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Liouville Type Theorem for Stable Solutions to Weighted Quasilinear Problems in $\mathbb{R}^N$

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Abstract
In this paper, we prove the Liouville type theorem for stable $W^{1,p}_{\mathrm{loc}}$ solutions of the weighted quasilinear problem

$$-\operatorname{div}\left(w_1(x)\left(s^2+|\nabla u|^2\right)^{\frac{p-2}{2}}\nabla u\right)=w_2(x)f(u)\quad \text{in } \mathbb{R}^N,$$

where $s\geq 0$ is a real number, $f(u)$ is either e^u or $-e^{\frac{1}{u}}$ and $w_1(x), w_2(x)\in L^1_{\mathrm{loc}}\left(\mathbb{R}^N\right)$ be nonnegative functions so that $w_1(x)\leq C_1|x|^m$ and $w_2(x)\geq C_2|x|^n$ when $|x|$ is big enough. Here we need $n>m$.

Keywords Stable solutions · Liouville theorem · Gelfand nonlinearity · Singular nonlinearity

Mathematics Subject Classification (2010) Primary 35J62 · Secondary 35A01 · 35B08 · 35B53

1 Introduction

In this work, we aim to study the nonexistence of stable solutions of weighted quasilinear equation

$$-\operatorname{div}\left(w_1(x)\left(s^2+|\nabla u|^2\right)^{\frac{p-2}{2}}\nabla u\right)=w_2(x)f(u)\quad \text{in } \mathbb{R}^N. \tag{1.1}$$

About the principal part of the equation, the special case is $s=0$, which yields the p-Laplace operator with weight.

We focus on the equations with two kinds of nonlinear terms. One is the so-called Gelfand nonlinearity with $f(u)=e^u$, and another is the singular nonlinearity with $f(u)=-e^{1/u}$ as the solution converges to 0. Because of the degenerate nature of the term $|\nabla u|^{p-2}$

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when $p > 2$, solutions to (1.1) must be understood in the weak sense. Moreover, solutions to elliptic equations with Hardy potentials may possess singularities. To overcome this difficulty, we need to define the weak solutions in a suitable weighted Sobolev space. On this account, for $\forall\ \varphi \in C_c^\infty\left(\mathbb{R}^N\right)$, let us define

$$\|\varphi\|_{w_1} = \left(\int_{\mathbb{R}^N} w_1(x)|\nabla\varphi|^p \mathrm{d}x\right)^{\frac{1}{p}}, \qquad w_1 > 0.$$

Because w_1 is strictly positive, it is obvious that the positive definiteness and positive homogeneity of the norm are satisfied. For subadditivity, we calculate

$$\begin{aligned}\|\varphi_1 + \varphi_2\|_{w_1} &= \left(\int_{\mathbb{R}^N} w_1(x)|\nabla\varphi_1 + \nabla\varphi_2|^p \mathrm{d}x\right)^{\frac{1}{p}}\\ &= \left(\int_{\mathbb{R}^N} |w_1(x)^{\frac{1}{p}}\nabla\varphi_1 + w_1(x)^{\frac{1}{p}}\nabla\varphi_2|^p \mathrm{d}x\right)^{\frac{1}{p}}\\ &\le \left(\int_{\mathbb{R}^N} |w_1(x)^{\frac{1}{p}}\nabla\varphi_1|^p \mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N} |w_1(x)^{\frac{1}{p}}\nabla\varphi_2|^p \mathrm{d}x\right)^{\frac{1}{p}}\\ &= \|\varphi_1\|_{w_1} + \|\varphi_1\|_{w_2},\end{aligned}$$

the above inequality holds on account of Minkowski's inequality. In conclusion, $\|\cdot\|_{w_1}$ is a norm.

Denote by $W_0^{1,p}\left(R^N, w_1\right)$ the closure of $C_c^\infty(R^N)$ with respect to the $\|\cdot\|_{w_1}$-norm. Remark that for $w_1 \in L^1_{loc}\left(R^N\right)$ we have $C_c^1\left(\mathbb{R}^N\right) \subset W_0^{1,p}\left(R^N, w_1\right)$ and $u \in W_{loc}^{1,p}\left(R^N, w_1\right)$ means that if for any $\varphi \in C_c^\infty\left(\mathbb{R}^N\right)$, there holds $u\varphi \in W_0^{1,p}\left(R^N, w_1\right)$.

We find the corresponding energy functional of (1.1) is

$$E(u) = \int_{R^N} \frac{w_1(x)}{p}\left[(s^2 + |\nabla u|^2)^{\frac{p}{2}} - s^p\right] - w_2(x)F(u)dx, \tag{1.2}$$

where $F(u) = \int_0^u f(t)dt$.

Intuitively, a system is in a stable state if it can recover from perturbations, a small change will not prevent the system from returning to equilibrium. Place a marble at the center of a smooth bowl and tap it slightly. After some rolling back and forth, the marble will return to its stable position. If instead you turned the bowl over and put the marble carefully on top at the center, then it would be in a rather unstable equilibrium, a slight breeze would suffice to make it fall. In particular, readers can find out about the physical motivation and recent development on the topic of stable solutions in monograph [10] by Dupaigne and references therein.

The above physical phenomena show that the minimum solution is a stable solution. Accordingly, thinking of solutions of PDEs as critical points of an energy functional, we say that a solution is stable when the second variation of energy functional is nonnegative. We do the first and second variations of the energy functional $E(u)$ as follows.

$$\begin{aligned}i'(\tau) = E'(u + \tau\varphi) = \int_{R^N} & w_1(x)(s^2 + |\nabla(u + \tau\varphi)|^2)^{\frac{p-2}{2}}\nabla(u + \tau\varphi)\nabla\varphi\\ & - w_2(x)F'(u + \tau\varphi)\varphi dx,\end{aligned}$$

$$i''(\tau)=E''(u+\tau\varphi)=(p-2)\int_{R^N}\left\{w_1(x)(s^2+|\nabla(u+\tau\varphi)|^2)^{\frac{p-4}{2}}(\nabla(u+\tau\varphi),\nabla\varphi)^2\right.$$
$$\left.+\int_{R^N}w_1(x)(s^2+|\nabla(u+\tau\varphi)|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2-w_2(x)F''(u+\tau\varphi)\varphi^2dx\right\}.$$

Finally, let $i'(\tau)|_{\tau=0}=0$ and $i''(\tau)|_{\tau=0}\geq 0$, then we have the following Definition 1.1 and 1.2

Definition 1.1 A function $u\in W^{1,p}_{loc}\left(\mathbb{R}^N,w_1\right)$ is said to be a weak solution of (1.1), if $w_2(x)f(u)\in L^1_{\mathrm{loc}}\left(\mathbb{R}^N\right)$ and

$$\int_{R^N}w_1(x)\left(s^2+|\nabla u|^2\right)^{\frac{p-2}{2}}(\nabla u,\nabla\varphi)\,dx=\int_{R^N}w_2(x)f(u)\varphi dx, \tag{1.3}$$

for all $\varphi\in C^1_c\left(\mathbb{R}^N\right)$.

Definition 1.2 We say a weak solution u of (1.1) is stable if

$$\int_{R^N}w_2(x)f'(u)\varphi^2dx\leq\int_{R^N}w_1(x)\,(p-2)\left(s^2+|\nabla u|^2\right)^{\frac{p-4}{2}}(\nabla u,\nabla\varphi)^2\,dx$$
$$+\int_{R^N}w_1(x)\left(s^2+|\nabla u|^2\right)^{\frac{p-2}{2}}|\nabla\varphi|^2dx, \tag{1.4}$$

for all $\varphi\in C^1_c\left(\mathbb{R}^N\right)$.

According to the Cauchy-Schwarz inequality, we know that if u is a stable solution of (1.1), then we have

$$\int_{R^N}w_2(x)f'(u)\varphi^2dx\leq\int_{R^N}w_1(x)\,(p-2)\left(s^2+|\nabla u|^2\right)^{\frac{p-4}{2}}|\nabla u|^2|\nabla\varphi|^2dx$$
$$+\int_{R^N}w_1(x)\left(s^2+|\nabla u|^2\right)^{\frac{p-2}{2}}|\nabla\varphi|^2dx, \tag{1.5}$$

for all $\varphi\in C^1_c\left(\mathbb{R}^N\right)$.

Remark 1.3 Since $w_2(x)f(u)$ is nonnegative, by density arguments, we know Definition 1.1, 1.2 and formula (1.5) hold for any test function $\varphi\in W^{1,p}_0\left(R^N,w_1\right)$.

