Exponential Stability in Mean Square of Stochastic Functional Differential Equations with Infinite Delay

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Abstract
A novel approach to the exponential stability in mean square of stochastic functional differential equations and neutral stochastic functional differential equations with infinite delay is presented. Consequently, some new criteria for the exponential stability in mean square of the considered equations are obtained. Lastly, some examples are investigated to illustrate the theory.

Keywords Exponential stability in mean square · Stochastic functional differential equations · Neutral stochastic functional differential equations · Infinite delay

Mathematics Subject Classification (2010) 60H15 · 60G15 · 60H05

1 Introduction

Stochastic functional differential equations (SFDEs) have drawn growing attentions owing to their applications in physical, biological, medical and social sciences, as well as in economics and finance, that is, in every where future states depend not only on the present state but also on the past states. On the other hand, neutral stochastic delay differential equations (NSFDEs) are often used to describe the dynamical systems which not only involve derivatives but also depend on present and past states. Neutral stochastic delay differential equations have attracted the increasing attentions due to the wide applications in the distributed networks containing lossless transmission lines, processes including steam or water pipes, heat exchanges, and other engineering systems and population ecology. For stochastic functional differential equation, we refer the reader to the books [8, 12] by Mao, among other things. For neutral stochastic functional differential equations, we refer to [2, 7, 19].

As particular interest, the stability is always one of the most important issues in the theory of SFDEs. One of the most important approach to stability for stochastic functional differential equations is a stochastic version of the Lyapunov direct method. Lyapunov functions and functionals have been successfully used to obtain the stability of stochastic differential

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equations ([1, 4, 5, 11, 18]). Another widely-used approach to stability of stochastic functional differential equations is the Razumikhin-type theorem. Razumikhin-type theorems for exponential stability of stochastic functional differential equations have been presented in [6, 10, 17, 23]. A Razumikhin-type theorem for the asymptotic stability of stochastic functional differential equations has been given in [9, 20, 22].

In fact, it is not easy to find a Lyapunov function or functional for stochastic differential equations and the stability conditions obtained by the Lyapunov's function method are often given in terms of differential inequalities, matrix inequalities and so on. The given conditions by Lyapunov function (functional) and Razumikhin-type theorems are not only a little bit strong but also general implicit and not easy to examine.

Very recently, we also observe that several authors have established some criteria for the exponential stability in mean square of solutions to stochastic differential equations by using the different technique from Lyapunov direct method. For example, Ngoc [13] and Ngoc and Hieu [15] presented some explicit criteria for the mean square exponential stability of general non-linear stochastic delay differential equations based on a spectral property of Metzler matrices; Ngoc [14] obtained some explicit criteria for the mean square exponential stability of stochastic delay differential equations by a comparison principle. However, there are still two faults in the above articles. First of all, although the conditions in [13–15] are explicit, they are seemly a little strong and difficult to be satisfied in practical application. Secondly, explicit criteria for the mean square exponential stability of stochastic functional differential equations and neutral stochastic functional differential equations with infinite delay are still an open question.

In this paper, we will present a novel approach to the exponential stability in mean square of stochastic functional differential equations and neutral stochastic functional differential equations with infinite delay. Our approach does not involve Lyapunov functions and complex calculations. Our approach is based on a comparison principle and a proof by reductio ad absurdum and our conditions are also feasible. Our results improve some known results.

The rest of this paper is organized as follows. In Sect. 2, we introduce some necessary notations and preliminaries. In Sect. 3, we present some criteria for the exponential stability in mean square of stochastic functional differential equations with infinite delay. In Sect. 4, we present some criteria for the exponential stability in mean square of neutral stochastic functional differential equations with infinite delay. In Sect. 5, we state some comparisons with existing results and present some examples to illustrate the advantage of our results.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, i.e., the filtration is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trac}(A^T A)}$. Moreover, let $w(t) = (w_1(t), \ldots, w_m(t))^T$ be an m-dimensional Brownian motion defined over $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $BC((-\infty, 0]; \mathbb{R}^n)$ the family of all bounded, continuous functions φ from $(-\infty, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\|_{BC} = \sup_{\theta \leq 0} |\varphi(\theta)|$.

