2}t)}{\sqrt{-\Gamma}}.
\end{aligned}
$$

We can infer from $|\sin x| \le |x|$ that

$$
\begin{aligned}
&|\widehat{G}_1(\xi,t)| \le te^{-\frac{1}{2}(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)t},\\
&|\widehat{G}_2(\xi,t)| \le \frac{1}{2}(e^{tRe\lambda_1}+e^{tRe\lambda_2}) \le Ce^{-\frac{1}{2}(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)t}.
\end{aligned}
$$

(b) For $\xi \in S_2$, λ_1 and λ_2 are real roots, we have

$$
\lambda_1 \le -\frac{3}{4}\left(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\right),
$$

and

$$
\lambda_2 = -\frac{1}{2}\left(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2-\sqrt{\Gamma}\right)
$$

$$
\begin{aligned}
&= -\frac{1}{2}\frac{|\xi|^{-2}(4\xi_1^2+4\eta\xi_1^2(\mu\xi_2^4+\kappa\xi_1^4))}{|\xi|^{-2}(\mu\xi_2^4+\kappa\xi_1^4)+\eta\xi_1^2+\sqrt{\Gamma}}\\
&\le -\frac{|\xi|^{-2}(\xi_1^2+\eta\xi_1^2(\mu\xi_2^4+\kappa\xi_1^4))}{|\xi|^{-2}(\mu\xi_2^4+\kappa\xi_1^4)+\eta\xi_1^2}.
\end{aligned}
$$

In order to control λ_2, we further divide S_2 into the following two regions:

$$
\begin{aligned}
S_{21} &= \{\xi \in S_2 : |\xi_1| \ge |\xi_2|\},\\
S_{22} &= \{\xi \in S_2 : |\xi_1| < |\xi_2|\}.
\end{aligned}
$$

For $\xi \in S_{21}$, we obtain

$$
\begin{aligned}
\lambda_2 &\le -\frac{1+\eta(\mu\xi_2^4+\kappa\xi_1^4)}{\mu\xi_2^4\xi_1^{-2}+\kappa\xi_1^2+\eta|\xi|^2} \le -\frac{1+\eta(\mu\xi_2^4+\kappa\xi_1^4)}{\mu\xi_2^2+\kappa\xi_1^2+\eta|\xi|^2}\\
&\le -\frac{1+\eta(\mu\xi_2^4+\kappa\xi_1^4)}{(\mu+\kappa+2\eta)\xi_1^2} \le -\frac{\eta\kappa}{\mu+\kappa+2\eta}\xi_1^2.
\end{aligned}
$$

For $\xi \in S_{22}$, one has

$$
\lambda_2 \le -\frac{\xi_1^2+\eta\xi_1^2(\mu\xi_2^4+\kappa\xi_1^4)}{\mu\xi_2^4+\kappa\xi_1^4+\eta\xi_1^2|\xi|^2} \le -\frac{\xi_1^2+\eta\xi_1^2(\mu\xi_2^4+\kappa\xi_1^4)}{(\mu+\kappa+2\eta)\xi_2^4} \le -\frac{\eta\mu}{\mu+\kappa+2\eta}\xi_1^2.
$$

Let $c_0 = \min\{\frac{\eta\kappa}{\mu+\kappa+2\eta}, \frac{\eta\mu}{\mu+\kappa+2\eta}\}$. Then we have

$$
\lambda_2 \le -c_0\xi_1^2, \quad \text{when} \quad \xi \in S_2.
$$

Thanks to $\xi \in S_2$, we have

$$
\lambda_2 - \lambda_1 = \sqrt{\Gamma} > \frac{1}{2}\left(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2\right).
$$

Consequently, we can easily obtain the upper bounds for $\widehat{G}_1(\xi,t)$ and $\widehat{G}_2(\xi,t)$ where $c = \min\{\frac{3}{4}\eta, c_0\}$. This completes the proof of Lemma 4.2. □

Now we are ready to prove Theorem 1.3 according to Lemma 4.1 and Lemma 4.2.