We recall that a Liouville type theorem addresses the nonexistence of nontrivial solution in the entire Euclidean space $\mathbb{R}^N$. Not only weak and positive solutions, convex solutions, periodic solutions but also other types of solutions, such as stable solutions have been widely studied by many scholars. In the past few years, Laplace's equation is the most studied and relatively well-studied equation with respect to stable solutions, which can be referred to the references [3, 8, 11, 12]. Liouville type theorems for stable solutions of nonlinear elliptic equations are usually guaranteed in low dimensional case. We should point out the work [13] by Farina for Gelfand equation

$$-\Delta u=e^u\quad\text{in }\mathbb{R}^N,$$

where he proved that the equation has no stable classical solutions for $2 \le N \le 9$. Afterwards, this nonexistence result was extended to stable C^1 solutions of the p-Laplace equation $-\triangle_p u = e^u$ when $N < \frac{p(p+3)}{p-1}$ in [18].

The weighted Laplace equations of Gelfand type are

$$-div(w(x)\nabla u) = f(x)e^u \quad \text{in } \mathbb{R}^N.$$

The weighted p-Laplace equations of Gelfand type are

$$-div(w(x)|\nabla u|^{p-2}\nabla u) = f(x)e^u \quad \text{in } \mathbb{R}^N.$$

They both have been studied recently by many authors. In [6] several Liouville type theorems for classical stable solutions of this equation were established under different assumptions on w and f. Wang and Ye [23] deal with more specific equation $-\triangle u = |x|^b e^u$ but for weak stable solutions, which covers solutions having singularities. Later, the result in [23] was extended to equation $-div(|x|^\alpha \nabla u) = |x|^\gamma e^u$ in [16] and equation $-\triangle_p u = f(x)e^u$ in [4].

Liouville type theorems were also established for elliptic equations with other type of nonlinearity, such as singular nonlinearity, Lane-Emden nonlinearity, MEMS nonlinearity and so on. In order to better understand the existence of stable solutions for elliptic equations with singular nonlinear terms, we can refer to the study of problems related to this type of equations, such as [2, 7, 17] for zero Dirichlet boundary value problem. For the nonexistence of stable solutions in the whole space, Ma-Wei [21] studied the equation

$$\triangle u = u^{-\delta} \quad \text{in } \mathbb{R}^N, \quad \forall \delta > 0. \tag{1.6}$$

They showed there are no stable positive solutions to (1.6) provided $2 \le N < 2 + \frac{4}{1+\delta}(\delta + \sqrt{\delta^2 + \delta})$. Guo and Mei [15] extended that result to the p-Laplace equation

$$\triangle_p u = u^{-\delta} \quad \text{in } \mathbb{R}^N. \tag{1.7}$$

Furthermore, they showed the nonexistence of stable solution for some range of $\delta > 0$ and $2 \le p < N$ among their results. Chen et al. [4] studied the problem

$$\triangle_p u = f(u)u^{-\delta} \quad \text{in } \mathbb{R}^N,$$

for some $f \in L^1_{loc}$ such that f behaves like a radial function for large enough x. This was recently generalized by Le et al. [20] for the weighted p-Laplace equation. Reader may also find the paper by Du and Guo [9] a good read related to singular problems.

In the latest paper [14], they studied the Liouville theorem of following equation

$$-\Delta_p u = |u|^{p^*-2}u$$

on the half space, the whole space and the bounded starshaped domain. It is worth putting the results of this paper in communication with recent results on quasilinear PDEs under higher order Morse index conditions.

For the critical points of the associated functional for (1.1), Silvia Cingolani et al. [5] provided the estimates of the corresponding critical polynomial. In addition, when $s = 1$, a priori estimates, existence and nonexistence of positive solutions for (1.1) in [22] was studied. Since (1.1) is a generalization of p-laplace equation with weight, the past research on the stable solution of p-laplace equation inspired us to study the stable solution of (1.1).

Throughout this paper let us define $\eta_R \in C_c^1(\mathbb{R}^N)$ to be a cut-off function such that:

$$\eta_R(x) = \begin{cases} 1, & |x| < R, \\ 0, & |x| > 2R, \end{cases} \tag{1.8}$$

which satisfies $0 \le \eta_R \le 1$ in $\mathbb{R}^N$ and $|\nabla \eta_R| \le \frac{C}{R}$ in $B_{2R} \backslash B_R$, for some positive constant C independent of R.

2 Main Results

The main purpose of this paper is to obtain a Liouville type theorem for stable solutions of class $W^{1,p}_{loc}$ to equation (1.1) with two different nonlinear terms.

About the Gelfand nonlinear equation, the nonexistence of the stable solution is given, while for the singular nonlinear equation, we only give the nonexistence of stable solutions when $\|u\|_{L^\infty(\mathbb{R}^N)} \le M$.

For Gelfand-type nonlinear equation

$$-\operatorname{div}\left(w_1(x)\left(s^2 + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u \right) = w_2(x) e^u \quad \text{in } \mathbb{R}^N, \tag{2.1}$$

we have the following nonexistence result for stable solution.

Theorem 2.1 *Let u be a weak solution of (2.1). Suppose that for a.e. $|x| \ge R_0$, we have that $w_1(x) \le C_1|x|^m$ and $w_2(x) \ge C_2|x|^n$, $C_1, C_2, R_0 > 0$ and $n > m$.*

(i) *When $2 \le p < 4$, there is no stable solution of equation (2.1) for $N < \frac{4}{\sqrt{2}^{4-p}(p-1)}(n - m) - m$.*
(ii) *When $p \ge 4$, the same conclusion is established for $N < \frac{4}{\sqrt{2}^{p-4}(p-1)}(n - m) - m$.*

For singular nonlinear equation

$$-\operatorname{div}\left(w_1(x)\left(s^2 + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u \right) = -w_2(x) e^{\frac{1}{u}} \quad \text{in } \mathbb{R}^N, \tag{2.2}$$

we have the corresponding Liouville results for stable solutions:

Theorem 2.2 *Let u be a weak solution of (2.2) such that $\|u\|_{L^\infty(\mathbb{R}^N)} \le M$ for some positive constant M. Suppose that for a.e. $|x| \ge R_0$, we have that $w_1(x) \le C_1|x|^m$ and $w_2(x) \ge C_2|x|^n$, $C_1, C_2, R_0 > 0$ and $n > m$. If $N < 4p(n - m) - m$, then there is no stable solution of (2.2) when $2 \le p \le \frac{m+n-1}{4(1+n-m)}$.*

Remark 2.3 In the future, it would be interesting to replace the condition $n > m$ with a more general asymptotic assumption that may allow a slower decay.

In the past forty years, the related problems have been studied by many scholars. The ideas of proving the theorem mainly come from article [1, 7, 18, 19]. In [1] the authors studied the nonexistence of the stable solution of p-Laplace equation with weight

$$div(w_1(x)|\nabla u|^{p-2}\nabla u) = w_2(x) f(u) \quad in\ \mathbb{R}^N. \tag{2.3}$$

The key of this paper is to select the appropriate cut-off function in Definition 1.1 and 1.2 to obtain the integral estimate of the solution of equation (1.1), that is, Proposition 3.2 and Proposition 3.3. When $s \neq 0$, equation (1.3) and (1.5) contain more non-homogeneous terms about $|\nabla u|$. Therefore, when doing the integral estimate of the solution of equation (1.1), the case of $s \neq 0$ is more difficult to deal with than that of $s = 0$.

We use the contradiction method to prove the main theorem of the paper. Firstly, the existence of stable solution is assumed. Then we choose appropriate test function in (1.3) and the stability condition (1.5) to get an inequality, so that one side of the inequality only depend on the cut-off function and the condition on weights. In the end, the contradiction is derived by selecting suitable cut-off function. The difficulty of this paper is how to choose the test function appropriately. We also point out here that the test function selected in Theorem 2.2 is quite different from what we used in Theorem 2.1.

3 Proof of Main Results

In the proof below, we always denote $B_R(0)$ to be the ball centered at 0 with radius $R > 0$, C as a ordinary constant whose exact values may change from line to line or so much as in the same line. If this constant relies on an arbitrary small number ε, then we may denote it by $C(\varepsilon)$. We are going to use Young inequality in the form $ab \leq \varepsilon a^p + C(\varepsilon) b^q$ for $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ in the later proof.

The proof of the theorem will be affected by different values of p. As a consequence, we will need the following lemma to discuss those cases.