Consider the following stochastic functional differential equation

$$dx(t) = f(t, x_t)dt + g(t, x_t)dw(t) \tag{2.1}$$

on $t \geq 0$ with initial data $x_0 = \xi = \{\xi(\theta) : \theta \leq 0\} \in BC((-\infty, 0]; \mathbb{R}^n)$, where $x_t = x(t + \theta)$, $-\infty < \theta \leq 0$ is regarded as a $BC((-\infty, 0]; \mathbb{R}^n)$-valued stochastic process, and

$$f : \mathbb{R}_+ \times BC((-\infty, 0]; \mathbb{R}^n) \to \mathbb{R}^n, \quad g : \mathbb{R}_+ \times BC((-\infty, 0]; \mathbb{R}^n) \to \mathbb{R}^{n \times m}.$$

An $\mathcal{F}_t$-adapted process $x(t)$ is said to be the solution of the equation (2.1) if it satisfies the initial condition above and moreover for each $t \geq 0$,

$$x(t) = \xi(0) + \int_0^t f(s, x_s) ds + \int_0^t g(s, x_s) dw(s), \tag{2.2}$$

where the stochastic integral is in the Itô's sense. The details on the existence and uniqueness of the solution to (2.1), we can refer [3]. For example, when f, g, G are uniformly Lipschitz continuous, or they are locally Lipschitz continuous and satisfy the linear growth condition, Kolmanovskii and Nosov [3] proved that there is a unique continuous solution to (2.1), and any moment of the solution is finite. For stability purpose, throughout the paper we always suppose that equation (2.1) has a unique solution for arbitrarily given initial data $\xi \in BC((-\infty, 0]; \mathbb{R}^n)$ and the solution is denoted by $x(t, \xi)$, or simply $x(t)$, when no confusion is possible. For the purposes of stability, we shall assume that

$$f(t, 0) \equiv 0, \qquad g(t, 0) \equiv 0 \quad \text{for any } t \geq 0.$$

It is well-known that for a given $\xi \in BC((-\infty, 0]; \mathbb{R}^n)$, equation (2.1) has a trivial solution when $\xi \equiv 0$.

Definition 2.1 The trivial solution $x(t, \xi)$ of (2.1) is said to be exponentially stable in mean square, if for any initial value ξ, there exists a pair of positive constants $\lambda > 0$ and C such that for all $t \geq 0$

$$\mathbb{E}|x(t, \xi)|^2 \leq C \|\xi\|_{BC} e^{-\lambda t},$$

or, equivalently,

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}|x(t, \xi)|^2 \leq -\lambda.$$

Definition 2.2 The trivial solution $x(t, \xi)$ of (2.1) is said to be almost surely exponentially stable if there exists a constant $\lambda > 0$ such that there is a finite random variable β such that for all $t \geq 0$

$$|x(t, \xi)| \leq \beta e^{-\lambda t} \quad a.s.$$

3 Exponential Stability for SFDEs with Infinite Delay

To state the main result of this section, let us define some functions. Let $\eta_i(t, \theta) : \mathbb{R}_+ \times (-\infty, 0] \to \mathbb{R}$, $(i = 1, 2)$ be non-decreasing in θ for each $t \in \mathbb{R}_+$. Furthermore, $\eta_i(t, \theta)$ is normalized to be continuous from the left in θ on $(-\infty, 0]$. Assume that

$$L_i(t, \phi) := \int_{-\infty}^0 \phi(\theta) d[\eta_i(t, \theta)], \quad t \in \mathbb{R}_+, \ i = 1, 2, \tag{3.1}$$

is a locally bounded Borel-measurable function in t for each $\phi \in BC((-\infty, 0]; \mathbb{R}^n)$. Here, the integral in (3.1) is the Riemann-Stieltjes integral.

Theorem 3.1 *Let $\gamma(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ be a locally bounded Borel-measurable function such that for any $t \in \mathbb{R}_+$, $\varphi \in BC(-\infty, 0]; \mathbb{R}^n)$,*

$$2\mathbb{E}\left(\varphi^T(0)f(t, \varphi)\right) \le \gamma(t)\mathbb{E}|\varphi(0)|^2 + \int_{-\infty}^{0} \mathbb{E}|\varphi(\theta)|^2 d[\eta_1(t, \theta)], \tag{3.2}$$

and

$$\mathbb{E}\left(\text{trac}[g^T(t, \varphi)g(t, \varphi)]\right) \le \int_{-\infty}^{0} \mathbb{E}|\varphi(\theta)|^2 d[\eta_2(t, \theta)]. \tag{3.3}$$

If there exists $\beta > 0$ such that for any $t \in \mathbb{R}_+$,

$$\gamma(t) + \int_{-\infty}^{0} e^{-\beta\theta} d[\eta_1(t, \theta)] + \int_{-\infty}^{0} e^{-\beta\theta} d[\eta_2(t, \theta)] \le -\beta, \tag{3.4}$$

then the trivial solution of (2.1) is exponentially stable in mean square. In particular, $\mathbb{E}|x(t, \xi)|^2$ exponentially decays with the rate β for any $\xi \in BC((-\infty, 0]; \mathbb{R}^n)$.

