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Existence of Invariant Densities and Time Asymptotics of Conservative Linear Kinetic Equations on the Torus Without Spectral Gaps

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Abstract
This work deals with general linear conservative neutron transport semigroups without spectral gaps in $L^1(\mathcal{T}^n \times \mathbb{R}^n)$ where $\mathcal{T}^n$ is the n-dimensional torus. We study the mean ergodicity of such semigroups and their strong convergence to their ergodic projections as time goes to infinity. Systematic functional analytic results are given.

Keywords Neutron transport equation · Invariant density · Absence of spectral gap · Compactness

Mathematics Subject Classification Primary 82C40 · Secondary 35F15 · 47D06

1 Introduction

This paper is a continuation of a previous work devoted to conservative neutron transport equations on the torus with spectral gaps [35]. The lack of spectral gaps leads to two key open problems which are the main concern of this paper: the existence of an invariant density and the strong convergence of neutron transport semigroups to their ergodic projections as time goes to infinity. The role of positivity in nuclear reactor theory was emphasized very early by Garrett Birkhoff (see e.g. [10–13]) and, since then, has not ceased to be taken into account in the mathematical literature on neutron transport. It turns out that *peripheral* spectral theory, the heart of asymptotics of discrete or continuous semigroups, is well established for positive operators on Banach lattices (see e.g. [7, 39]).

We show here how positivity, combined to compactness arguments, allows to build a general theory of qualitative time asymptotics for L^1-conservative neutron transport equations without spectral gaps; we provide systematic functional analytic results. While our analysis is purely qualitative, a simpler situation on the torus with space homogeneous cross sections is dealt with in a paper devoted to the delicate problem of (algebraic) rates of convergence to equilibrium [24]. We mention that a quantitative version of the present paper relying on a different construction is given in a forthcoming work [26].

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Note that beyond neutron transport theory, time asymptotics have been intensively analyzed the last two decades in different areas of kinetic theory. We cannot comment on a considerable literature which deals with so many kinetic models (different vector fields or scattering operators, different boundary conditions, L^1-conservative models or L^2-dissipative models, bounded or unbounded geometries etc.) with quantitative or just qualitative convergence results relying on different mathematical tools, mainly spectral, hypocoercivity or entropy methods. For information only, and without claiming to be complete, we refer e.g. to the following works which provide us with a sample of linear kinetic problems involved in time asymptotics [2, 5, 6, 8, 15–20, 23–25, 27–30, 32–38, 41, 42].

In [35], we characterized the existence of a spectral gap, i.e. the strict inequality

$$\omega_{ess} < \omega,$$

for a general class of conservative neutron transport semigroups $(W(t))_{t\geq 0}$ on the n-dimensional torus, where ω and ω_{ess} are respectively the type and the *essential* type of $(W(t))_{t\geq 0}$, (see the definition below). This characterization is based upon *two* ingredients: the computation of the type of collisionless (i.e. advection) kinetic semigroups and a general L^1-compactness theorem implying a stability of essential type of perturbed semigroups. The presence of a spectral gap provides us automatically with an invariant density from which we can derive, by standard functional analytic arguments (see e.g. [7, 39]), the exponential trend of such semigroups to the spectral projection associated to the zero eigenvalue of their generators.

Our aim here is to analyze this class of kinetic equations in the absence of spectral gap, i.e. when

$$\omega_{ess} = \omega.$$

The lack of spectral gap leads to two open problems which are the subject matter of this paper. Before explaining the nature of such problems and outlining our main results, we need to review quickly some results from [35]. Let $n \in \mathbb{N}$ and let

$$\mathcal{T}^n := \mathbb{R}^n/(\mathbb{Z})^n$$

be the n-dimensional torus. We will identify any function

$$p : x \in \mathcal{T}^n \to p(x) \in \mathbb{R}$$

to a $[0,1]^n$-periodic function on $\mathbb{R}^n$.

We are concerned with time asymptotics of conservative neutron transport equations

$$\frac{\partial f}{\partial t} + v.\frac{\partial f}{\partial x} + \sigma(x,v) f(t,x,v) = \int_V k(x,v,v') f(t,x,v')\mu(dv') \qquad (1)$$

on

$$L^1(\mathcal{T}^n \times V) \quad (n \geq 1)$$

where V is the support of a σ-finite measure $\mu(dv)$ on $\mathbb{R}^n$ such that

$$\text{the affine hyperplanes have zero } \mu\text{-measure} \qquad (2)$$

even if this assumption is *not* necessary in all our statements (while a key compactness result needs a stronger assumption; see below). The *conservativity* assumption refers to the condition

$$\sigma(x, v) = \int_V k(x, v', v)\mu(dv'). \tag{3}$$

Here $L^1(\mathcal{T}^n \times V)$ is identified isometrically to the space of measurable and $[0, 1]^n$-periodic (with respect to $x \in \mathbb{R}^n$) functions

$$\varphi : \mathbb{R}^n \times V \ni (x, v) \to \varphi(x, v) \in \mathbb{R}$$

with finite norm

$$\int_{[0,1]^n \times V} |\varphi(x, v)|\, dx \mu(dv).$$

We refer to $\sigma(.,.)$ as the collision frequency and assume (for simplicity) that

$$\sigma \in L^\infty(\mathcal{T}^n \times V).$$

The partial integral operator

$$K : L^1(\mathcal{T}^n \times V) \ni \varphi(.,.) \to \int_V k(x, v, v')\varphi(x, v')\mu(dv') \in L^1(\mathcal{T}^n \times V)$$

is called the scattering (or collision) operator; we refer to its kernel $k(.,.,.)$ as the scattering kernel. We note that (3) implies that K is a bounded operator.

For each $v \in V$, $\sigma(., v)$ is identified to a $[0, 1]^n$-periodic function on $\mathbb{R}^n$. Similarly, for each $v, v' \in V$, $k(., v', v)$ is identified to a $[0, 1]^n$-periodic function on $\mathbb{R}^n$. The "collisionless" equation on $\mathcal{T}^n \times V$

$$\frac{\partial f}{\partial t} + v.\frac{\partial f}{\partial x} + \sigma(x, v) f(t, x, v) = 0, \quad f(0, x, v) = f_0(x, v)$$

is solved explicitly by the method of characteristics by means of a weighted shift C_0-semigroup $(U(t))_{t \geq 0}$ acting as

$$L^1(\mathcal{T}^n \times V) \ni \varphi \to e^{-\int_0^t \sigma(x-sv,v)ds}\varphi(x - tv, v) \in L^1(\mathcal{T}^n \times V) \ \ (t \geq 0). \tag{4}$$

We denote by T its generator. The type (or growth bound) of $(U(t))_{t\geq 0}$, i.e.

$$\omega(U) := \lim_{t\to+\infty} \frac{\ln(\|U(t)\|)}{t},$$

is equal to

$$\omega(U) = -\lim_{t\to+\infty} \inf_{(x,v)\in\mathcal{T}^n\times V} t^{-1} \int_0^t \sigma(x + sv, v)ds,$$

(see [35] Theorem 1). In particular

$$\omega(U) < 0$$

if and only if there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\int_0^{C_1} \sigma(x + sv, v)ds \geq C_2 \ \text{ a.e. on } \mathcal{T}^n \times V, \tag{5}$$

(see [35] Corollary 2; see also [8] for an earlier result in this direction).

The full dynamics (1) on $L^1(\mathcal{T}^n \times V)$ is governed by a C_0-semigroup $(W(t))_{t\geq 0}$ generated by

$$A := T + K, \quad D(A) = D(T). \tag{6}$$

We note that under (3)

$$\int_{\mathcal{T}^n \times V} A\varphi = 0, \quad \forall \varphi \in D(A)$$

so

$$\frac{d}{dt} \int_{\mathcal{T}^n \times V} W(t)\varphi = \int_{\mathcal{T}^n \times V} AW(t)\varphi = 0 \ \ (t \geq 0)$$

for all $\varphi \in D(A)$ and (by a density argument)

$$\int_{\mathcal{T}^n \times V} W(t)\varphi = \int_{\mathcal{T}^n \times V} \varphi \ \ (t \geq 0)$$

for all $\varphi \in L^1(\mathcal{T}^n \times V)$. Thus $(W(t))_{t\geq 0}$ is a *stochastic* (or Markov) semigroup, i.e. $W(t)$ is mass-preserving on the positive cone

$$\|W(t)\varphi\| = \|\varphi\| \quad \forall \varphi \in L^1_+(\mathcal{T}^n \times V),$$

in particular its type is equal to zero

$$\omega(W) = 0.$$

This perturbed semigroup $(W(t))_{t\geq 0}$ is given by a Dyson-Phillips series

$$W(t) = \sum_{j=0}^{\infty} U_j(t) \tag{7}$$

where

$$U_0(t) = U(t) \text{ and } U_{j+1}(t) = \int_0^t U(t-s)KU_j(s)ds \ \ (j \geq 0). \tag{8}$$

We recall that any C_0-semigroup $(Z(t))_{t\geq 0}$ in a Banach space X admits a type $\omega(Z)$ such that

$$r(Z(t)) = e^{\omega(Z)t} \ \ (t > 0)$$

($r(Z(t))$ is the spectral radius of $Z(t)$) and an *essential* type $\omega_{ess}(Z)$ such that

$$r_{ess}(Z(t)) = e^{\omega_{ess}(Z)t} \ \ (t > 0)$$

where

$$r_{ess}(Z(t)) := \sup\{|\lambda|;\ \lambda \in \sigma_{ess}(Z(t))\}$$

is the *essential* spectral radius of $Z(t)$ and $\sigma_{ess}(Z(t))$ is the essential spectrum of $Z(t)$; in particular

$$\omega_{ess}(Z) \le \omega(Z).$$

We recall also that $(U(t))_{t\ge 0}$ and $(W(t))_{t\ge 0}$ have the *same* essential type provided that *some* $U_j(t)$ is a compact operator for all $t > 0$, (see [31] Theorem 2.10); in this case, $\omega_{ess}(W)$, the essential type of $(W(t))_{t\ge 0}$ is such that

$$\omega_{ess}(W) \le \omega(U).$$

In particular, if $\omega(U) < 0$ (or equivalently if (5) is satisfied) then

$$\omega_{ess}(W) < 0 = \omega(W)$$

i.e. $(W(t))_{t\ge 0}$ exhibits a *spectral gap* and 0 is an isolated eigenvalue of $T + K$ with finite algebraic multiplicity. If $(W(t))_{t\ge 0}$ is irreducible then 0 is algebraically simple and is associated to a unique positive normalized eigenfunction u, i.e. a unique *invariant density*. In this case, there exist $\varepsilon > 0$ and $C > 0$ such that for any density φ

$$\|W(t)\varphi - u\| \le Ce^{-\varepsilon t}\ \ (t \ge 0).$$

Let us recall the so-called *regular* scattering operators used in [35]. Since a scattering operator is local in the space variable, we may regard it as a bounded mapping

$$K : \mathcal{T}^n \ni x \to K(x) \in \mathcal{L}(L^1(V))$$

acting on $L^1(\mathcal{T}^n \times V)$ as

$$K\varphi = K(x)\varphi(x)$$

where we identify $L^1(\mathcal{T}^n \times V)$ to $L^1(\mathcal{T}^n;\ L^1(V))$. In this case

$$\|K\|_{\mathcal{L}(L^1(\mathcal{T}\times V))} = \sup_{x\in\mathcal{T}^n} \|K(x)\|_{\mathcal{L}(L^1(V))}\,.$$

