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Well-Posedness, Blow-up Criteria and Stability for Solutions of the Generalized MHD Equations in Sobolev-Gevrey Spaces

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Abstract
This work presents the existence of local in time solutions for the generalized Magnetohydrodynamics equations in Sobolev-Gevrey (and Sobolev) spaces. Moreover, we establish the behavior of these solutions at potential blow-up times. In addition, if the initial data is assumed to be small enough, this paper proves the existence of global in time solutions, which are stable, in these same type of spaces.

Keywords Generalized MHD equations · Existence of solutions · Blow-up criteria · Stability for global solutions · Sobolev-Gevrey spaces

Mathematics Subject Classification 35Q30 · 76W05 · 35B44 · 35A01 · 35A07 · 35B35

1 Introduction

In this paper, we study the existence and behavior of solutions for the following generalized incompressible Magnetohydrodynamic (GMHD) equations in Sobolev-Gevrey (and

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Sobolev) spaces:

$$\begin{cases} u_t + u\cdot\nabla u + \nabla(p+\frac{1}{2}|b|^2) + \mu(-\Delta)^\alpha u = b\cdot\nabla b, & x\in\mathbb{R}^3, \quad t>0,\\ b_t + u\cdot\nabla b + \nu(-\Delta)^\beta b = b\cdot\nabla u, & x\in\mathbb{R}^3, \quad t>0,\\ \operatorname{div} u = \operatorname{div} b = 0, & x\in\mathbb{R}^3, \quad t>0,\\ u(x,0)=u_0(x),\ b(x,0)=b_0(x), & x\in\mathbb{R}^3, \end{cases} \tag{1}$$

where $u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))\in\mathbb{R}^3$ denotes the incompressible velocity field, $b(x,t)=(b_1(x,t),b_2(x,t),b_3(x,t))\in\mathbb{R}^3$ the magnetic field and $p(x,t)\in\mathbb{R}$ the hydrostatic pressure. The positive constants μ and ν are associated with specific properties of the fluid: The constant μ is the kinematic viscosity and ν^{-1} is the magnetic Reynolds number. Here α and β belong to $(\frac{1}{2},\frac{5}{3})$. The initial data for the velocity and magnetic fields, given by u_0 and b_0 in (1), are assumed to be divergence free, i.e., $\operatorname{div} u_0=\operatorname{div} b_0=0$.

The system (1) is of interest for many reasons. In fact, physically, the GMHD equations (1) describe the dynamics of electrically conducting fluids in the presence of magnetic fields and has broad applications in applied sciences such as astrophysics, geophysics and plasma physics. More specifically, some applications are the following: the machinery of the sun, black holes with the formation of extragalactic jets, interstellar clouds, and planetary magnetospheres (we refer to [12, 14, 27, 28] for more details). Despite their wide physical applicability, the GMHD equations are of great interest in Mathematics as well. It is also worth to emphasize that the GMHD equations become the famous Magnetohydrodynamics (MHD) equations if $\alpha=\beta=1$. Moreover, the GMHD equations also present as particular cases the generalized Navier-Stokes (GNS) equations whether $b=0$, and the usual Navier-Stokes (NS) equations if it is assumed that $b=0$ and $\alpha=1$. In order to refer some papers, see [1–8, 11–29] and references therein.

Let us list our main results as follows (for the essential notations and definitions, see Section 2 and footnotes throughout the paper).

Our first theorem presents an extension for the local solution obtained by J. Lorenz, W.G. Melo, and N.F. Rocha [23] from the MHD equations to the GMHD system (1) (see also [1, 3–5, 7, 8, 24–27, 29] and references therein).

Theorem 1.1 *Assume that* $a\geq 0$, $\sigma\geq 1$, $\alpha>\frac{1}{2}$, $\beta>\frac{1}{2}$ *and* $\max\{\frac{5}{2}-2\alpha,\frac{5}{2}-2\beta,0\}<s<\frac{3}{2}$. *Let* $(u_0,b_0)\in\dot{H}^s_{a,\sigma}$ *such that* $\operatorname{div} u_0=\operatorname{div} b_0=0$. *Then, there exist an instant* $T=T_{s,a,\sigma,\alpha,\beta\mu,\nu,u_0,b_0}>0$ *and a unique solution* $(u,b)\in C_T(\dot{H}^s_{a,\sigma})$ *for the GMHD equations* (1).

The second theorem establishes improvements as well as generalizations for some results obtained by H. Orf [26] (see also [1, 3–5, 7, 8, 11, 16–21, 25, 27, 29] and references included).

Theorem 1.2 *The following statements hold*:

i) *Assume that* $a\geq 0$, $\sigma\geq 1$, $\frac{2}{3}<\alpha=\beta<\frac{5}{3}$, *and* $\alpha-\frac{5}{2}<s<\frac{3-\alpha}{2}$. *Let* $(u_0,b_0)\in\dot{H}^{s,\alpha}_{a,\sigma}$ *such that* $\operatorname{div} u_0=\operatorname{div} b_0=0$. *There exists a constant* $C_{s,\alpha}>0$ *such that if the initial data* (u_0,b_0) *satisfies the inequality* $\|(u_0,b_0)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}<C_{s,\alpha}\min\{\mu,\nu\}$; *then, we obtain a unique solution* $(u,b)\in L^4_T(\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma})\cap L^4_T(\dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma})$, *for all* $T>0$, *for the GMHD equations* (1) *that satisfies*

$$\|u\|^4_{L^4_T(\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma})}+\|b\|^4_{L^4_T(\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma})}+\|u\|^4_{L^4_T(\dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma})}+\|b\|^4_{L^4_T(\dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma})}\leq C_{\mu,\nu}\|(u_0,b_0)\|^4_{\dot{H}^{s,\alpha}_{a,\sigma}}.$$

Moreover, we have

$$(u,b)\in C_T(\dot H^s_{a,\sigma})\cap C_T(\dot H^{\frac52-2\alpha}_{a,\sigma})\cap L^p_T(\dot H^{s+\frac{2\alpha}{p}}_{a,\sigma})\cap L^p_T(\dot H^{\frac{(5-4\alpha)p+4\alpha}{2p}}_{a,\sigma}),\quad \forall p\ge 2. \tag{2}$$

ii) *Consider that* $a\ge 0$, $\sigma\ge 1$, $\frac23<\alpha=\beta<\frac53$, *and* $\alpha-\frac52<s<\frac{3-\alpha}{2}$. *Let* $(u_0,b_0)\in \dot H^{s,\alpha}_{a,\sigma}$ *such that* $\operatorname{div}u_0=\operatorname{div}b_0=0$. *There exist an instant* $\overline{T}=\overline{T}_{s,\alpha,a,\sigma,\mu,\nu,u_0,b_0}>0$ *and a unique solution* $(u,b)\in L^{\frac4{\overline{T}}}(\dot H^{s+\frac{\alpha}{2}}_{a,\sigma})\cap L^{\frac4{\overline{T}}}(\dot H^{\frac{5-3\alpha}{2}}_{a,\sigma})$ *for the GMHD equations* (1). *Moreover, we have*

$$(u,b)\in C_{\overline{T}}(\dot H^s_{a,\sigma})\cap C_{\overline{T}}(\dot H^{\frac52-2\alpha}_{a,\sigma})\cap L^{\frac{p}{\overline{T}}}(\dot H^{s+\frac{2\alpha}{p}}_{a,\sigma})\cap L^{\frac{p}{\overline{T}}}(\dot H^{\frac{(5-4\alpha)p+4\alpha}{2p}}_{a,\sigma}),\quad \forall p\ge 2.$$

iii) *Let* $a>0$, $\sigma>1$, $\frac12<\alpha=\beta<1$, $\alpha-1\le s<\frac52-2\alpha$, *and* $(u_0,b_0)\in\dot H^s_{a,\sigma}$, *with* $\operatorname{div}u_0=\operatorname{div}b_0=0$. *There exists a constant* $C_{a,\sigma,s,\alpha,\mu,\nu}>0$ *such that if* $\|(u_0,b_0)\|_{\dot H^s_{a,\sigma}}<C_{a,\sigma,s,\alpha,\mu,\nu}$; *then, we obtain a unique solution* $(u,b)\in L_T^{\frac{2\alpha}{1-\alpha}}(\dot H^{s+1-\alpha}_{a,\sigma})\cap L_T^{\frac{2\alpha}{1-\alpha}}(\dot H^{s+2\alpha-1}_{a,\sigma})$, *for all* $T>0$, *for the GMHD equations* (1) *that satisfies*

$$\begin{aligned}&\|u\|_{L_T^{\frac{2\alpha}{1-\alpha}}(\dot H^{s+1-\alpha}_{a,\sigma})}+\|b\|_{L_T^{\frac{2\alpha}{1-\alpha}}(\dot H^{s+1-\alpha}_{a,\sigma})}+\|u\|_{L_T^{\frac{2\alpha}{1-\alpha}}(\dot H^{s+2\alpha-1}_{a,\sigma})}+\|b\|_{L_T^{\frac{2\alpha}{1-\alpha}}(\dot H^{s+2\alpha-1}_{a,\sigma})}\\&\quad\le C_{\mu,\nu,\alpha}\|(u_0,b_0)\|_{\dot H^s_{a,\sigma}}.\end{aligned}$$

Furthermore, one infers

$$(u,b)\in C_T(\dot H^s_{a,\sigma})\cap L^p_T(\dot H^{s+\frac{2\alpha}{p}}_{a,\sigma}),\quad \forall p\ge 2. \tag{3}$$

Below, we present new blow-up criteria for the local solution obtained in Theorem 1.2 (see also [1, 5, 7, 8, 16, 23–26] and references therein).

Theorem 1.3 *Assume that* $a>0$, $\sigma>1$, $1\le\alpha=\beta<\frac53$, *and* $\alpha-\frac52<s<\frac{3-\alpha}{2}$ *if* $\alpha\ge\frac32$ *or* $-1\le s<\frac{3-\alpha}{2}$ *provided that* $\alpha<\frac32$. *Let* $(u_0,b_0)\in\dot H^{s,\alpha}_{a,\sigma}$ *such that* $\operatorname{div}u_0=\operatorname{div}b_0=0$. *Assume that* $(u,b)\in C([0,T^*);\dot H^s_{a,\sigma})\cap C([0,T^*);\dot H^{\frac52-2\alpha}_{a,\sigma})$ *is the solution for the GMHD equations* (1) *in the maximal time interval* $0\le t<T^*$ *given in Theorem* 1.2. *If* $T^*<\infty$, *then the following statements hold*:

i) $\limsup_{t\nearrow T^*}\|(u,b)(t)\|_{\dot H^{s,\alpha}_{\frac{a}{(\sqrt{\sigma})^{(n-1)}},\sigma}}=\infty$;

ii) $\displaystyle\int_t^{T^*}\|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac1\sigma}}(\widehat{u},\widehat{b})(\tau)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}\,d\tau=\infty$;

iii) $\|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac1\sigma}}(\widehat{u},\widehat{b})(\tau)\|_{L^1}\ge C_{\mu,\nu,\alpha}(T^*-t)^{-\frac{2\alpha}{2\alpha-1}}$;

iv) $\|(u,b)(t)\|_{\dot H^s_{\frac{a}{(\sqrt{\sigma})^n},\sigma}}\ge C_{a,\sigma,\mu,\nu,s,\alpha}(T^*-t)^{-\frac{2\alpha}{2\alpha-1}}$;

v) $\|(u,b)(t)\|_{\dot H^{\frac52-2\alpha}_{\frac{a}{(\sqrt{\sigma})^n},\sigma}}\ge C_{a,\sigma,\mu,\nu,\alpha}(T^*-t)^{-\frac{2\alpha}{2\alpha-1}}$;

vi) $\displaystyle\frac{a^{\sigma_0+\frac12}C_{\mu,\nu,s,\alpha,\sigma,\sigma_0,u_0,b_0}}{(T^*-t)^{\frac{(2\alpha-1)[2(s\sigma+\sigma_0)+1]}{6\alpha\sigma}}}\exp\left\{\frac{aC'_{\mu,\nu,\sigma,s,\alpha,u_0,b_0}}{(T^*-t)^{\frac{2\alpha-1}{3\alpha\sigma}}}\right\}\le\|(u,b)(t)\|_{\dot H^s_{a,\sigma}}$, *if* $(u_0,b_0)\in L^2$,

vii) $\displaystyle\frac{a^{\sigma_0+\frac12}C_{\mu,\nu,\alpha,\sigma,\sigma_0,u_0,b_0}}{(T^*-t)^{\frac{(2\alpha-1)[2((\frac52-2\alpha)\sigma+\sigma_0)+1]}{6\alpha\sigma}}}\exp\left\{\frac{aC'_{\mu,\nu,\sigma,\alpha,u_0,b_0}}{(T^*-t)^{\frac{2\alpha-1}{3\alpha\sigma}}}\right\}\le\|(u,b)(t)\|_{\dot H^{\frac52-2\alpha}_{a,\sigma}}$, *if* $(u_0,b_0)\in L^2$, *for all* $t \in [0, T^*), n \in \mathbb{N}$. *Here* $2\sigma_0$ *is the integer part of* $2\sigma \max\{\frac{5}{2}, 4-\alpha\}$.

We also establish a study on the properties at potential blow-up times satisfied by solution obtained in Theorem 1.1. The theorem below extends the blow-up criteria obtained by J. Lorenz, W.G. Melo, N.F. Rocha [23] from the MHD equations to the GMHD system (1) (see also [1, 5, 7, 8, 16, 24–26] and references therein).

Theorem 1.4 *Assume that* $a > 0, \sigma > 1, \alpha \geq 1, \beta \geq 1$ *and* $\max\{\frac{5}{2} - 2\alpha, \frac{5}{2} - 2\beta, 0\} < s < \frac{3}{2}$. *Let* $(u_0, b_0) \in \dot{H}^s_{a,\sigma}$ *such that* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. *Assume that* $(u, b) \in C([0, T^*); \dot{H}^s_{a,\sigma})$ *is the solution for the GMHD equations* (1) *in the maximal time interval* $0 \leq t < T^*$ *given in Theorem* 1.1. *If* $T^* < \infty$, *then the following statements hold*:

i) $\limsup_{t \nearrow T^*} \|(u, b)(t)\|_{\dot{H}^s_{\frac{a}{(\sqrt{\sigma})^{(n-1)}},\sigma}} = \infty$;

ii) $\int_t^{T^*} [\|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}} (\widehat{u}, \widehat{b})(\tau)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}} + \|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}} (\widehat{u}, \widehat{b})(\tau)\|_{L^1}^{\frac{2\beta}{2\beta-1}}]\, d\tau = \infty$;

iii) $\|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}} (\widehat{u}, \widehat{b})(t)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}} + \|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}} (\widehat{u}, \widehat{b})(t)\|_{L^1}^{\frac{2\beta}{2\beta-1}} \geq C_{\mu,\nu,\alpha,\beta}(T^* - t)^{-1}$;

iv) $\|(u, b)(t)\|_{\dot{H}^s_{\frac{a}{(\sqrt{\sigma})^n},\sigma}}^{\frac{2\alpha}{2\alpha-1}} + \|(u, b)(t)\|_{\dot{H}^s_{\frac{a}{(\sqrt{\sigma})^n},\sigma}}^{\frac{2\beta}{2\beta-1}} \geq C_{a,\sigma,\mu,\nu,s,\alpha,\beta}(T^* - t)^{-1}$;

v) $\dfrac{a^{\sigma_0+\frac{1}{2}} C_{\mu,\nu,s,\alpha,\sigma,\sigma_0,u_0,b_0}}{(T^* - t)^{\frac{(2\alpha-1)[2(s\sigma+\sigma_0)+1]}{6\alpha\sigma}}} \exp\left\{ \dfrac{aC'_{\mu,\nu,\sigma,s,\alpha,u_0,b_0}}{(T^* - t)^{\frac{2\alpha-1}{3\alpha\sigma}}} \right\} \leq \|(u, b)(t)\|_{\dot{H}^s_{a,\sigma}}$, *if* $(u_0, b_0) \in L^2$, *and* $\alpha = \beta$,

for all $t \in [0, T^*), n \in \mathbb{N}$. *Here* $2\sigma_0$ *is the integer part of* $2\sigma(2\alpha - 1)$.

Our last result proves the existence of stable global solutions for the GMHD equations (1). The next theorem is our extended version for the equivalent study presented by H. Orf [26] (see also [1–8, 11, 15–21, 23, 25, 29] and references included).