Proof Fix $K > 1$ and let $\xi \in BC((-\infty, 0]; \mathbb{R}^n)$ such that $\mathbb{E}\|\xi\|_{BC}^2 > 0$. For the sake of simplicity, we denote $x(t) := x(t, \xi)$, where $x(t, \xi)$ is the solution to (2.1). Let $Z(t) := Ke^{-\beta t}\mathbb{E}\|\xi\|_{BC}^2$, $t \ge 0$. Then, we deduce that from $K > 1$ and $\mathbb{E}\|\xi\|_{BC}^2 > 0$ that $X(t) := \mathbb{E}|x(t)|^2 \le Z(t)$, $t \in (-\infty, 0]$. We will show

$$\mathbb{E}|x(t)|^2 \le Z(t), \quad \forall t \ge 0. \tag{3.5}$$

Assume on the contrary that there exists $t_1 > 0$ such that $X(t_1) > Z(t_1)$. Let $t_* := \inf\{t > 0 : X(t) > Z(t)\}$. By continuity of $X(t)$ and $Z(t)$,

$$X(t) \le Z(t), \ t \in [0, t_*], \quad X(t_*) = Z(t_*), \tag{3.6}$$

and

$$\mathbb{E}|x(t_m)|^2 > Ke^{-\beta t_m}\mathbb{E}\|\xi\|_{BC}^2,$$

for some $t_m \in (t_*, t_* + \frac{1}{m})$, $m \in \mathbb{N}$.

Applying the Itô's formula to the function $V(t, x) = e^{\alpha t}|x(t)|^2$, (3.2)–(3.3) and the Fubini's theorem, we have

$$\mathbb{E}(e^{\alpha t}|x(t)|^2)$$

$$= \mathbb{E}|\xi(0)|^2 + \mathbb{E}\int_0^t \alpha e^{\alpha s}|x(s)|^2 ds + 2\mathbb{E}\int_0^t e^{\alpha s}x^T(s)f(s, x_s)ds$$

$$= +\mathbb{E}\int_0^t e^{\alpha s}\text{trac}[g^T(s, x_s)g(s, x_s)]ds$$

$$\le \mathbb{E}|\xi(0)|^2 + \mathbb{E}\int_0^t \alpha e^{\alpha s}|x(s)|^2 ds + \int_0^t \gamma(s)e^{\alpha s}\mathbb{E}|x(s)|^2 ds$$

$$+ \int_0^t e^{\alpha s}\left(\int_{-\infty}^{0}\mathbb{E}|x(s+\theta)|^2 d[\eta_1(s, \theta)]\right)ds$$

$$+ \int_0^t e^{\alpha s}\left(\int_{-\infty}^{0}\mathbb{E}|x(s+\theta)|^2 d[\eta_2(s, \theta)]\right)ds.$$

Let $K_1 := K\mathbb{E}\|\xi\|_{BC}^2$. Since $\eta_1(s,\theta)$ and $\eta_2(s,\theta)$ are increasing in θ on $(-\infty, 0]$, we derive that from (3.6)

$$\int_{-\infty}^0 \mathbb{E}|x(s+\theta)|^2 d[\eta_1(s,\theta)] \leq K_1 e^{-\beta s} \int_{-\infty}^0 e^{-\beta\theta} d[\eta_1(s,\theta)],$$

and

$$\int_{-\infty}^0 \mathbb{E}|x(s+\theta)|^2 d[\eta_2(s,\theta)] \leq K_1 e^{-\beta s} \int_{-\infty}^0 e^{-\beta\theta} d[\eta_2(s,\theta)],$$

for any $s \leq t_*$. Then, it follows that

$$e^{\alpha t_*}\mathbb{E}|x(t_*)|^2$$

$$\leq \mathbb{E}|\xi(0)|^2 + \int_0^{t_*} e^{\alpha s} e^{-\beta s}(K_1\alpha + K_1\gamma(s))ds$$

$$+ \int_0^{t_*} e^{\alpha s} K_1 e^{-\beta s} \int_{-\infty}^0 e^{-\beta\theta} d[\eta_1(s,\theta)]ds + \int_0^{t_*} e^{\alpha s} K_1 e^{-\beta s} \int_{-\infty}^0 e^{-\beta\theta} d[\eta_2(s,\theta)]ds$$

$$= \mathbb{E}|\xi(0)|^2 + \int_0^{t_*} K_1 e^{\alpha s} e^{-\beta s} \cdot \left[\alpha + \gamma(s) + \int_{-\infty}^0 e^{-\beta\theta} d[\eta_1(s,\theta)] + \int_{-\infty}^0 e^{-\beta\theta} d[\eta_2(s,\theta)]\right]ds.$$