Lemma 3.1 *x, y are non-negative numbers, then*

(i) *If $0 \leq \gamma < 1$, there hold $2^{\gamma-1}(x^\gamma + y^\gamma) \leq (x+y)^\gamma \leq x^\gamma + y^\gamma$.*
(ii) *If $\gamma \geq 1$, there hold $(x^\gamma + y^\gamma) \leq (x+y)^\gamma \leq 2^{\gamma-1}(x^\gamma + y^\gamma)$.*

Proof (i) On the one hand, we obtain from the properties of concave functions that $\frac{x^\gamma + y^\gamma}{2} \leq (\frac{x+y}{2})^\gamma$, where $0 \leq \gamma < 1$. On the other hand

$$\begin{aligned}
\frac{x^\gamma + y^\gamma}{(x+y)^\gamma} &= (\frac{x}{x+y})^\gamma + (\frac{y}{x+y})^\gamma \\
&\geq \frac{x}{x+y} + \frac{y}{x+y} \\
&= 1,
\end{aligned}$$

that is $(x+y)^\gamma \leq x^\gamma + y^\gamma$.

(ii) According to the features of convex functions we have $(\frac{x+y}{2})^\gamma \leq \frac{x^\gamma + y^\gamma}{2}$. And when $x = 0$, the left side of the inequality obviously holds. When $x \neq 0$, set $s := \frac{y}{x}$ and let $g(s) := (1+s)^\gamma$, $f(s) := 1 + s^\gamma$. We verify that $g(0) = f(0) = 1$ and $g'(s) \geq f'(s)$ for all $s \geq 0$, thus we get the required inequality. □

The following propositions play an important role in proving Theorem 2.1. We use the same cut-off functions as in [19].

Proposition 3.2 *Fix* α *and* p *to satisfy either of the following conditions*: (i) $2 \le p < 4$ *and* $\alpha \in (0, \frac{4}{\sqrt{2}^{4-p}(p-1)})$; *or* (ii) $p \ge 4$ *and* $\alpha \in (0, \frac{4}{\sqrt{2}^{p-4}(p-1)})$. *Let* u *be a stable solution of equation* (2.1). *Then, there exists a positive constant* $C = C(p, \alpha)$ *for which there holds*

$$\int_{R^N} w_2(x)e^{(\alpha+1)u}\eta^p dx \le C\Bigg[\int_{R^N} w_1(x)^{\alpha+1}w_2(x)^{-\alpha}\Bigg(|\nabla\eta|^{\frac{p(\alpha+1)}{p-1}}\eta^{\frac{p^2-p\alpha-2p}{p-1}} + \eta^{-p\alpha}|\nabla\eta|^{p(\alpha+1)} + \eta^p + \eta^{p-2\alpha-2}|\nabla\eta|^{2(\alpha+1)}\Bigg)dx\Bigg], \tag{3.1}$$

for any $\eta \in C_c^1(\mathbb{R}^N)$ *with* $0 \le \eta \le 1$ *and such that* $\nabla\eta = 0$ *in a neighborhood of* $\{x \in \mathbb{R}^N : w_2(x) = 0\}$.

Proof We need pay more attention to $W_{loc}^{1,p}$ solution, because u is not assumed to be bounded, $e^{\beta u}\eta$ may not belong to $W_0^{1,p}(\mathbb{R}^N, w_1)$ for any $\beta > 0$ even with $\eta \in C_c^1(\Omega)$. The idea is to truncate $e^{\beta u}$ in a small region. More specifically, for each $k \in \mathbb{N}$, let us define positive $C^1(\mathbb{R})$ functions

$$a_k(u) = \begin{cases} e^{\frac{\alpha u}{2}}, & u < k, \\ (\frac{\alpha}{2}(u-k)+1)e^{\frac{\alpha k}{2}}, & u \ge k, \end{cases}$$

and

$$b_k(u) = \begin{cases} e^{\alpha u}, & u < k, \\ (\alpha(u-k)+1)e^{\alpha k}, & u \ge k. \end{cases}$$

Since $u \in W_{loc}^{1,p}(\mathbb{R}^N, w_1)$, $a_k(u)$ and $b_k(u)$ defined in this way are obviously $W_{loc}^{1,p}(\mathbb{R}^N, w_1)$ for $k \in \mathbb{N}$. Then for any $\eta \in C_c^1(\mathbb{R}^N)$, we have $a_k(u)\eta^{\frac{p}{2}}$ and $b_k(u)\eta^p$ belonging to $W_0^{1,p}(R^N, w_1)$. Using Remark 1.3, it is easy to see that $a_k(u)\eta^{\frac{p}{2}}$ and $b_k(u)\eta^p$ are legitimate test function.

A direct computation shows that

$$b_k(u) \le a_k^2(u), \quad \frac{\alpha}{4}b_k'(u) = a_k'(u)^2, \quad a_k(u)^p a_k'(u)^{2-p}, \quad b_k(u)^p b_k'(u)^{1-p}, a_k^2(u) \quad \text{and} \quad a_k'(u)^2 \le Ce^{\alpha u}, \tag{3.2}$$

for all $u \in \mathbb{R}$, where C depend only on p and α.

Next we divide the proof of the proposition into two cases.

Case 1 When $2 \le p < 4$, we select $\varphi = b_k(u)\eta^p$ as a test function in formula (1.3). Since

$$\nabla\varphi = b_k'(u)\eta^p\nabla u + pb_k(u)\eta^{p-1}\nabla\eta\ ,$$

using Cauchy-Schwarz inequality we get

$$\int_{R^N} w_1(x)(s^2+|\nabla u|^2)^{\frac{p-2}{2}}b_k'(u)\eta^p|\nabla u|^2dx \le p\int_{R^N} w_1(x)(s^2+|\nabla u|^2)^{\frac{p-2}{2}}b_k(u)\eta^{p-1}|\nabla u||\nabla\eta|dx$$

$$+\int_{R^N} w_2(x)e^u b_k(u)\eta^p dx. \tag{3.3}$$

In the (i) of Lemma 3.1, we set $x := s^2$, $y := |\nabla u|^2$ and $\gamma := \frac{p-2}{2}$, then we have

$$\text{the left side of (3.3)} \ge \int_{R^N} w_1(x)2^{\frac{p-4}{2}}(s^{p-2}+|\nabla u|^{p-2})|\nabla u|^2 b_k'(u)\eta^p dx,$$

and

$$\text{the right side of (3.3)} \le p\int_{R^N} w_1(x)(s^{p-2}+|\nabla u|^{p-2})|\nabla u||\nabla\eta| b_k(u)\eta^{p-1}dx$$
$$+\int_{R^N} w_2(x)e^u b_k(u)\eta^p dx.$$

Moreover, by Young inequality, (3.3) becomes

$$\begin{aligned}
&\int_{R^N} w_1(x)|\nabla u|^p b_k'(u)\eta^p dx\\
&\le 2^{\frac{4-p}{2}}\left[p\int_{R^N} w_1(x)\left(s^{p-2}b_k(u)\eta^{p-1}|\nabla u||\nabla\eta| + |\nabla u|^{p-1}b_k(u)\eta^{p-1}|\nabla\eta|\right)dx\right.\\
&\quad\left.+\int_{R^N} w_2(x)e^u b_k(u)\eta^p dx\right] - \int_{R^N} w_1(x)s^{p-2}b_k'(u)\eta^p|\nabla u|^2 dx\\
&\le \varepsilon\int_{R^N} w_1(x)|\nabla u|^p b_k'(u)\eta^p dx + C(p,\varepsilon)\left(\int_{R^N} w_1(x)b_k'(u)^{1-p}b_k(u)^p|\nabla\eta|^p)dx\right.\\
&\quad\left.+\int_{R^N} w_1(x)(b_k(u)^{\frac{p}{p-1}}b_k'(u)^{-\frac{1}{p-1}}\eta^{\frac{p(p-2)}{p-1}}|\nabla\eta|^{\frac{p}{p-1}}dx\right)\\
&\quad+2^{\frac{4-p}{2}}\int_{R^N} w_2(x)e^u b_k(u)\eta^p dx.
\end{aligned}$$

It implies for any $\varepsilon\in(0,1)$, any $k\in\mathbb{N}$ and any nonnegative function $\eta\in C_c^1(\mathbb{R}^N)$, there exists a constant $C(p,\varepsilon)>0$ such that

$$\begin{aligned}
&(1-\varepsilon)\int_{R^N} w_1(x)|\nabla u|^p b_k'(u)\eta^p dx\\
&\quad\le 2^{\frac{4-p}{2}}\int_{R^N} w_2(x)e^u b_k(u)\eta^p dx\\
&\qquad+C(p,\varepsilon)\left(\int_{R^N} w_1(x)b_k'(u)^{\frac{1}{1-p}}b_k(u)^{\frac{p}{p-1}}|\nabla\eta|^{\frac{p}{p-1}}\eta^{\frac{p(p-2)}{p-1}}dx\right.\\
&\qquad\left.+\int_{R^N} w_1(x)b_k(u)^p b_k'(u)^{1-p}|\nabla\eta|^p dx\right).
\end{aligned}\tag{3.4}$$