Theorem 1.5 *Assume that* $a \geq 0$, $\sigma \geq 1$, $\frac{2}{3} < \alpha = \beta < \frac{5}{3}$, *and* $\alpha - \frac{5}{2} < s < \frac{3-\alpha}{2}$. *Let* $(u_0, b_0) \in \dot{H}^{s,\alpha}_{a,\sigma}$ *such that* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. *Assume that* $(u, b) \in C([0, T^*); \dot{H}^s_{a,\sigma}) \cap C([0, T^*); \dot{H}^{\frac{5}{2}-2\alpha}_{a,\sigma}) \cap L^p([0, T^*); \dot{H}^{s+\frac{2\alpha}{p}}_{a,\sigma}) \cap L^p([0, T^*); \dot{H}^{\frac{(5-2\alpha)p-4\alpha}{2p}}_{a,\sigma})$, *for all* $p \geq 2$, *is the solution for the GMHD equations* (1) *in the maximal time interval* $0 \leq t < T^*$ *given in Theorem* 1.2. *Then, the following statements hold*:

i) *If* $T^* < \infty$, *we obtain* $\int_0^{T^*} \|(u, b)(\tau)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\, d\tau = \infty$;

ii) *There is a constant* $C_{s,\alpha} > 0$ *such that if the initial data* (u_0, b_0) *satisfies* $\|(u_0, b_0)\|_{\dot{H}^{s,\alpha}_{a,\sigma}} < C_{s,\alpha}[\mu + \nu]$, *we conclude that* $T^* = \infty$. *Moreover, we have*

$$\|(u, b)(t)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + [\mu + \nu] \int_0^t \|(u, b)(\tau)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\, d\tau \leq \|(u_0, b_0)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}}, \quad \forall t \geq 0;$$

iii) *Let* $(v_0, w_0) \in \dot{H}^{s,\alpha}_{a,\sigma}$ *such that* $\operatorname{div} v_0 = \operatorname{div} w_0 = 0$. *Consider that* $T^* = \infty$. *Then, there are constants* $C_{s,\alpha}, C'_{s,\alpha} > 0$ *such that if* (u_0, b_0) *and* (v_0, w_0) *verify the inequality*

$\|(u_0, b_0) - (v_0, w_0)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}$

$$< C_{s,\alpha}[\mu+\nu]^4 \exp\{-C'_{s,\alpha}[\mu+\nu]^{-3}\int_0^\infty [\|(u,b)(\tau)\|^4_{\dot H_{a,\sigma}^{s+\frac{\alpha}{2}}} + \|(u,b)(\tau)\|^4_{\dot H_{a,\sigma}^{\frac{5-3\alpha}{2}}}]\,d\tau\},$$

we conclude that the GMHD equations (1) *admit global solution* (v,w), *with initial data* (v_0,w_0), *that satisfies*

$$\|[(u,b)-(v,w)](t)\|^2_{\dot H^{s,\alpha}_{a,\sigma}} + \frac{\mu+\nu}{4}\int_0^t \|[(u,b)-(v,w)](\tau)\|^2_{\dot H_{a,\sigma}^{s+\alpha,\frac{\alpha}{2}}}\,d\tau$$

$$\le \|(u_0,b_0)-(v_0,w_0)\|^2_{\dot H^{s,\alpha}_{a,\sigma}} \exp\{C'_{s,\alpha}[\mu+\nu]^{-3}\int_0^\infty [\|[(u,b)-(v,w)](\tau)\|^4_{\dot H_{a,\sigma}^{s+\frac{\alpha}{2}}}$$

$$+ \|[(u,b)-(v,w)](\tau)\|^4_{\dot H_{a,\sigma}^{\frac{5-3\alpha}{2}}}]\,d\tau\},$$

for all $t \ge 0$.

Remark 1.6 In this observation, our goal is to expose the reason why we have chosen the ranges for a,σ,α,β and s presented in Theorems 1.1, 1.2, 1.3, 1.4 and 1.5. It is important to emphasize that all the explanations presented below are related to the technics and the preliminary results applied in this paper. Thus, let us examine the hypotheses of our main results one by one.

• In the proof of Theorem 1.1, we applied Lemma 2.3, and resolved the elementary integral obtained in (17) that made us assume the conditions for the values a,σ,α, β and s given in this same result. More specifically, in order to utilize Lemma 2.3 (with $s_1 = s_2 = s$, $s_1 < \frac{3}{2}$, $s_2 < \frac{3}{2}$, $s_1+s_2>0$, $a\ge 0$ and $\sigma\ge 1$), we were obligated to consider that $0<s<\frac{3}{2}$ (for more details, see the inequality (17)). Furthermore, it was necessary to suppose that $s>\frac{5}{2}-2\alpha$ to calculate the integral established in (17). Similarly, we had to admit that $s>\frac{5}{2}-2\beta$ (see (19)). Therefore, our assumptions for s in Theorem 1.1 are the following: $\max\{\frac{5}{2}-2\alpha, \frac{5}{2}-2\beta, 0\} < s < \frac{3}{2}$. Notice that these last two inequalities involving s only make sense if $\alpha > \frac{1}{2}$ and $\beta > \frac{1}{2}$.

Notice that if we observe the proof established for Theorem 1.1, in order to prove this result, by using the same technics and considering $\alpha = \frac{1}{2}$, one would have in (16) the following:

$$\|B_1((w,v),(\gamma,\phi))(t)\|_{\dot H^s_{a,\sigma}}$$

$$\le \int_0^t \Big(\int_{\mathbb{R}^3} [|\xi|^2 e^{-2\mu(t-\tau)|\xi|}]|\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}(w\otimes\gamma-\phi\otimes v)(\xi)|^2\,d\xi\Big)^{\frac{1}{2}}\,d\tau$$

$$\le C_\mu \int_0^t (t-\tau)^{-1}\|[w\otimes\gamma-\phi\otimes v](\tau)\|_{\dot H^s_{a,\sigma}}\,d\tau.$$

(See Lemma 2.6.) Then, by Lemma 2.9 in [23], one would obtain

$$\|B_1((w,v),(\gamma,\phi))(t)\|_{\dot H^s_{a,\sigma}} \le C_{a,s,\sigma,\mu}\|(w,v)\|_{L^\infty_T(\dot H^s_{a,\sigma})}\|(\gamma,\phi)\|_{L^\infty_T(\dot H^s_{a,\sigma})}\int_0^t (t-\tau)^{-1}d\tau.$$

However, this last integral is not finite. That is the reason why we have not done the case $\alpha = \frac{1}{2}$ in Theorem 1.1 so far.

• In the proof of Theorem 1.2 i) and ii), we used Lemma 2.3 (with $s_1 = \frac{5-3\alpha}{2}$, $s_2 = s + \frac{\alpha}{2}$, $s_1 < \frac{3}{2}$, $s_2 < \frac{3}{2}$, $s_1 + s_2 > 0$, $a \geq 0$ and $\sigma \geq 1$) in (24). Thus, it was assumed that $\alpha > \frac{2}{3}$ and $\alpha - \frac{5}{2} < s < \frac{3-\alpha}{2}$. Moreover, we needed to consider that $\alpha < \frac{5}{3}$ since Lemma 2.3 (with $s_1 = s_2 = \frac{5-3\alpha}{2}$, $s_1 < \frac{3}{2}$, $s_2 < \frac{3}{2}$, $s_1 + s_2 > 0$, $a \geq 0$ and $\sigma \geq 1$) was utilized in (25) as well. Therefore, by summarizing, one infers $a \geq 0, \sigma \geq 1, \frac{2}{3} < \alpha < \frac{5}{3}$ and $\alpha - \frac{5}{2} < s < \frac{3-\alpha}{2}$. In addition, Theorem 1.2 iii) was established by applying Lemma 2.5. That is the reason we supposed $a > 0, \sigma > 1$ and $\alpha - 1 \leq s < \frac{5}{2} - 2\alpha$. Besides, by analyzing (34), the reader can notice where we needed to take $\frac{1}{2} < \alpha < 1$.

Now, let us clarify why we have chosen $\alpha = \beta$ in Theorem 1.2. Thus, the absence of the case more general in our paper is due to some technical issues. More precisely, in this general situation, by following the proof of Theorem 1.2 i) and ii), we would have, for instance, the term $\|\phi \otimes v\|_{\dot{H}^{s+1-\alpha}_{a,\sigma}}$ (with $\phi, v \in L^4_T(\dot{H}^{s+\frac{\beta}{2}}_{a,\sigma}(\mathbb{R}^3)) \cap L^4_T(\dot{H}^{\frac{5-3\beta}{2}}_{a,\sigma}(\mathbb{R}^3))$) in (23). A similar difficulty was found in Theorem 1.2 iii) (see (33)). Therefore, in order to apply Lemmas 2.3 and 2.5, and by observing the proof of Theorem 1.2, we decided to consider $\alpha = \beta$.

• In Theorems 1.3 and 1.4, we supposed at first that $a > 0$, $\sigma > 1$, $\alpha \geq 1$, $\beta \geq 1$ and $-1 \leq s < \frac{3}{2}$ due to the use of Lemmas 2.2 and 2.1 i) (see, for example, (43), (44) and (46) for more details) in their proofs. Thereby, because of the hypotheses of Theorems 1.1 and 1.2, we assumed $a > 0$, $\sigma > 1$, $1 \leq \alpha = \beta < \frac{5}{3}$ and $\alpha - \frac{5}{2} < s < \frac{3-\alpha}{2}$ if $\alpha \geq \frac{3}{2}$ or $-1 \leq s < \frac{3-\alpha}{2}$ provided that $\alpha < \frac{3}{2}$ in Theorem 1.3, and also $a > 0$, $\sigma > 1$, $\alpha \geq 1, \beta \geq 1$ and $\max\{\frac{5}{2} - 2\alpha, \frac{5}{2} - 2\beta, 0\} < s < \frac{3}{2}$ in Theorem 1.4. Specifically, by applying Dominated Convergence Theorem to Theorem 1.4 iii), we would have

$$C_{\mu,\nu,\alpha,\beta}(T^* - t)^{-1} \leq \|(\widehat{u}, \widehat{b})(t)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}} + \|(\widehat{u}, \widehat{b})(t)\|_{L^1}^{\frac{2\beta}{2\beta-1}}, \quad \forall t \in [0, T^*).$$

(Compare with (65).) Thus, by following a analogous process as in the proof of Theorem 1.4 iii), one would deduce

$$\frac{C_{\mu,\nu,s,\alpha,\beta,u_0,b_0}}{(T^* - t)^{\frac{2s}{3}}} \frac{\left(\frac{a^{\zeta} C'_{\mu,\nu,\sigma,s,\alpha,\beta,u_0,b_0}}{(T^*-t)^{\frac{1}{3\sigma}}}\right)^k}{k!} \leq \left[\int_{\mathbb{R}^3} \frac{(2a|\xi|^{\frac{1}{\sigma}})^k}{k!} |\xi|^{2s} |(\widehat{u}, \widehat{b})(t)|^2 \, d\xi\right]^{\frac{\alpha}{2\alpha-1}} + \left[\int_{\mathbb{R}^3} \frac{(2a|\xi|^{\frac{1}{\sigma}})^k}{k!} |\xi|^{2s} |(\widehat{u}, \widehat{b})(t)|^2 \, d\xi\right]^{\frac{\beta}{2\beta-1}},$$

for all $t \in [0, T^*)$ and $k \geq 2\sigma(2\alpha - 1)$ in $\mathbb{N}$ (see (68)). Here $\zeta = \max\{\frac{\alpha}{2\alpha-1}, \frac{\beta}{2\beta-1}\}$ if $a \leq 1$, and $\zeta = \min\{\frac{\alpha}{2\alpha-1}, \frac{\beta}{2\beta-1}\}$ if $a > 1$. Thereby, we decided to consider $\alpha = \beta$ in order to sum over the set $\{k \in \mathbb{N}; k \geq 2\sigma(2\alpha - 1)\}$ and proceed with the proof of Theorem 1.4 iii) as it is done in this paper. Moreover, Theorem 1.3 assumes $\alpha = \beta$ because of Theorem 1.2.

• The solution obtained in Theorem 1.2 ii) was studied in Theorem 1.5; as a result, all the conditions given for the value of a, σ, s, α and β in Theorem 1.2 also must hold in Theorem 1.5. Thus, it was supposed that $a \geq 0$, $\sigma \geq 1$, $\frac{2}{3} < \alpha = \beta < \frac{5}{3}$, and $\alpha - \frac{5}{2} < s < \frac{3-\alpha}{2}$ in this last result.

Remark 1.7 It is important to point out that our results consider the usual Laplacian operator as well (it is enough to take $\alpha = \beta = 1$). In addition, Theorems 1.2, 1.3, and 1.5 determine new information even in the critical case $s = \frac{5}{2} - 2\alpha$. Moreover, our main results still hold for the MHD and Navier-Stokes equations. Lastly, let us emphasize that Theorems 1.1, 1.2, and 1.5 take into account the usual Sobolev spaces (by assuming $a = 0$) (see [1–8, 11–29] and references included).

2 Notations, Definitions and Preliminary Results

This section presents the most important notations, definitions and lemmas that will play important roles throughout our paper.

Primordial notations and definitions:

- Fractional Laplacian[1] $(-\Delta)^\alpha$, $\alpha \in \mathbb{R}$, is defined by $\mathcal{F}[(-\Delta)^\alpha f](\xi) = |\xi|^{2\alpha} \hat{f}(\xi)$.
- Let $a \geq 0, \sigma \geq 1$ and $s \in \mathbb{R}$. Sobolev-Gevrey space is defined by $\dot{H}^s_{a,\sigma} = \{f \in S' : \widehat{f} \in L^1_{loc}, \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi)|^2 \, d\xi < \infty\}$. $\dot{H}^s_{a,\sigma}$-norm and $\dot{H}^s_{a,\sigma}$-inner product are given, respectively, by

$$\|f\|_{\dot{H}^s_{a,\sigma}} := \Big(\int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi)|^2 \, d\xi \Big)^{\frac{1}{2}}, \text{ and}$$

$$\langle f, g \rangle_{\dot{H}^s_{a,\sigma}} := \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} \widehat{f}(\xi) \cdot \widehat{g}(\xi) \, d\xi,$$

- Let $a \geq 0$, $\sigma \geq 1$, $s \in \mathbb{R}$, and $\alpha \in \mathbb{R}$. We consider the spaces $\dot{H}^{s,\alpha}_{a,\sigma} := \dot{H}^s_{a,\sigma} \cap \dot{H}^{\frac{5}{2}-2\alpha}_{a,\sigma}$ which are endowed with the norms

$$\|f\|_{\dot{H}^{s,\alpha}_{a,\sigma}} := [\|f\|^2_{\dot{H}^s_{a,\sigma}} + \|f\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\sigma}}]^{\frac{1}{2}}.$$

- Let $T > 0$. Assuming that $(X, \|\cdot\|_X)$ is a normed space and $I \subseteq \mathbb{R}$ is an interval, we define $C(I; X) = \{f : I \to X \text{ continuous function}\}$. $C(I; X)$-norm is given by

$$\|f\|_{L^\infty(I;X)} := \sup_{t \in I} \{\|f(t)\|_X\}.$$

  We denote $C_T(X) = C([0, T]; X)$ and $\|\cdot\|_{L^\infty_T(X)} = \|\cdot\|_{L^\infty([0,T];X)}$.
- Let $1 \leq p < \infty$ and $T > 0$. Assuming that $(X, \|\cdot\|_X)$ is a normed space and $I \subseteq \mathbb{R}$ is an interval, we define $L^p(I; X) = \{f : I \to X \text{ mensurable function: } \int_I \|f(t)\|^p_X \, dt < \infty\}$. $L^p(I; X)$-norm is given by

$$\|f\|_{L^p(I;X)} := \Big(\int_I \|f(t)\|^p_X \, dt \Big)^{\frac{1}{p}}.$$

  We denote $L^p_T(X) = L^p([0, T]; X)$.
- Constants that appear in this paper may change their values from line to line without change of notation. Thus, C_q will denote constants that depend on q, for example. Here C is always an absolute positive constant. Other notations and definitions are given in footnotes throughout the paper.

[1] Here S' is the space of tempered distributions, $\mathcal{F}(f)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^3} e^{-i\xi \cdot x} f(x) \, dx$, and $\mathcal{F}^{-1}(g)(x) := (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\xi \cdot x} g(\xi) \, d\xi$.

Auxiliary lemmas:

The lemma below establishes interpolation inequalities related to Sobolev-Gevrey (Sobolev) spaces.

Lemma 2.1 *Let $a \geq 0$, $\sigma \geq 1$, $s \in \mathbb{R}$, $\varpi \in \mathbb{R}$, $p \geq 2$, and $\theta \geq 1$. The following inequalities hold*:

i) $\|f\|_{\dot{H}^{s+1}_{a,\sigma}} \leq \|f\|^{1-\frac{1}{\theta}}_{\dot{H}^{s}_{a,\sigma}} \|f\|^{\frac{1}{\theta}}_{\dot{H}^{s+\theta}_{a,\sigma}}$;

ii) $\|f\|_{\dot{H}^{s+\frac{2\varpi}{p}}_{a,\sigma}} \leq \|f\|^{1-\frac{2}{p}}_{\dot{H}^{s}_{a,\sigma}} \|f\|^{\frac{2}{p}}_{\dot{H}^{s+\varpi}_{a,\sigma}}$.