Taking (3.4) into account, we get

$$e^{\alpha t_*}\mathbb{E}|x(t_*)|^2 \leq \mathbb{E}|\xi(0)|^2 + \int_0^{t_*} e^{\alpha s} K_1 e^{-\beta s}(\alpha - \beta)ds$$

$$= \mathbb{E}|\xi(0)|^2 + K_1(e^{\alpha t_*}e^{-\beta t_*} - 1)$$

$$= \mathbb{E}|\xi(0)|^2 - K_1 + K_1 e^{\alpha t_*}e^{-\beta t_*}$$

$$= \mathbb{E}|\xi(0)|^2 - K\mathbb{E}\|\xi\|_{BC}^2 + Ke^{\alpha t_*}e^{-\beta t_*}\mathbb{E}\|\xi\|_{BC}^2$$

$$< Ke^{\alpha t_*}e^{-\beta t_*}\mathbb{E}\|\xi\|_{BC}^2,$$

which conflicts with (3.6). Therefore

$$\mathbb{E}|x(t)|^2 \leq Ke^{-\beta t}\mathbb{E}\|\xi\|_{BC}^2, \quad t \geq 0.$$

So, we know that the trivial solution to (2.1) is exponentially stable in mean square and $\mathbb{E}|x(t,\xi)|^2$ exponentially decays with the rate β. The proof is complete. □

Corollary 3.1 *Let $\Upsilon_1(\cdot,\cdot), \Upsilon_2(\cdot,\cdot) : \mathbb{R}_+ \times (-\infty, 0] \to \mathbb{R}_+$, $\gamma_i(\cdot), \zeta_i(\cdot), h_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}$, $i = 0, 1, 2, \ldots n$ with $0 := h_0(t) \leq h_1(t) \leq h_2(t) \leq \cdots \leq h_n(t) < +\infty$, $t \in \mathbb{R}_+$, be locally bounded Borel measurable functions such that for any $t \in \mathbb{R}_+$, $\varphi \in BC((-\infty, 0]; \mathbb{R}^n)$,*

$$2\mathbb{E}\left(\varphi^T(0)f(t,\varphi)\right) \leq \sum_{i=0}^n \gamma_i(t)\mathbb{E}|\varphi(-h_i(t))|^2 + \int_{-\infty}^0 \Upsilon_1(t,s)\mathbb{E}|\varphi(s)|^2 ds, \qquad (3.7)$$

$$\mathbb{E}\left(\text{trac}[g^T(t,\varphi)g(t,\varphi)]\right) \leq \sum_{i=0}^n \zeta_i(t)\mathbb{E}|\varphi(-h_i(t))|^2 + \int_{-\infty}^0 \Upsilon_2(t,s)\mathbb{E}|\varphi(s)|^2 ds. \qquad (3.8)$$

If there exists $\beta > 0$ such that for any $t \in \mathbb{R}_+$,

$$\sum_{i=0}^{n} e^{\beta h_i(t)} \gamma_i(t) + \int_{-\infty}^{0} e^{-\beta s} \Upsilon_1(t, s) ds + \sum_{i=0}^{n} e^{\beta h_i(t)} \zeta_i(t) + \int_{-\infty}^{0} e^{-\beta s} \Upsilon_2(t, s) ds \leq -\beta, \quad (3.9)$$

then the trivial solution of (2.1) is exponentially stable in mean square. In particular, $\mathbb{E}|x(t, \xi)|^2$ exponentially decays with the rate β for any $\xi \in BC((-\infty, 0]; \mathbb{R}^n)$.

Proof Define the following functions for $t \geq 0$, $s \in (-\infty, 0]$

$$u_i(t, s) := \begin{cases} 0, & \text{if } s \in (-\infty, -h_i(t)], \\ \gamma_i(t), & \text{if } s \in (-h_i(t), 0], \end{cases}$$

$$\eta_1(t, s) := \sum_{i=1}^{n} u_i(t, s) + \int_{-\infty}^{s} \Upsilon_1(t, r) dr,$$

and

$$v_i(t, s) := \begin{cases} 0, & \text{if } s \in (-\infty, -h_i(t)], \\ \zeta_i(t), & \text{if } s \in (-h_i(t), 0], \end{cases}$$

$$\eta_2(t, s) := \sum_{i=1}^{n} v_i(t, s) + \int_{-\infty}^{s} \Upsilon_2(t, r) dr.$$