The main feature of a regular scattering operator K is that the family of operators

$$L^1(V) \ni \varphi \to \int_V k(x, v, v')\varphi(v')\mu(dv') \in L^1(V)$$

(indexed by the space variable $x \in \mathcal{T}^n$) is *collectively* weakly compact, i.e.

$$\left\{k(x, ., v');\ (x, v') \in \mathcal{T}^n \times V\right\} \text{ is relatively weakly compact in } L^1(V).$$

For instance, a sufficient condition for K to be regular is the existence of $p \in L^1_+(V)$ such that

$$k(x, v, v') \le p(v).$$

Finally, we used a general class of measures $\mu(dv)$ defined by the existence of $\alpha > 0$ such that for any bounded set $S \subset V$ there exists $c_S > 0$ and

$$\sup_{e \in S^{n-1}} d\mu \otimes d\mu \left\{ (v, v') \in S \times S; \ \left| (v - v').e \right| < \varepsilon \right\} \leq c_S \varepsilon^{\alpha}. \tag{9}$$

This assumption is stronger than (2) but is satisfied by the Lebesgue measure on $\mathbb{R}^n$ or on spheres (i.e. for multigroup models). We showed that if the scattering operator K is regular and if the measure $\mu(dv)$ satisfies (9) then there exists an integer $\widehat{j}$ depending on α such that

$$U_j(t) \text{ is a compact operator}, \quad (t \geq 0) \ (j \geq \widehat{j})$$

(see [35] Theorem 13). Thus, if the scattering operator is regular and if the measure $\mu(dv)$ satisfies (9) then the semigroups $(W(t))_{t\geq 0}$ exhibits a spectral gap if and only if

$$\lim_{t \to +\infty} \inf_{(x,v) \in \mathcal{T}^n \times V} t^{-1} \int_0^t \sigma(x + sv, v) ds > 0.$$

The purpose of this paper is to deal with the *critical case*

$$\lim_{t \to +\infty} \inf_{(x,v) \in \mathcal{T}^n \times V} t^{-1} \int_0^t \sigma(x + sv, v) ds = 0. \tag{10}$$

We are thus faced with two key open questions:

(i) *Does $(W(t))_{t\geq 0}$ admit an invariant density or more generally is $(W(t))_{t\geq 0}$ mean ergodic?*
(ii) *If so, does $(W(t))_{t\geq 0}$ converge strongly to its ergodic projection as $t \to +\infty$?*

A systematic functional analytic treatment of these problems is provided. We deal mainly with the relevant case corresponding to non trivial scattering operators, i.e.

$$K \neq 0, \tag{11}$$

and will comment briefly on the case $K = 0$ in Sect. 10.

We point out that apart from the special case of kinetic equations obeying a *detailed balance principle* where the existence of an invariant density is obtained for free (see Remark 26), no existence result in kinetic theory is available up to now. We note also that our construction is based on many preliminary results of independent interest. As far as we know, most of our results are new and appear here for the first time. The author thanks the referees for useful remarks and suggestions.

1.1 The General Strategy for the Existence of an Invariant Density

The existence of an invariant density is the cornerstone of this work. To deal with this key question, our strategy consists in approximating the semigroup $(W(t))_{t\geq 0}$ by a sequence $\left\{ \left(W^j(t)\right)_{t\geq 0} \right\}_j$ of stochastic semigroups *with spectral gaps*. To this end, we choose non trivial $f \in L^1_+(V)$ and $g \in L^\infty_+(V)$ with

$$\eta := \inf g > 0$$

where inf refers to the essential infimum. We will define in Sect. 2 a *compact* set $\Pi \subset V$ (see (26)) such that $\mu(\Pi^c) > 0$ where Π^c is the complement of Π in V. We choose f such that

$$f = 0 \text{ a.e. on a neighborhood of } \Pi. \tag{12}$$

Let

$$k_j(x, v, v') := k(x, v, v') + \frac{1}{j} f(v) g(v') \tag{13}$$

and let

$$\sigma_j(x, v) := \int_V k_j(x, v', v)\mu(dv') = \sigma(x, v) + \frac{1}{j}\left(\int_V f(v')\mu(dv')\right) g(v). \tag{14}$$

Note that

$$\sigma_j(x, v) \geq \sigma(x, v) \ \ (j \in \mathbb{N}).$$

We consider the C_0-semigroup $\left(U^j(t)\right)_{t\geq 0}$ of contractions on $L^1(\mathcal{T}^n \times V)$

$$L^1(\mathcal{T}^n \times V) \ni \varphi \to e^{-\int_0^t \sigma_j(x-sv,v)ds}\varphi(x - tv, v) \in L^1(\mathcal{T}^n \times V) \ \ (t \geq 0)$$

and denote by T_j its generator. Let $\left(W^j(t)\right)_{t\geq 0}$ be the perturbed *stochastic* semigroup generated by

$$A_j := T_j + K_j \tag{15}$$

where K_j is the scattering operator with kernel $k_j(x, v, v')$. Since

$$\sigma_j(x, v) \geq \frac{\eta \, \|f\|_{L^1}}{j}$$

then the type ω_j of $\left\{U^j(t); t \geq 0\right\}$ is *negative*

$$\omega_j \leq -\frac{\eta \, \|f\|_{L^1}}{j}.$$

In particular 0 belongs to the resolvent *set* of T_j

$$0 \in \rho(T_j).$$

We note that K_j is also a regular scattering operator. According to the theory given in [35], $\left(W^j(t)\right)_{t\geq 0}$ is a conservative semigroup having a spectral gap so there exists a non trivial nonnegative eigenfunction φ_j of $T_j + K_j$ relative to the isolated eigenvalue 0

$$T_j\varphi_j + K_j\varphi_j = 0$$

or equivalently

$$(0 - T_j)^{-1} K_j \varphi_j = \varphi_j.$$

A key result of the paper is that, under a suitable assumption on the scattering operator (see below), the normalized sequence $\{\varphi_j\}_j$ is compact and then a convergent subsequence provides us with an invariant density. This compactness property follows from a fundamental collective compactness theorem (see below).

We note that beyond kinetic equations (i.e. in the general setting of abstract perturbed substochastic semigroups in L^1-spaces), $K(\lambda - T)^{-1}$ $(\lambda > 0)$ is a contraction. Since $(0, +\infty) \ni \lambda \to (\lambda - T)^{-1}$ is nonincreasing then a strong limit $\lim_{\lambda \to 0_+} K(\lambda - T)^{-1}$ exists we denote symbolically by $K(0_+ - T)^{-1}$ even if $0 \in \sigma(T)$. There exists a connection between the fact that 0 is an eigenvalue of A and the fact that 1 is an eigenvalue of $K(0_+ - T)^{-1}$ [36]; (see also [9] for more results in this direction). We follow here a different strategy which exploits fully the properties of kinetic equations.

1.2 The Results

Our main results are the existence of an invariant density and the strong convergence of $(W(t))_{t \geq 0}$ to its ergodic projection as $t \to +\infty$ (see Theorem 19 and Theorem 24). However, to state them understandably, we need to explain first several facts. The lack of a spectral gap steems from a *degeneracy* of the collision frequency, i.e. when σ vanishes on suitable sets. In order to avoid additional technicalities when V is unbounded, we assume in this case that

$$\lim_{|v| \to \infty} \inf \left(ess\text{-} \inf_{x \in \mathcal{T}^n} \sigma(x, v) \right) > 0. \tag{16}$$

Despite the restriction (16) when V is unbounded, we build a very rich mathematical theory for stochastic kinetic semigroups without spectral gaps.

Section 2 and Sect. 4 are devoted to the *characterization of the lack of spectral gap* and to various related technical results. The lack of spectral gap is due to the *non emptiness* of the set

$$\Xi := \{(x, v) \in \mathcal{T}^n \times V; \sigma_t(x, v) = 0,\ t > 0\}$$

where

$$\sigma_t : \mathcal{T}^n \times V \ni (x, v) \to \int_0^t \sigma(x + sv, v) ds.$$

We show that if

$$\sigma_t : \mathcal{T}^n \times V \ni (x, v) \to \int_0^t \sigma(x + sv, v) ds \ \text{ is continuous } (t > 0)$$

then Ξ consists of

$$\{(x, v) \in \mathcal{T}^n \times V;\ \sigma(x + sv, v) = 0 \quad \text{a.e. } s > 0\}$$

and (if $0 \in V$) of

$$\{(x, 0);\ x \in \mathcal{T}^n;\ \sigma(x, 0) = 0\},$$

(see Corollary 3).

Let Π be the projection of Ξ on V along $\mathcal{T}^n$, i.e.

$$\Pi := \{v \in V;\ \exists\, (x, v) \in \Xi\}.$$

We show that if the C_0-semigroup $(W(t))_{t\geq 0}$ is *irreducible* and has an invariant density then $(U(t))_{t\geq 0}$ must be *strongly stable*, i.e.

$$U(t)\varphi \to 0\ \ (t \to +\infty),\ \ \varphi \in L^1(\mathcal{T}^n \times V),$$

(see Theorem 11). Thus the strong stability of $(U(t))_{t\geq 0}$ appears as a *prerequisit* of our construction. It turns out that this strong stability is characterized by

$$\int_0^{+\infty} \sigma(x + sv, v)ds = +\infty\ \text{ a.e.}$$

(see Theorem 12). It follows that if (2) is satisfied, i.e. if the hyperplanes of $\mathbb{R}^n$ have zero μ-measure, and if

$$\int_{\mathcal{T}^n} \sigma(x, v)dx > 0\ \ \mu\text{-a.e.}$$

then $(U(t))_{t\geq 0}$ is strongly stable (see Corollary 14). We show that if $(U(t))_{t\geq 0}$ is strongly stable then

$$\Xi \text{ has } \textit{zero } dx \otimes \mu(dv) \text{ measure,}$$

(see Theorem 15).