Proof Note that Hölder's inequality implies that

$$\begin{aligned}\|f\|^2_{\dot{H}^{s+1}_{a,\sigma}} &\leq \left(\int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi)|^2\, d\xi\right)^{1-\frac{1}{\theta}} \left(\int_{\mathbb{R}^3} |\xi|^{2s+2\theta} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi)|^2\, d\xi\right)^{\frac{1}{\theta}} \\ &= \|f\|^{2(1-\frac{1}{\theta})}_{\dot{H}^{s}_{a,\sigma}} \|f\|^{\frac{2}{\theta}}_{\dot{H}^{s+\theta}_{a,\sigma}}.\end{aligned}$$

This proves i). Similarly, by Hölder's inequality once more, we have

$$\begin{aligned}\|f\|^2_{\dot{H}^{s+\frac{2\varpi}{p}}_{a,\sigma}} &\leq \left(\int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi)|^2\, d\xi\right)^{1-\frac{2}{p}} \left(\int_{\mathbb{R}^3} |\xi|^{2s+2\varpi} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi)|^2\, d\xi\right)^{\frac{2}{p}} \\ &= \|f\|^{2(1-\frac{2}{p})}_{\dot{H}^{s}_{a,\sigma}} \|f\|^{\frac{4}{p}}_{\dot{H}^{s+\varpi}_{a,\sigma}}.\end{aligned}$$

This proves ii). □

The lemma below is the key point in the proofs of Theorems 1.3 and 1.4.

Lemma 2.2 *Let $a > 0$, $\sigma > 1$, $-1 \leq s < \frac{3}{2}$, $\alpha \geq 1$, and $\beta \geq 1$. For every $f \in \dot{H}^{s}_{a,\sigma} \cap \dot{H}^{s+\alpha}_{a,\sigma}$ and $g \in \dot{H}^{s}_{a,\sigma} \cap \dot{H}^{s+\beta}_{a,\sigma}$, we have that $fg \in \dot{H}^{s+1}_{a,\sigma}$. More precisely, we have*

$$\|fg\|_{\dot{H}^{s+1}_{a,\sigma}} \leq C_{a,\sigma,s}[\|f\|_{\dot{H}^{s}_{a,\sigma}} \|g\|^{1-\frac{1}{\beta}}_{\dot{H}^{s}_{a,\sigma}} \|g\|^{\frac{1}{\beta}}_{\dot{H}^{s+\beta}_{a,\sigma}} + \|g\|_{\dot{H}^{s}_{a,\sigma}} \|f\|^{1-\frac{1}{\alpha}}_{\dot{H}^{s}_{a,\sigma}} \|f\|^{\frac{1}{\alpha}}_{\dot{H}^{s+\alpha}_{a,\sigma}}].$$

Proof By Cauchy-Schwarz's inequality, one infers

$$\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{f}\|_{L^1} \leq \left(\int_{\mathbb{R}^3} |\xi|^{-2s} e^{2(\frac{a}{\sigma}-a)|\xi|^{\frac{1}{\sigma}}}\, d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}|^2\, d\xi\right)^{\frac{1}{2}} =: C_{a,\sigma,s}\|f\|_{\dot{H}^{s}_{a,\sigma}}, \tag{4}$$

where $C^2_{a,\sigma,s} = \frac{4\pi\sigma\Gamma(\sigma(3-2s))}{[2(a-\frac{a}{\sigma})]^{\sigma(3-2s)}}$ (Γ is the standard gamma function). Consequently, by applying Lemma 2.9 in [23], (4) and Lemma 2.1, we deduce

$$\begin{aligned}\|fg\|_{\dot{H}^{s+1}_{a,\sigma}} &\leq C_s[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{f}\|_{L^1} \|g\|_{\dot{H}^{s+1}_{a,\sigma}} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{g}\|_{L^1} \|f\|_{\dot{H}^{s+1}_{a,\sigma}}] \\ &\leq C_{a,\sigma,s}[\|f\|_{\dot{H}^{s}_{a,\sigma}} \|g\|_{\dot{H}^{s+1}_{a,\sigma}} + \|g\|_{\dot{H}^{s}_{a,\sigma}} \|f\|_{\dot{H}^{s+1}_{a,\sigma}}] \\ &\leq C_{a,\sigma,s}[\|f\|_{\dot{H}^{s}_{a,\sigma}} \|g\|^{1-\frac{1}{\beta}}_{\dot{H}^{s}_{a,\sigma}} \|g\|^{\frac{1}{\beta}}_{\dot{H}^{s+\beta}_{a,\sigma}} + \|g\|_{\dot{H}^{s}_{a,\sigma}} \|f\|^{1-\frac{1}{\alpha}}_{\dot{H}^{s}_{a,\sigma}} \|f\|^{\frac{1}{\alpha}}_{\dot{H}^{s+\alpha}_{a,\sigma}}].\end{aligned}$$

This completes the proof. □

Another important product estimate is added as follows.

Lemma 2.3 (see [5, 10]) *Let $a \geq 0, \sigma \geq 1$ and $(s_1, s_2) \in \mathbb{R}^2$ such that $s_1 < \frac{3}{2}$ and $s_1 + s_2 > 0$. Then, there exists a positive constant C_{s_1,s_2} such that, for all $f, g \in \dot{H}^{s_1}_{a,\sigma}(\mathbb{R}^3) \cap \dot{H}^{s_2}_{a,\sigma}(\mathbb{R}^3)$, we have*

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)} \leq C_{s_1,s_2}\Big(\|f\|_{\dot{H}^{s_1}_{a,\sigma}(\mathbb{R}^3)}\|g\|_{\dot{H}^{s_2}_{a,\sigma}(\mathbb{R}^3)} + \|f\|_{\dot{H}^{s_2}_{a,\sigma}(\mathbb{R}^3)}\|g\|_{\dot{H}^{s_1}_{a,\sigma}(\mathbb{R}^3)}\Big).$$

If $s_1 < \frac{3}{2}$, $s_2 < \frac{3}{2}$ and $s_1 + s_2 > 0$, then there is a positive constant C_{s_1,s_2} such that

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)} \leq C_{s_1,s_2}\|f\|_{\dot{H}^{s_1}_{a,\sigma}(\mathbb{R}^3)}\|g\|_{\dot{H}^{s_2}_{a,\sigma}(\mathbb{R}^3)}.$$

Proof For details, see Lemma 2.2 in [5] and Lemma 2.8 in [10]. □

Motivated by H. Orf [26], the next result will be useful in the proof of Theorem 1.2.

Lemma 2.4 *Let $a \geq 0, \sigma \geq 1, \theta \in \mathbb{R}, \omega > 0, T > 0$ and $s \in \mathbb{R}$. Assume that $f \in L^2_T(\dot{H}^{s-\theta}_{a,\sigma})$ and $v_0 \in \dot{H}^s_{a,\sigma}$. Consider that $v \in C_T(S')$ solves the system*

$$\begin{cases} v_t + \omega(-\Delta)^\theta v = f; \\ v(\cdot, 0) = v_0. \end{cases} \tag{5}$$

Then, we have $v \in C_T(\dot{H}^s_{a,\sigma}) \cap L^p_T(\dot{H}^{s+\frac{2\theta}{p}}_{a,\sigma})$, for all $p \geq 2$. Moreover, we obtain

i) $\left(\int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} \sup_{0\leq\tau\leq t}\{|\widehat{v}(\tau)|^2\}\, d\xi\right)^{\frac{1}{2}} \leq \|v_0\|_{\dot{H}^s_{a,\sigma}} + \frac{1}{\sqrt{2\omega}}\|f\|_{L^2_t(\dot{H}^{s-\theta}_{a,\sigma})}$, *for all $t \in [0, T]$;*

ii) $\|v\|_{L^p_T(\dot{H}^{s+\frac{2\theta}{p}}_{a,\sigma})} \leq [\|v_0\|_{\dot{H}^s_{a,\sigma}} + \frac{1}{\sqrt{2\omega}}\|f\|_{L^2_T(\dot{H}^{s-\theta}_{a,\sigma})}]^{1-\frac{2}{p}}[\frac{1}{\sqrt{2\omega}}\|v_0\|_{\dot{H}^s_{a,\sigma}} + \frac{1}{\omega}\|f\|_{L^2_T(\dot{H}^{s-\theta}_{a,\sigma})}]^{\frac{2}{p}}$.

Proof By applying $e^{-\omega(\tau-\varrho)(-\Delta)^\theta}$ (with $0 \leq \varrho \leq \tau \leq t$), Fourier transform to the system (5), and integrating over $[0, \tau]$ the result obtained, we deduce that

$$|\widehat{v}(\tau)| \leq e^{-\omega\tau|\xi|^{2\theta}}|\widehat{v}_0| + \int_0^\tau e^{-\omega(\tau-\varrho)|\xi|^{2\theta}}|\widehat{f}(\varrho)|\, d\varrho \leq |\widehat{v}_0| + \frac{1}{\sqrt{2\omega}}|\xi|^{-\theta}\|\widehat{f}(\xi)\|_{L^2([0,t])},$$
$$\forall\tau \in [0, t], \tag{6}$$

where we have used Cauchy-Schwarz's inequality. Now, multiply the inequality above by $|\xi|^s e^{a|\xi|^{\frac{1}{\sigma}}}$ and take $L^2(\mathbb{R}^3)$-norm to get

$$\left(\int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} \sup_{0\leq\tau\leq t}\{|\widehat{v}(\tau)|^2\}\, d\xi\right)^{\frac{1}{2}} \leq \|v_0\|_{\dot{H}^s_{a,\sigma}} + \frac{1}{\sqrt{2\omega}}\|f\|_{L^2_t(\dot{H}^{s-\theta}_{a,\sigma})}, \quad \forall t \in [0, T]. \tag{7}$$

This proves i). Moreover, since $v \in C_T(S')$, then by applying Dominated Convergence Theorem and (7) we can conclude that $v \in C_T(\dot{H}^s_{a,\sigma})$.

On the other hand, by multiplying (6) (with $\tau = t$) by $|\xi|^{s+\theta}e^{a|\xi|^{\frac{1}{\sigma}}}$, taking $L^2([0,T])$-norm, and using Young's inequality, we deduce

$$
\begin{aligned}
&\left(\int_0^T |\xi|^{2(s+\theta)}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{v}(t)|^2\,dt\right)^{\frac{1}{2}}\\
&\quad\le \frac{1}{\sqrt{2\omega}}|\xi|^s e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{v}_0| + |\xi|^{s+\theta}e^{a|\xi|^{\frac{1}{\sigma}}}\|[e^{-\omega t|\xi|^{2\theta}}]*[|\widehat{f}(t)|]\|_{L^2([0,T])}\\
&\quad\le \frac{1}{\sqrt{2\omega}}|\xi|^s e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{v}_0| + \frac{1}{\omega}|\xi|^{s-\theta}e^{a|\xi|^{\frac{1}{\sigma}}}\left(\int_0^T |\widehat{f}(t)|^2\,dt\right)^{\frac{1}{2}}].
\end{aligned}
$$

Apply $L^2(\mathbb{R}^3)$-norm in order to obtain

$$
\|v\|_{L^2_T(\dot{H}^{s+\theta}_{a,\sigma})} \le \frac{1}{\sqrt{2\omega}}\|v_0\|_{\dot{H}^s_{a,\sigma}} + \frac{1}{\omega}\|f\|_{L^2_T(\dot{H}^{s-\theta}_{a,\sigma})}. \tag{8}
$$

Therefore, by using Lemma 2.1, (7) and (8), it yields that

$$
\begin{aligned}
\|v\|^p_{L^p_T(\dot{H}^{s+\frac{2\theta}{p}}_{a,\sigma})} &\le \int_0^T \|v(t)\|^{p-2}_{\dot{H}^s_{a,\sigma}}\|v(t)\|^2_{\dot{H}^{s+\theta}_{a,\sigma}}\,dt\\
&\le \left(\int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}\sup_{0\le\tau\le T}\{|\widehat{v}(\tau)|^2\}\,d\xi\right)^{\frac{p}{2}-1}\|v\|^2_{L^2_T(\dot{H}^{s+\theta}_{a,\sigma})}\\
&\le [\|v_0\|_{\dot{H}^s_{a,\sigma}} + \frac{1}{\sqrt{2\omega}}\|f\|_{L^2_T(\dot{H}^{s-\theta}_{a,\sigma})}]^{p-2}[\frac{1}{\sqrt{2\omega}}\|v_0\|_{\dot{H}^s_{a,\sigma}} + \frac{1}{\omega}\|f\|_{L^2_T(\dot{H}^{s-\theta}_{a,\sigma})}]^2.
\end{aligned}
$$

This proves ii). As a result, we can conclude that $v \in L^p_T(\dot{H}^{s+\frac{2\theta}{p}}_{a,\sigma})$, for all $p \ge 2$, since $f \in L^2_T(\dot{H}^{s-\theta}_{a,\sigma})$ and $v_0 \in \dot{H}^s_{a,\sigma}$. □

The next lemma will be useful in the proof of Theorem 1.2 iii).

Lemma 2.5 *Let $a > 0$, $\sigma > 1$, $\alpha < \frac{7}{6}$, $\alpha - 1 \le s < \frac{5}{2} - 2\alpha$. If $f, g \in \dot{H}^{s+1-\alpha}_{a,\sigma}(\mathbb{R}^3) \cap \dot{H}^{s+2\alpha-1}_{a,\sigma}(\mathbb{R}^3)$, then there exists a positive constant $C_{a,\sigma,\alpha,s}$ such that*

$$
\|fg\|_{\dot{H}^{s+1-\alpha}_{a,\sigma}(\mathbb{R}^3)} \le C_{a,\sigma,\alpha,s}[\|f\|_{\dot{H}^{s+1-\alpha}_{a,\sigma}(\mathbb{R}^3)}\|g\|_{\dot{H}^{s+2\alpha-1}_{a,\sigma}(\mathbb{R}^3)} + \|f\|_{\dot{H}^{s+2\alpha-1}_{a,\sigma}(\mathbb{R}^3)}\|g\|_{\dot{H}^{s+1-\alpha}_{a,\sigma}(\mathbb{R}^3)}].
$$

Proof Notice that, by applying Lemma 2.9 in [23], we deduce that

$$
\|fg\|_{\dot{H}^{s+1-\alpha}_{a,\sigma}(\mathbb{R}^3)} \le C_{s,\alpha}[\|f\|_{\dot{H}^{s+1-\alpha}_{a,\sigma}(\mathbb{R}^3)}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|_{L^1(\mathbb{R}^3)} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{f}\|_{L^1(\mathbb{R}^3)}\|g\|_{\dot{H}^{s+1-\alpha}_{a,\sigma}(\mathbb{R}^3)}]. \tag{9}
$$

By Cauchy-Schwarz's inequality, we have

$$
\begin{aligned}
\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|^2_{L^1(\mathbb{R}^3)} &\le \left(\int_{\mathbb{R}^3}|\xi|^{2(1-s-2\alpha)}e^{2(\frac{a}{\sigma}-a)|\xi|^{\frac{1}{\sigma}}}\,d\xi\right)\left(\int_{\mathbb{R}^3}|\xi|^{2(s+2\alpha-1)}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{g}(\xi)|^2\,d\xi\right)\\
&= \frac{4\pi\sigma\Gamma(\sigma(5-2s-4\alpha))}{[2(a-\frac{a}{\sigma})]^{\sigma(5-2s-4\alpha)}}\|g\|^2_{\dot{H}^{s+2\alpha-1}_{a,\sigma}(\mathbb{R}^3)},
\end{aligned} \tag{10}
$$

and, consequently, the desired result follows from (9) and (10). □

Finally, we present a result which comes from Calculus.

Lemma 2.6 (see [23]) *Let $a, b > 0$. Then, $\lambda^a e^{-b\lambda} \leq a^a (eb)^{-a}$ for all $\lambda > 0$.*

Proof For details, see Lemma 2.10 in [23] (and references therein). □

As follows, we shall prove our main results.

3 Proof of Theorem 1.1

The proof of Theorem 1.1 was motivated by [23]. Thus, by applying $e^{-\mu(t-\tau)(-\Delta)^\alpha}$ (with $\tau \in [0, t]$) to the first equation in (1), and integrating the result obtained over $[0, t]$, we deduce[2]

$$u(t) = e^{-\mu t(-\Delta)^\alpha} u_0 - \int_0^t e^{-\mu(t-\tau)(-\Delta)^\alpha} P(u \cdot \nabla u - b \cdot \nabla b)\, d\tau.$$

It is known that this operator satisfies

$$\mathcal{F}[P(f)](\xi) = \widehat{f}(\xi) - \frac{\widehat{f}(\xi) \cdot \xi}{|\xi|^2} \xi, \quad \forall \xi \in \mathbb{R}^3. \tag{11}$$

Analogously, apply $e^{-\nu(t-\tau)(-\Delta)^\beta}$ (with $\tau \in [0, t]$) to the second equation in (1), and integrate the result obtained to get

$$b(t) = e^{-\nu t(-\Delta)^\beta} b_0 - \int_0^t e^{-\nu(t-\tau)(-\Delta)^\beta} [u \cdot \nabla b - b \cdot \nabla u]\, d\tau.$$

Therefore, one has

$$(u, b)(t) = (e^{-\mu t(-\Delta)^\alpha} u_0, e^{-\nu t(-\Delta)^\beta} b_0) + B((u, b), (u, b))(t), \tag{12}$$

where

$$B((w, v), (\gamma, \phi))(t) = (B_1((w, v), (\gamma, \phi))(t), B_2((w, v), (\gamma, \phi))(t)) \tag{13}$$

with

$$B_1((w, v), (\gamma, \phi))(t) = -\int_0^t e^{-\mu(t-\tau)(-\Delta)^\alpha} P(\gamma \cdot \nabla w - v \cdot \nabla \phi)\, d\tau \tag{14}$$

and also

$$B_2((w, v), (\gamma, \phi))(t) = -\int_0^t e^{-\nu(t-\tau)(-\Delta)^\beta} [w \cdot \nabla \phi - v \cdot \nabla \gamma]\, d\tau, \tag{15}$$

for all $w, v, \gamma, \phi \in C_T(\dot{H}^s_{a,\sigma})$ ($T > 0$ will be determined as follows). It is easy to check that $B : C_T(\dot{H}^s_{a,\sigma}) \times C_T(\dot{H}^s_{a,\sigma}) \to C_T(\dot{H}^s_{a,\sigma})$ is a bilinear operator. Thus, our imminent goal is to prove that this operator is also continuous.