By the properties of the Riemann-Stieltjes integrals, one has for each $i = 1, 2$ that

$$\int_{-\infty}^{0} \phi(s) d \left[\int_{-\infty}^{s} \Upsilon_i(t, r) dr \right] = \int_{-\infty}^{0} \phi(s) \Upsilon_i(t, s) ds, \quad t \in \mathbb{R}_+,$$

for any $\phi(\cdot) \in BC((-\infty, 0]; \mathbb{R}^n)$. Then for any $t \in \mathbb{R}_+$, $\phi(\cdot) \in BC((-\infty, 0]; \mathbb{R}^n)$,

$$\int_{-\infty}^{0} \phi(s) d[\eta_1(t, s)] = \sum_{i=1}^{n} \gamma_i(t) \phi(-h_i(t)) + \int_{-\infty}^{0} \phi(s) \Upsilon_1(t, s) ds,$$

$$\int_{-\infty}^{0} \phi(s) d[\eta_2(t, s)] = \sum_{i=1}^{n} \zeta_i(t) \phi(-h_i(t)) + \int_{-\infty}^{0} \phi(s) \Upsilon_2(t, s) ds.$$

Therefore, (3.7)–(3.8) imply that (3.2)–(3.3) hold and (3.9) ensures that (3.4) holds. By the Theorem 3.1 we can obtain our desired results. The proof is complete.   □

4 Exponential Stability for NSFDEs with Infinite Delay

Consider the following neutral stochastic functional differential equation with infinite delay

$$d[x(t) - G(x_t)] = f(t, x_t) dt + g(t, x_t) dw(t) \tag{4.1}$$

on $t \geq 0$ with initial data $x_0 = \xi = \{\xi(\theta) : \theta \leq 0\} \in BC((-\infty, 0]; \mathbb{R}^n)$, where

$$G : BC((-\infty, 0]; \mathbb{R}^n) \to \mathbb{R}^n, \qquad f : \mathbb{R}_+ \times BC((-\infty, 0]; \mathbb{R}^n) \to \mathbb{R}^n,$$

$$g : \mathbb{R}_+ \times BC((-\infty, 0]; \mathbb{R}^n) \to \mathbb{R}^{n \times m}.$$

An $\mathcal{F}_t$-adapted process $x(t)$, $-\infty < t < +\infty$ is said to be the solution of the equation (4.1) if it satisfies the initial condition above and moreover for each $t \geq 0$,

$$x(t) - G(x_t) = \xi(0) - G(x_0) + \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)dw(s), \qquad (4.2)$$

where the stochastic integral is in the Itô's sense. The details on the existence and uniqueness of the solution to (4.1), we can refer to [3]. For example, when f, g, G are uniformly Lipschitz continuous, or they are locally Lipschitz continuous and satisfy the linear growth condition, Kolmanovskii and Nosov [3] proved that there is unique continuous solution to (4.1), and any moment of the solution is finite. For stability purpose, throughout the paper we always suppose that equation (4.1) has a unique solution for arbitrarily given initial data $\xi \in C_{\mathcal{F}_0}^b([-\tau, 0]; \mathbb{R}^n)$ and the solution is denoted by $x(t, \xi)$, or simply $x(t)$, when no confusion is possible. For the purposes of stability, we shall assume that

$$G(0) \equiv 0, \quad f(t, 0) \equiv 0, \quad g(t, 0) \equiv 0 \quad \text{for any } t \geq 0.$$

It is well-known that for a given $\xi \in BC((-\infty, 0]; \mathbb{R}^n)$, equation (4.1) has a trivial solution when $\xi \equiv 0$.

We assume that there is a constant $k \in (0, 1)$ such that for all $\varphi \in BC((-\infty, 0]; \mathbb{R}^n)$

$$\mathbb{E}|G(\varphi)|^2 \leq k \sup_{-\infty < \theta \leq 0} \mathbb{E}|\varphi(\theta)|^2. \qquad (4.3)$$

Lemma 4.1 *Let* (4.3) *hold with* $0 < k < 1$ *and* $\rho \geq 0$, $\delta > 0$, $K > 1$. *If*

$$e^{\delta t}\mathbb{E}|x(t) - G(x_t)|^2 \leq K \sup_{-\infty < \theta \leq 0} \mathbb{E}|x(\theta)|^2 \qquad (4.4)$$

for all $0 \leq t \leq \rho$, *then*

$$e^{\delta t}\mathbb{E}|x(t)|^2 \leq \frac{K}{(1 - \sqrt{k})^2} \sup_{-\infty < \theta \leq 0} \mathbb{E}|x(\theta)|^2.$$