We take advantage of the stability assumption (1.5) with $\varphi = a_k(u)\eta^{\frac{p}{2}}$, note that

$$\nabla\varphi = a_k'(u)\eta^{\frac{p}{2}}\nabla u + \frac{p}{2}a_k(u)\eta^{\frac{p-2}{2}}\nabla\eta,$$

from equation (1.5) and Lemma 3.1, we can obtain

$$\begin{aligned}
&\int_{R^N} w_2(x)e^u a_k^2(u)\eta^p dx \\
&\le \int_{R^N} w_1(x)\,(p-1)\left(s^2+|\nabla u|^2\right)^{\frac{p-2}{2}} |a_k'(u)\eta^{\frac{p}{2}}\nabla u+\frac{p}{2}a_k(u)\eta^{\frac{p-2}{2}}\nabla\eta|^2 dx \\
&\le \int_{R^N} w_1(x)\,(p-1)\left(s^{p-2}+|\nabla u|^{p-2}\right) |a_k'(u)\eta^{\frac{p}{2}}\nabla u+\frac{p}{2}a_k(u)\eta^{\frac{p-2}{2}}\nabla\eta|^2 dx \\
&\le (p-1)\int_{R^N} w_1(x)\Bigg[s^{p-2}\Bigg(a_k'(u)^2\eta^p|\nabla u|^2+pa_k(u)a_k'(u)\eta^{p-1}|\nabla u||\nabla\eta| \\
&+\frac{p^2}{4}a_k^2(u)\eta^{p-2}|\nabla\eta|^2\Bigg)+a_k'(u)^2\eta^p|\nabla u|^p+pa_k(u)a_k'(u)\eta^{p-1}|\nabla u|^{p-1}|\nabla\eta| \\
&+\frac{p^2}{4}a_k^2(u)\eta^{p-2}|\nabla\eta|^2|\nabla u|^{p-2}\Bigg]dx.
\end{aligned} \tag{3.5}$$

Now we concentrate on the right side of (3.5). Firstly, according to Young inequality we have

$$\begin{aligned}
&(p-1)\int_{R^N} s^{p-2}w_1(x)\left(a_k'(u)^2\eta^p|\nabla u|^2+p|a_k(u)|a_k'(u)\eta^{p-1}|\nabla u||\nabla\eta|\right)dx \\
&\le \frac{\varepsilon}{4}\int_{R^N} w_1(x)a_k'(u)^2\eta^p|\nabla u|^p dx + C(p,\varepsilon)\int_{R^N} w_1(x)a_k'(u)^2\eta^p dx \\
&+\frac{\varepsilon}{4}\int_{R^N} w_1(x)a_k'(u)^2\eta^p|\nabla u|^p dx \\
&+C(p,\varepsilon)\int_{R^N} w_1(x)(a_k'(u))^{\frac{p-2}{p-1}}a_k(u)^{\frac{p}{p-1}}\eta^{\frac{p(p-2)}{p-1}}|\nabla\eta|^{\frac{p}{p-1}}dx.
\end{aligned}$$

By the same way, we conclude that

$$\begin{aligned}
&p(p-1)\int_{R^N} w_1(x)\left(a_k(u)a_k'(u)|\nabla u|^{p-1}\eta^{p-1}|\nabla\eta|+\frac{p}{4}a_k^2(u)\eta^{p-2}|\nabla u|^{p-2}|\nabla\eta|^2\right)dx \\
&\le \frac{\varepsilon}{4}\int_{R^N} w_1(x)a_k'(u)^2\eta^p|\nabla u|^p dx + C(p,\varepsilon)\int_{R^N} w_1(x)(a_k'(u))^{2-p}a_k(u)^p|\nabla\eta|^p dx \\
&+\frac{\varepsilon}{4}\int_{R^N} w_1(x)a_k'(u)^2\eta^p|\nabla u|^p dx + C(p,\varepsilon)\int_{R^N} w_1(x)(a_k(p))^p|\nabla\eta|^p a_k'(u)^{2-p}dx.
\end{aligned}$$

Putting these two estimates into the right side of (3.5), we obtain that for any $\varepsilon\in(0,1)$, any $k\in\mathbb{N}$ and any nonnegative function $\eta\in C_c^1(\mathbb{R}^N)$, there exists a constant $C(p,\varepsilon)>0$ such that

$$\begin{aligned}
\int_{R^N} w_2(x)e^u {a_k}^2(u)\eta^p dx \le (p-1+\varepsilon)\int_{R^N} w_1(x)a_k'(u)^2\eta^p|\nabla u|^p dx \\
+C(p,\varepsilon)\Bigg[\int_{R^N} w_1(x)\Bigg(a_k'(u)^2\eta^p+a_k'(u)^{2-p}a_k(u)^p|\nabla\eta|^p
\end{aligned}$$

$$+ a_k'(u)^{\frac{p-2}{p-1}} a_k(u)^{\frac{p}{p-1}} \eta^{\frac{p(p-2)}{p-1}} |\nabla \eta|^{\frac{p}{p-1}} \Big) dx \Big]$$

$$+ \frac{p^2(p-1)}{4} \int_{R^N} w_1(x) s^{p-2} a_k^2(u) \eta^{p-2} |\nabla \eta|^2 dx. \tag{3.6}$$

Substitute the first term of the right side of (3.6) by (3.4), and with the help of (3.2) we obtain

$$\begin{aligned}
&\int_{R^N} w_2(x) e^u a_k^2(u) \eta^p dx \\
&\le \frac{\alpha(p-1+\varepsilon)}{4(1-\varepsilon)} 2^{\frac{4-p}{2}} \int_{R^N} w_2(x) e^u b_k(u) \eta^p dx \\
&+ \frac{p^2(p-1)}{4} \int_{R^N} w_1(x) s^{p-2} a_k^2(u) \eta^{p-2} |\nabla \eta|^2 dx \\
&+ C(p,\varepsilon) \Big[\int_{R^N} w_1(x) \Big(b_k(u)^{\frac{p}{p-1}} b_k'(u)^{\frac{1}{1-p}} \eta^{\frac{p(p-2)}{p-1}} |\nabla \eta|^{\frac{p}{p-1}} + b_k'(u)^{1-p} b_k(u)|^p \nabla \eta|^p \\
&+ a_k'(u)^{2-p} a_k(u)^p |\nabla \eta|^p + a_k'(u)^2 \eta^p + a_k'(u)^{\frac{p-2}{p-1}} a_k(u)^{\frac{p}{p-1}} \eta^{\frac{p(p-2)}{p-1}} |\nabla \eta|^{\frac{p}{p-1}} \Big) dx \Big] \\
&\le \frac{\alpha(p-1+\varepsilon)}{4(1-\varepsilon)} 2^{\frac{4-p}{2}} \int_{R^N} w_2(x) e^u a_k^2(u) \eta^p dx \\
&+ \frac{p^2(p-1)}{4} \int_{R^N} w_1(x) s^{p-2} e^{\alpha u} \eta^{p-2} |\nabla \eta|^2 dx \\
&+ C(p,\varepsilon) \Big[\int_{R^N} w_1(x) \Big(e^{\alpha u} \eta^{\frac{p(p-2)}{p-1}} |\nabla \eta|^{\frac{p}{p-1}} + e^{\alpha u} |\nabla \eta|^p + e^{\alpha u} \eta^p \Big) dx \Big].
\end{aligned} \tag{3.7}$$

We denote $\beta_\varepsilon = 1 - \frac{\alpha(p-1+\varepsilon)}{4(1-\varepsilon)} 2^{\frac{4-p}{2}}$, then $\lim_{\varepsilon \to 0^+} \beta_\varepsilon = 1 - \frac{\alpha(p-1)}{4} 2^{\frac{4-p}{2}} > 0$ for $\alpha \in (0, \frac{4}{\sqrt{2}^{4-p}(p-1)})$. Hence there exists an $\varepsilon > 0$ small enough, depending on p and α, such that $\beta_\varepsilon > 0$. In that way, (3.7) is transformed into

$$\begin{aligned}
&\beta_\varepsilon \int_{R^N} w_2(x) e^u a_k^2(u) \eta^p dx \\
&\le C(p,\varepsilon,\alpha) \Big[\int_{R^N} w_1(x) \Big(e^{\alpha u} \eta^{\frac{p(p-2)}{p-1}} |\nabla \eta|^{\frac{p}{p-1}} + e^{\alpha u} |\nabla \eta|^p + e^{\alpha u} \eta^p \Big) dx \Big] \\
&+ \frac{p^2(p-1)}{4} \int_{R^N} w_1(x) s^{p-2} e^{\alpha u} \eta^{p-2} |\nabla \eta|^2 dx.
\end{aligned}$$