[2] P is Helmholtz's projector (see [22]).

As a consequence of (11) and Lemma 2.6, we deduce[3]

$$
\begin{aligned}
&\|B_1((w,v),(\gamma,\phi))(t)\|_{\dot{H}^s_{a,\sigma}}\\
&\leq \int_0^t \Big(\int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^{2\alpha}}|\xi|^{2s+2}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(w\otimes\gamma-\phi\otimes v)(\xi)|^2\,d\xi\Big)^{\frac12}d\tau\\
&=\int_0^t \Big(\int_{\mathbb{R}^3} [(|\xi|^{2\alpha})^{\frac{5-2s}{2\alpha}}e^{-2\mu(t-\tau)|\xi|^{2\alpha}}]|\xi|^{4s-3}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(w\otimes\gamma-\phi\otimes v)(\xi)|^2\,d\xi\Big)^{\frac12}d\tau\\
&\leq C_{s,\mu,\alpha}\int_0^t (t-\tau)^{-\frac{5-2s}{4\alpha}}\Big(\int_{\mathbb{R}^3}|\xi|^{4s-3}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(w\otimes\gamma-\phi\otimes v)(\xi)|^2\,d\xi\Big)^{\frac12}d\tau\\
&\leq C_{s,\mu,\alpha}\int_0^t (t-\tau)^{-\frac{5-2s}{4\alpha}}\|[w\otimes\gamma-\phi\otimes v](\tau)\|_{\dot{H}^{2s-\frac32}_{a,\sigma}}\,d\tau. \qquad (16)
\end{aligned}
$$

By Lemma 2.3, it follows that

$$
\|B_1((w,v),(\gamma,\phi))(t)\|_{\dot{H}^s_{a,\sigma}} \leq C_{s,\mu,\alpha}\|(w,v)\|_{L^\infty_T(\dot{H}^s_{a,\sigma})}\|(\gamma,\phi)\|_{L^\infty_T(\dot{H}^s_{a,\sigma})}\int_0^t (t-\tau)^{-\frac{5-2s}{4\alpha}}d\tau \qquad (17)
$$

$$
\leq C_{s,\mu,\alpha}T^{\frac{2s+4\alpha-5}{4\alpha}}\|(w,v)\|_{L^\infty_T(\dot{H}^s_{a,\sigma})}\|(\gamma,\phi)\|_{L^\infty_T(\dot{H}^s_{a,\sigma})}, \qquad (18)
$$

for all $t\in[0,T]$. Analogously, we can write

$$
\|B_2((w,v),(\gamma,\phi))(t)\|_{\dot{H}^s_{a,\sigma}} \leq C_{s,\nu,\beta}T^{\frac{2s+4\beta-5}{4\beta}}\|(w,v)\|_{L^\infty_T(\dot{H}^s_{a,\sigma})}\|(\gamma,\phi)\|_{L^\infty_T(\dot{H}^s_{a,\sigma})}, \qquad (19)
$$

for all $t\in[0,T]$. By (13), (18) and (19), one infers

$$
\begin{aligned}
&\|B((w,v),(\gamma,\phi))(t)\|_{\dot{H}^s_{a,\sigma}}\\
&\leq C_{s,\mu,\nu,\alpha,\beta}[T^{\frac{2s+4\alpha-5}{4\alpha}}+T^{\frac{2s+4\beta-5}{4\beta}}]\|(w,v)\|_{L^\infty_T(\dot{H}^s_{a,\sigma})}\|(\gamma,\phi)\|_{L^\infty_T(\dot{H}^s_{a,\sigma})}, \qquad (20)
\end{aligned}
$$

for all $t\in[0,T]$.

On the other hand, it is easy to check that

$$
\|e^{-\mu t(-\Delta)^\alpha}u_0\|^2_{\dot{H}^s_{a,\sigma}} = \int_{\mathbb{R}^3} e^{-2\mu t|\xi|^{2\alpha}}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{u}_0(\xi)|^2\,d\xi \leq \|u_0\|^2_{\dot{H}^s_{a,\sigma}}, \quad \forall t\in[0,T]. \qquad (21)
$$

Similarly, we deduce

$$
\|e^{-\nu t(-\Delta)^\beta}b_0\|_{\dot{H}^s_{a,\sigma}} \leq \|b_0\|_{\dot{H}^s_{a,\sigma}}, \quad \forall t\in[0,T].
$$

Therefore, we have shown that

$$
\|(e^{-\mu t(-\Delta)^\alpha}u_0, e^{-\nu t(-\Delta)^\beta}b_0)\|_{\dot{H}^s_{a,\sigma}} \leq \|(u_0,b_0)\|_{\dot{H}^s_{a,\sigma}}, \quad \forall t\in[0,T].
$$

Choose $0<T<\min\{1,[8C_{s,\mu,\nu,\alpha,\beta}\|(u_0,b_0)\|_{\dot{H}^s_{a,\sigma}}]^{-[\min\{\frac{2s+4\alpha-5}{4\alpha},\frac{2s+4\beta-5}{4\beta}\}]^{-1}}\}$, where $C_{s,\mu,\nu,\alpha,\beta}$ is given in (20), in order to obtain, by Lemma 2.1 in [25] (for more details, see [9]), a unique solution $(u,b)\in C_T(\dot{H}^s_{a,\sigma})$ for the equation (12). □

[3] The tensor product is given by $f\otimes g := (g_1 f, g_2 f, g_3 f)$.

4 Proof of Theorem 1.2

The proof of Theorem 1.2 was motivated by [23, 26]. Thus, by observing (13), (14) and (15) (with $\alpha=\beta$), we have

$$\begin{cases}\partial_t B_1+\mu(-\Delta)^\alpha B_1=-P(\gamma\cdot\nabla w-v\cdot\nabla\phi);\\ B_1(0)=0,\end{cases}$$
$$\begin{cases}\partial_t B_2+\nu(-\Delta)^\alpha B_2=-[w\cdot\nabla\phi-v\cdot\nabla\gamma];\\ B_2(0)=0.\end{cases}\tag{22}$$

Now, assume that $T>0$. Thus, by using (11), Lemma 2.3 and Hölder's inequality, one obtains[4]

$$\|P(\gamma\cdot\nabla w-v\cdot\nabla\phi)\|^2_{L^2_T(\dot H^{s-\alpha}_{a,\sigma})}\le\int_0^T[\|w\otimes\gamma\|_{\dot H^{s+1-\alpha}_{a,\sigma}}+\|\phi\otimes v\|_{\dot H^{s+1-\alpha}_{a,\sigma}}]^2\,dt\tag{23}$$
$$\le C_{s,\alpha}\int_0^T[\|w\|^2_{\dot H^{\frac{5-3\alpha}{2}}_{a,\sigma}}\|\gamma\|^2_{\dot H^{s+\frac{\alpha}{2}}_{a,\sigma}}+\|\phi\|^2_{\dot H^{\frac{5-3\alpha}{2}}_{a,\sigma}}\|v\|^2_{\dot H^{s+\frac{\alpha}{2}}_{a,\sigma}}]\,dt\tag{24}$$
$$\le C_{s,\alpha}[\|w\|^2_{L^4_T(\dot H^{\frac{5-3\alpha}{2}}_{a,\sigma})}\|\gamma\|^2_{L^4_T(\dot H^{s+\frac{\alpha}{2}}_{a,\sigma})}+\|\phi\|^2_{L^4_T(\dot H^{\frac{5-3\alpha}{2}}_{a,\sigma})}\|v\|^2_{L^4_T(\dot H^{s+\frac{\alpha}{2}}_{a,\sigma})}]$$
$$\le C_{s,\alpha}\|(w,v)\|^2_{Z_T}\|(\gamma,\phi)\|^2_{Z_T}.$$

Analogously, we can write

$$\|w\cdot\nabla\phi-v\cdot\nabla\gamma\|_{L^2_T(\dot H^{s-\alpha}_{a,\sigma})}\le C_{s,\alpha}\|(w,v)\|_{Z_T}\|(\gamma,\phi)\|_{Z_T}$$

and also

$$\|P(\gamma\cdot\nabla w-v\cdot\nabla\phi)\|_{L^2_T(\dot H^{\frac{5}{2}-3\alpha}_{a,\sigma})},\ \|w\cdot\nabla\phi-v\cdot\nabla\gamma\|_{L^2_T(\dot H^{\frac{5}{2}-3\alpha}_{a,\sigma})}\le C_\alpha\|(w,v)\|_{Z_T}\|(\gamma,\phi)\|_{Z_T},\tag{25}$$

for all $(w,v),(\gamma,\phi)\in Z_T$. As a consequence, one has[5]

$$\|P(\gamma\cdot\nabla w-v\cdot\nabla\phi)\|_{Y_T},\ \|w\cdot\nabla\phi-v\cdot\nabla\gamma\|_{Y_T}\le C_{s,\alpha}\|(w,v)\|_{Z_T}\|(\gamma,\phi)\|_{Z_T},$$
$$\forall(w,v),(\gamma,\phi)\in Z_T.\tag{26}$$

[4]Let $Z_T=L^4_T(\dot H^{s+\frac{\alpha}{2}}_{a,\sigma})\cap L^4_T(\dot H^{\frac{5-3\alpha}{2}}_{a,\sigma})$, where $\|\cdot\|^4_{Z_T}=\|\cdot\|^4_{L^4_T(\dot H^{s+\frac{\alpha}{2}}_{a,\sigma})}+\|\cdot\|^4_{L^4_T(\dot H^{\frac{5-3\alpha}{2}}_{a,\sigma})}$. Moreover, $\|(\cdot,\star)\|^4_{L^4(I;X)}=\|\cdot\|^4_{L^4(I;X)}+\|\star\|^4_{L^4(I;X)}$ and $\|(\cdot,\star)\|^4_{Z_T}=\|\cdot\|^4_{Z_T}+\|\star\|^4_{Z_T}$.

[5]Denote $Y_T=L^2_T(\dot H^{s-\alpha}_{a,\sigma})\cap L^2_T(\dot H^{\frac{5}{2}-3\alpha}_{a,\sigma})$, where $\|\cdot\|^2_{Y_T}=\|\cdot\|^2_{L^2_T(\dot H^{s-\alpha}_{a,\sigma})}+\|\cdot\|^2_{L^2_T(\dot H^{\frac{5}{2}-3\alpha}_{a,\sigma})}$. Moreover, $\|(\cdot,\star)\|^2_{L^2(I;X)}=\|\cdot\|^2_{L^2(I;X)}+\|\star\|^2_{L^2(I;X)}$ and $\|(\cdot,\star)\|^2_{Y_T}=\|\cdot\|^2_{Y_T}+\|\star\|^2_{Y_T}$.

By applying Lemma 2.4 ii) (with $p=4$) to the systems in (22) and using (26), we deduce

$$\|B((w,v),(\gamma,\phi))\|_{Z_T} \leq C_{s,\alpha}\varsigma^{-\frac{3}{4}}\|(w,v)\|_{Z_T}\|(\gamma,\phi)\|_{Z_T}, \quad \forall (w,v),(\gamma,\phi)\in Z_T, \tag{27}$$

where $\varsigma=\min\{\mu,\nu\}$. On the other hand, it is easy to check that

$$\begin{aligned}\|e^{-\mu t(-\Delta)^\alpha}u_0\|_{L^4_T(\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma})} &\leq \Big(\int_{\mathbb{R}^3}|\xi|^{2s+\alpha}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{u}_0(\xi)|^2\Big(\int_0^T e^{-4\mu t|\xi|^{2\alpha}}\,dt\Big)^{\frac{1}{2}}d\xi\Big)^{\frac{1}{2}}\\ &\leq C\mu^{-\frac{1}{4}}\|u_0\|_{\dot{H}^{s,\alpha}_{a,\sigma}}.\end{aligned} \tag{28}$$

Similarly, we obtain

$$\|e^{-\mu t(-\Delta)^\alpha}u_0\|_{L^4_T(\dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma})} \leq C\mu^{-\frac{1}{4}}\|u_0\|_{\dot{H}^{s,\alpha}_{a,\sigma}}$$

$$\|e^{-\nu t(-\Delta)^\alpha}b_0\|_{L^4_T(\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma})} \text{ and } \|e^{-\nu t(-\Delta)^\alpha}b_0\|_{L^4_T(\dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma})} \leq C\nu^{-\frac{1}{4}}\|b_0\|_{\dot{H}^{s,\alpha}_{a,\sigma}},$$

and also

$$\|e^{-\mu t(-\Delta)^\alpha}u_0\|_{Z_T} \leq C\mu^{-\frac{1}{4}}\|u_0\|_{\dot{H}^{s,\alpha}_{a,\sigma}}, \text{ and } \|e^{-\nu t(-\Delta)^\alpha}b_0\|_{Z_T} \leq C\nu^{-\frac{1}{4}}\|b_0\|_{\dot{H}^{s,\alpha}_{a,\sigma}}.$$

As a consequence, we have

$$\|(e^{-\mu t(-\Delta)^\alpha}u_0, e^{-\nu t(-\Delta)^\alpha}b_0)\|_{Z_T} \leq C\varsigma^{-\frac{1}{4}}\|(u_0,b_0)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}. \tag{29}$$

Assume that $\|(u_0,b_0)\|_{\dot{H}^{s,\alpha}_{a,\sigma}} < \dfrac{\varsigma}{4CC_{s,\alpha}}$ (where C and $C_{s,\alpha}$ are given in (29) and (27), respectively) in order to get, by Lemma 2.1 in [25] (for more details, see [9]), a unique solution $(u,b)\in Z_T$ for the equation (12) (with $\alpha=\beta$) such that

$$\|(u,b)\|_{Z_T} \leq 2C\varsigma^{-\frac{1}{4}}\|(u_0,b_0)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}.$$

Analogously to (26) and by applying Lemma 2.4 to the systems (see (1))

$$\begin{cases} u_t+\mu(-\Delta)^\alpha u = -P[u\cdot\nabla u - b\cdot\nabla b],\\ u(\cdot,0)=u_0,\end{cases} \qquad \begin{cases} b_t+\nu(-\Delta)^\alpha b = b\cdot\nabla u - u\cdot\nabla b,\\ b(\cdot,0)=b_0,\end{cases}$$

we conclude that (2) holds. This proves i).

Now, let us prove the item ii) (we shall assume that the initial data may be large). Thus, as $(u_0,b_0)\in\dot{H}^{s,\alpha}_{a,\sigma}$; then, for $\epsilon>0$ there is $\rho_0>0$ such that

$$\int_{|\xi|>\rho_0}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|(\widehat{u}_0,\widehat{b}_0)(\xi)|^2\,d\xi \leq \epsilon, \text{ and } \int_{|\xi|>\rho_0}|\xi|^{5-4\alpha}e^{2a|\xi|^{\frac{1}{\sigma}}}|(\widehat{u}_0,\widehat{b}_0)(\xi)|^2\,d\xi\leq\epsilon. \tag{30}$$

Define $U_0=\mathcal{F}^{-1}\{\chi_{\{|\xi|\leq\rho_0\}}\widehat{u}_0\}$. Thereby, we deduce

$$\|e^{-\mu t(-\Delta)^\alpha}u_0\|^2_{\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma}} = \int_{|\xi|>\rho_0}e^{-2\mu t|\xi|^{2\alpha}}|\xi|^{2s+\alpha}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{u}_0(\xi)|^2\,d\xi + \|e^{-\mu t(-\Delta)^\alpha}U_0\|^2_{\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma}}, \tag{31}$$

for all $t \in [0, T]$. On the other hand, by (30) (similarly to (28)), one infers

$$\int_0^T \Big(\int_{|\xi|>\rho_0} e^{-2\mu t|\xi|^{2\alpha}} |\xi|^{2s+\alpha} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{u}_0(\xi)|^2 \, d\xi \Big)^2 dt \le C_\mu \epsilon^2,$$

and, moreover,

$$\begin{aligned} \|e^{-\mu t(-\Delta)^\alpha} U_0\|^4_{L^4_T(\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma})} &= \int_0^T \Big(\int_{|\xi|\le\rho_0} |\xi|^{2s+\alpha} e^{2a|\xi|^{\frac{1}{\sigma}}} e^{-4\mu t|\xi|^{2\alpha}} |\widehat{u}_0(\xi)|^2 \, d\xi \Big)^2 dt \\ &\le \rho_0^{2\alpha} T \|u_0\|^4_{\dot{H}^s_{a,\sigma}}. \end{aligned}$$

Therefore, by taking $L^4([0, T])$-norm in (31), one concludes

$$\|e^{-\mu t(-\Delta)^\alpha} u_0\|^4_{L^4_T(\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma})} \le C_\mu [\epsilon^2 + \rho_0^{2\alpha} T \|u_0\|^4_{\dot{H}^{s,\alpha}_{a,\sigma}}].$$

Analogously, we can write

$$\|e^{-\mu t(-\Delta)^\alpha} u_0\|^4_{L^4_T(\dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma})} \le C_\mu [\epsilon^2 + \rho_0^{2\alpha} T \|u_0\|^4_{\dot{H}^{s,\alpha}_{a,\sigma}}]$$

and, consequently,

$$\|e^{-\mu t(-\Delta)^\alpha} u_0\|_{Z_T} \le C_\mu [\sqrt{\epsilon} + \rho_0^{\frac{\alpha}{2}} T^{\frac{1}{4}} \|u_0\|_{\dot{H}^{s,\alpha}_{a,\sigma}}], \text{ and}$$

$$\|e^{-\nu t(-\Delta)^\alpha} b_0\|_{Z_T} \le C_\nu [\sqrt{\epsilon} + \rho_0^{\frac{\alpha}{2}} T^{\frac{1}{4}} \|b_0\|_{\dot{H}^{s,\alpha}_{a,\sigma}}].$$

As a result, one has

$$\|(e^{-\mu t(-\Delta)^\alpha} u_0, e^{-\nu t(-\Delta)^\alpha} b_0)\|_{Z_T} \le C_{\mu,\nu} [\sqrt{\epsilon} + \rho_0^{\frac{\alpha}{2}} T^{\frac{1}{4}} \|(u_0, b_0)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}]. \tag{32}$$

Now, choose $0 < \epsilon < [8C_{s,\alpha}\varsigma^{-\frac{3}{4}} C_{\mu,\nu}]^{-2}$ and $0 < T < [8C_{s,\alpha}\varsigma^{-\frac{3}{4}} C_{\mu,\nu}\rho_0^{\frac{\alpha}{2}} \|(u_0, b_0)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}]^{-4}$ (where $C_{\mu,\nu}$ and $C_{s,\alpha}$ are given in (32) and (27), respectively) in order to get, by Lemma 2.1 in [25] (for more details, see [9]), a unique solution $(u, b) \in Z_T$ for the equation (12) (with $\alpha = \beta$).