Proof Applying Lemma 4.1 to (1.14) leads to

$$
\begin{cases}
u(x,t) = G_1\left(\partial_t u(x,0) - \frac{1}{2}(\Delta^{-1}(\mu\partial_2^4+\kappa\partial_1^4)+\eta\partial_{11})u_0\right) + G_2 u_0,\\
\theta(x,t) = G_1\left(\partial_t \theta(x,0) - \frac{1}{2}(\Delta^{-1}(\mu\partial_2^4+\kappa\partial_1^4)+\eta\partial_{11})\theta_0\right) + G_2\theta_0.
\end{cases} \tag{4.17}
$$

Setting $t=0$ in the linearized equations (1.9), we get

$$
\begin{cases}
\partial_t u_1(x,0) = \Delta^{-1}(\mu\partial_2^4+\kappa\partial_1^4)u_{10} - \partial_1\partial_2\Delta^{-1}\theta_0,\\
\partial_t u_2(x,0) = \Delta^{-1}(\mu\partial_2^4+\kappa\partial_1^4)u_{20} + \partial_1\partial_1\Delta^{-1}\theta_0,\\
\partial_t \theta(x,0) = \eta\partial_{11}\theta_0 - u_{20}.
\end{cases} \tag{4.18}
$$

Then, inserting (4.18) into (4.17) yields

$$\begin{cases} u_1(x,t) = \frac{1}{2}G_1\left(\Delta^{-1}(\mu\partial_2^4 + \kappa\partial_1^4) - \eta\partial_{11}\right)u_{10} - \partial_1\partial_2\Delta^{-1}G_1\theta_0 + G_2u_{10}, \\ u_2(x,t) = \frac{1}{2}G_1\left(\Delta^{-1}(\mu\partial_2^4 + \kappa\partial_1^4) - \eta\partial_{11}\right)u_{20} + \partial_1\partial_1\Delta^{-1}G_1\theta_0 + G_2u_{20}, \\ \theta(x,t) = -\frac{1}{2}G_1\left(\Delta^{-1}(\mu\partial_2^4 + \kappa\partial_1^4) - \eta\partial_{11}\right)\theta_0 + G_2\theta_0 - G_1u_{20}. \end{cases} \tag{4.19}$$

(i) To estimate $\|u_1\|_{L^2}$, by Plancherel's Theorem and dividing the spatial domain $\mathbf{R}^2$ as in Lemma 4.2, we have

$$\begin{aligned} \|u_1\|_{L^2} = \|\widehat{u}_1\|_{L^2} \le & \frac{1}{2}\|\left(\frac{\mu\xi_2^4 + \kappa\xi_1^4}{|\xi|^2} - \eta\xi_1^2\right)\widehat{G}_1\widehat{u}_{10}\|_{L^2(S_1)} + \|\frac{\xi_1\xi_2}{|\xi|^2}\widehat{G}_1\widehat{\theta}_0\|_{L^2(S_1)} \\ & + \|\widehat{G}_2\widehat{u}_{10}\|_{L^2(S_1)} + \frac{1}{2}\|\left(\frac{\mu\xi_2^4 + \kappa\xi_1^4}{|\xi|^2} - \eta\xi_1^2\right)\widehat{G}_1\widehat{u}_{10}\|_{L^2(S_2)} \\ & + \|\frac{\xi_1\xi_2}{|\xi|^2}\widehat{G}_1\widehat{\theta}_0\|_{L^2(S_2)} + \|\widehat{G}_2\widehat{u}_{10}\|_{L^2(S_2)} \\ = & K_1 + K_2 + K_3 + K_4 + K_5 + K_6. \end{aligned}$$

Thanks to (4.15) and the fact that $x^n e^{-x} \le C(n)$ for any $n \ge 0$ and $x \ge 0$.