We show that if $K \neq 0$ then Π *cannot* be of full measure, i.e.

$$\mu(\Pi^c) > 0$$

is always true where Π^c is the complement of Π in V, (see Proposition 27). We note that a priori the μ-measure of the compact set Π need not be zero. However, for the sake of simplicity, we restrict ourselves to the case

$$\mu(\Pi) = 0. \tag{17}$$

Remark 1 If $\mu(\Pi) > 0$, we need to assume additionally that

$$k(x, v, v') = 0 \text{ a.e. on } \mathcal{T}^n \times \Pi \times V.$$

In this case, $(W(t))_{t\geq 0}$ leaves invariant the closed subspace $L^1(\mathcal{T}^n \times \Pi^c)$ of

$$L^1(\mathcal{T}^n \times V) = L^1(\mathcal{T}^n \times \Pi^c) \oplus L^1(\mathcal{T}^n \times \Pi)$$

and the construction of this paper can be done in $L^1(\mathcal{T}^n \times \Pi^c)$ while the action of $(W(t))_{t\geq 0}$ on $L^1(\mathcal{T}^n \times \Pi)$ reduces to a shift semigroup. We did not try to elaborate on this point here.

We show also the *key* result that for any closed set

$$\Lambda \subset \Pi^c,$$

the restriction $\left(U^{\Lambda}(t)\right)_{t\geq 0}$ of the advection semigroup $(U(t))_{t\geq 0}$ to the closed subspace $L^1(\mathcal{T}^n \times \Lambda)$ (which is invariant under $(U(t))_{t\geq 0}$) has a negative type, i.e.

$$c_{\Lambda} := \lim_{t\to+\infty} \inf_{(x,v)\in\mathcal{T}^n\times\Lambda} t^{-1}\int_0^t \sigma(x+sv,v)ds > 0$$

or equivalently there exists $t_{\Lambda} > 0$ such that

$$t^{-1}\int_0^t \sigma(x+sv,v)ds \geq \frac{c_{\Lambda}}{2} \ \ \forall(x,v)\in\mathcal{T}^n\times\Lambda \ \ (\forall t\geq t_{\Lambda}), \tag{18}$$

(see Proposition 4). Thus, the *unbounded* function

$$\Theta : \mathcal{T}^n \times \Pi^c \ni (x,v) \to \int_0^{+\infty} e^{-\int_0^t \sigma(x+sv,v)ds} dt$$

(note that $\Theta(.,.) = +\infty$ on $\Xi \subset \mathcal{T}^n \times \Pi$) is such that for any closed set $\Lambda \subset \Pi^c$

$$\Theta(x,v) \leq t_{\Lambda} + \frac{2}{c_{\Lambda}} \ \ \forall(x,v)\in\mathcal{T}^n\times\Lambda,$$

i.e. $\Theta(x,v)$ gets large near $\mathcal{T}^n \times \Pi$ only.

In Sect. 3, we give a general sufficient criterion of irreducibility of $(W(t))_{t\geq 0}$, (see Proposition 7).

Our proof of the existence of an invariant density is based on the key assumption

$$\sup_{(y,v')\in\mathcal{T}^n\times V}\left(\int_{\{v;\ 0<dist(v,\Pi)<\varepsilon\}} \Theta(y,v)k(y,v,v')\mu(dv)\right) \to 0 \ \ (\varepsilon\to 0) \tag{19}$$

which expresses, in a suitable way, that

"$k(y,v,v') \to 0$" uniformly in (y,v') as $dist(v,\Pi) \to 0$.

The main statement of this paper is:

Main Theorem *Let the type of $(U(t))_{t\geq 0}$ be equal to zero, i.e. (10). We assume that $(W(t))_{t\geq 0}$ is irreducible and that (17) (19) are satisfied. Then $(W(t))_{t\geq 0}$ has an invariant density and converges strongly to its ergodic projection as $t \to +\infty$.*

This result follows from a series of preliminary results scattered in the different sections. In Sect. 5, we show that under (19) and (9) there exists $N \in \mathbb{N}$ such that the sequence

$$\left(\left((0-T_j)^{-1}K_j\right)^N\right)_j$$

is collectively compact in $\mathcal{L}(L^1(\mathcal{T}^n \times V))$, (see Theorem 18) in the sense that the image by $\left((0-T_j)^{-1}K_j\right)^N$ of the unit ball of $L^1(\mathcal{T}^n \times V)$ is included in a compact set independent of $j \in \mathbb{N}$. This important theorem is based on a key technical result (see Lemma 17).

In Sect. 6, we show the existence of an invariant density under (19) and (9), (see Theorem 19). The proof follows from the above collective compactness theorem.

Section 7 is devoted to the analysis of the key assumption (19) when the degeneracy of the collision frequency "*is not spatial*" in the sense that

$$\widehat{\sigma}(v) := \inf_{x\in\mathcal{T}^n} \sigma(x,v) > 0 \text{ a.e.}$$

and $\inf_{v\in V} \widehat{\sigma}(v) = 0$. In this case, (19) holds provided that

$$\int_V \frac{\widehat{k}(v,v')}{\widehat{\sigma}(v)} dv < +\infty$$

(where $\widehat{k}(v,v') := \sup_{x\in\mathcal{T}^n} k(x,v,v')$) and the convergence of this integral is uniform in $v' \in V$ (see Proposition 20).

In Sect. 8, we give two results for space homogeneous cross sections. Indeed, under an irreducibility condition, we show that the invariant density (if any) must be space homogeneous; we show also how to derive from the previous results the existence of an invariant density for *space homogeneous* equations i.e. $\phi \in L^1_+(V)$ such that

$$-\sigma(v)\phi(v) + \int_V k(v,v')\phi(v')\mu(dv') = 0,$$

(see Theorem 22). Section 9 is devoted to time asymptotics when $(W(t))_{t\geq 0}$ is irreducible. If an invariant density exists (i.e. the kernel of the generator is not trivial), the one-dimensional ergodic projection is given by

$$\mathcal{P} : L^1(\mathcal{T}^n \times V) \ni h \to \left(\int_{\mathcal{T}^n\times V} h\right)\varphi.$$

In this case, it is well known that an irreducible substochastic semigroup having an invariant density is mean ergodic, i.e. the Cesaro convergence

$$\frac{1}{t}\int_0^t W(s)\varphi ds \to \mathcal{P}\varphi \;\; (t \to +\infty) \;\; \forall \varphi \in L^1(\mathcal{T}^n \times V)$$

holds (see e.g. [3] Chap. 4). In fact, we show here the stronger result

$$W(t)\varphi \to \mathcal{P}\varphi \;\; (t \to +\infty) \;\; \forall \varphi \in L^1(\mathcal{T}^n \times V),$$

by means of a general functional analytic result, relying on a 0-2 law for C_0-semigroups, given in [36]; (see Theorem 24).

We note that if the *detailed balance* principle holds, i.e. there exists $M \in L^1(V)$ such that $M(v) > 0$ a.e. and

$$k(x,v',v)M(v) = k(x,v,v')M(v'), \tag{20}$$

(this may occur e.g. in nuclear reactor theory where M is typically a Maxwellian function, see e.g. [1, 44]) then an invariant density is given for free. Indeed, it is easy to see that

$$\left(\int_V M\right)^{-1} M$$

is an invariant density of $(W(t))_{t\geq 0}$ since (20) and (3) imply

$$AM = TM + KM = -\sigma(x,v)M(v) + \int_V k(x,v,v')M(v')d\mu(v') = 0.$$

In this case, under the assumption that $(W(t))_{t\geq 0}$ is irreducible, $(W(t))_{t\geq 0}$ converges strongly (as $t \to +\infty$) to its ergodic projection without Assumption (19).

Finally, Sect. 10 is devoted to some comments related to $K = 0$. In particular, in this case $(W(t))_{t\geq 0}$ is nothing but the translation semigroup

$$L^1(\mathcal{T}^n \times V) \ni \varphi \to \varphi(x - tv, v) \in L^1(\mathcal{T}^n \times V) \ \ (t \geq 0).$$

Under (2) $(W(t))_{t\geq 0}$ is mean ergodic with infinite rank ergodic projection

$$\psi \in L^1(\mathcal{T}^n \times V) \to \mathcal{P}\varphi = \int_{\mathcal{T}^n} \psi(x,v)dx \in L^1(V)$$

but $(W(t))_{t\geq 0}$ does not converge strongly in $L^1(\mathcal{T}^n \times V)$ as $t \to +\infty$.

2 Characterization of Lack of Spectral Gap

Since $\omega(U) < 0$ if and only if there exist two constants $C_1 > 0$ and $C_2 > 0$ such that

$$\int_0^{C_1} \sigma(x+sv,v)ds \geq C_2 \ \text{ a.e. on } \mathcal{T}^n \times V \tag{21}$$

then, at least formally, $\omega(U) = 0$ if and only if σ vanishes (almost everywhere) on some characteristic curve, i.e.

$$(\sigma(\overline{x} + s\overline{v}, \overline{v}) = 0 \ \text{ a.e. } s > 0 \ \ (\overline{v} \neq 0),$$

or on a point of the form $(\overline{x}, 0)$, i.e.

$$\sigma(\overline{x}, 0) = 0 \text{ (if } 0 \in V).$$

This can be shown rigorously if σ is "smooth" in a suitable sense. Indeed, we have the following results which improve some ones given in [35].

Proposition 2 *We assume that V is either bounded or is unbounded and*

$$\liminf_{|v|\to\infty} \left(ess\text{-}\inf_{x\in\mathcal{T}^n} \sigma(x,v) \right) > 0. \tag{22}$$

(i) *If*

$$\sigma_t : \mathcal{T}^n \times V \ni (x,v) := \int_0^t \sigma(x+sv,v)ds \text{ is lower semi continuous} \tag{23}$$

and if $\omega(U) = 0$ then there exists $(\overline{x}, \overline{v}) \in \mathcal{T}^n \times V$ such that $\overline{v} \neq 0$ and

$$\sigma(\overline{x} + s\overline{v}, \overline{v}) = 0 \quad a.e.\ s > 0$$

(or there exists $\overline{x} \in \mathcal{T}^n$ such that $\sigma(\overline{x}, 0) = 0$ if $0 \in V$).