Now, observe that

$$\begin{cases} (u_t, b_t) + (\mu(-\Delta)^\alpha u, \nu(-\Delta)^\alpha b) = (P(b \cdot \nabla b - u \cdot \nabla u), b \cdot \nabla u - u \cdot \nabla b); \\ (u, b)(\cdot, 0) = (u_0, b_0). \end{cases}$$

Therefore, analogously to (26), by using the fact that $(u_0, b_0) \in \dot{H}^{s,\alpha}_{a,\sigma}$ and $(u, b) \in Z_T$, and applying Lemma 2.4, the proof of ii) is given.

Now, we re ready to verify iii). Thus, similarly to (24); however, by using Lemma 2.5, it follows from Hölder's inequality that[6]

$$\|P(\gamma \cdot \nabla w - v \cdot \nabla \phi)\|^2_{L^2_T(\dot{H}^{s-\alpha}_{a,\sigma})} \le \int_0^T [\|w \otimes \gamma\|_{\dot{H}^{s+1-\alpha}_{a,\sigma}} + \|\phi \otimes v\|_{\dot{H}^{s+1-\alpha}_{a,\sigma}}]^2 \, dt \tag{33}$$

[6]Denote $X_T = L_T^{\frac{2\alpha}{1-\alpha}}(\dot{H}^{s+1-\alpha}_{a,\sigma}) \cap L_T^{\frac{2\alpha}{2\alpha-1}}(\dot{H}^{s+2\alpha-1}_{a,\sigma})$, where $\|\cdot\|^2_{X_T} = \|\cdot\|^2_{L_T^{\frac{2\alpha}{1-\alpha}}(\dot{H}^{s+1-\alpha}_{a,\sigma})} + \|\cdot\|^2_{L_T^{\frac{2\alpha}{2\alpha-1}}(\dot{H}^{s+2\alpha-1}_{a,\sigma})}$.

$$\leq C_{a,\sigma,s,\alpha}\int_0^T [\|w\|^2_{\dot{H}^{s+1-\alpha}_{a,\sigma}}\|\gamma\|^2_{\dot{H}^{s+2\alpha-1}_{a,\sigma}}+\|\gamma\|^2_{\dot{H}^{s+1-\alpha}_{a,\sigma}}\|w\|^2_{\dot{H}^{s+2\alpha-1}_{a,\sigma}}+\|\phi\|^2_{\dot{H}^{s+1-\alpha}_{a,\sigma}}\|v\|^2_{\dot{H}^{s+2\alpha-1}_{a,\sigma}}$$

$$+\|v\|^2_{\dot{H}^{s+1-\alpha}_{a,\sigma}}\|\phi\|^2_{\dot{H}^{s+2\alpha-1}_{a,\sigma}}]\,dt$$

$$\leq C_{a,\sigma,s,\alpha}[\|(w,v)\|^2_{L_T^{\frac{2\alpha}{1-\alpha}}(\dot{H}^{s+1-\alpha}_{a,\sigma})}\|(\gamma,\phi)\|^2_{L_T^{\frac{2\alpha}{2\alpha-1}}(\dot{H}^{s+2\alpha-1}_{a,\sigma})}$$

$$+\|(\gamma,\phi)\|^2_{L_T^{\frac{2\alpha}{1-\alpha}}(\dot{H}^{s+1-\alpha}_{a,\sigma})}\|(w,v)\|^2_{L_T^{\frac{2\alpha}{2\alpha-1}}(\dot{H}^{s+2\alpha-1}_{a,\sigma})}] \tag{34}$$

$$\leq C_{a,\sigma,s,\alpha}\|(w,v)\|^2_{X_T}\|(\gamma,\phi)\|^2_{X_T}. \tag{35}$$

Likewise, we conclude that

$$\|w\cdot\nabla\phi - v\cdot\nabla\gamma\|_{L^2_T(\dot{H}^{s-\alpha}_{a,\sigma})}\leq C_{a,\sigma,s,\alpha}\|(w,v)\|_{X_T}\|(\gamma,\phi)\|_{X_T}, \tag{36}$$

for all $(w,v),(\gamma,\phi)\in X_T$. By applying Lemma 2.4 ii) (for the following values of p: $\frac{2\alpha}{1-\alpha}$ and $\frac{2\alpha}{2\alpha-1}$) to the systems in (22) and, by using (35) and (36), we reach

$$\|B((w,v),(\gamma,\phi))\|_{X_T}\leq C_{a,\sigma,s,\alpha,\mu,\nu}\|(w,v)\|_{X_T}\|(\gamma,\phi)\|_{X_T},\quad \forall (w,v),(\gamma,\phi)\in X_T. \tag{37}$$

On the other hand, by Minkowski's inequality, one infers

$$\|e^{-\mu t(-\Delta)^\alpha}u_0\|_{L_T^{\frac{2\alpha}{1-\alpha}}(\dot{H}^{s+1-\alpha}_{a,\sigma})}\leq\Big(\int_{\mathbb{R}^3}|\xi|^{2s+2-2\alpha}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{u}_0(\xi)|^2\Big(\int_0^T e^{\frac{-4\mu\alpha t|\xi|^{2\alpha}}{1-\alpha}}\,dt\Big)^{\frac{1-\alpha}{\alpha}}d\xi\Big)^{\frac{1}{2}}$$

$$\leq C_{\mu,\alpha}\|u_0\|_{\dot{H}^s_{a,\sigma}}.$$

Similarly, we obtain

$$\|e^{-\mu t(-\Delta)^\alpha}u_0\|_{L_T^{\frac{2\alpha}{2\alpha-1}}(\dot{H}^{s+2\alpha-1}_{a,\sigma})}\leq C_{\mu,\alpha}\|u_0\|_{\dot{H}^s_{a,\sigma}}$$

$$\|e^{-\nu t(-\Delta)^\alpha}b_0\|_{L_T^{\frac{2\alpha}{1-\alpha}}(\dot{H}^{s+1-\alpha}_{a,\sigma})}\ \text{and}\ \|e^{-\nu t(-\Delta)^\alpha}b_0\|_{L_T^{\frac{2\alpha}{2\alpha-1}}(\dot{H}^{s+2\alpha-1}_{a,\sigma})}\leq C_{\nu,\alpha}\|b_0\|_{\dot{H}^{s,\alpha}_{a,\sigma}}.$$

Therefore, one concludes that

$$\|(e^{-\mu t(-\Delta)^\alpha}u_0, e^{-\nu t(-\Delta)^\alpha}b_0)\|_{X_T}\leq C_{\mu,\nu,\alpha}\|(u_0,b_0)\|_{\dot{H}^s_{a,\sigma}}. \tag{38}$$

Finally, it is enough to assume that $\|(u_0,b_0)\|_{\dot{H}^s_{a,\sigma}}<\dfrac{1}{4C_{\mu,\nu,\alpha}C_{a,\sigma,s,\alpha\mu,\nu}}$ (where $C_{\mu,\nu,\alpha}$ and $C_{a,\sigma,s,\alpha\mu,\nu}$ are given in (38) and (37), respectively) and follow the arguments established in the proof of i) to determine the veracity of iii). □

5 Proofs of Theorems 1.3 and 1.4

In this section, motivated by [23, 26], we are going to prove Theorems 1.3 and 1.4. Due to their similarities, we have decided to establish the proof of Theorem 1.4 and show the specificities of the proof of Theorem 1.3 in between parenthesis. Let us point out that the proofs below extend the respective particular cases established in [23] and, thus, part of them are similar. Nevertheless, for the convenience of the reader, we will argue Theorems 1.3 and 1.4 exposing their details.

Proofs of Theorems 1.3 i) and 1.4 i) with $n=1$ Suppose, by contradiction, that $\limsup_{t\nearrow T^*}\|(u,b)(t)\|_{\dot{H}^s_{a,\sigma}}<\infty$ (respectively, $\limsup_{t\nearrow T^*}\|(u,b)(t)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}<\infty$). By Theorem 1.1 (respectively, Theorem 1.2), there exists a positive constant $C_{a,\sigma,s}$

$$\|(u,b)(t)\|_{\dot{H}^s_{a,\sigma}}\le C_{a,\sigma,s},\quad \forall t\in[0,T^*). \tag{39}$$

(Respectively, we infer

$$\|(u,b)(t)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}\le C_{a,\sigma,s},\quad \forall t\in[0,T^*).) \tag{40}$$

On the other hand, by taking $\dot{H}^s_{a,\sigma}$-inner product of the first equation of (1) with $u(t)$, one deduces

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{\dot{H}^s_{a,\sigma}}+\mu\|u(t)\|^2_{\dot{H}^{s+\alpha}_{a,\sigma}}\le|\langle u,u\cdot\nabla u\rangle_{\dot{H}^s_{a,\sigma}}|+|\langle u,b\cdot\nabla b\rangle_{\dot{H}^s_{a,\sigma}}|. \tag{41}$$

Analogously, by applying $\dot{H}^s_{a,\sigma}$-inner product to the second equation with $b(t)$, it follows that

$$\frac{1}{2}\frac{d}{dt}\|b(t)\|^2_{\dot{H}^s_{a,\sigma}}+\nu\|b(t)\|^2_{\dot{H}^{s+\beta}_{a,\sigma}}\le|\langle b,u\cdot\nabla b\rangle_{\dot{H}^s_{a,\sigma}}|+|\langle b,b\cdot\nabla u\rangle_{\dot{H}^s_{a,\sigma}}|. \tag{42}$$

By (41) and (42), we infer that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}}+\mu\|u(t)\|^2_{\dot{H}^{s+\alpha}_{a,\sigma}}+\nu\|b(t)\|^2_{\dot{H}^{s+\beta}_{a,\sigma}}\\&\le\|u\|_{\dot{H}^s_{a,\sigma}}\|u\otimes u\|_{\dot{H}^{s+1}_{a,\sigma}}+\|u\|_{\dot{H}^s_{a,\sigma}}\|b\otimes b\|_{\dot{H}^{s+1}_{a,\sigma}}\\&\quad+\|b\|_{\dot{H}^s_{a,\sigma}}\|b\otimes u\|_{\dot{H}^{s+1}_{a,\sigma}}+\|b\|_{\dot{H}^s_{a,\sigma}}\|u\otimes b\|_{\dot{H}^{s+1}_{a,\sigma}}.\end{aligned} \tag{43}$$

By applying the proof of Lemma 2.2 and Lemma 2.1 i), we conclude

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}}+\mu\|u(t)\|^2_{\dot{H}^{s+\alpha}_{a,\sigma}}+\nu\|b(t)\|^2_{\dot{H}^{s+\beta}_{a,\sigma}}\\&\le C_s[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|_{L^1}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}\|_{L^1}][\|(u,b)\|^{2-\frac{1}{\alpha}}_{\dot{H}^s_{a,\sigma}}\|u\|^{\frac{1}{\alpha}}_{\dot{H}^{s+\alpha}_{a,\sigma}}\\&\quad+\|(u,b)\|^{2-\frac{1}{\beta}}_{\dot{H}^s_{a,\sigma}}\|b\|^{\frac{1}{\beta}}_{\dot{H}^{s+\beta}_{a,\sigma}}].\end{aligned} \tag{44}$$

By Young's inequality, one deduces

$$\begin{aligned}&\frac{d}{dt}\|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}}+\mu\|u(t)\|^2_{\dot{H}^{s+\alpha}_{a,\sigma}}+\nu\|b(t)\|^2_{\dot{H}^{s+\beta}_{a,\sigma}}\le C_{s,\alpha,\beta,\mu,\nu}\{[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|_{L^1}\\&+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}\|_{L^1}]^{\frac{2\alpha}{2\alpha-1}}+[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|_{L^1}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}\|_{L^1}]^{\frac{2\beta}{2\beta-1}}\}\|(u,b)\|^2_{\dot{H}^s_{a,\sigma}}.\end{aligned} \tag{45}$$

On the other hand, by (4), one infers

$$\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})\|_{L^1}\le C_{a,\sigma,s}\|(u,b)\|_{\dot{H}^s_{a,\sigma}}. \tag{46}$$

Now, integrate over $[0, t]$ the inequality (45) and apply (39) (respectively, (40)) and (46) to obtain

$$\|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}} + \mu \int_0^t \|u(\tau)\|^2_{\dot{H}^{s+\alpha}_{a,\sigma}} + \nu \int_0^t \|b(\tau)\|^2_{\dot{H}^{s+\beta}_{a,\sigma}} d\tau$$
$$\leq \|(u_0,b_0)\|^2_{\dot{H}^s_{a,\sigma}} + C_{s,a,\sigma,\mu,\nu,\alpha,\beta} T^*,$$

for all $t \in [0, T^*)$. Consequently,

$$\mu \int_0^t \|u(\tau)\|^2_{\dot{H}^{s+\alpha}_{a,\sigma}} d\tau + \nu \int_0^t \|b(\tau)\|^2_{\dot{H}^{s+\beta}_{a,\sigma}} d\tau \leq C_{s,a,\sigma,\mu,\nu,\alpha,\beta,u_0,b_0,T^*}, \quad \forall t \in [0, T^*). \tag{47}$$

(Respectively, if $\alpha = \beta$, we have

$$\|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}} + [\mu+\nu] \int_0^t \|(u,b)(\tau)\|^2_{\dot{H}^{s+\alpha}_{a,\sigma}} d\tau \leq \|(u_0,b_0)\|^2_{\dot{H}^s_{a,\sigma}} + C_{s,a,\sigma,\mu,\nu,\alpha} T^*.$$

In particular, it follows that

$$\|(u,b)(t)\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\sigma}} + [\mu+\nu] \int_0^t \|(u,b)(\tau)\|^2_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\sigma}} d\tau \leq \|(u_0,b_0)\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\sigma}} + C_{a,\sigma,\mu,\nu,\alpha} T^*.$$

As a result, we obtain

$$\int_0^t \|(u,b)(\tau)\|^2_{\dot{H}^{s+\alpha}_{a,\sigma}} d\tau + \int_0^t \|(u,b)(\tau)\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\sigma}} d\tau \leq C_{a,\sigma,\mu,\nu,\alpha,u_0,b_0,T^*}, \quad \forall t \in [0, T^*).) \tag{48}$$

On the other hand, denote by $(\kappa_n)_{n\in\mathbb{N}}$ a sequence such that $0 < \kappa_n < T^*$ and $\kappa_n \nearrow T^*$. We shall prove that

$$\lim_{n,m\to\infty} \|(u,b)(\kappa_n) - (u,b)(\kappa_m)\|_{\dot{H}^s_{a,\sigma}} = 0. \tag{49}$$

(Respectively, we are going to show that

$$\lim_{n,m\to\infty} \|(u,b)(\kappa_n) - (u,b)(\kappa_m)\|_{\dot{H}^{s,\alpha}_{a,\sigma}} = 0.) \tag{50}$$

By (12), (13), (14) and (15), one obtains

$$(u,b)(\kappa_n) - (u,b)(\kappa_m) = I_1 + I_2 + I_3, \tag{51}$$

where

$$I_1 = (I_{11}, I_{12}) := ([e^{-\mu\kappa_n(-\Delta)^\alpha} - e^{-\mu\kappa_m(-\Delta)^\alpha}]u_0, [e^{-\nu\kappa_n(-\Delta)^\beta} - e^{-\nu\kappa_m(-\Delta)^\beta}]b_0), \tag{52}$$

and

$$I_2 = (I_{21}, I_{22}) := \Big(\int_0^{\kappa_m} [e^{-\mu(\kappa_m-\tau)(-\Delta)^\alpha} - e^{-\mu(\kappa_n-\tau)(-\Delta)^\alpha}] P[u\cdot\nabla u - b\cdot\nabla b]\, d\tau,$$
$$\int_0^{\kappa_m} [e^{-\nu(\kappa_m-\tau)(-\Delta)^\beta} - e^{-\nu(\kappa_n-\tau)(-\Delta)^\beta}](u\cdot\nabla b - b\cdot\nabla u)\, d\tau \Big), \tag{53}$$

and also

$$I_3 = -\Big(\int_{\kappa_m}^{\kappa_n} e^{-\mu(\kappa_n-\tau)(-\Delta)^\alpha} P[u\cdot\nabla u - b\cdot\nabla b]\,d\tau, \int_{\kappa_m}^{\kappa_n} e^{-\nu(\kappa_n-\tau)(-\Delta)^\beta}(u\cdot\nabla b - b\cdot\nabla u)\,d\tau\Big). \tag{54}$$

Thus, it is easy to check that

$$\|I_{12}\|^2_{\dot{H}^s_{a,\sigma}} \le \int_{\mathbb{R}^3} [e^{-\nu\kappa_n|\xi|^{2\beta}} - e^{-\nu T^*|\xi|^{2\beta}}]^2 |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{b_0}(\xi)|^2\,d\xi.$$

Since $b_0 \in \dot{H}^s_{a,\sigma}$, Dominated Convergence Theorem implies that $\lim\limits_{n,m\to\infty} \|I_{12}\|_{\dot{H}^s_{a,\sigma}} = 0$. Similarly, we have $\lim\limits_{n,m\to\infty} \|I_{11}\|_{\dot{H}^s_{a,\sigma}} = 0$. As a result, $\lim\limits_{n,m\to\infty} \|I_1\|_{\dot{H}^s_{a,\sigma}} = 0$. (Respectively, we have $\lim\limits_{n,m\to\infty} \|I_1\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\sigma}} = 0$.)