$$\begin{aligned} K_1 \le & \|\left(\frac{\mu\xi_2^4 + \kappa\xi_1^4}{|\xi|^2} + \eta\xi_1^2\right)te^{-\frac{1}{4}(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)t}\widehat{u}_{10}\|_{L^2(S_1)} \\ \le & C\|e^{-\frac{1}{8}(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)t}\widehat{u}_{10}\|_{L^2} \\ \le & C\||\xi_1|^\sigma e^{-\frac{\eta}{8}\xi_1^2t}|\xi_1|^{-\sigma}\widehat{u}_{10}\|_{L^2} \\ \le & Ct^{-\frac{\sigma}{2}}\|\Lambda_1^{-\sigma}u_{10}\|_{L^2}, \end{aligned}$$

where $\sigma > 0$. By L^p-boundedness of the Riesz transform and (4.15), we get

$$\begin{aligned} K_2 \le & \|\widehat{G}_1\widehat{\theta}_0\|_{L^2(S_1)} \\ \le & \|te^{-\frac{1}{4}(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)t}\widehat{\theta}_0\|_{L^2} \\ \le & \|te^{-\frac{\eta}{4}\xi_1^2t}\widehat{\theta}_0\|_{L^2} \\ \le & C\|e^{-\frac{\eta}{8}\xi_1^2t}\xi_1^{-2}\widehat{\theta}_0\|_{L^2} \\ \le & C\||\xi_1|^\sigma e^{-\frac{\eta}{8}\xi_1^2t}\xi_1^{-(\sigma+2)}\widehat{\theta}_0\|_{L^2} \\ \le & Ct^{-\frac{\sigma}{2}}\|\Lambda_1^{-(\sigma+2)}\theta_0\|_{L^2}. \end{aligned}$$

From (4.15), one can follow that

$$K_3 \le C\|e^{-\frac{\eta}{4}\xi_1^2t}\widehat{u}_{10}\|_{L^2}$$

$$\leq Ct^{-\frac{\sigma}{2}}\|\Lambda_1^{-\sigma}u_{10}\|_{L^2}.$$

Similarly, due to (4.16), one gets

$$\begin{aligned}K_4 \leq& C\|\left(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}-\eta\xi_1^2\right)\frac{1}{\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2}\left(e^{-\frac{3}{4}(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)t}+e^{-c_0\xi_1^2t}\right)\widehat{u}_{10}\|_{L^2},\\ \leq& C\|e^{-c\xi_1^2t}\widehat{u}_{10}\|_{L^2}\\ \leq& Ct^{-\frac{\sigma}{2}}\|\Lambda_1^{-\sigma}u_{10}\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}K_5 \leq& C\|\frac{1}{\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2}\left(e^{-\frac{3}{4}(\frac{\mu\xi_2^4+\kappa\xi_1^4}{|\xi|^2}+\eta\xi_1^2)t}+e^{-c_0\xi_1^2t}\right)\widehat{\theta}_0\|_{L^2},\\ \leq& C\||\xi_1|^{-2}e^{-c\xi_1^2t}\widehat{\theta}_0\|_{L^2}\\ \leq& Ct^{-\frac{\sigma}{2}}\|\Lambda_1^{-(\sigma+2)}\theta_0\|_{L^2}.\end{aligned}$$

The estimates for K_6 are similar to those for K_4 and the bound is

$$\begin{aligned}K_6 \leq& C\|e^{-c\xi_1^2t}\widehat{u}_{10}\|_{L^2}\\ \leq& Ct^{-\frac{\sigma}{2}}\|\Lambda_1^{-\sigma}u_{10}\|_{L^2}.\end{aligned}$$

Combining the estimates from K_1 and K_6, we can obtain

$$\|u_1\|_{L^2}\leq Ct^{-\frac{\sigma}{2}}(\|\Lambda_1^{-\sigma}u_{10}\|_{L^2}+\|\Lambda_1^{-(\sigma+2)}\theta_0\|_{L^2}).$$

Similarly,

$$\|u_2\|_{L^2}\leq Ct^{-\frac{\sigma}{2}}(\|\Lambda_1^{-\sigma}u_{20}\|_{L^2}+\|\Lambda_1^{-(\sigma+2)}\theta_0\|_{L^2}),$$

and

$$\|\theta\|_{L^2}\leq Ct^{-\frac{\sigma}{2}}\|\Lambda_1^{-\sigma}\theta_0\|_{L^2}.$$

(ii) The bound for $\|\partial_1^m u\|_{L^2}$ and $\|\partial_1^m\theta\|_{L^2}$ are similar to case (i). This completes the proof of Theorem 1.3. □

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