(ii) *If*

$$\sigma_t : \mathcal{T}^n \times V \ni (x, v) := \int_0^t \sigma(x + sv, v) ds \text{ is upper semi continuous}$$

and if $\omega(U) < 0$ *then there exist no* $(\overline{x}, \overline{v}) \in \mathcal{T}^n \times V$ *such that* $\overline{v} \neq 0$ *and* $\sigma(\overline{x} + s\overline{v}, \overline{v}) = 0$ *a.e.* $s > 0$ (*and there exist no* $\overline{x} \in \mathcal{T}^n$ *such that* $\sigma(\overline{x}, 0) = 0$ *if* $0 \in V$).

Proof (i) Suppose that $\omega(U) = 0$ or equivalently for any constant $C > 0$

$$ess \inf \int_0^C \sigma(x + sv, v) ds = 0.$$

Then there exists a sequence $((x_k, v_k))_k \subset \mathcal{T}^n \times V$ such that

$$\int_0^k \sigma(x_k + sv_k, v_k) ds \leq k^{-1}.$$

It follows that any $t > 0$

$$\sigma_t(x_k, v_k) := \int_0^t \sigma(x_k + sv_k, v_k) ds \leq k^{-1} \quad \forall k \geq t.$$

Note that $\{v_k\}_k$ is always bounded. Indeed, if V is unbounded and if a subsequence of $\{v_k\}_k$ tends to infinity then this last estimate is not compatible with (22). Hence there exists a subsequence $\left((x_{\varphi(k)}, v_{\varphi(k)})\right)_k$ converging to some $(\overline{x}, \overline{v})$. By passing to the limit in

$$0 \leq \sigma_t(x_{\varphi(k)}, v_{\varphi(k)}) \leq \varphi(k)^{-1}$$

and using the lower semicontinuity of σ_t we get

$$0 \leq \sigma_t(\overline{x}, \overline{v}) \leq \lim \inf_{k \to \infty} \sigma_t(x_{\varphi(k)}, v_{\varphi(k)}) = 0$$

whence

$$(0, +\infty) \ni s \to \sigma(\overline{x} + s\overline{v}, \overline{v})$$

vanishes almost everywhere.

(ii) Suppose there exists some $(\overline{x}, \overline{v}) \in \mathcal{T}^n \times V$ such that

$$(0, +\infty) \ni s \to \sigma(\overline{x} + s\overline{v}, \overline{v})$$

vanishes almost everywhere. Then, for *any* $t > 0$ we have $\sigma_t(\overline{x}, \overline{v}) = 0$. Hence, by the upper semicontinuity of σ_t, for *any* $\varepsilon > 0$ there exists a *neighborhood* $\mathcal{V}(\overline{x}, \overline{v})$ of $(\overline{x}, \overline{v})$ on which $\sigma_t(x, v) \leq \varepsilon$, i.e.

$$\int_0^t \sigma(x + sv, v) ds \leq \varepsilon \quad (x, v) \in \mathcal{V}(\overline{x}, \overline{v})$$

so

$$ess\text{-} \inf_{(x,v)} \int_0^t \sigma(x + sv, v) ds \leq \varepsilon \;\; (\forall \varepsilon > 0)$$

i.e.

$$ess\text{-}\inf_{(x,v)} \int_0^t \sigma(x+sv,v)ds = 0 \ \ (\forall t>0).$$

This contradicts (21) so $\omega(U)=0$. □

Corollary 3 *If*

$$\sigma_t : \mathcal{T}^n \times V \ni (x,v) := \int_0^t \sigma(x+sv,v)ds \ \text{ is continuous } (t>0) \tag{24}$$

(*in particular if* σ *is continuous*) *then* $\omega(U)=0$ *if and only if there exists* $(\overline{x},\overline{v}) \in \mathcal{T}^n \times V$ *such that* $\overline{v} \neq 0$ *and*

$$\sigma(\overline{x}+s\overline{v},\overline{v}) = 0 \quad a.e.\ s>0$$

or there exists $\overline{x} \in \mathcal{T}^n$ *such that* $\sigma(\overline{x},0)=0$ *if* $0 \in V$.

We introduce the set (of "curves")

$$\Xi \subset \mathcal{T}^n \times V \tag{25}$$

consisting of those $(x,v) \in \mathcal{T}^n \times V$ such that

$$\sigma(x+sv,v) = 0 \quad \text{a.e. } s>0.$$

Note that the set Ξ contains also "*stationary points*"

$$\Xi_0 := \{(x,0);\ \sigma(x,0)=0\} \quad (\text{if } 0 \in V).$$

We define the set

$$\Pi := \{v \in V;\ \exists\ (x,v) \in \Xi\}. \tag{26}$$

We note that if $K \neq 0$ then

$$\mu(\Pi^c) > 0,$$

(see Proposition 27). We note that under (23)

$$\{(x,v) \in \mathcal{T}^n \times V;\ \sigma_t(x,v)=0\} = \cap_{n\in\mathbb{N}} \left\{(x,v) \in \mathcal{T}^n \times V;\ \sigma_t(x,v) \leq n^{-1}\right\}$$

and

$$\{(x,0);\ \sigma(x,0)=0\} = \cap_{n\in\mathbb{N}} \left\{(x,0);\ \sigma(x,0) \leq n^{-1}\right\}$$

are closed sets so that the set Ξ is closed too since

$$\Xi = \cap_{t>0} \{(x,v) \in \mathcal{T}^n \times V;\ \sigma_t(x,v)=0\} \cup \{(x,0);\ \sigma(x,0)=0\}.$$

This set is also bounded if we assume (22) when V is not bounded. For the sake of definiteness, we will assume in all the paper that (23) is satisfied as well as (22) when V is not bounded (even if such assumptions are not necessary in all our statements). Thus

$$\Xi \text{ is a compact subset of } \mathcal{T}^n \times V.$$

It follows that

$$\Pi \text{ is a compact subset of } V.$$

Since the lack of spectral gap for $(W(t))_{t\geq 0}$ is due to the vanishing of the collision frequency σ on the set Ξ, it is essential to have as much information as possible on this set. Note that the compact set $\Xi \subset \mathcal{T}^n \times V$ is the union of the *characteristic curves* on which the collision frequency vanishes so that a priori

$$\Xi \subset \{(x, v) \in \mathcal{T}^n \times V;\ \sigma(x, v) = 0\} \tag{27}$$

where the inclusion may be proper.

Let V_d be the set of velocities v whose coordinates are rationally dependent and let

$$\widehat{V} = \{v \in V;\ v \notin V_d\}$$

be the set of velocities v whose coordinates are rationally independent. The set Ξ may be decomposed into at most three disjoint parts

$$\Xi = \Xi_0 \cup \Xi_1 \cup \Xi_2$$

where

$$\Xi_1 := \{(x, v);\ v \in V_d;\ \sigma(x + sv, v) = 0 \text{ a.e. } s > 0\}$$

and

$$\Xi_2 := \left\{(x, v);\ v \in \widehat{V};\ \sigma(x + sv, v) = 0 \text{ a.e. } s > 0\right\}.$$

Since we assume in all the paper that the hyperplanes have zero μ-measure then

$$\Xi_0 \cup \Xi_1 \text{ has zero } dx\mu(dv)\text{-measure.}$$

Indeed

$$\Xi_0 \cup \Xi_1 \subset (\mathcal{T}^n \times \{0\}) \cup (\mathcal{T}^n \times V_d)$$

and

$$\mu(V_d) = \mu(\{0\}) = 0$$

because any velocity $\overline{v} \in V$ with rationally dependent coordinates belongs to some hyperplane

$$H_\xi := \{v;\ v.\xi = 0\}$$

with $\xi \in \mathbb{Q}^n$ and therefore

$$\mu(V_d) \subset \mu(\cup_{\xi \in \mathbb{Q}^n} H_\xi) = 0$$

since $\mu(H_\xi) = 0$ and $\mathbb{Q}^n$ is countable. We end this section with a *key* observation.

Proposition 4 *We have*

$$\Xi \subset \mathcal{T}^n \times \Pi. \tag{28}$$

We assume that (22) *is satisfied and for all* $t > 0$

$$\sigma_t : \mathcal{T}^n \times \Pi^c \ni (x, v) := \int_0^t \sigma(x + sv, v)ds \ \text{ is lower semi continuous} \tag{29}$$

(*e.g.* σ *is continuous on* $\mathcal{T}^n \times \Pi^c$). *Then, for any closed set* $\Lambda \subset \Pi^c$ *the advection semigroup* (4) *leaves invariant the closed subspace* $L^1(\mathcal{T}^n \times \Lambda)$ *and the type of its restriction to* $L^1(\mathcal{T}^n \times \Lambda)$ *is negative, i.e.*

$$c_\Lambda := \lim_{t \to +\infty} \inf_{(x,v) \in \mathcal{T}^n \times \Lambda} t^{-1} \int_0^t \sigma(x + sv, v)ds > 0.$$

Proof Note that we identify $L^1(\mathcal{T}^n \times \Lambda)$ to the elements of $L^1(\mathcal{T}^n \times V)$ vanishing a.e. on $\mathcal{T}^n \times \Lambda^c$ so the fact that $(U(t))_{t \geq 0}$ leaves invariant $L^1(\mathcal{T}^n \times \Lambda)$ is clear. We denote by $\left(U^\Lambda(t)\right)_{t \geq 0}$ the restriction of $(U(t))_{t \geq 0}$ to $L^1(\mathcal{T}^n \times \Lambda)$ whose type is given by

$$\omega(U^\Lambda) := - \lim_{t \to +\infty} \inf_{(x,v) \in \mathcal{T}^n \times \Lambda} t^{-1} \int_0^t \sigma(x + sv, v)ds.$$

According to Proposition 2 (i), $\omega(U^\Lambda) = 0$ would imply the existence of $(\overline{x}, \overline{v}) \in \mathcal{T}^n \times \Lambda$ such that

$$(0, +\infty) \ni s \to \sigma(\overline{x} + s\overline{v}, \overline{v}) = 0 \ \text{ a.e. } s > 0$$

so $(\overline{x}, \overline{v}) \in \Xi$. On the other hand, (28) implies that $\overline{v} \in \Pi$ which is a contradiction. □

Remark 5 Proposition 4 is an important ingredient of the proof of the key collective convergence result given in Lemma 17.