By applying (11) and Cauchy-Schwarz's inequality, we obtain

$$\|I_{21}\|_{\dot{H}^s_{a,\sigma}} \le \sqrt{T^*}\Big(\int_0^{T^*}\int_{\mathbb{R}^3} [1 - e^{-\mu(T^*-\kappa_m)|\xi|^{2\alpha}}]^2 |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}[u\cdot\nabla u - b\cdot\nabla b](\xi)|^2 d\xi\,d\tau\Big)^{\frac{1}{2}}.$$

Observe that, by using Lemma 2.2, (39) (respectively, (40)), Hölder's inequality, and (47) (respectively, (48)), we have

$$\begin{aligned}
&\int_0^{T^*} \|u\cdot\nabla u - b\cdot\nabla b\|^2_{\dot{H}^s_{a,\sigma}}\,d\tau \\
&\le \int_0^{T^*} [\|u\otimes u\|_{\dot{H}^{s+1}_{a,\sigma}} + \|b\otimes b\|_{\dot{H}^{s+1}_{a,\sigma}}]^2 d\tau \\
&\le C_{a,\sigma,s,\alpha,\beta} \int_0^{T^*} [\|u\|^{\frac{1}{\alpha}}_{\dot{H}^{s+\alpha}_{a,\sigma}} + \|b\|^{\frac{1}{\beta}}_{\dot{H}^{s+\beta}_{a,\sigma}}]^2 d\tau \\
&\le C_{a,\sigma,s,\alpha,\beta}\Big[\Big(\int_0^{T^*} \|u\|^2_{\dot{H}^{s+\alpha}_{a,\sigma}} d\tau\Big)^{\frac{1}{\alpha}} (T^*)^{1-\frac{1}{\alpha}} + \Big(\int_0^{T^*} \|b\|^2_{\dot{H}^{s+\beta}_{a,\sigma}} d\tau\Big)^{\frac{1}{\beta}} (T^*)^{1-\frac{1}{\beta}}\Big] \\
&\le C_{a,\sigma,s,\alpha,\beta,u_0,b_0,T^*} < \infty.
\end{aligned} \tag{55}$$

Thus, by applying Dominated Convergence Theorem, one obtains that $\lim\limits_{n,m\to\infty} \|I_{21}\|_{\dot{H}^s_{a,\sigma}} = 0$. Analogously, we have $\lim\limits_{n,m\to\infty} \|I_{22}\|_{\dot{H}^s_{a,\sigma}} = 0$. Therefore, $\lim\limits_{n,m\to\infty} \|I_2\|_{\dot{H}^s_{a,\sigma}} = 0$. (Respectively, we obtain $\lim\limits_{n,m\to\infty} \|I_2\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\sigma}} = 0$.)

Now, note that, by applying (11), Lemma 2.2, (39) (respectively, (40)), Hölder's inequality, and (47) (respectively, (48)), one gets

$$\begin{aligned}
\|I_3\|_{\dot{H}^s_{a,\sigma}} &\le \int_{\kappa_m}^{\kappa_n} \|e^{-\mu(\kappa_n-\tau)(-\Delta)^\alpha} P(u\cdot\nabla u - b\cdot\nabla b)\|_{\dot{H}^s_{a,\sigma}}\,d\tau \\
&\quad + \int_{\kappa_m}^{\kappa_n} \|e^{-\nu(\kappa_n-\tau)(-\Delta)^\beta}(u\cdot\nabla b - b\cdot\nabla u)\|_{\dot{H}^s_{a,\sigma}}\,d\tau \\
&\le C_{a,\sigma,s,\alpha,\beta}\Big[\Big(\int_0^{T^*} \|u\|^2_{\dot{H}^{s+\alpha}_{a,\sigma}} d\tau\Big)^{\frac{1}{2\alpha}} (T^* - \kappa_m)^{1-\frac{1}{2\alpha}}
\end{aligned}$$

$$+\Big(\int_0^{T^*}\|b\|^2_{\dot H^{s+\beta}_{a,\sigma}}d\tau\Big)^{\frac{1}{2\beta}}(T^*-\kappa_m)^{1-\frac{1}{2\beta}}\Big]$$

$$\leq C_{a,\sigma,s,\alpha,\beta,u_0,b_0,T^*}[(T^*-\kappa_m)^{1-\frac{1}{2\alpha}}+(T^*-\kappa_m)^{1-\frac{1}{2\beta}}].$$

This implies that $\lim_{n,m\to\infty}\|I_3\|_{\dot H^s_{a,\sigma}}=0$ (respectively, we deduce $\lim_{n,m\to\infty}\|I_3\|_{\dot H^{\frac52-2\alpha}_{a,\sigma}}=0$). Thus, we have proved (49) (respectively, (50)); that is, we have shown that $((u,b)(\kappa_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Banach space $\dot H^s_{a,\sigma}$ (respectively, $\dot H^{s,\alpha}_{a,\sigma}$). Therefore, there is $(u_1,b_1)\in\dot H^s_{a,\sigma}$ (respectively, $\dot H^{s,\alpha}_{a,\sigma}$) with

$$\lim_{n\to\infty}\|(u,b)(\kappa_n)-(u_1,b_1)\|_{\dot H^s_{a,\sigma}}=0.$$

(Respectively, $\lim_{n\to\infty}\|(u,b)(\kappa_n)-(u_1,b_1)\|_{\dot H^{s,\alpha}_{a,\sigma}}=0$.) It is true that (u_1,b_1) is independent on the sequence $(\kappa_n)_{n\in\mathbb{N}}$. In fact, suppose that $(\rho_n)_{n\in\mathbb{N}}\subseteq(0,T^*)$ satisfies $\rho_n\nearrow T^*$ and $\lim_{n\to\infty}\|(u,b)(\rho_n)-(u_2,b_2)\|_{\dot H^s_{a,\sigma}}=0$ (respectively, $\lim_{n\to\infty}\|(u,b)(\rho_n)-(u_2,b_2)\|_{\dot H^{s,\alpha}_{a,\sigma}}=0$) for some $(u_2,b_2)\in\dot H^s_{a,\sigma}$ (respectively, $\dot H^{s,\alpha}_{a,\sigma}$). It is easy to notice that $(u_2,b_2)=(u_1,b_1)$. Indeed, let $(\varsigma_n)_{n\in\mathbb{N}}\subseteq(0,T^*)$ be defined by $\varsigma_{2n}=\kappa_n$ and $\varsigma_{2n-1}=\rho_n$, for all $n\in\mathbb{N}$. As a consequence, one concludes $\varsigma_n\nearrow T^*$ and there exists $(u_3,b_3)\in\dot H^s_{a,\sigma}$ (respectively, $\dot H^{s,\alpha}_{a,\sigma}$) with $\lim_{n\to\infty}\|(u,b)(\varsigma_n)-(u_3,b_3)\|_{\dot H^s_{a,\sigma}}=0$ (respectively, $\lim_{n\to\infty}\|(u,b)(\varsigma_n)-(u_3,b_3)\|_{\dot H^{s,\alpha}_{a,\sigma}}=0$). Therefore, $(u_1,b_1)=(u_3,b_3)=(u_2,b_2)$. All the proof above establishes that $\lim_{t\nearrow T^*}\|(u,b)(t)-(u_1,b_1)\|_{\dot H^s_{a,\sigma}}=0$ (respectively, $\lim_{t\nearrow T^*}\|(u,b)(t)-(u_1,b_1)\|_{\dot H^{s,\alpha}_{a,\sigma}}=0$).

Thereby, consider the GMHD equations (1) (with the initial data (u_1,b_1)) and apply Theorem 1.1. As usual, we can put the two solutions together to get a solution for (1) that is defined beyond T^*. This is an absurd. Therefore, the proofs of Theorems 1.3 i) and 1.4 i) with $n=1$ are given. □

Proofs of Theorems 1.3 ii) and 1.4 ii) with $n=1$ Consider $0\leq t\leq T<T^*$ and apply Gronwall's inequality to (45) in order to obtain

$$\|(u,b)(T)\|^2_{\dot H^s_{a,\sigma}}\leq\|(u,b)(t)\|^2_{\dot H^s_{a,\sigma}}\exp\{C_{s,\alpha,\beta,\mu,\nu}\int_t^T\{[\|e^{\frac{a}{\sigma}|\cdot|^{\frac1\sigma}}\widehat{u}(\tau)\|_{L^1}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac1\sigma}}\widehat{b}(\tau)\|_{L^1}]^{\frac{2\alpha}{2\alpha-1}}$$

$$+[\|e^{\frac{a}{\sigma}|\cdot|^{\frac1\sigma}}\widehat{u}(\tau)\|_{L^1}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac1\sigma}}\widehat{b}(\tau)\|_{L^1}]^{\frac{2\beta}{2\beta-1}}\}d\tau\}.$$

(In particular, one deduces

$$\|(u,b)(T)\|^2_{\dot H^{\frac52-2\alpha}_{a,\sigma}}$$

$$\leq\|(u,b)(t)\|^2_{\dot H^{\frac52-2\alpha}_{a,\sigma}}\exp\{C_{\alpha,\mu,\nu}\int_t^T[\|e^{\frac{a}{\sigma}|\cdot|^{\frac1\sigma}}\widehat{u}(\tau)\|_{L^1}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac1\sigma}}\widehat{b}(\tau)\|_{L^1}]^{\frac{2\alpha}{2\alpha-1}}\,d\tau\}.)$$

Passing to the limit superior, as $T\nearrow T^*$, Theorem 1.4 i) (with $n=1$) implies that

$$\int_t^{T^*}\{[\|e^{\frac{a}{\sigma}|\cdot|^{\frac1\sigma}}\widehat{u}(\tau)\|_{L^1}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac1\sigma}}\widehat{b}(\tau)\|_{L^1}]^{\frac{2\alpha}{2\alpha-1}}+[\|e^{\frac{a}{\sigma}|\cdot|^{\frac1\sigma}}\widehat{u}(\tau)\|_{L^1}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac1\sigma}}\widehat{b}(\tau)\|_{L^1}]^{\frac{2\beta}{2\beta-1}}\}d\tau$$

$$=\infty,$$

for all $t\in[0,T^*)$. This completes the proofs of Theorems 1.3 ii) and 1.4 ii) with $n=1$. □

Proofs of Theorems 1.3 iii) and 1.4 iii) with $n = 1$ By applying Fourier transform to the first equation of (1) and taking the scalar product in $\mathbb{C}^3$ with $\widehat{u}(t)$ of the result obtained, one infers

$$\frac{1}{2}\partial_t|\widehat{u}(t)|^2 + \mu|\xi|^{2\alpha}|\widehat{u}|^2 \le |\widehat{u}\cdot\widehat{u\cdot\nabla u}| + |\widehat{u}\cdot\widehat{b\cdot\nabla b}|. \tag{56}$$

For $\delta > 0$ arbitrary, it is easy to check that

$$\partial_t\sqrt{|\widehat{u}(t)|^2+\delta} + \mu\frac{|\xi|^{2\alpha}|\widehat{u}|^2}{\sqrt{|\widehat{u}|^2+\delta}} \le |\widehat{u\cdot\nabla u}| + |\widehat{b\cdot\nabla b}|.$$

By integrating from t to T (where $0 \le t \le T < T^* < \infty$), one obtains

$$\begin{aligned}&\sqrt{|\widehat{u}(T)|^2+\delta} + \mu|\xi|^{2\alpha}\int_t^T \frac{|\widehat{u}(\tau)|^2}{\sqrt{|\widehat{u}(\tau)|^2+\delta}}\,d\tau \\ &\le \sqrt{|\widehat{u}(t)|^2+\delta} + \int_t^T [|(\widehat{u\cdot\nabla u})(\tau)| + |(\widehat{b\cdot\nabla b})(\tau)|]\,d\tau.\end{aligned}$$

Passing to the limit, as $\delta \to 0$, multiplying by $e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}$ and integrating over $\xi \in \mathbb{R}^3$, we obtain

$$\begin{aligned}&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(T)\|_{L^1} + \mu\int_t^T \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\mathcal{F}[(-\Delta)^\alpha u](\tau)\|_{L^1}\,d\tau \le \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|_{L^1} \\ &+ \int_t^T\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}[|(\widehat{u\cdot\nabla u})(\tau)| + |(\widehat{b\cdot\nabla b})(\tau)|]\,d\xi\,d\tau.\end{aligned}$$

Similarly, we deduce

$$\begin{aligned}&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}(T)\|_{L^1} + \nu\int_t^T \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\mathcal{F}[(-\Delta)^\beta b](\tau)\|_{L^1}\,d\tau \le \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}(t)\|_{L^1} \\ &+ \int_t^T\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}[|(\widehat{u\cdot\nabla b})(\tau)| + |(\widehat{b\cdot\nabla u})(\tau)|]\,d\xi\,d\tau.\end{aligned}$$

Consequently, one obtains

$$\begin{aligned}&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(T)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}(T)\|_{L^1} + \mu\int_t^T \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\mathcal{F}[(-\Delta)^\alpha u](\tau)\|_{L^1}\,d\tau \\ &+ \nu\int_t^T \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\mathcal{F}[(-\Delta)^\beta b](\tau)\|_{L^1}\,d\tau \\ &\le \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}(t)\|_{L^1} + \int_t^T\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}[|(\widehat{u\cdot\nabla u})(\tau)| + |(\widehat{b\cdot\nabla b})(\tau)| \\ &+ |(\widehat{u\cdot\nabla b})(\tau)| + |(\widehat{b\cdot\nabla u})(\tau)|]\,d\xi\,d\tau.\end{aligned}$$

By applying (17) in [1], and Young and Hölder's inequalities, we obtain

$$\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|(\widehat{u\cdot\nabla b})(\xi)|\,d\xi \le (2\pi)^{-3}\|[e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}|\widehat{u}|]*[e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}|\widehat{\nabla b}|]\|_{L^1}$$

$$
\begin{aligned}
&\le (2\pi)^{-3} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}\|_{L^1} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{\nabla b}\|_{L^1} \\
&\le (2\pi)^{-3} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}\|_{L^1} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}\|_{L^1}^{1-\frac{1}{2\beta}} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \mathcal{F}[(-\Delta)^{\beta} b]\|_{L^1}^{\frac{1}{2\beta}}.
\end{aligned} \tag{57}
$$

Consequently, by Young's inequality, one deduces

$$
\begin{aligned}
&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(T)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(T)\|_{L^1} + \frac{\mu}{2}\int_t^T \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \mathcal{F}[(-\Delta)^{\alpha} u](\tau)\|_{L^1}\, d\tau \\
&+ \frac{\nu}{2}\int_t^T \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \mathcal{F}[(-\Delta)^{\beta} b](\tau)\|_{L^1}\, d\tau \\
&\le \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(t)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(t)\|_{L^1} + C_{\mu,\nu,\alpha,\beta}\int_t^T [\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(\tau)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(\tau)\|_{L^1}] \\
&\times \{[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(\tau)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(\tau)\|_{L^1}]^{\frac{2\alpha}{2\alpha-1}} + [\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(\tau)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(\tau)\|_{L^1}]^{\frac{2\beta}{2\beta-1}}\}\, d\tau.
\end{aligned}
$$