Let $k \to \infty$, by Fatou's lemma, these exists a constant $C = C(p,\alpha) > 0$ such that for any nonnegative function $\eta \in C_c^1(\mathbb{R}^N)$ we have

$$\int_{R^N} w_2(x) e^{(\alpha+1)u} \eta^p dx \le C \int_{R^N} w_1(x) e^{\alpha u} \Big(|\nabla \eta|^{\frac{p}{p-1}} \eta^{\frac{p(p-2)}{p-1}} + |\nabla \eta|^p + \eta^p + \eta^{p-2} |\nabla \eta|^2 \Big) dx. \tag{3.8}$$

Using Young inequality, we obtain

$$\int_{R^N} w_2(x)e^{(\alpha+1)u}\eta^p dx \le C\Big[\int_{R^N} w_1(x)^{\alpha+1}w_2(x)^{-\alpha}\Big(|\nabla\eta|^{\frac{p(\alpha+1)}{p-1}}\eta^{\frac{p^2-p\alpha-2p}{p-1}} + \eta^{-p\alpha}|\nabla\eta|^{p(\alpha+1)} + \eta^p + \eta^{p-2\alpha-2}|\nabla\eta|^{2(\alpha+1)}\Big)dx\Big] + \frac{1}{2}\int_{R^N} w_2(x)e^{(\alpha+1)u}\eta^p dx. \tag{3.9}$$

Hence, (3.1) follows at once.

Case 2 When $p \ge 4$, as in case 1, we choose $\varphi = b_k(u)\eta^p$ as the test function in (1.3) and obtain formula (3.3). Set $x := s^2$, $y := |\nabla u|^2$ and $\gamma := \frac{p-2}{2}$ in (ii) of Lemma 3.1, we have

$$\text{the left side of (3.3)} \ge \int_{R^N} w_1(x)(s^{p-2} + |\nabla u|^{p-2})|\nabla u|^2 b_k'(u)\eta^p dx,$$

and

$$\text{the right side of (3.3)} \le \int_{R^N} \sqrt{2}^{p-4} p w_1(x)(s^{p-2} + |\nabla u|^{p-2})|\nabla u||\nabla\eta|b_k(u)\eta^{p-1}dx + \int_{R^N} w_2(x)e^u b_k(u)\eta^p dx.$$

Therefore,

$$\int_{R^N} w_1(x)|\nabla u|^p b_k'(u)\eta^p dx \le p\int_{R^N} w_1(x)(\sqrt{2})^{p-4}s^{p-2}|\nabla u||\nabla\eta|b_k(u)\eta^{p-1}dx + \int_{R^N} w_2(x)e^u b_k(u)\eta^p dx + p\int_{R^N} w_1(x)(\sqrt{2})^{p-4}|\nabla u|^{p-1}|\nabla\eta|b_k(u)\eta^{p-1}dx.$$

Furthermore using Young inequality, it yields

$$\int_{R^N} w_1(x)|\nabla u|^p b_k'(u)\eta^p dx \le C(\varepsilon,p)\Big[\int_{R^N} w_1(x)|\nabla\eta|^{\frac{p}{p-1}}b_k'(u)^{-\frac{1}{p-1}}b_k(u)^{\frac{p}{p-1}}\eta^{\frac{p(p-2)}{p-1}}dx + \int_{R^N} w_1(x)|\nabla\eta|^p b_k(u)^p b_k'(u)^{1-p}dx\Big] + \int_{R^N} w_2(x)e^u b_k(u)\eta^p dx + \varepsilon\int_{R^N} w_1(x)|\nabla u|^p b_k'(u)\eta^p dx,$$

which implies

$$(1-\varepsilon)\int_{R^N} w_1(x)|\nabla u|^p b_k'(u)\eta^p dx$$

$$\leq \int_{R^N} w_2(x)e^u b_k(u)\eta^p dx + C(\varepsilon, p)\bigg(\int_{R^N} w_1(x)b_k'(u)^{\frac{1}{1-p}} b_k(u)^{\frac{p}{p-1}}|\nabla\eta|^{\frac{p}{p-1}}\eta^{\frac{p(p-2)}{p-1}}dx$$
$$+\int_{R^N} w_1(x)b_k(u)^p b_k'(u)^{1-p}|\nabla\eta|^p dx\bigg). \tag{3.10}$$

We apply the test function $\varphi = a_k(u)\eta^{\frac{p}{2}}$ in the stability assumption (1.5) and get

$$\int_{R^N} w_2(x)e^u a_k^2(u)\eta^p dx \leq (\sqrt{2})^{p-4}(p-1)\bigg[\int_{R^N} w_1(x)\bigg(p|\nabla u|^{p-1}a_k(u)|a_k'(u)||\nabla\eta|\eta^{p-1}$$
$$+\frac{p^2}{4}|\nabla u|^{p-2}a_k^2(u)\eta^{p-2}|\nabla\eta|^2 + |\nabla u|^p a_k'(u)^2\eta^p\bigg)dx$$
$$+\int_{R^N} w_1(x)s^{p-2}\bigg(a_k'(u)^2\eta^p|\nabla u|^2 + pa_k(u)a_k'(u)\eta^{p-1}|\nabla u||\nabla\eta|$$
$$+\frac{p^2}{4}a_k'(u)\eta^{p-2}|\nabla\eta|^2\bigg)dx\bigg].$$

By Young inequality, it is true that

$$\int_{R^N} w_2(x)e^u a_k(u)^2\eta^p dx \leq C(\varepsilon, p)\bigg[\int_{R^N} w_1(x)\bigg(|a_k'(u)|^{2-p}|\nabla\eta|^p|a_k(u)|^p + a_k'(u)^2\eta^p$$
$$+a_k'(u)^{\frac{p-2}{p-1}}a_k(u)^{\frac{p}{p-1}}\eta^{\frac{p(p-2)}{p-1}}|\nabla\eta|^{\frac{p}{p-1}}\bigg)dx\bigg]$$
$$+\frac{p^2}{4}(\sqrt{2})^{p-4}(p-1)\int_{R^N} w_1(x)s^{p-2}a_k(u)^2\eta^{p-2}|\nabla\eta|^2 dx$$
$$+((\sqrt{2})^{p-4}(p-1)+\varepsilon)\int_{R^N} w_1(x)a_k'(u)^2\eta^p|\nabla u|^p dx. \tag{3.11}$$

Combining (3.10), (3.11) and (3.2), we obtain

$$\int_{R^N} w_2(x)e^u a_k^2(u)\eta^p dx \leq C(\varepsilon, p)\bigg[\int_{R^N} w_1(x)\bigg(|\nabla\eta|^{\frac{p}{p-1}}b_k'(u)^{-\frac{1}{p-1}}b_k(u)^{\frac{p}{p-1}}\eta^{\frac{p(p-2)}{p-1}}$$
$$+|\nabla\eta|^p b_k(u)^p b_k'(u)^{1-p} + |a_k'(u)|^{2-p}|\nabla\eta|^p a_k(u)^p$$
$$+a_k'(u)^2\eta^p + a_k'(u)^{\frac{p-2}{p-1}}a_k(u)^{\frac{p}{p-1}}\eta^{\frac{p(p-2)}{p-1}}|\nabla\eta|^{\frac{p}{p-1}}\bigg)dx\bigg]$$
$$+\frac{p^2}{4}(\sqrt{2})^{p-4}(p-1)\int_{R^N} w_1(x)s^{p-2}a_k^2(u)\eta^{p-2}|\nabla\eta|^2 dx$$
$$+\frac{(\sqrt{2}^{p-4}(p-1)+\varepsilon)\alpha}{4(1-\varepsilon)}\int_{R^N} w_2(x)e^u b_k(u)\eta^p dx.$$

We set $\beta_\varepsilon = 1 - \frac{(\sqrt{2}^{p-4}(p-1)+\varepsilon)\alpha}{4(1-\varepsilon)}$. Since $\alpha \in (0, \frac{4}{\sqrt{2}^{p-4}(p-1)})$, then $\lim_{\varepsilon\to 0^+}\beta_\varepsilon = 1 - \frac{\sqrt{2}^{p-4}(p-1)\alpha}{4} > 0$. Thus we can find some $\varepsilon > 0$ small enough, depending on p and α, such that $\beta_\varepsilon > 0$. Hence

$$\beta_\varepsilon \int_{R^N} w_2(x)e^u a_k^2(u)\eta^p dx \le C(\varepsilon, p, \alpha)\left[\int_{R^N} w_1(x)e^{\alpha u}\left(\eta^{\frac{p(p-2)}{p-1}}|\nabla\eta|^{\frac{p}{p-1}} + |\nabla\eta|^p + \eta^p\right)dx\right] + \frac{p^2}{4}(\sqrt{2})^{p-4}(p-1)\int_{R^N} w_1(x)e^{\alpha u}s^{p-2}\eta^{p-2}|\nabla\eta|^2 dx. \quad (3.12)$$

Let $k \to \infty$, by Fatou's lemma we obtain (3.8), and in combination with Young inequality, we get formula (3.1). □

Now we prepare to prove the first main result.