3 An Irreducibility Criterion for $(W(t))_{t \geq 0}$

We start with:

Definition 6 We say that $(W(t))_{t \geq 0}$ is *irreducible* if there is no *non trivial* closed subspace $L^1(\Omega) \subset L^1(\mathcal{T}^n \times V)$ invariant under $(W(t))_{t \geq 0}$.

Note that $L^1(\Omega)$ is identified to the elements of $L^1(\mathcal{T}^n \times V)$ vanishing a.e. on Ω^c where Ω^c is the complement of Ω in $\mathcal{T}^n \times V$. We have the sufficient criterion:

Proposition 7 *We assume that for any measurable set $\Omega \subset \mathcal{T}^n \times V$ such that Ω and Ω^c have positive $dy\mu(dv)$-measure we have*

$$\int_{\mathcal{T}^n}\left[\int_{\{v;\ (x,v)\in\Omega^c\}\times\{v';\ (x,v')\in\Omega\}} k(x,v,v')\mu(dv)\mu(dv')\right]dx > 0. \tag{30}$$

Then $(W(t))_{t\geq 0}$ is irreducible.

Proof According to ([39] Proposition 3.3, p. 307) it *suffices* to show that K is irreducible. Let us check that K is irreducible if and only if (30) holds for any measurable set $\Omega \subset \mathcal{T}^n \times V$ such that Ω and Ω^c have positive $dy\mu(dv)$-measure. If K is *not* irreducible then there exists a *non trivial* subspace $L^1(\Omega) \subset L^1(\mathcal{T}^n \times V)$ invariant under K. Let $\varphi \in L^1(\Omega)$, i.e. φ vanishes a.e. on Ω^c, then

$$\begin{aligned} K\varphi(x,v) &= \int_V k(x,v,v')\varphi(x,v')\mu(dv') \\ &= \int_{\{v';\ (x,v')\in\Omega\}} k(x,v,v')\varphi(x,v')\mu(dv'). \end{aligned}$$

Since $K\varphi \in L^1(\Omega)$, i.e. $K\varphi$ vanishes a.e. on Ω^c, then

$$\int_{\Omega^c}\left[\int_{\{v';\ (x,v')\in\Omega\}} k(x,v,v')\varphi(x,v')\mu(dv')\right]dx\mu(dv) = 0$$

i.e.

$$\int_{\mathcal{T}^n}\left[\int_{\{v;\ (x,v)\in\Omega^c\}\times\{v';\ (x,v')\in\Omega\}} k(x,v,v')\varphi(x,v')\mu(dv)\mu(dv')\right]dx = 0$$

or

$$\chi_{\{v;\ (x,v)\in\Omega^c\}}\chi_{\{v';\ (x,v')\in\Omega\}}k(x,v,v')\varphi(x,v') = 0 \text{ a.e.}$$

By choosing $\varphi > 0$ a.e. on Ω one sees that

$$\chi_{\{v;\ (x,v)\in\Omega^c\}}\chi_{\{v';\ (x,v')\in\Omega\}}k(x,v,v') = 0 \text{ a.e.}$$

which contradicts (30). Thus (30) implies that K is irreducible. Conversely, let there exists a measurable subset $\Omega \subset \mathcal{T}^n \times V$ such that Ω and Ω^c have positive $dy\mu(dv)$-measure and

$$\int_{\mathcal{T}^n}\left[\int_{\{v;\ (x,v)\in\Omega^c\}\times\{v';\ (x,v')\in\Omega\}} k(x,v,v')\mu(dv)\mu(dv')\right]dx = 0$$

or equivalently

$$\chi_{\{v;\ (x,v)\in\Omega^c\}}\chi_{\{v';\ (x,v')\in\Omega\}}k(x,v,v') = 0 \text{ a.e.}$$

This implies that for any $\varphi \in L^1(\Omega)$

$$K\varphi(x,v) = \int_V k(x,v,v')\varphi(x,v')\mu(dv') = \int_V \chi_{\{v';\ (x,v')\in\Omega\}}k(x,v,v')\varphi(x,v')\mu(dv')$$

vanishes a.e. on Ω^c, i.e. $K\varphi \in L^1(\Omega)$ and therefore K is not irreducible. □

Remark 8 In particular, $(W(t))_{t\geq 0}$ is irreducible if $k(x,v,v')>0$ a.e.

We give now a simple case where $(W(t))_{t\geq 0}$ is *not* irreducible.

Proposition 9 *Let there exist* $\Omega\subset V$ *such that* $\mu(\Omega)>0$, $\mu(\Omega^c)>0$ *and*

$$\int_{\Omega^c}\mu(dv')\int_{\Omega}k(x,v,v')\mu(dv')=0\ \ (x\in\mathcal{T}^n).$$

Then $(W(t))_{t\geq 0}$ *is not irreducible.*

Proof For any $\psi\in L^1(\mathcal{T}^n\times\Omega)$ (i.e. ψ vanishes on $\mathcal{T}^n\times\Omega^c$) we have

$$K\varphi(x,v)=\int_V k(x,v,v')\varphi(x,v')\mu(dv')=\int_\Omega k(x,v,v')\varphi(x,v')\mu(dv')$$

so that the restriction of $K\varphi$ to $\mathcal{T}^n\times\Omega^c$ vanishes since

$$k(x,v,v')=0 \text{ on } \mathcal{T}^n\times\Omega^c\times\Omega$$

Hence K leaves invariant the closed subspace $L^1(\mathcal{T}^n\times\Omega)$. On the other hand $L^1(\mathcal{T}^n\times\Omega)$ is trivially invariant under $(U(t))_{t\geq 0}$. It follows easily, e.g. from the Dyson-Phillips expansion (7) and (8), that $L^1(\mathcal{T}^n\times\Omega)$ is invariant under $(W(t))_{t\geq 0}$. □

4 On Strong Stability of Advection Semigroups

We first recall some basic notions on mean ergodic semigroups we can find e.g. in [4] Chap. 4, p. 261. A C_0-semigroup of contractions $(Z(t))_{t\geq 0}$ with generator A on a Banach space X is said to be mean ergodic if for any $x\in X$,

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t Z(s)x\,ds \text{ exists}$$

for the norm of X. In this case, this limit is a projection (the so-called ergodic projection) P on $Ker(A)$ along $\overline{R(A)}$ where $R(A)$ is the range of A. In particular

$$X=Ker(A)\oplus\overline{R(A)}\ \ \text{(direct sum)}.$$

Remark 10 A sufficient condition of mean ergodicity of a positive contraction C_0-semigroup $(Z(t))_{t\geq 0}$ on $L^1(\nu)$-space is the existence of some $\varphi\in L^1(\nu)$ such that $\varphi>0$ a.e. and $Z(t)\varphi\leq\varphi$ $(t\geq 0)$, (see e.g. [4] Proposition 4.3.14).

We observe first if $(W(t))_{t\geq 0}$ is *irreducible* then the existence of an invariant density (or equivalently the ergodicity of $(W(t))_{t\geq 0}$) implies a specific *constraint* on $(U(t))_{t\geq 0}$.

Theorem 11 *Let* $K\neq 0$ *and let* $(W(t))_{t\geq 0}$ *be irreducible. If* $(W(t))_{t\geq 0}$ *is ergodic then* $(U(t))_{t\geq 0}$ *is strongly stable, i.e.*

$$\lim_{t\to+\infty}\|U(t)\varphi\|_{L^1(\mathcal{T}^n\times V)}=0\ \forall\varphi\in L^1(\mathcal{T}^n\times V).$$

Proof Note that

$$0 \leq U(t) \leq W(t) \ \ (t \geq 0)$$

so that the ergodicity of $(W(t))_{t\geq 0}$ implies the ergodicity of $(U(t))_{t\geq 0}$, (see [3] Thm. 1.1). Since $K \neq 0$ then

$$(U(t))_{t\geq 0} \neq (W(t))_{t\geq 0}.$$

If $(W(t))_{t\geq 0}$ is irreducible then the kernel of T' (the dual of T) must be trivial (see [3] Thm. 1.3). Since $(U(t))_{t\geq 0}$ is ergodic (with ergodic projection P) then for any $\varphi \in L^1_+(\mathcal{T}^n \times V)$

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t U(s)\varphi ds = P\varphi.$$

In particular

$$\lim_{t\to+\infty} \frac{1}{t} \left\| \int_0^t U(s)\varphi ds \right\| = \|P\varphi\|.$$

By the additivity of the norm on the positive cone $L^1_+(\mathcal{T}^n \times V)$

$$\left\| \int_0^t U(s)\varphi ds \right\| = \int_0^t \|U(s)\varphi\| ds$$

whence

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \|U(s)\varphi\| ds = \|P\varphi\|.$$

Since

$$[0, +\infty[\ni t \to \|U(t)\varphi\| \text{ is non increasing}$$

$((U(t))_{t\geq 0}$ is a contraction semigroup) then

$$\lim_{t\to+\infty} \|U(t)\varphi\| \ \text{ exists}$$

and therefore

$$\lim_{t\to+\infty} \|U(t)\varphi\| = \|P\varphi\|.$$

We decompose any $\varphi \in L^1(\mathcal{T}^n \times V)$ into positive and negative parts

$$\varphi = \varphi_+ - \varphi_- \in L^1(\mathcal{T}^n \times V)$$

so

$$\begin{aligned}\int_{\mathcal{T}^n\times V} U(t)\varphi &= \langle U(t)\varphi_+, 1\rangle_{L^1,\, L^\infty} - \langle U(t)\varphi_-, 1\rangle_{L^1,\, L^\infty}\\ &= \langle \varphi, U'(t)1\rangle_{L^1,\, L^\infty}\\ &= \|U(t)\varphi_+\| - \|U(t)\varphi_-\| \to \|P\varphi_+\| - \|P\varphi_-\|\end{aligned}$$

i.e.

$$\zeta := \lim_{t\to\infty} U'(t)1 \text{ exists in weak star topology}$$

and

$$\lim_{t\to\infty}\int_{\mathcal{T}^n\times V} U(t)\varphi = \langle \varphi, \zeta\rangle_{L^1,\ L^\infty}.$$

The fact that

$$U'(s)\zeta = U'(s)\lim_{t\to\infty} U'(t)1 = \lim_{t\to\infty} U'(t+s)1 = \lim_{t\to\infty} U'(t)1 = \zeta \quad \forall s \geq 0$$

implies

$$\zeta \in D(T') \text{ and } T'\zeta = 0$$

and then $\zeta = 0$ because the kernel of T' is trivial. Finally, for any $\varphi \in L^1(\mathcal{T}^n \times V)$

$$\|U(t)\varphi\| \leq \|U(t)\,|\varphi|\| = \int_{\mathcal{T}^n\times V} U(t)\,|\varphi| \to \langle |\varphi|\,, \zeta\rangle_{L^1,\ L^\infty} = 0.$$

□

We characterize now the strong stability of $(U(t))_{t\geq 0}$ in terms of properties of the collision frequency.