By Gronwall's inequality, it follows that

$$
\begin{aligned}
&[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(T)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(T)\|_{L^1}]^{\frac{2\alpha}{1-2\alpha}} + [\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(T)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(T)\|_{L^1}]^{\frac{2\beta}{1-2\beta}} \\
&\le \{[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(t)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(t)\|_{L^1}]^{\frac{2\alpha}{1-2\alpha}} + [\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(t)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(t)\|_{L^1}]^{\frac{2\beta}{1-2\beta}}\} \\
&\times \exp\{C_{\mu,\nu,\alpha,\beta}\int_t^T \{[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(\tau)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(\tau)\|_{L^1}]^{\frac{2\alpha}{2\alpha-1}} + [\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(\tau)\|_{L^1} \\
&+ \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(\tau)\|_{L^1}]^{\frac{2\beta}{2\beta-1}}\}\, d\tau\},
\end{aligned}
$$

for all $0 \le t \le T < T^*$, that is,

$$
\begin{aligned}
&-C_{\mu,\nu,\alpha,\beta}^{-1}\frac{d}{dT}[\exp\{-C_{\mu,\nu,\alpha,\beta}\int_t^T \{[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(\tau)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(\tau)\|_{L^1}]^{\frac{2\alpha}{2\alpha-1}} \\
&+ [\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(\tau)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(\tau)\|_{L^1}]^{\frac{2\beta}{2\beta-1}}\}d\tau\}] \le [\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(t)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(t)\|_{L^1}]^{\frac{2\alpha}{2\alpha-1}} \\
&+ [\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(t)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(t)\|_{L^1}]^{\frac{2\beta}{2\beta-1}}.
\end{aligned}
$$

Integrate over $[t, t_0]$ (with $0 \le t \le t_0 < T^*$) to obtain

$$
\begin{aligned}
&-C_{\mu,\nu,\alpha,\beta}^{-1}\exp\{-C_{\mu,\nu,\alpha,\beta}\int_t^{t_0} \{[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(\tau)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(\tau)\|_{L^1}]^{\frac{2\alpha}{2\alpha-1}} + [\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(\tau)\|_{L^1} \\
&+ \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(\tau)\|_{L^1}]^{\frac{2\beta}{2\beta-1}}\}d\tau\} + C_{\mu,\nu,\alpha,\beta}^{-1} \le \{[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(t)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(t)\|_{L^1}]^{\frac{2\alpha}{2\alpha-1}} \\
&+ [\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(t)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(t)\|_{L^1}]^{\frac{2\beta}{2\beta-1}}\}(t_0 - t).
\end{aligned}
$$

By passing to the limit, as $t_0 \nearrow T^*$, and using Theorem 1.4 ii) with $n = 1$ (respectively, Theorem 1.3 ii) with $n = 1$), we have

$$
C_{\mu,\nu,\alpha,\beta}^{-1} \le \{[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(t)\|_{L^1} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}(t)\|_{L^1}]^{\frac{2\alpha}{2\alpha-1}}
$$

$$+[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|_{L^1}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}(t)\|_{L^1}]^{\frac{2\beta}{2\beta-1}}\}(T^*-t),$$

for all $t\in[0,T^*)$. This completes the proofs of Theorems 1.3 iii) and 1.4 iii) for $n=1$. □

Proofs of Theorem 1.3 iv) and v), and Theorem 1.4 iv) with $n=1$ It is a fact that $\dot{H}^s_{a,\sigma}\hookrightarrow\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}$ (since $\sigma>1$). More precisely, we have $\|\cdot\|_{\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}}\le\|\cdot\|_{\dot{H}^s_{a,\sigma}}$. Therefore, we conclude that $(u,b)\in C([0,T_a^*),\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma})$[7] (respectively, $(u,b)\in C([0,T_a^*),\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma})\cap C([0,T_a^*),\dot{H}^{\frac{5}{2}-2\alpha}_{\frac{a}{\sqrt{\sigma}},\sigma})$). As a consequence, we obtain $T^*_{\frac{a}{\sqrt{\sigma}}}\ge T_a^*$. Moreover, by applying Theorem 1.4 iii) with $n=1$ (respectively, Theorem 1.3 iii) with $n=1$) and Cauchy-Schwarz's inequality (analogously to (46)), it follows that

$$\begin{aligned}C_{\mu,\nu,\alpha,\beta}(T_a^*-t)^{-1}&\le\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1}^{\frac{2\beta}{2\beta-1}}\\&\le C_{a,\sigma,s,\alpha,\beta}[\|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}}^{\frac{2\alpha}{2\alpha-1}}+\|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}}^{\frac{2\beta}{2\beta-1}}],\end{aligned}\tag{58}$$

for all $t\in[0,T_a^*)$. (Respectively, we have

$$C_{\mu,\nu,\alpha}(T_a^*-t)^{-\frac{2\alpha-1}{2\alpha}}\le C_{a,\sigma,\alpha}\|(u,b)(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{\frac{a}{\sqrt{\sigma}},\sigma}},\quad\forall t\in[0,T_a^*).)\tag{59}$$

This proves Theorem 1.3 iv) and v), and Theorem 1.4 iv) with $n=1$. □

Proofs of Theorem 1.3 i), ii), iii), iv) and v), and Theorem 1.4 i), ii), iii) and iv) with $n>1$ First of all, (58) implies that

$$\limsup_{t\nearrow T_a^*}[\|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}}^{\frac{2\alpha}{2\alpha-1}}+\|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}}^{\frac{2\beta}{2\beta-1}}]=\infty.\tag{60}$$

(Respectively, by applying (59), one concludes

$$\limsup_{t\nearrow T_a^*}\|(u,b)(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{\frac{a}{\sqrt{\sigma}},\sigma}}=\infty.)\tag{61}$$

Notice that (60) implies directly that

$$\limsup_{t\nearrow T_a^*}\|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}}=\infty.\tag{62}$$

This proves Theorems 1.3 i) and 1.4 i) with $n=2$. By following the process that we have done so far, one obtains

$$\int_t^{T_a^*}[\|e^{\frac{a}{\sigma\sqrt{\sigma}}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}+\|e^{\frac{a}{\sigma\sqrt{\sigma}}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|_{L^1}^{\frac{2\beta}{2\beta-1}}]\,d\tau=\infty,\quad\forall t\in[0,T_a^*).$$

[7]From now on $T_\omega^*<\infty$ denotes the first blow-up time for the solution $(u,b)\in C([0,T_\omega^*);\dot{H}^s_{\omega,\sigma})$ (respectively, $(u,b)\in C([0,T_\omega^*);\dot{H}^s_{\omega,\sigma})\cap C([0,T_\omega^*);\dot{H}^{\frac{5}{2}-2\alpha}_{\omega,\sigma})$), where $\omega>0$.

This proves Theorem 1.3 ii) and 1.4 ii) with $n=2$ and Theorem 1.3 iii) and 1.4 iii) with $n=2$ follows analogously to the case $n=1$. As an immediate consequence of (62) (and respectively, (61)), one obtains that $T_a^* \geq T^*_{\frac{a}{\sqrt{\sigma}}}$. As a result, we infer that $T_a^* = T^*_{\frac{a}{\sqrt{\sigma}}}$. Consequently, rewriting the arguments above with $\frac{a}{\sqrt{\sigma}}$ instead of a, we obtain, analogously to (58), that

$$C_{\mu,\nu,\alpha,\beta}(T_a^*-t)^{-1} \leq C_{a,\sigma,s,\alpha,\beta}[\|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sigma},\sigma}}^{\frac{2\alpha}{2\alpha-1}} + \|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sigma},\sigma}}^{\frac{2\beta}{2\beta-1}}], \quad \forall t\in[0,T_a^*). \tag{63}$$

(Respectively, analogously to (59), one infers

$$C_{\mu,\nu,\alpha,\beta}(T_a^*-t)^{-\frac{2\alpha-1}{2\alpha}} \leq C_{a,\sigma,\alpha}\|(u,b)(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{\frac{a}{\sigma},\sigma}}, \quad \forall t\in[0,T_a^*).) \tag{64}$$

Thus, (63) (and respectively, (64)) completes the proofs of Theorem 1.3 iv) and v), and Theorem 1.4 iv) with $n=2$. By passing to the limit, as $t \nearrow T_a^*$, we deduce that $\limsup_{t\nearrow T_a^*} \|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)} = \infty$. Consequently, Theorem 1.4 i) holds with $n=3$. Notice that, replacing a by $\frac{a}{\sqrt{\sigma}}$, one obtains that $T_a^* = T^*_{\frac{a}{\sqrt{\sigma}}} = T^*_{\frac{a}{\sigma}}$. Therefore, inductively, one concludes that $T_a^* = T^*_{\frac{a}{(\sqrt{\sigma})^n}}$ for all $n\in\mathbb{N}\cup\{0\}$. Theorem 1.3 i), ii), iii), iv) and v), and Theorem 1.4 i), ii), iii) and iv) hold for all $n\geq 1$. □

Proofs of Theorem 1.3 vi) and vii), and Theorem 1.4 v) First of all, by applying Dominated Convergence Theorem to Theorem 1.4 iii) with $\alpha=\beta$ (respectively, Theorem 1.3 iii)), one obtains

$$C_{\mu,\nu,\alpha}(T^*-t)^{-\frac{2\alpha-1}{2\alpha}} \leq \|(\widehat{u},\widehat{b})(t)\|_{L^1}, \quad \forall t\in[0,T^*). \tag{65}$$

Choose $\delta = s+\frac{k}{2\sigma}$, with $k\in\mathbb{N}$ and $k\geq 2\sigma(2\alpha-1)$ (respectively, $k\geq 2\sigma\max\{\frac{5}{2},4-\alpha\}$), and $\delta_0 = s+2\alpha-1$ (respectively, $\delta_0 = s+\max\{\frac{5}{2},4-\alpha\}$). By using Lemma 2.1 in [1], Lemma 2.4 in [23], and (65), we obtain

$$C_{\mu,\nu,\alpha}(T^*-t)^{-\frac{2\alpha-1}{2\alpha}} \leq \|(\widehat{u},\widehat{b})(t)\|_{L^1} \leq C_s\|(u,b)(t)\|_{L^2}^{1-\frac{3}{2(s+\frac{k}{2\sigma})}}\|(u,b)(t)\|_{\dot{H}^{s+\frac{k}{2\sigma}}}^{\frac{3}{2(s+\frac{k}{2\sigma})}}.$$

On the other hand, by taking L^2-inner product of the first and second equations of (1), with u and b, respectively, and integrating the results obtained over $[0,t]$, we have

$$\|(u,b)(t)\|_{L^2} \leq \|(u_0,b_0)\|_{L^2}, \quad \forall 0\leq t<T^*. \tag{66}$$

As a consequence,

$$\frac{C_{\mu,\nu,s,\alpha,u_0,b_0}}{(T^*-t)^{\frac{2s(2\alpha-1)}{3\alpha}}}\left(\frac{C'_{\mu,\nu,\sigma,s,\alpha,u_0,b_0}}{(T^*-t)^{\frac{2\alpha-1}{3\alpha\sigma}}}\right)^k \leq \|(u,b)(t)\|^2_{\dot{H}^{s+\frac{k}{2\sigma}}}. \tag{67}$$

Multiplying (67) by $\dfrac{(2a)^k}{k!}$, one concludes that

$$\frac{C_{\mu,\nu,s,\alpha,u_0,b_0}}{(T^*-t)^{\frac{2s(2\alpha-1)}{3\alpha}}}\frac{\left(\frac{2aC'_{\mu,\nu,\sigma,s,\alpha,u_0,b_0}}{(T^*-t)^{\frac{2\alpha-1}{3\alpha\sigma}}}\right)^k}{k!} \leq \int_{\mathbb{R}^3}\frac{(2a|\xi|^{\frac{1}{\sigma}})^k}{k!}|\xi|^{2s}|(\widehat{u},\widehat{b})(t)|^2\,d\xi. \tag{68}$$

Finally, if we define

$$f(x)=\Big[e^x-\sum_{k=0}^{2\sigma_0}\frac{x^k}{k!}\Big][x^{-(2\sigma_0+1)}e^{-\frac{x}{2}}],\quad \forall x\in(0,\infty),$$

where $2\sigma_0$ is the integer part of $2\sigma(2\alpha-1)$ (respectively, $2\sigma\max\{\frac{5}{2},4-\alpha\}$), then there is a positive constant C_{σ_0} with $f(x)\geq C_{\sigma_0}$ for all $x>0$ (for more details, see [23]). Therefore, by summing over the set $\{k\in\mathbb{N};k\geq 2\sigma(2\alpha-1)\}$ (respectively, $\{k\in\mathbb{N};k\geq 2\sigma\max\{\frac{5}{2},4-\alpha\}\}$) and applying Monotone Convergence Theorem to (68), one obtains

$$\|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}}\geq\frac{a^{2\sigma_0+1}C_{\mu,\nu,s,\alpha,\sigma,\sigma_0,u_0,b_0}}{(T^*-t)^{\frac{(2\alpha-1)[2(s\sigma+\sigma_0)+1]}{3\alpha\sigma}}}\exp\left\{\frac{aC'_{\mu,\nu,\sigma,s,\alpha,u_0,b_0}}{(T^*-t)^{\frac{2\alpha-1}{3\alpha\sigma}}}\right\},\quad\forall t\in[0,T^*).$$

(Respectively, we have

$$\|(u,b)(t)\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\sigma}}\geq\frac{a^{2\sigma_0+1}C_{\mu,\nu,\alpha,\sigma,\sigma_0,u_0,b_0}}{(T^*-t)^{\frac{(2\alpha-1)[2((\frac{5}{2}-2\alpha)\sigma+\sigma_0)+1]}{3\alpha\sigma}}}\exp\left\{\frac{aC'_{\mu,\nu,\sigma,\alpha,u_0,b_0}}{(T^*-t)^{\frac{2\alpha-1}{3\alpha\sigma}}}\right\},\quad\forall t\in[0,T^*).)$$

The proofs of Theorem 1.3 vi) and vii), and Theorem 1.4 v) are given. □

6 Proof of Theorem 1.5

The proof of Theorem 1.5 was motivated by [23, 26].

Proof of Theorem 1.5 i) Suppose, by contradiction, that $\int_0^{T^*}\|(u,b)(\tau)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\,d\tau<\infty$. Thus, for $\epsilon>0$ there is $T\in(0,T^*)$ such that

$$\int_T^{T^*}\|(u,b)(\tau)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\,d\tau\leq\epsilon. \tag{69}$$

By applying (41) and (42) (with $\alpha=\beta$), we have

$$\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}}+[\mu+\nu]\|(u,b)(t)\|^2_{\dot{H}^{s+\alpha}_{a,\sigma}}\leq|\langle u,u\cdot\nabla u\rangle_{\dot{H}^s_{a,\sigma}}|+|\langle u,b\cdot\nabla b\rangle_{\dot{H}^s_{a,\sigma}}|$$
$$+|\langle b,u\cdot\nabla b\rangle_{\dot{H}^s_{a,\sigma}}|+|\langle b,b\cdot\nabla u\rangle_{\dot{H}^s_{a,\sigma}}|. \tag{70}$$

By applying Cauchy-Schwarz's inequality, we have

$$\begin{aligned}|\langle b,u\cdot\nabla b\rangle_{\dot{H}^s_{a,\sigma}}|&\leq\int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{b}||\mathcal{F}[u\cdot\nabla b]|\,d\xi\\&\leq\left(\int_{\mathbb{R}^3}|\xi|^{2(s+\alpha)}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{b}|^2\,d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|\xi|^{2(s-\alpha)}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}[u\cdot\nabla b]|^2\,d\xi\right)^{\frac{1}{2}}\\&\leq\|b\|_{\dot{H}^{s+\alpha}_{a,\sigma}}\|b\otimes u\|_{\dot{H}^{s+1-\alpha}_{a,\sigma}}.\end{aligned} \tag{71}$$

By Lemma 2.3, we infer that

$$\|b\otimes u\|_{\dot{H}^{s+1-\alpha}_{a,\sigma}} \le C_{s,\alpha}[\|b\|_{\dot{H}^{s}_{a,\sigma}}\|u\|_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\sigma}} + \|b\|_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\sigma}}\|u\|_{\dot{H}^{s}_{a,\sigma}}]$$
$$\le C_{s,\alpha}\|(u,b)\|_{\dot{H}^{s}_{a,\sigma}}\|(u,b)\|_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\sigma}}. \tag{72}$$

As a result, we can write

$$\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{\dot{H}^{s}_{a,\sigma}} + [\mu+\nu]\|(u,b)(t)\|^2_{\dot{H}^{s+\alpha}_{a,\sigma}} \le C_{s,\alpha}\|(u,b)\|_{\dot{H}^{s+\alpha}_{a,\sigma}}\|(u,b)\|_{\dot{H}^{s}_{a,\sigma}}\|(u,b)\|_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\sigma}}. \tag{73}$$