Proof of Theorem 2.1 By contradiction, we assume that equation (2.1) exists a stable solution u. Because we finally get the Liouville theorem by making $R \to \infty$, we can choose R large enough so that $B_R(0)$ contains the set $V := \{x \in \mathbb{R}^N | w_2(x) = 0\}$. According to the selection of η_R in equation 1.8, we know that when $|x| < R$, we have $\nabla\eta_R = 0$. That is, $\nabla\eta_R = 0$ in a neighbourhood of $V := \{x \in \mathbb{R}^N | w_2(x) = 0\}$.

(i) When $2 \le p < 4$, $N < \frac{4}{\sqrt{2}^{4-p}(p-1)}(n-m) - m$, we apply cut-off function $\eta_R(x)$ defined in (1.8) to (3.1). Consequently, for all $R > R_0$ there exists a constant C independent of R such that

$$\int_{B_R} w_2(x)e^{(\alpha+1)u}dx \le CR^{m\alpha+m-n\alpha+N}. \quad (3.13)$$

Since $\lim_{\alpha\to\frac{4}{\sqrt{2}^{4-p}(p-1)}} N + (\alpha+1)m - n\alpha < 0$, we may find some $\alpha \in (0, \frac{4}{\sqrt{2}^{4-p}(p-1)})$, such that $N + (\alpha+1)m - n\alpha < 0$. Letting $R \to \infty$ in (3.13), we can get $\int_{R^N} w_2(x)e^{(\alpha+1)u}dx = 0$, this is a contradiction.

(ii) While $p \ge 4$, we suppose that (2.1) admits a stable solution u in dimension $N < \frac{4}{\sqrt{2}^{p-4}(p-1)}(n-m) - m$. Then we apply cut-off function (1.8) to Proposition 3.2. There exists a constant C independent of R making inequality (3.13) true for all $R > R_0$, we may find some $\alpha \in (0, \frac{4}{\sqrt{2}^{p-4}(p-1)})$ such that $m\alpha + m - n\alpha + N < 0$. Letting $R \to \infty$, we obtain $\int_{R^N} w_2(x)e^{(\alpha+1)u}dx = 0$, a contradiction. □

This completes the proof of Theorem 2.1.

For the proof of Theorem 2.2, we need another proposition substituting Proposition 3.2.

Proposition 3.3 *Suppose that u is a stable solution of equation (2.2). Then there exists a constant $C = C(p, \alpha) > 0$ such that for any function $\eta \in C_c^1(\mathbb{R}^N)$ with $0 \le \eta \le 1$, $\nabla\eta = 0$ in a neighborhood of $\{x \in \mathbb{R}^N : w_2(x) = 0\}$, there hold*

$$\int_{R^N} w_2(x)(\frac{1}{u})^{4p+1}\eta^p dx$$
$$\le C\left[\int_{R^N} w_1(x)^{\frac{4p+1}{3}} w_2(x)^{\frac{2-4p}{3}} \eta^{\frac{4p^2-7p-2}{3}} |\nabla\eta|^{\frac{8p+2}{3}} dx\right.$$
$$+ \int_{R^N} w_1(x)^{\frac{4p+1}{p+1}} w_2(x)^{\frac{-3p}{p+1}} |\nabla\eta|^{\frac{4p^2+p}{p+1}} dx + \int_{R^N} w_1(x)^{4p+1} w_2(x)^{-4p}\eta^{4p^2+p} dx$$
$$\left.+ \int_{R^N} w_1(x)^{\frac{4p^2-3p-1}{2p-1}} w_2(x)^{\frac{-4p^2+5p}{2p-1}} \eta^{\frac{(p^2-2p)(4p+1)}{2p-1}} |\nabla\eta|^{\frac{4p^2+p}{2p-1}} dx\right].$$

Proof Since the nonlinear part contains singular term, the truncation functions $a_k(u)$ and $b_k(u)$ which we picked in the proof of Theorem 2.1 are no longer valid. By the fact that $u \in W^{1,p}_{loc}\left(\mathbb{R}^N, w_1\right)$ is a positive stable bounded solution of equation (2.2), then $\rho(u) = u^{1-2p}$, $\sigma(u) = \frac{1}{1-4p}u^{1-4p}$ are obviously $W^{1,p}_{loc}\left(\mathbb{R}^N, w_1\right)$. According to Remark 1.3, we know that $\rho(u)\eta^{\frac{p}{2}}$ and $\sigma(u)\eta^p$ are legitimate test function.

It is straightforward to figure out that

$$\begin{aligned} &|\sigma'(u)|^{1-p}|\sigma(u)|^p = C_1 u^{-3p}, \quad |\sigma'(u)|^{\frac{1}{1-p}}|\sigma(u)|^{\frac{p}{p-1}} = C_2 u^{\frac{-4p^2+5p}{p-1}}, \\ &\rho'(u)^2 = (1-2p)^2\sigma'(u), \end{aligned} \tag{3.14}$$

and furthermore

$$|\rho'(u)|^{2-p}|\rho(u)|^p = C_3 u^{-3p}, \quad |\rho'(u)|^{\frac{p-2}{p-1}}|\rho(u)|^{\frac{p}{p-1}} = C_4 u^{\frac{-4p^2+5p}{p-1}}, \tag{3.15}$$

where C_1, C_2, C_3, C_4 are positive constants depending only on p.

Step 1 Let $\eta \in C^1_c\left(\mathbb{R}^N\right)$ and select $\varphi = \sigma(u)\eta^p$ be nonnegative as a test function in (1.3). Since

$$\nabla\varphi = \sigma'(u)\eta^p\nabla u + p\sigma(u)\eta^{p-1}\nabla\eta,$$

using Cauchy-Schwarz inequality we obtain

$$\begin{aligned} &\int_{R^N} w_1(x)(s^2+|\nabla u|^2)^{\frac{p-2}{2}}\sigma'(u)\eta^p|\nabla u|^2 dx \\ &\le p\int_{R^N} w_1(x)(s^2+|\nabla u|^2)^{\frac{p-2}{2}}\sigma(u)\eta^{p-1}|\nabla u||\nabla\eta| dx \\ &\quad - \int_{R^N} w_2(x)e^{\frac{1}{u}}\sigma(u)\eta^p dx. \end{aligned} \tag{3.16}$$

When $2 \le p < 4$, in the (i) of Lemma 3.1, we set $x := s^2$, $y := |\nabla u|^2$ and $\gamma := \frac{p-2}{2}$, then we have

$$\text{the left side of (3.16)} \ge \int_{R^N} w_1(x)2^{\frac{p-4}{2}}(s^{p-2}+|\nabla u|^{p-2})|\nabla u|^2\sigma'(u)\eta^p dx,$$

and

$$\begin{aligned} \text{the right side of (3.16)} &\le p\int_{R^N} w_1(x)(s^{p-2}+|\nabla u|^{p-2})|\nabla u||\nabla\eta|\sigma(u)\eta^{p-1} dx \\ &- \int_{R^N} w_2(x)e^{\frac{1}{u}}\sigma(u)\eta^p dx. \end{aligned}$$