Proof Let $k < \varepsilon < 1$. For $0 \leq t \leq \rho$, we have

$$\mathbb{E}|x(t) - G(x_t)|^2 \geq \mathbb{E}|x(t)|^2 - 2\mathbb{E}(|x(t)||G(x_t)|) + \mathbb{E}|G(x_t)|^2$$

$$\geq (1 - \varepsilon)\mathbb{E}|x(t)|^2 - (\varepsilon^{-1} - 1)\mathbb{E}|G(x_t)|^2.$$

Then, by (4.3) we have

$$\mathbb{E}|x(t)|^2 \leq \frac{1}{1 - \varepsilon}\mathbb{E}|x(t) - G(x_t)|^2 + \frac{k}{\varepsilon} \sup_{-\infty < \theta \leq 0} \mathbb{E}|x(t + \theta)|^2.$$

Using the condition (4.4), we derive that for all $0 \leq t \leq \rho$

$$e^{\delta t}\mathbb{E}|x(t)|^2 \leq \frac{K}{1 - \varepsilon} \sup_{-\infty < \theta \leq 0} \mathbb{E}|x(\theta)|^2 + \frac{k}{\varepsilon} \sup_{-\infty < \theta \leq 0} \left[e^{\delta t}\mathbb{E}|x(t + \theta)|^2 \right]$$

$$\leq \frac{K}{1 - \varepsilon} \sup_{-\infty < \theta \leq 0} \mathbb{E}|x(\theta)|^2 + \frac{k}{\varepsilon} \sup_{-\infty < t \leq \rho} \left[e^{\delta t}\mathbb{E}|x(t)|^2 \right].$$

Moreover, this holds for $-\infty < t \leq 0$. Thus,

$$\sup_{-\infty < t \leq \rho} \left[e^{\delta t} \mathbb{E}|x(t)|^2 \right] \leq \frac{K}{1-\varepsilon} \sup_{-\infty < \theta \leq 0} \mathbb{E}|x(\theta)|^2 + \frac{k}{\varepsilon} \sup_{-\infty < t \leq \rho} \left[e^{\delta t} \mathbb{E}|x(t)|^2 \right].$$

Since $1 > \frac{k}{\varepsilon}$, we can obtain

$$\sup_{-\infty < t \leq \rho} \left[e^{\delta t} \mathbb{E}|x(t)|^2 \right] \leq \frac{K\varepsilon}{(1-\varepsilon)(\varepsilon - k)} \sup_{-\infty < \theta \leq 0} \mathbb{E}|x(\theta)|^2.$$

Lastly, letting $\varepsilon = \sqrt{k}$ we can obtain our desired result. The proof is complete. $\square$

Theorem 4.1 *Assume that* (4.3) *holds with* $0 < k < 1$. *Let* $\gamma(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ *be a locally bounded Borel-measurable function such that for any* $t \in \mathbb{R}_+$, $\varphi \in BC((-\infty, 0]; \mathbb{R}^n)$,

$$2\mathbb{E}\left((\varphi(0) - G(\varphi))^T f(t, \varphi) \right) \leq \gamma(t)\mathbb{E}|\varphi(0)|^2 + \int_{-\infty}^{0} \mathbb{E}|\varphi(\theta)|^2 d[\eta_1(t, \theta)], \qquad (4.5)$$

and

$$\mathbb{E}\left(\text{trac}[g^T(t, \varphi)g(t, \varphi)] \right) \leq \int_{-\infty}^{0} \mathbb{E}|\varphi(\theta)|^2 d[\eta_2(t, \theta)]. \qquad (4.6)$$

If there exists $\beta > 0$ *such that for any* $t \in \mathbb{R}_+$,

$$\gamma(t) + \int_{-\infty}^{0} e^{-\beta\theta} d[\eta_1(t, \theta)] + \int_{-\infty}^{0} e^{-\beta\theta} d[\eta_2(t, \theta)] \leq -(1 - \sqrt{k})^2 \beta, \qquad (4.7)$$

then the trivial solution of (4.1) *is exponentially stable in mean square. In particular,* $\mathbb{E}|x(t, \xi)|^2$ *exponentially decays with the rate* β *for any* $\xi \in BC((-\infty, 0]; \mathbb{R}^n)$.