Similarly, we have

$$\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\sigma}} + [\mu+\nu]\|(u,b)(t)\|^2_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\sigma}} \le C_{\alpha}\|(u,b)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\sigma}}\|(u,b)\|^2_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\sigma}},$$

and, consequently

$$\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + [\mu+\nu]\|(u,b)(t)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}} \le C_{s,\alpha}\|(u,b)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}\|(u,b)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}. \tag{74}$$

Consider that $0\le T\le\tau\le t<T^*$, integrate the inequality above over $[T,\tau]$, and use (69) in order to obtain

$$\begin{aligned}&\|(u,b)(\tau)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + 2[\mu+\nu]\int_T^\tau \|(u,b)(r)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\,dr\\ &\le \|(u,b)(T)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + C_{s,\alpha}\sup_{T\le r\le t}\{\|(u,b)(r)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}\}\\ &\times\int_T^{T^*}\|(u,b)(r)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\,dr \le \|(u,b)(T)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + C_{s,\alpha}\epsilon\sup_{T\le r\le t}\{\|(u,b)(r)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}\}.\end{aligned}$$

By taking $0<\epsilon<\frac{1}{2C_{s,\alpha}}$ (where $C_{s,\alpha}$ is given in the inequality above), one concludes

$$\|(u,b)(\tau)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} \le \|(u,b)(T)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + \frac{1}{2}\sup_{T\le r\le t}\{\|(u,b)(r)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}\},$$

for all $\tau\in[T,t]$. As a consequence, one infers

$$\sup_{T\le\tau\le t}\{\|(u,b)(\tau)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}}\} \le \|(u,b)(T)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + \frac{1}{2}\sup_{T\le\tau\le t}\{\|(u,b)(\tau)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}\}.$$

Therefore, one has

$$\sup_{T\le\tau\le t}\{\|(u,b)(\tau)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}\} \le \frac{1}{4} + \left[\frac{1}{16} + \|(u,b)(T)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}}\right]^{\frac{1}{2}} =: C_{s,a,\sigma,\alpha,T}, \quad \forall t\in[T,T^*).$$

Now, let us denote $M_{s,a,\sigma,\alpha,T} = \max\{C_{s,a,\sigma,\alpha,T}, \sup_{0\le\tau\le T}\{\|(u,b)(\tau)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}\}\}$. Notice that $M_{s,a,\sigma,\alpha,T}$ is finite since $(u,b)\in C_T(\dot{H}^s_{a,\sigma})\cap C_T(\dot{H}^{\frac{5}{2}-2\alpha}_{a,\sigma})$. Thereby, we obtain

$$\|(u,b)(\tau)\|_{\dot{H}^{s,\alpha}_{a,\sigma}} \le M_{s,a,\sigma,\alpha,T}, \quad \forall\tau\in[0,T^*). \tag{75}$$

On the other hand, by applying Lemma 2.1 ii) (with $p = 4$) and (75), we have

$$\int_0^{T^*} \|(u,b)(\tau)\|^4_{\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma}}\, d\tau \le M^2_{s,a,\sigma,\alpha,T} \int_0^{T^*} \|(u,b)(\tau)\|^2_{\dot{H}^{s+\alpha}_{a,\sigma}}\, d\tau < \infty.$$

Similarly, we deduce that $\int_0^{T^*} \|(u,b)(\tau)\|^4_{\dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma}}\, d\tau < \infty$. Consequently, $(u,b) \in L^4([0,T^*); \dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma}) \cap L^4([0,T^*); \dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma})$. Thus, for $\varepsilon > 0$ there is $t_0 \in (0,T^*)$ such that

$$\|(u,b)\|^4_{L^4([t_0,T^*);\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma})} + \|(u,b)\|^4_{L^4([t_0,T^*);\dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma})} \le \varepsilon. \tag{76}$$

Now, consider the following system:

$$\begin{cases} U_t + U\cdot\nabla U + \nabla(P + \frac{1}{2}|B|^2) + \mu(-\Delta)^\alpha U = B\cdot\nabla B, \\ B_t + U\cdot\nabla B + \nu(-\Delta)^\alpha B = B\cdot\nabla U, \\ \operatorname{div} U = \operatorname{div} B = 0, \\ U(x,0) = u(x,t_0),\ \ B(x,0) = b(x,t_0). \end{cases} \tag{77}$$

Notice that $U(x,t) = u(x,t+t_0)$, $B(x,t) = b(x,t+t_0)$ (and $P(x,t) = p(x,t+t_0)$) solve the system (77) in $L^4([0,T^*-t_0); \dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma}) \cap L^4([0,T^*-t_0); \dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma})$. Moreover, by using (76), it is easy to check that

$$\begin{aligned} &\|(U,B)\|^4_{L^4([0,T^*-t_0);\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma})} + \|(U,B)\|^4_{L^4([0,T^*-t_0);\dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma})} \\ &= \|(u,b)\|^4_{L^4([t_0,T^*);\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma})} + \|(u,b)\|^4_{L^4([t_0,T^*);\dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma})} \le \epsilon. \end{aligned} \tag{78}$$

By choosing $0 < \varepsilon < \frac{\varsigma^3}{16C^4_{s,\alpha}}$ (where $C_{s,\alpha}$ is given in (27)), we find, by applying (78) and Lemma 2.1 in [25] (for more details, see [9]), a unique solution in the maximal interval $[0,T^*-t_0)$ which can be extended beyond T^*-t_0. In fact, it is enough to consider the following solution for the system (77) in $L^4([0,T^*); \dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma}) \cap L^4([0,T^*); \dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma})$:

$$(\tilde{u},\tilde{b})(t) = \begin{cases} (U,B)(t), & t \in [0,T^*-t_0); \\ (u,b)(t-(T^*-t_0)), & t \in [T^*-t_0,T^*). \end{cases}$$

This is an absurd. □

Proof of Theorem 1.5 ii) By integrating (74) from 0 to t (with $0 \le t < T^*$), we obtain

$$\begin{aligned} &\|(u,b)(t)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + 2[\mu+\nu]\int_0^t \|(u,b)(\tau)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\, d\tau \le \|(u_0,b_0)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + C_{s,\alpha} \\ &\times \int_0^t \|(u,b)(\tau)\|_{\dot{H}^{s,\alpha}_{a,\sigma}} \|(u,b)(\tau)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\, d\tau. \end{aligned}$$

Now, define $T := \sup_{t\in[0,T^*)} \{ \sup_{0\le\tau\le t} \{\|(u,b)(t)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}\} < \dfrac{\mu+\nu}{C_{s,\alpha}}\}$ (where $C_{s,\alpha}$ is given in (74)). By applying the fact that $\|(u_0,b_0)\|_{\dot{H}^{s,\alpha}_{a,\sigma}} < \frac{\mu+\nu}{C_{s,\alpha}}$ and $(u,b) \in C([0,T^*); \dot{H}^s_{a,\sigma}) \cap C([0,T^*);$

$\dot{H}_{a,\sigma}^{\frac{5}{2}-2\alpha}$), we have that $T \in (0, T^*]$. Thus, we conclude that

$$\|(u,b)(t)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + [\mu+\nu]\int_0^t \|(u,b)(\tau)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\,d\tau \le \|(u_0,b_0)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}}, \quad \forall t \in [0,T). \tag{79}$$

Assume, by contradiction, that $T < T^*$. Then, $(u,b) \in C_T(\dot{H}^s_{a,\sigma}) \cap C_T(\dot{H}_{a,\sigma}^{\frac{5}{2}-2\alpha})$. Thereby, by passing to the limit, as $t \nearrow T$ in (79), we infer

$$\|(u,b)(T)\|_{\dot{H}^{s,\alpha}_{a,\sigma}} \le \|(u_0,b_0)\|_{\dot{H}^{s,\alpha}_{a,\sigma}} < \frac{\mu+\nu}{C_{s,\alpha}}. \tag{80}$$

As result, there is $\overline{T} \in (T,T^*)$ (by using $(u,b) \in C([0,T^*);\dot{H}^s_{a,\sigma}) \cap C([0,T^*);\dot{H}_{a,\sigma}^{\frac{5}{2}-2\alpha})$ once more, (79) and (80)) such that $\sup\limits_{0\le\tau\le\overline{T}} \{\|(u,b)(\tau)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}\} < \dfrac{\mu+\nu}{C_{s,\alpha}}$. This is an absurd. Hence, $T = T^*$ and, consequently, by (79), one has

$$\int_0^{T^*} \|(u,b)(\tau)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\,d\tau \le \frac{1}{\mu+\nu}\|(u_0,b_0)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} < \frac{\mu+\nu}{C^2_{s,\alpha}} < \infty.$$

By Theorem 1.5 i), we have that $T^* = \infty$ and, as a result, by (79), one deduces

$$\|(u,b)(t)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + [\mu+\nu]\int_0^t \|(u,b)(\tau)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\,d\tau \le \|(u_0,b_0)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}}, \quad \forall t \ge 0. \qquad \square$$

Proof of Theorem 1.5 iii) By applying Theorem 1.2, the GMHD equations (1) (with initial data (v_0,w_0)) admit maximal solution $(v,w) \in C([0,\tilde{T});\dot{H}^s_{a,\sigma}) \cap C([0,\tilde{T});\dot{H}_{a,\sigma}^{\frac{5}{2}-2\alpha})$. Thus, let us prove that $\tilde{T} = \infty$.

Define $\overline{u}(x,t) = u(x,t) - v(x,t)$ and $\overline{b}(x,t) = b(x,t) - w(x,t)$ (and $\overline{p}(x,t) = p(x,t) - p_1(x,t)$) for all $x \in \mathbb{R}^3$ and $t \in [0,\tilde{T})$. (Here p_1 is the pressure associated with (v,w).) As a result, we conclude that $(\overline{u},\overline{b}) \in C([0,\tilde{T});\dot{H}^s_{a,\sigma}) \cap C([0,\tilde{T});\dot{H}_{a,\sigma}^{\frac{5}{2}-2\alpha})$ (since $(u,b) \in C([0,\infty);\dot{H}^s_{a,\sigma}) \cap C([0,\infty);\dot{H}_{a,\sigma}^{\frac{5}{2}-2\alpha})$). Moreover, we have the following system:

$$\begin{cases} \overline{u}_t + u\cdot\nabla\overline{u} + \overline{u}\cdot\nabla u + \overline{b}\cdot\nabla\overline{b} + \nabla(\overline{p} + \frac{1}{2}|b|^2 - \frac{1}{2}|w|^2) + \mu(-\Delta)^\alpha\overline{u} \\ \quad = \overline{u}\cdot\nabla\overline{u} + b\cdot\nabla\overline{b} + \overline{b}\cdot\nabla b, \\ \overline{b}_t + u\cdot\nabla\overline{b} + \overline{u}\cdot\nabla b + \overline{b}\cdot\nabla\overline{u} + \nu(-\Delta)^\alpha\overline{b} = \overline{u}\cdot\nabla\overline{b} + b\cdot\nabla\overline{u} + \overline{b}\cdot\nabla u, \\ \operatorname{div}\overline{u} = \operatorname{div}\overline{b} = 0, \\ \overline{u}(\cdot,0) = \overline{u}_0 := u_0(\cdot) - v_0(\cdot), \ \overline{b}(\cdot,0) = \overline{b}_0 := b_0(\cdot) - w_0(\cdot). \end{cases} \tag{81}$$

On the other hand, analogously to (72), we obtain

$$\|b\otimes u\|_{\dot{H}^{s+1-\alpha}_{a,\sigma}} \le C_{s,\alpha}\|b\|_{\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma}}\|u\|_{\dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma}} \le C_{s,\alpha}\|(u,b)\|_{\dot{H}^{s+\frac{\alpha}{2}}_{a,\sigma}}\|(u,b)\|_{\dot{H}^{\frac{5-3\alpha}{2}}_{a,\sigma}}. \tag{82}$$

By applying (71) and (82) and arguing as in (74), one concludes

$$\frac{1}{2}\frac{d}{dt}\|(\overline{u},\overline{b})(t)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + [\mu+\nu]\|(\overline{u},\overline{b})(t)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}} \le C_{s,\alpha}[\|(\overline{u},\overline{b})\|_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\|(\overline{u},\overline{b})\|^2_{\dot{H}^{s+\frac{\alpha}{2},\frac{3\alpha}{4}}_{a,\sigma}}$$

$$+ \|(\overline{u},\overline{b})\|_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\|(u,b)\|_{\dot{H}^{s+\frac{\alpha}{2},\frac{3\alpha}{4}}_{a,\sigma}}\|(\overline{u},\overline{b})\|_{\dot{H}^{s+\frac{\alpha}{2},\frac{3\alpha}{4}}_{a,\sigma}}].$$

By Lemma 2.1 ii) (with $p = 4$) and Young's inequality, we deduce

$$\frac{1}{2}\frac{d}{dt}\|(\overline{u},\overline{b})(t)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + [\mu+\nu]\|(\overline{u},\overline{b})(t)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}} \leq C_{s,\alpha}[\|(\overline{u},\overline{b})\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\|(\overline{u},\overline{b})\|_{\dot{H}^{s,\alpha}_{a,\sigma}}$$

$$+ \|(u,b)\|_{\dot{H}^{s+\frac{\alpha}{2},\frac{3\alpha}{4}}_{a,\sigma}}\|(\overline{u},\overline{b})\|^{\frac{3}{2}}_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\|(\overline{u},\overline{b})\|^{\frac{1}{2}}_{\dot{H}^{s,\alpha}_{a,\sigma}}] \leq C_{s,\alpha}[\mu+\nu]^{-3}[\|(\overline{u},\overline{b})\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\|(\overline{u},\overline{b})\|_{\dot{H}^{s,\alpha}_{a,\sigma}}$$

$$+ \|(u,b)\|^4_{\dot{H}^{s+\frac{\alpha}{2},\frac{3\alpha}{4}}_{a,\sigma}}\|(\overline{u},\overline{b})\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}}] + \frac{\mu+\nu}{2}\|(\overline{u},\overline{b})\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}.$$

By integrating the inequality above from 0 to t (with $0 \leq t < \tilde{T}$), we obtain

$$\|(\overline{u},\overline{b})(t)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + \frac{\mu+\nu}{2}\int_0^t \|(\overline{u},\overline{b})(\tau)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\,d\tau \leq \|(\overline{u}_0,\overline{b}_0)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}}$$

$$+ C_{s,\alpha}[\mu+\nu]^{-3}\int_0^t [\|(\overline{u},\overline{b})(\tau)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}$$

$$\times \|(\overline{u},\overline{b})(\tau)\|_{\dot{H}^{s,\alpha}_{a,\sigma}} + \|(u,b)(\tau)\|^4_{\dot{H}^{s+\frac{\alpha}{2},\frac{3\alpha}{4}}_{a,\sigma}}\|(\overline{u},\overline{b})(\tau)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}}]\,d\tau. \tag{83}$$

Now, define $T := \sup_{t\in[0,\tilde{T})}\{\sup_{0\leq\tau\leq t}\{\|(\overline{u},\overline{b})(t)\|_{\dot{H}^{s,\alpha}_{a,\sigma}}\} < \frac{[\mu+\nu]^4}{4C_{s,\alpha}}\}$ (where $C_{s,\alpha}$ is given in (83)). Thus, $T \in (0,\tilde{T}]$ (see the proof of Theorem 1.5 ii) and our assumptions). Therefore, (83) implies that

$$\|(\overline{u},\overline{b})(t)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + \frac{\mu+\nu}{4}\int_0^t \|(\overline{u},\overline{b})(\tau)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\,d\tau \leq \|(\overline{u}_0,\overline{b}_0)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}}$$

$$+ C_{s,\alpha}[\mu+\nu]^{-3}\int_0^t \|(u,b)(\tau)\|^4_{\dot{H}^{s+\frac{\alpha}{2},\frac{3\alpha}{4}}_{a,\sigma}}\|(\overline{u},\overline{b})(\tau)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}}\,d\tau, \quad \forall t \in [0,T). \tag{84}$$

By applying Gronwall's inequality and our hypothesis, one infers

$$\|(\overline{u},\overline{b})(t)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}} + \frac{\mu+\nu}{4}\int_0^t \|(\overline{u},\overline{b})(\tau)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\,d\tau \leq \|(\overline{u}_0,\overline{b}_0)\|^2_{\dot{H}^{s,\alpha}_{a,\sigma}}$$

$$\times \exp\{C_{s,\alpha}[\mu+\nu]^{-3}\int_0^\infty \|(u,b)(t)\|^4_{\dot{H}^{s+\frac{\alpha}{2},\frac{3\alpha}{4}}_{a,\sigma}}\,dt\} < \frac{[\mu+\nu]^8}{16C^2_{s,\alpha}}, \tag{85}$$

for all $t \in [0,T)$. Similarly to the proof of Theorem 1.5 ii), we conclude that $T = \tilde{T}$ and, consequently, $\int_0^{\tilde{T}} \|(\overline{u},\overline{b})(\tau)\|^2_{\dot{H}^{s+\alpha,\frac{\alpha}{2}}_{a,\sigma}}\,d\tau < \infty$. By applying Theorem 1.5 i), we obtain that $\tilde{T} = \infty$. Then, the result follows from (84). □

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