Moreover, by Young inequality, (3.16) becomes

$$\begin{aligned} &\int_{R^N} w_1(x)|\nabla u|^p\sigma'(u)\eta^p dx \\ &\le 2^{\frac{4-p}{2}}\Bigg[p\int_{R^N} w_1(x)\Big(s^{p-2}\sigma(u)\eta^{p-1}|\nabla u||\nabla\eta| + |\nabla u|^{p-1}\sigma(u)\eta^{p-1}|\nabla\eta|\Big)dx \end{aligned}$$

$$
\begin{aligned}
&- \int_{R^N} w_2(x) e^{\frac{1}{u}} \sigma(u) \eta^p dx \Big] - \int_{R^N} w_1(x) s^{p-2} \sigma'(u) \eta^p |\nabla u|^2 dx \\
&\leq \varepsilon \int_{R^N} w_1(x) |\nabla u|^p \sigma'(u) \eta^p dx + C(p,\varepsilon) \Big(\int_{R^N} w_1(x) \sigma'(u)^{1-p} \sigma(u)^p |\nabla \eta|^p) dx \\
&+ \int_{R^N} w_1(x) (\sigma(u)^{\frac{p}{p-1}} \sigma'(u)^{-\frac{1}{p-1}} \eta^{\frac{p(p-2)}{p-1}} |\nabla \eta|^{\frac{p}{p-1}} dx \Big) \\
&- 2^{\frac{4-p}{2}} \int_{R^N} w_2(x) e^{\frac{1}{u}} \sigma(u) \eta^p dx.
\end{aligned}
$$

When $p \geq 4$, we set $x := s^2$, $y := |\nabla u|^2$ and $\gamma := \frac{p-2}{2}$ in (ii) of Lemma 3.1, we have

$$
\text{the left side of (3.16)} \geq \int_{R^N} w_1(x) (s^{p-2} + |\nabla u|^{p-2}) |\nabla u|^2 \sigma'(u) \eta^p dx,
$$

and

$$
\begin{aligned}
&\text{the right side of (3.16)} \leq \int_{R^N} \sqrt{2}^{p-4} p w_1(x) (s^{p-2} + |\nabla u|^{p-2}) |\nabla u| |\nabla \eta| \sigma(u) \eta^{p-1} dx \\
&- \int_{R^N} w_2(x) e^{\frac{1}{u}} \sigma(u) \eta^p dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_{R^N} w_1(x) |\nabla u|^p \sigma'(u) \eta^p dx \\
&\leq p \int_{R^N} w_1(x) (\sqrt{2})^{p-4} s^{p-2} |\nabla u| |\nabla \eta| \sigma(u) \eta^{p-1} dx - \int_{R^N} w_2(x) e^{\frac{1}{u}} \sigma(u) \eta^p dx \\
&+ p \int_{R^N} w_1(x) (\sqrt{2})^{p-4} |\nabla u|^{p-1} |\nabla \eta| \sigma(u) \eta^{p-1} dx.
\end{aligned}
$$

Furthermore using Young inequality, it yields

$$
\begin{aligned}
&\int_{R^N} w_1(x) |\nabla u|^p \sigma'(u) \eta^p dx \\
&\leq C(\varepsilon, p) \Big[\int_{R^N} w_1(x) |\nabla \eta|^{\frac{p}{p-1}} \sigma'(u)^{-\frac{1}{p-1}} \sigma(u)^{\frac{p}{p-1}} \eta^{\frac{p(p-2)}{p-1}} dx \\
&+ \int_{R^N} w_1(x) |\nabla \eta|^p \sigma(u)^p \sigma'(u)^{1-p} dx \Big] - \int_{R^N} w_2(x) e^{\frac{1}{u}} \sigma(u) \eta^p dx \\
&+ \varepsilon \int_{R^N} w_1(x) |\nabla u|^p \sigma'(u) \eta^p dx.
\end{aligned}
$$

We finally get that for ε small enough, and any nonnegative function $\eta \in C_c^1(\mathbb{R}^N)$, there exists a constant $C(p,\varepsilon) > 0$ such that for any $p \geq 2$, there have

$$
(1-\varepsilon) \int_{R^N} w_1(x) |\nabla u|^p \sigma'(u) \eta^p dx
$$

$$
\begin{aligned}
&\leq C(p,\varepsilon)\Big[\int_{R^N} w_1(x)\sigma'(u)^{\frac{1}{1-p}}\sigma(u)^{\frac{p}{p-1}}|\nabla\eta|^{\frac{p}{p-1}}\eta^{\frac{p(p-2)}{p-1}}dx \\
&\quad+\int_{R^N} w_1(x)\sigma(u)^p\sigma'(u)^{1-p}|\nabla\eta|^p dx\Big].
\end{aligned}
\tag{3.17}
$$

Step 2 Utilizing the stability assumption (1.5), we choose $\varphi=\rho(u)\eta^{\frac{p}{2}}$ as the test function, note that

$$
\nabla\varphi=\rho'(u)\eta^{\frac{p}{2}}\nabla u+\frac{p}{2}\rho(u)\eta^{\frac{p-2}{2}}\nabla\eta,
$$

when $2\leq p<4$, we get

$$
\begin{aligned}
&\int_{R^N}\frac{1}{u^2}w_2(x)e^{\frac{1}{u}}\rho(u)^2\eta^p dx \\
&\leq(p-1)\int_{R^N} w_1(x)\Big[s^{p-2}\Big(\rho'(u)^2\eta^p|\nabla u|^2+p\rho(u)\rho(u)'\eta^{p-1}|\nabla u|^{|}\nabla\eta| \\
&+\frac{p^2}{4}\rho(u)^2\eta^{p-2}|\nabla\eta|^2\Big)+\rho'(u)^2\eta^p|\nabla u|^p+p\rho(u)\rho'(u)\eta^{p-1}|\nabla u|^{p-1}|\nabla\eta| \\
&+\frac{p^2}{4}\rho(u)^2\eta^{p-2}|\nabla\eta|^2|\nabla u|^{p-2}\Big]dx.
\end{aligned}
$$

By Young inequality, we have

$$
\begin{aligned}
&\int_{R^N}\frac{1}{u^2}w_2(x)e^{\frac{1}{u}}\rho(u)^2\eta^p dx \\
&\leq(p-1+\varepsilon)\int_{R^N} w_1(x)\rho'(u)^2\eta^p|\nabla u|^p dx+C(p,\varepsilon)\Big[\int_{R^N} w_1(x)\Big(\rho'(u)^2\eta^p \\
&+\rho'(u)^{2-p}\rho(u)^p|\nabla\eta|^p+\rho'(u)^{\frac{p-2}{p-1}}\rho(u)^{\frac{p}{p-1}}\eta^{\frac{p(p-2)}{p-1}}|\nabla\eta|^{\frac{p}{p-1}}\Big)dx\Big] \\
&+\frac{p^2(p-1)}{4}\int_{R^N} w_1(x)s^{p-2}\rho(u)^2\eta^{p-2}|\nabla\eta|^2 dx.
\end{aligned}
$$

When $p\geq 4$, we obtain

$$
\begin{aligned}
&\int_{R^N}\frac{1}{u^2}w_2(x)e^{\frac{1}{u}}\rho(u)^2\eta^p dx\leq(\sqrt{2})^{p-4}(p-1)\int_{R^N} w_1(x) \\
&\Big[s^{p-2}\Big(\rho'(u)^2\eta^p|\nabla u|^2+p\rho(u)\rho(u)'\eta^{p-1}|\nabla u|^{|}\nabla\eta|+\frac{p^2}{4}\rho(u)^2\eta^{p-2}|\nabla\eta|^2\Big) \\
&+\rho'(u)^2\eta^p|\nabla u|^p+p\rho(u)\rho'(u)\eta^{p-1}|\nabla u|^{p-1}|\nabla\eta|+\frac{p^2}{4}\rho(u)^2\eta^{p-2}|\nabla\eta|^2|\nabla u|^{p-2}\Big]dx.
\end{aligned}
$$

By Young inequality, we have

$$
\int_{R^N}\frac{1}{u^2}w_2(x)e^{\frac{1}{u}}\rho(u)^2\eta^p dx
$$

$$\leq (\sqrt{2})^{p-4}(p-1+\varepsilon)\int_{R^N} w_1(x)\rho'(u)^2\eta^p|\nabla u|^p dx + C(p,\varepsilon)\Big[\int_{R^N} w_1(x)\Big(\rho'(u)^2\eta^p + \rho'(u)^{2-p}\rho(u)^p|\nabla\eta|^p + \rho'(u)^{\frac{p-2}{p-1}}\rho(u)^{\frac{p}{p-1}}\eta^{\frac{p(p-2)}{p-1}}|\nabla\eta|^{\frac{p}{p-1}}\Big)dx\Big] + \frac{(\sqrt{2})^{p-4}p^2(p-1)}{4}\int_{R^N} w_1(x)s^{p-2}\rho(u)^2\eta^{p-2}|\nabla\eta|^2 dx.$$