Proof Fix $K > 1$ sufficient large and let $\xi \in BC((-\infty, 0]; \mathbb{R}^n)$ such that $\mathbb{E}\|\xi\|_{BC}^2 > 0$. For the sake of simplicity, we denote $x(t) := x(t, \xi)$, where $x(t, \xi)$ is the solution to (4.1). Let $Z(t) := Ke^{-\beta t}\mathbb{E}\|\xi\|_{BC}^2$, $t \geq 0$. Then, we deduce that from $K > 1$ sufficient large and $\mathbb{E}\|\xi\|_{BC}^2 > 0$ that $X(t) := \mathbb{E}|x(t) - G(t, x_t)|^2 \leq Z(t)$, $t \in (-\infty, 0]$. We will show

$$\mathbb{E}|x(t) - G(t, x_t)|^2 \leq Z(t), \quad \forall t \geq 0. \qquad (4.8)$$

Assume on the contrary that there exists $t_1 > 0$ such that $X(t_1) > Z(t_1)$. Let $t_* := \inf\{t > 0 : X(t) > Z(t)\}$. By continuity of $X(t)$ and $Z(t)$,

$$X(t) \leq Z(t), \ t \in [0, t_*], \ X(t_*) = Z(t_*), \qquad (4.9)$$

and

$$\mathbb{E}|x(t_m) - G(x_{t_m})|^2 > Ke^{-\beta t_m}\mathbb{E}\|\xi\|_{BC}^2,$$

for some $t_m \in (t_*, t_* + \frac{1}{m})$, $m \in \mathbb{N}$.

Applying the Itô's formula to the function $V(t, x) = e^{\alpha t}|x(t) - G(x_t)|^2$, (4.5)–(4.6) and the Fubini's theorem, we have

$$\mathbb{E}(e^{\alpha t}|x(t) - G(x_t)|^2)$$

$$= \mathbb{E}|\xi(0) - G(\xi)|^2 + \mathbb{E}\int_0^t \alpha e^{\alpha s}|x(s) - G(x_s)|^2 ds$$

$$+ 2\mathbb{E}\int_0^t e^{\alpha s}(x(s) - G(x_s))^T f(s, x_s)ds + \mathbb{E}\int_0^t e^{\alpha s}\text{trac}[g^T(s, x_s)g(s, x_s)]ds$$

$$\leq \mathbb{E}|\xi(0) - G(\xi)|^2 + \mathbb{E}\int_0^t \alpha e^{\alpha s}|x(s) - G(x_s)|^2 ds + \int_0^t \gamma(s)e^{\alpha s}\mathbb{E}|x(s)|^2 ds$$

$$+ \int_0^t e^{\alpha s}\left(\int_{-\tau}^0 \mathbb{E}|x(s + \theta)|^2 d[\eta_1(s, \theta)]\right)ds + \int_0^t e^{\alpha s}\left(\int_{-\tau}^0 \mathbb{E}|x(s + \theta)|^2 d[\eta_2(s, \theta)]\right)ds.$$

Let $K_1 := K\mathbb{E}\|\xi\|_{BC}^2$ and $K_2 = \frac{K_1}{(1 - \sqrt{k})^2}$. Since $\eta_1(s, \theta)$ and $\eta_2(s, \theta)$ are increasing in θ on $(-\infty, 0]$, we derive that from (4.9) and the Lemma 4.1

$$\int_{-\infty}^0 \mathbb{E}|x(s + \theta)|^2 d[\eta_1(s, \theta)] \leq K_2 e^{-\beta s}\int_{-\infty}^0 e^{-\beta \theta} d[\eta_1(s, \theta)],$$

and

$$\int_{-\infty}^0 \mathbb{E}|x(s + \theta)|^2 d[\eta_2(s, \theta)] \leq K_2 e^{-\beta s}\int_{-\infty}^0 e^{-\beta \theta} d[\eta_2(s, \theta)],$$

for any $s \leq t_*$. Then, it follows that

$$e^{\alpha t_*}\mathbb{E}|x(t_*) - G(x_{t_*})|^2$$

$$\leq \mathbb{E}|\xi(0) - G(\xi)|^2 + \int_0^{t_*} e^{\alpha s}e^{-\beta s}(K_1\alpha + K_2\gamma(s))ds$$

$$+ \int_0^{t_*} e^{\alpha s}K_2 e^{-\beta s}\int_{-\infty}^0 e^{-\beta \theta} d[\eta_1(s, \theta)]ds + \int_0^{t_*} e^{\alpha s}K_2 e^{-\beta s}\int_{-\infty}^0 e^{-\beta \theta} d[\eta_2(s, \theta)]ds$$

$$= \mathbb{E}|\xi(0) - G(\xi)|^2 + \int_0^{t_*} K_1 e^{\alpha s}e^{-\beta s}$$

$$\cdot \left[\alpha + \frac{1}{(1 - \sqrt{k})^2}\left(\gamma(s) + \int_{-\infty}^0 e^{-\beta \theta} d[\eta_1(s, \theta)] + \int_{-\infty}^0 e^{-\beta \theta} d[\eta_2(s, \theta)]\right)\right]ds.$$