Theorem 12 $(U(t))_{t\geq 0}$ *is strongly stable if and only if*

$$\int_0^{+\infty} \sigma(y+sv, v)ds = +\infty \ \text{ a.e. } (y, v) \in \mathcal{T}^n \times V. \tag{31}$$

Proof Let $\varphi \in L^1(\mathcal{T}^n \times V)$. Then, the monotone convergence theorem and

$$\|U(t)\varphi\|_{L^1(\mathcal{T}^n\times V)} = \int_{V\times\mathcal{T}^n} e^{-\int_0^t \sigma(y+sv,v)ds}\,|\varphi(y, v)|\,dy\mu(dv)$$

show that

$$\lim_{t\to+\infty}\|U(t)\varphi\|_{L^1(\mathcal{T}^n\times V)} = \int_{V\times\mathcal{T}^n} e^{-\int_0^{+\infty} \sigma(y+sv,v)ds}\,|\varphi(y, v)|\,dy\mu(dv).$$

Thus (31) is sufficient for the strong stability. Conversely, if

$$\int_0^{+\infty} \sigma(y+sv, v)ds < +\infty$$

on a measurable set $\Omega \subset \mathcal{T}^n \times V$ of positive measure then

$$\lim_{t\to+\infty}\|U(t)\varphi\|_{L^1(\mathcal{T}^n\times V)} \geq \int_\Omega e^{-\int_0^{+\infty} \sigma(y+sv,v)ds}\,|\varphi(y, v)|\,dy\mu(dv) > 0$$

for any non trivial $\varphi \in L^1(\Omega)$. □

Remark 13 Another characterization of the strong stability of $(U(t))_{t\geq 0}$ is that the strong limit $\lim_{\lambda\to 0_+} K(\lambda - T)^{-1}$ is a stochastic operator, (see [43] Theorem 3.6).

A practical condition of strong stability is given by:

Corollary 14 *Let the affine hyperplanes have zero μ-measure. If*

$$\int_{\mathcal{T}^n} \sigma(x,v)dx > 0 \ \ \mu\text{-}a.e. \tag{32}$$

then (31) *is satisfied.*

Proof We know that the set of velocities $v \in V$ with rationally dependent coordinates is a μ-null set. On the other hand, if $v \in V$ has rationally independent coordinates then the ergodicity of the flow shows that

$$\frac{1}{t}\int_0^t \sigma(y+sv,v)ds \to \int_{\mathcal{T}^n} \sigma(x,v)dx > 0 \ (t \to +\infty),$$

(see e.g. [35]). It follows that

$$\lim_{t\to+\infty} \int_0^t \sigma(y+sv,v)ds = \lim_{t\to+\infty} \left(\frac{1}{t}\int_0^t \sigma(y+sv,v)ds\right) t = +\infty.$$

This shows (31). □

We note that in full generality Assumption (31) implies a condition on the set of characteristic curves.

Theorem 15 *Let* (24) *be satisfied. If $(U(t))_{t\geq 0}$ is strongly stable then*

$$\{(y,v);\ v \neq 0,\ \ \sigma(y+sv,v) = 0,\ s > 0\ \} \textit{ has zero } dx\mu(dv)\textit{-measure.}$$

Proof We note that

$$\begin{aligned}\lim_{t\to+\infty} \|U(t)\varphi\|_{L^1(\mathcal{T}^n\times V)} &= \int_{V\times\mathcal{T}^n} e^{-\int_0^{+\infty}\sigma(y+sv,v)ds}\, |\varphi(y,v)|\, dy\mu(dv) \\ &\geq \int_{\{(y,v);\ \sigma(y+sv,v)=0,\ s>0\ \}} |\varphi(y,v)|\, dy\mu(dv) \\ &\geq \int_{\{(y,v);\ v\neq 0,\ \ \sigma(y+sv,v)=0,\ s>0\ \}} |\varphi(y,v)|\, dy\mu(dv).\end{aligned}$$

Thus, the strong stability of $(U(t))_{t\geq 0}$ implies that the set

$$\{(y,v);\ v \neq 0,\ \ \sigma(y+sv,v) = 0,\ s > 0\ \}$$

has zero $dx\mu(dv)$-measure. This ends the proof. □

Remark 16 Note that in the case $0 \in V$, the strong stability of $(U(t))_{t\geq 0}$ provides no information on the dx-measure of the set of equilibrium points

$$\{y \in \mathcal{T}^n;\ \sigma(y,0) = 0\}$$

unless $\mu\{0\} > 0$ which was excluded from the beginning.

5 A Collective Compactness Theorem

We need first a key *collectively* uniform convergence result. Let T_j and K_j be the approximate operators defined in Sect. 1.1. We recall that

$$\mathcal{T}^n \times \Pi^c \ni (y, v) \to \Theta(y, v) := \int_0^{+\infty} e^{-\int_0^t \sigma(y+sv,v)ds} dt.$$

Lemma 17 *Let* Π *be the set defined by* (26) *and let*

$$\Lambda_\varepsilon := \{v;\ dist(v, \Pi) \geq \varepsilon\}.$$

If

$$\sup_{(y,v')\in\mathcal{T}^n\times V} \left(\int_{\Lambda_\varepsilon^c} \Theta(y, v) k(y, v, v') \mu(dv) \right) \to 0 \ \ (\varepsilon \to 0) \tag{33}$$

then

$$\lim_{\lambda\to 0} \left(\lambda - T_j\right)^{-1} K_j \text{ exists in } \mathcal{L}\left(L^1(\mathcal{T}^n \times V)\right)$$

uniformly in $j \in \mathbb{N}$.

Proof *Step 1*: Let $\varepsilon > 0$ and P_ε be the restriction operator

$$\varphi \in L^1(\mathcal{T}^n \times V) \to 1_{\Lambda_\varepsilon}\varphi \in L^1(\mathcal{T}^n \times V).$$

Let us show first that for any $\varepsilon > 0$

$$\lim_{\lambda\to 0} P_\varepsilon \left(\lambda - T_j\right)^{-1} K_j \text{ exists in } \mathcal{L}\left(L^1(\mathcal{T}^n \times V)\right) \textit{ uniformly} \text{ in } j \in \mathbb{N}. \tag{34}$$

Since $K_j \to K \ (j \to \infty)$ in operator norm, it suffices to show that

$$\lim_{\lambda\to 0} P_\varepsilon \left(\lambda - T_j\right)^{-1} \text{exists in } \mathcal{L}\left(L^1(\mathcal{T}^n \times V)\right) \ \ (\varepsilon > 0) \textit{ uniformly} \text{ in } j \in \mathbb{N}.$$

To this end, we note first that

$$\lim_{\lambda\to 0} P_\varepsilon (\lambda - T)^{-1} = P_\varepsilon (0 - T)^{-1} \text{ exists in } \mathcal{L}\left(L^1(\mathcal{T}^n \times V)\right) \ \ (\varepsilon > 0)$$

because $P_\varepsilon (\lambda - T)^{-1}$ is identified to $(\lambda - T^\varepsilon)^{-1}$ where T^ε is nothing but the transport operator on $L^1(\mathcal{T}^n \times \Lambda_\varepsilon)$ and we know by Proposition 4 that $0 \notin \sigma(T^\varepsilon)$ and

$$(\lambda - T^\varepsilon)^{-1} \to (0 - T^\varepsilon)^{-1} \text{ in } \mathcal{L}\left(L^1(\mathcal{T}^n \times \Lambda_\varepsilon)\right) \ \ (\lambda \to 0).$$

Since $\sigma_j(.) \geq \sigma(.)$ then the spectral bounds

$$s(T_j^\varepsilon) := \sup\left\{\operatorname{Re}\lambda,\ \lambda \in \sigma(T_j^\varepsilon)\right\},\ \ s(T^\varepsilon) := \sup\left\{\operatorname{Re}\lambda,\ \lambda \in \sigma(T^\varepsilon)\right\}$$

of T_j^ε and T^ε are such that

$$s(T_j^\varepsilon) \leq s(T^\varepsilon) < 0$$

so $0 \notin \sigma(T_j^{\varepsilon})$. Thus

$$\begin{aligned} & 1_{\Lambda_\varepsilon}\left(0-T_j\right)^{-1}\varphi - 1_{\Lambda_\varepsilon}\left(\lambda-T_j\right)^{-1}\varphi \\ &= 1_{\Lambda_\varepsilon}\int_0^{+\infty}\left(1-e^{-\lambda t}\right)e^{-\int_0^t \sigma_j(x-sv,v)ds}\varphi(x-tv,v)dt \end{aligned}$$

and

$$\begin{aligned} & \left\| 1_{\Lambda_\varepsilon}\left(0-T_j\right)^{-1}\varphi - 1_{\Lambda_\varepsilon}\left(\lambda-T_j\right)^{-1}\varphi \right\|_{L^1(\mathcal{T}^n\times V)} \\ &\leq \int_{\Lambda_\varepsilon}\mu(dv)\int_{\mathcal{T}^n}dx\int_0^{+\infty}\left(1-e^{-\lambda t}\right)e^{-\int_0^t \sigma_j(x-sv,v)ds}\left|\varphi(x-tv,v)\right|dt \\ &\leq \int_{\Lambda_\varepsilon}\mu(dv)\int_{\mathcal{T}^n}dx\int_0^{+\infty}\left(1-e^{-\lambda t}\right)e^{-\int_0^t \sigma(x-sv,v)ds}\left|\varphi(x-tv,v)\right|dt \\ &= \left\| 1_{\Lambda_\varepsilon}(0-T)^{-1}|\varphi| - 1_{\Lambda_\varepsilon}(\lambda-T)^{-1}|\varphi| \right\|_{L^1(\mathcal{T}^n\times V)} \\ &\leq \left\| 1_{\Lambda_\varepsilon}(0-T)^{-1} - 1_{\Lambda_\varepsilon}(\lambda-T)^{-1} \right\|_{\mathcal{L}(L^1(\mathcal{T}^n\times V))}\left\|\varphi\right\|_{L^1(\mathcal{T}^n\times V)} \end{aligned}$$

hence

$$\begin{aligned} & \left\| P_\varepsilon\left(0-T_j\right)^{-1} - P_\varepsilon\left(\lambda-T_j\right)^{-1} \right\|_{\mathcal{L}(L^1(\mathcal{T}^n\times V))} \\ &\leq \left\| P_\varepsilon(0-T)^{-1} - P_\varepsilon(\lambda-T)^{-1} \right\|_{\mathcal{L}(L^1(\mathcal{T}^n\times V))} \to 0 \quad (\lambda\to 0) \end{aligned}$$

ends the proof of (34).