Hence, for any $p \geq 2$, there exists an ε small enough such that

$$\int_{R^N} w_2(x)\frac{1}{u^2}e^{\frac{1}{u}}\rho(u)^2\eta^p dx \leq C\int_{R^N} w_1(x)\Big[\rho(u)^2\eta^{p-2}|\nabla\eta|^2 + \rho(u)^p\rho'(u)^{2-p}|\nabla\eta|^p + \rho'(u)^2\eta^p + \rho'(u)^2\eta^p|\nabla u|^p + \rho(u)^{\frac{p}{p-1}}\rho'(u)^{\frac{p-2}{p-1}}\eta^{\frac{p(p-2)}{p-1}}|\nabla\eta|^{\frac{p}{p-1}}\Big]dx. \tag{3.18}$$

Step 3 For the fourth term on the right-hand side of equation (3.18), we use (3.14) and (3.17) to get

$$\int_{R^N} w_1(x)\rho'(u)^2\eta^p|\nabla u|^p dx \leq C\int_{R^N} w_1(x)\Big[\sigma'(u)^{\frac{1}{1-p}}\sigma(u)^{\frac{p}{p-1}}|\nabla\eta|^{\frac{p}{p-1}}\eta^{\frac{p(p-2)}{p-1}} + \sigma(u)^p\sigma'(u)^{1-p}|\nabla\eta|^p\Big]dx. \tag{3.19}$$

We combine (3.18) with (3.19) to obtain

$$\int_{R^N} w_2(x)\frac{1}{u^2}e^{\frac{1}{u}}\rho(u)^2\eta^p dx \leq C\int_{R^N} w_1(x)\Big[\rho(u)^2\eta^{p-2}|\nabla\eta|^2 + \rho(u)^p\rho'(u)^{2-p}|\nabla\eta|^p + \rho'(u)^2\eta^p + \sigma'(u)^{\frac{1}{1-p}}\sigma(u)^{\frac{p}{p-1}}|\nabla\eta|^{\frac{p}{p-1}}\eta^{\frac{p(p-2)}{p-1}} + \sigma(u)^p\sigma'(u)^{1-p}|\nabla\eta|^p + \rho(u)^{\frac{p}{p-1}}\rho'(u)^{\frac{p-2}{p-1}}\eta^{\frac{p(p-2)}{p-1}}|\nabla\eta|^{\frac{p}{p-1}}\Big]dx. \tag{3.20}$$

Now using the fact $e^{1/u} > \frac{1}{u}$ and employing (3.15) in the left side of inequality (3.20), it yields

$$\int_{R^N} w_2(x)u^{-4p-1}\eta^p dx \leq C\int_{R^N} w_1(x)\Big[u^{-4p+2}\eta^{p-2}|\nabla\eta|^2 + u^{-3p}|\nabla\eta|^p + u^{-4p}\eta^p + u^{\frac{-4p^2+5p}{p-1}}\eta^{\frac{p(p-2)}{p-1}}|\nabla\eta|^{\frac{p}{p-1}}\Big]dx. \tag{3.21}$$

By Young inequality we have for small ε that

$$\int_{R^N} w_2(x)(\frac{1}{u})^{4p+1}\eta^p dx$$
$$\leq C\Big[\int_{R^N} w_1(x)^{\frac{4p+1}{3}} w_2(x)^{\frac{2-4p}{3}} \eta^{\frac{4p^2-7p-2}{3}} |\nabla\eta|^{\frac{8p+2}{3}} dx$$
$$+\int_{R^N} w_1(x)^{\frac{4p+1}{p+1}} w_2(x)^{\frac{-3p}{p+1}} |\nabla\eta|^{\frac{4p^2+p}{p+1}} dx$$
$$+\int_{R^N} w_1(x)^{4p+1} w_2(x)^{-4p}\eta^{4p^2+p} dx$$
$$+\int_{R^N} w_1(x)^{\frac{4p^2-3p-1}{2p-1}} w_2(x)^{\frac{-4p^2+5p}{2p-1}} \eta^{\frac{(p^2-2p)(4p+1)}{2p-1}} |\nabla\eta|^{\frac{4p^2+p}{2p-1}} dx\Big].$$

Then we complete the proof of Proposition 3.3. □

At last, we are going to prove the second main result.

Proof of Theorem 2.2 By contradiction, let us suppose that u is a bounded stable solution to (2.2) such that $0 < u \leq M$ in $\mathbb{R}^N$. We have known that $w_1(x) \leq C_1|x|^m$ and $w_2(x) \geq C_2|x|^n$ when $|x|$ is large enough. Then by Proposition 3.3, we have

$$\int_{R^N} w_2(x)(\frac{1}{u})^{4p+1}\eta^p dx$$
$$\leq C\Big[\int_{R^N} |x|^{\frac{(4p+1)m-(4p-2)n}{3}} \eta^{\frac{4p^2-7p-2}{3}} |\nabla\eta|^{\frac{8p+2}{3}} dx + \int_{R^N} |x|^{\frac{(4p+1)m-3pn}{p+1}} |\nabla\eta|^{\frac{4p^2+p}{p+1}} dx$$
$$+\int_{R^N} |x|^{(4p+1)m-4pn}\eta^{4p^2+p} dx + \int_{R^N} |x|^{\frac{(4p^2-3p-1)m-(4p^2-5p)n}{2p-1}} \eta^{\frac{(p^2-2p)(4p+1)}{2p-1}} |\nabla\eta|^{\frac{4p^2+p}{2p-1}} dx\Big]. \tag{3.22}$$

Choosing $\eta = \eta_R$ in (3.19) and according to the conditions of the theorem, we have

$$\int_{B_R(0)} w_2(x)(\frac{1}{u})^{4p+1} dx$$
$$\leq C\Big[R^{\frac{(4p+1)m-(4p-2)n-8p-2}{3}+N} + R^{\frac{(4p+1)m-3pn-p(4p+1)}{p+1}+N} + R^{(4p+1)m-4pn+N}$$
$$+R^{\frac{(4P^2-3p-1)m-(4p^2-5p)n-p(4p+1)}{2p-1}+N}\Big]$$
$$\leq CR^{(4p+1)m-4pn+N}. \tag{3.23}$$

Therefore, letting $R \to \infty$ in (3.23), we obtain

$$\int_{R^N} w_2(x)(\frac{1}{u})^{4p+1} dx = 0,$$

which is a contradiction.

At the end of the paper, we discuss the stable solution of equation (1.1) in special case. In fact, let $s = 0$ and $p = 2$ in equation (1.1), and we consider the Hardy-Hénon equation

$$-div(|x|^m \nabla u) = |x|^n e^u \quad \text{in } \mathbb{R}^N. \tag{3.24}$$

By solving ordinary differential equations, we find that

$$\bar{u}(x) = \ln \frac{(2-m+n)(N+m-2)}{|x|^{2-m+n}}, \qquad n > m-2$$

is a radial solution of equation (3.24). Next, we verify the stability condition (1.5), for all $\varphi \in C_c^1\left(\mathbb{R}^N\right)$, it holds

$$\int_{R^N} |x|^m |\nabla\varphi|^2 - |x|^n e^{\bar{u}} \varphi^2 dx = \int_{R^N} |x|^m |\nabla\varphi|^2 - (2-m+n)(N+m-2)\frac{\varphi^2}{|x|^{2-m}} dx \tag{3.25}$$

According to Lemma 2.1 in [16], for all $\varphi \in C_c^1\left(\mathbb{R}^N\right)$, we derive

$$\int_{R^N} |x|^m |\nabla\varphi|^2 \geq \left(\frac{N+m-2}{2}\right)^2 \int_{R^N} \frac{\varphi^2}{|x|^{2-m}} dx \tag{3.26}$$

With the help of (3.25), (3.26) yields that

$$\int_{R^N} |x|^m |\nabla\varphi|^2 - |x|^n e^{\bar{u}} \varphi^2 dx \geq \frac{(N+m-2)(N+5m-4n-10)}{4} \int_{R^N} \frac{\varphi^2}{|x|^2} dx \tag{3.27}$$

Under the assumption of $N \geq 10 - 5m + 4n$, we have

$$\int_{R^N} |x|^m |\nabla\varphi|^2 - |x|^n e^{\bar{u}} \varphi^2 dx \geq 0 \tag{3.28}$$

Therefore, when $N \geq 10 - 5m + 4n$, and $n > m - 2$, $\bar{u}$ is a stable solution of equation (3.24).
□

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