Step 2. Let us show that

$$\lim_{\varepsilon\to 0}\left\| (I-P_\varepsilon)\left(\lambda-T_j\right)^{-1}K_j \right\|_{\mathcal{L}\left(L^1(\mathcal{T}^n\times V)\right)} = 0$$

uniformly in $\lambda \geq 0$ and $j \in \mathbb{N}$. We recall that

$$k_j(x,v,v') := k(x,v,v') + \frac{1}{j}f(v)g(v')$$

and $f = 0$ a.e. on a neighborhood of Π. We write $K_j = K + \widehat{K}_j$ where the kernel of $\widehat{K}_j$ is given by $\frac{1}{j}f(v)g(v')$ i.e.

$$\widehat{K}_j\varphi = \frac{1}{j}f(v)\int_V \varphi(x,v')g(v')\mu(dv')$$

Consider first the part

$$\begin{aligned} & \left\| (I-P_\varepsilon)\left(\lambda-T_j\right)^{-1}\widehat{K}_j\varphi \right\|_{L^1(\mathcal{T}^n\times V)} \\ &\leq \int_{\mathcal{T}^n\times\Lambda_\varepsilon^c}\int_0^{+\infty} e^{-\lambda t}e^{-\int_0^t \sigma(x-sv,v)ds}\widehat{K}_j\left|\varphi\right|(x-tv,v)dt \\ &\leq \int_{\Lambda_\varepsilon^c}\mu(dv)\int_{\mathcal{T}^n}dx\int_0^{+\infty} e^{-\int_0^t \sigma(x-sv,v)ds}\widehat{K}_j\left|\varphi\right|(x-tv,v)dt \end{aligned}$$

$$
\begin{aligned}
&= \int_{\Lambda_\varepsilon^c} \mu(dv) \int_{\mathcal{T}^n} dy \int_0^{+\infty} e^{-\int_0^t \sigma(y+sv,v)ds} \widehat{K}_j \left|\varphi\right|(y,v)dt \\
&= \frac{1}{j} \int_{\Lambda_\varepsilon^c} f(v)\mu(dv) \int_{\mathcal{T}^n} dy \int_0^{+\infty} e^{-\int_0^t \sigma(y+sv,v)ds} \left(\int_V \left|\varphi\right|(y,v')g(v')\mu(dv') \right) dt \\
&\leq \int_{\Lambda_\varepsilon^c} f(v)\mu(dv) \int_{\mathcal{T}^n} dy \int_0^{+\infty} e^{-\int_0^t \sigma(y+sv,v)ds} \left(\int_V \left|\varphi\right|(y,v')g(v')\mu(dv') \right) dt = 0
\end{aligned}
$$

because f vanishes in a neighborhood of Π in particular on Λ_ε^c for ε *small enough*. Consider now

$$
\begin{aligned}
&\left\| (I - P_\varepsilon) \left(\lambda - T_j\right)^{-1} K\varphi \right\|_{L^1(\mathcal{T}^n \times V)} \\
&\leq \left\| (I - P_\varepsilon) (\lambda - T)^{-1} K \left|\varphi\right| \right\|_{L^1(\mathcal{T}^n \times V)} \\
&= \int_{\Lambda_\varepsilon^c} \mu(dv) \int_{\mathcal{T}^n} dy \left(\int_0^{+\infty} e^{-\int_0^t \sigma(y+sv,v)ds} dt \right) K \left|\varphi\right|(y,v) \\
&= \int_{\Lambda_\varepsilon^c} \mu(dv) \int_{\mathcal{T}^n} \Theta(y,v) K \left|\varphi\right|(y,v)dy \\
&= \int_{\Lambda_\varepsilon^c} \mu(dv) \int_{\mathcal{T}^n} \Theta(y,v) \left(\int_V k(y,v,v') \left|\varphi(y,v')\right| \mu(dv') \right) dy \\
&= \int_{\mathcal{T}^n \times V} \left(\int_{\Lambda_\varepsilon^c} \Theta(y,v)k(y,v,v')\mu(dv) \right) \left|\varphi(y,v')\right| dy\mu(dv') \\
&\leq \sup_{(y,v') \in \mathcal{T}^n \times V} \left(\int_{\Lambda_\varepsilon^c} \Theta(y,v)k(y,v,v')\mu(dv) \right) \|\varphi\|_{L^1(\mathcal{T}^n \times V)}.
\end{aligned}
$$

This ends the proof. □

We are ready to show:

Theorem 18 *Let* (9) (33) *be satisfied. Then there exists* $N \in \mathbb{N}$ *such that the sequence of operators* $\left(\left((0 - T_j)^{-1} K_j\right)^N\right)_j$ *is collectively compact in* $\mathcal{L}\left(L^1(\mathcal{T}^n \times V)\right)$.

Proof This perturbed semigroup $\left(W_j(t)\right)_{t \geq 0}$ with generator $T_j + K_j$ is given by a Dyson-Phillips series

$$
W_j(t) = \sum_{n=0}^{\infty} U_n^{(j)}(t) \tag{35}
$$

where

$$
U_0^{(j)}(t) = U_j(t) \text{ and } U_{n+1}^{(j)}(t) = \int_0^t U_j(t-s) K_j U_n^{(j)}(s)ds \ \ (n \geq 0). \tag{36}
$$

Since K_j is a regular scattering operator then there exists an integer $\widehat{n}$ *independent* of j such that

$$
U_n^{(j)}(t) \text{ is a compact operator}, \quad (t \geq 0)\ (n \geq \widehat{n}),
$$

(see [35] Theorem 13). It follows that for $n \geq \widehat{n}+1$

$$t \to U_n^{(j)}(t)$$

is continuous in operator norm, (see [31] Corollary 2.2). Since the following integral

$$\int_0^\infty e^{-\lambda t} U_n^{(j)}(t)dt = \left(\left(\lambda - T_j\right)^{-1} K_j\right)^n \left(\lambda - T_j\right)^{-1} \quad (\lambda > 0)$$

converges in operator norm then

$$\left(\left(\lambda - T_j\right)^{-1} K_j\right)^n \left(\lambda - T_j\right)^{-1} \text{ is a compact operator } (n \geq \widehat{n}+1).$$

The choice $N = \widehat{n}+1$ shows that $\left(\left(\lambda - T_j\right)^{-1} K_j\right)^N$ is a compact operator. By Lemma 17

$$\lim_{\lambda \to 0} \left(\left(\lambda - T_j\right)^{-1} K_j\right)^N$$

exists in $\mathcal{L}\left(L^1(\mathcal{T}^n \times V)\right)$ uniformly in j and consequently the set of operators

$$\left\{\left(\left(\lambda - T_j\right)^{-1} K_j\right)^N,\ j \in \mathbb{N},\ 0 < \lambda \leq 1\right\}$$

is collectively compact. A simple calculation shows that $\left(((0 - T_j)^{-1} K_j)^N\right)_j$ is also collectively compact in $\mathcal{L}\left(L^1(\mathcal{T}^n \times V)\right)$. □

6 Existence of an Invariant Density

The main theorem in this paper is:

Theorem 19 *Let (9) (33) be satisfied. Then $(W(t))_{t \geq 0}$ has an invariant density.*

Proof To show that $(W(t))_{t\geq 0}$ admits an invariant density, as noted in the introduction, we introduce the approximate semigroup $\left(W^j(t)\right)_{t\geq 0}$ with generator (15) corresponding to approximate scattering kernel (13) and approximate collision frequency (14). Since $\left(W^j(t)\right)_{t\geq 0}$ is conservative semigroup and has a *spectral gap* then there exists a nonnegative eigenfunction φ_j of $T_j + K_j$ relative to the isolated eigenvalue 0

$$A_j \varphi_j = T_j \varphi_j + K_j \varphi_j = 0 \quad (j \geq 1) \tag{37}$$

or equivalently

$$(0 - T_j)^{-1} K_j \varphi_j = \varphi_j \quad (j \geq 1) \tag{38}$$

since $\omega(U^j) < 0$. We normalize $\left\{\varphi_j\right\}_j$ as

$$\int_{\mathcal{T}^n \times V} \varphi_j = 1 \quad (j \geq 1). \tag{39}$$

According to Theorem 18

$$\left(\left((0-T_j)^{-1}K_j\right)^N\right)_j \text{ is collectively compact}$$

i.e.

$$\cup_{j\in\mathbb{N}}\left((0-T_j)^{-1}K_j\right)^N B \text{ is relatively compact in } L^1(\mathcal{T}^n\times V) \tag{40}$$

where B is the unit ball of $L^1(\mathcal{T}^n\times V)$. It follows from (38) that

$$\left((0-T_j)^{-1}K_j\right)^N\varphi_j=\varphi_j \quad (j\geq 1)$$

and (40) implies that $\left(\varphi_j\right)_j$ is contained in a compact set and then has a subsequence converging in norm toward φ with norm one. For the simplicity of notations, we still denote it by $\left(\varphi_j\right)_j$, so

$$\varphi_j\to\varphi \ \ (j\to\infty) \text{ in } L^1(\mathcal{T}^n\times V).$$

On the other hand

$$K_j\varphi_j=(0-T_j)\varphi_j \ =-T_j\varphi_j=v.\frac{\partial\varphi_j}{\partial x}+\sigma_j(x,v)\varphi_j$$

and (14) show that

$$v.\frac{\partial\varphi_j}{\partial x}\to K\varphi-\sigma(x,v)\varphi.$$

By a closedness argument $\varphi\in D(T)$ and

$$T\varphi+K\varphi=0$$

so φ is an invariant density. □

7 A Class of Cross-Sections

This section is devoted to the analysis of the key assumption (33) when the degeneracy of the collision frequency "*is not spatial*" in the sense that

$$\widehat{\sigma}(v):=\inf_{x\in\mathcal{T}^n}\sigma(x,v)>0 \text{ a.e.} \tag{41}$$

and

$$\inf_{v\in V}\widehat{\sigma}(v)=0. \tag{42}$$

We assume for simplicity that

$$\widehat{\sigma}(.) \text{ is continuous} \tag{43}$$

so that

$$\Pi=\{v\in V;\ \widehat{\sigma}(v)=0\}.$$

Proposition 20 *Let* (41) (42) (43) *be satisfied. We set* $\widehat{k}(v, v') := \sup_{x \in \mathcal{T}^n} k(x, v, v')$ *and assume that*

$$\int_V \frac{\widehat{k}(v, v')}{\widehat{\sigma}(v)} dv < +\infty$$

and the convergence of this integral is uniform in $v' \in V$. *Then* (33) *is satisfied.*

Proof Under (41)

$$\Theta(y, v) = \int_0^{+\infty} e^{-\int_0^t \sigma(y+sv, v)ds} dt \leq \int_0^{+\infty} e^{-t\widehat{\sigma}(v)} dt = \frac{1}{\widehat{\sigma}(v)}.$$

Hence condition (33) holds if

$$\sup_{(y,v') \in \mathcal{T}^n \times V} \left(\int_{\Lambda_\varepsilon^c} \Theta(y, v) \widehat{k}(v, v') \mu(dv) \right) \to 0 \ \ (\varepsilon \to 0),$$

in particular if

$$\sup_{v' \in V} \int_{\Lambda_\varepsilon^c} \frac{\widehat{k}(v, v')}{\widehat{\sigma}(v)} \mu(dv) \to 0 \ \ (\varepsilon \to 0).$$

This ends the proof. □

Remark 21 A typical illustration is the separable scattering kernel

$$k(x, v, v') = \alpha(x) \widehat{k}(v, v')$$

where $\alpha(.) \in L_+^\infty(T^n)$ is *bounded away from zero. The analysis of spatial degeneracy is more tricky and deserves further study.*

8 On Space Homogeneous Cross Sections

In this section, we consider space homogeneous cross sections σ and k such that

$$\sigma(v) = \int_V k(v', v) \mu(dv').$$

Theorem 22 *Let the cross sections be space homogeneous and let* (9) (33) *be satisfied. Then*:

(i) *there exists a nontrivial* $\phi \in L_+^1(V)$ *such that*

$$-\sigma(v)\phi(v) = \int_V k(v, v')\phi(v')\mu(dv'). \tag{44}$$

(ii) *If* $(W(t))_{t \geq 0}$ *is irreducible then the (unique) invariant density of* $(W(t))_{t \geq 0}$ *is space homogeneous.*

Proof (i) According to Theorem 19 there exists an invariant density ψ so

$$-v.\frac{\partial \psi}{\partial x} - \sigma(v)\psi + K\psi = 0$$

and consequently

$$\phi(v) := \int_{\mathcal{T}^n} \psi(x, v)dx$$

satisfies (44).

(ii) Let ψ be the invariant density of $(W(t))_{t\geq 0}$ (given by Theorem 19). Note that

$$\phi(v) := \int_{\mathcal{T}^n} \psi(x, v)dx$$

is also an invariant density of $(W(t))_{t\geq 0}$. Hence the irreducibility of $(W(t))_{t\geq 0}$ implies the *uniqueness* of the invariant density which must be space homogeneous. □

9 On Time Asymptotics

The object of this section is to show how to pass from Cesaro time asymptotics of $(W(t))_{t\geq 0}$ to strong time asymptotics. To this end, we recall a particular version of a known abstract result on $L^1(\nu)$ spaces:

Theorem 23 ([36] Theorem 4) *Let $(U(t))_{t\geq 0}$ be a positive contraction C_0-semigroup on $L^1(\nu)$ with generator T and let $K \in \mathcal{L}_+(L^1(\nu))$. Let $(W(t))_{t\geq 0}$ be the C_0-semigroup (generated by $A = T + K$) given by the Dyson-Phillips expansion*

$$\sum_{j=0}^{+\infty} U_j(t) = W(t)$$

where $U_0(t) = U(t)$ and $U_j(t)$ is defined inductively by (8). We assume that $(W(t))_{t\geq 0}$ is a contraction C_0-semigroup, is irreducible with $Ker(A) \neq \{0\}$ and denote by P its ergodic projection on $Ker(A)$. If there exists some positive integer m such that

$$(0, +\infty) \ni t \to R_m(t) = \sum_{j=m}^{+\infty} U_j(t) \ \in \mathcal{L}(L^1(\nu))$$

is continuous in operator norm, then

$$\lim_{t\to+\infty} W(t)f = Pf \quad \forall f \in L^1(\nu).$$

This result is based on a *0-2 law* for C_0-semigroups by G. Greiner [22], (see also [39], p. 346). We are ready to show:

Theorem 24 *Let (9) (33) be satisfied and let $(W(t))_{t\geq 0}$ be irreducible. Then $(W(t))_{t\geq 0}$ admits a unique invariant density φ and*

$$\lim_{t\to+\infty} \left\| W(t)\psi - \left(\int_{\mathcal{T}^n\times V} \psi\right)\varphi \right\|_{L^1(\mathcal{T}^n\times V)} = 0 \tag{45}$$

for any $\psi \in L^1(\mathcal{T}^n \times V)$.

Proof According to Theorem 19, $(W(t))_{t\geq 0}$ admits a unique invariant density φ. On the other hand, for j large enough, $U_j(t)$ (for all $t \geq 0$) is a compact operator, (see [35] Theorem 13). It follows that

$$[0, +\infty[\ni t \to U_j(t) \in \mathcal{L}(L^1(\mathcal{T}^n \times V))$$

is continuous in *operator norm* (see [31] Corollary 2.2, p. 19 or [14]) and consequently so is

$$[0, +\infty[\ni t \to R_j(t) := \sum_{m=j}^{\infty} U_m(t) \in \mathcal{L}(L^1(\mathcal{T}^n \times V))$$

because the series converges in operator norm uniformly in t bounded. Finally, the strong convergence (45) follows from Theorem 23. □

Remark 25 We could prove Theorem 24 differently. Indeed, since for j large enough $U_j(t)$ (for all $t \geq 0$) is a compact operator so is $R_j(t)$. Hence $R_j(t)$ is an integral operator (see [21], p. 508). It follows that $(W(t))_{t\geq 0}$ partially integral in the sense that $W(t)$ dominates an integral operator ($W(t) \geq R_j(t)$). Finally, the conclusion follows from [40].

Remark 26 We noted in Sect. 1.2 that under the detailed balance condition (20), $(W(t))_{t\geq 0}$ admits (automatically) an invariant density $\widehat{M}$. By arguing as in the proof of Theorem 24, we can show

$$\left\| W(t)\psi - \left(\int_{\mathcal{T}^n \times V} \psi \right) \widehat{M} \right\|_{L^1(\mathcal{T}^n \times V)} \to 0 \ \ (t \to +\infty), \quad \psi \in L^1(\mathcal{T}^n \times V)$$

under Assumption (9) only; Assumption (33) is no longer necessary.

10 Comments on Trivial Scattering Operators

We start with

Proposition 27 *If $K \neq 0$ then $\mu(\Pi^c) > 0$.*

Proof Arguing by contradiction, if $\mu(\Pi^c) = 0$, i.e. if $\mu(\Pi) = \mu(V)$ then for almost all $v \in V$ there exists a characteristic curve

$$(0, +\infty) \ni s \to (x + sv, v) \in \mathcal{T}^n \times V$$

on which $\sigma(.,.)$ vanishes. Since

$$\lim_{a \to +\infty} \frac{1}{a} \int_0^a \sigma(x + sv, v) ds = \int_{\mathcal{T}^n} \sigma(y, v) dy$$

for any v with rationally independent coordinates then

$$\int_{\mathcal{T}^n} \sigma(x, v) dy = 0 \ \text{ a.e. } v \in V$$

i.e. $\sigma(.,.) = 0$. and then (3) implies that $k(.,.,.) = 0$ i.e. $K = 0$. □

We end this paper by some comments on the case

$$K = 0.$$

In this case $\sigma(x, v) = 0$ a.e. and then $(W(t))_{t\geq 0}$ is nothing but the translation semigroup

$$L^1(\mathcal{T}^n \times V) \ni \varphi \to \varphi(x - tv, v) \in L^1(\mathcal{T}^n \times V) \ \ (t \geq 0).$$

One sees that

$$W(t)\varphi = \varphi \ \ \forall \varphi \in L^1(V).$$

According to Remark 10, $(W(t))_{t\geq 0}$ is mean ergodic (and admits infinitely many invariant densities, actually all the subspace $L^1(V)$). If we assume that the hyperplanes have zero μ-measure then the ergodic projection

$$\mathcal{P} : L^1(\mathcal{T}^n \times V) \to L^1(V)$$

is given by

$$\mathcal{P}\psi := \lim_{t\to+\infty} \frac{1}{t} \int_0^t W(s)\psi ds = \lim_{t\to+\infty} \frac{1}{t} \int_0^t \psi(y - sv, v)ds = \int_{\mathcal{T}^n} \psi(x, v)dx.$$

We note that for any measurable subset $\Omega \subset V$ with positive μ-measure, the subspace $L^1(\mathcal{T}^n \times \Omega)$ is invariant under $(W(t))_{t\geq 0}$. Thus $(W(t))_{t\geq 0}$ is *not* irreducible. Finally, we observe that for any initial data ψ of the form

$$\psi(x, v) = e^{2i\pi k.x}\varphi(v)$$

(with a non zero $\varphi \in L^1(V)$ and a non zero multiindex $k \in \mathbb{N}^n$),

$$W(t)\psi = e^{-2i\pi tk.v}\psi$$

does not converge in $L^1(\mathcal{T}^n \times V)$ as $t \to \infty$.

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