Taking (4.7) into account, we get for sufficient large K,

$$e^{\alpha t_*}\mathbb{E}|x(t_*) - G(x_{t_*})|^2 \leq \mathbb{E}|\xi(0) - G(\xi)|^2 + \int_0^{t_*} e^{\alpha s}K_1 e^{-\beta s}(\alpha - \beta)ds$$

$$= \mathbb{E}|\xi(0) - G(\xi)|^2 + K_1(e^{\alpha t_*}e^{-\beta t_*} - 1)$$

$$= \mathbb{E}|\xi(0) - G(\xi)|^2 - K_1 + K_1 e^{\alpha t_*}e^{-\beta t_*}$$

$$= \mathbb{E}|\xi(0) - G(\xi)|^2 - K\mathbb{E}\|\xi\|_{BC}^2 + Ke^{\alpha t_*}e^{-\beta t_*}\mathbb{E}\|\xi\|_C^2$$

$$< Ke^{\alpha t_*}e^{-\beta t_*}\mathbb{E}\|\xi\|_{BC}^2,$$

which conflicts with (4.9). Therefore

$$\mathbb{E}|x(t) - G(x_t)|^2 \le K e^{-\beta t} \mathbb{E}\|\xi\|_{BC}^2, \quad t \ge 0.$$

So, by applying the Lemma 4.1, we know that the trivial solution to (4.1) is exponentially stable in mean square and $\mathbb{E}|x(t, \xi)|^2$ exponentially decays with the rate β. The proof is complete. □

Corollary 4.1 *Assume that* (4.3) *holds with* $0 < k < 1$. *Let* $\Upsilon_1(\cdot, \cdot), \Upsilon_2(\cdot, \cdot) : \mathbb{R}_+ \times (-\infty, 0] \to \mathbb{R}_+, \gamma_i(\cdot), \zeta_i(\cdot), h_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}, i = 0, 1, 2, \ldots n$ *with* $0 := h_0(t) \le h_1(t) \le h_2(t) \le \cdots \le h_n(t) < +\infty, t \in \mathbb{R}_+,$ *be locally bounded Borel measurable functions such that for any* $t \in \mathbb{R}_+, \varphi \in BC((-\infty, 0]; \mathbb{R}^n),$

$$2\mathbb{E}\Big((\varphi(0) - G(\varphi))^T f(t, \varphi)\Big) \le \sum_{i=0}^{n} \gamma_i(t)\mathbb{E}|\varphi(-h_i(t))|^2 + \int_{-\infty}^{0} \Upsilon_1(t, s)\mathbb{E}|\varphi(s)|^2 ds, \quad (4.10)$$

$$\mathbb{E}\Big(\mathrm{trac}[g^T(t, \varphi)g(t, \varphi)]\Big) \le \sum_{i=0}^{n} \zeta_i(t)\mathbb{E}|\varphi(-h_i(t))|^2 + \int_{-\infty}^{0} \Upsilon_2(t, s)\mathbb{E}|\varphi(s)|^2 ds. \quad (4.11)$$

If there exists $\beta > 0$ *such that for any* $t \in \mathbb{R}_+,$

$$\sum_{i=0}^{n} e^{\beta h_i(t)} \gamma_i(t) + \int_{-\infty}^{0} e^{-\beta s} \Upsilon_1(t, s) ds + \sum_{i=0}^{n} e^{\beta h_i(t)} \zeta_i(t) + \int_{-\infty}^{0} e^{-\beta s} \Upsilon_2(t, s) ds$$

$$\le -(1 - \sqrt{k})^2 \beta, \quad (4.12)$$

then the trivial solution of (4.1) *is exponentially stable in mean square. In particular,* $\mathbb{E}|x(t, \xi)|^2$ *exponentially decays with the rate* β *for any* $\xi \in BC((-\infty, 0]; \mathbb{R}^n)$.

Proof Define the following functions for $t \ge 0, s \in (-\infty, 0]$

$$u_i(t, s) := \begin{cases} 0, & \text{if } s \in (-\infty, -h_i(t)], \\ \gamma_i(t), & \text{if } s \in (-h_i(t), 0], \end{cases}$$

$$\eta_1(t, s) := \sum_{i=1}^{n} u_i(t, s) + \int_{-\infty}^{s} \Upsilon_1(t, r) dr,$$

and

$$v_i(t, s) := \begin{cases} 0, & \text{if } s \in (-\infty, -h_i(t)], \\ \zeta_i(t), & \text{if } s \in (-h_i(t), 0], \end{cases}$$

$$\eta_2(t, s) := \sum_{i=1}^{n} v_i(t, s) + \int_{-\infty}^{s} \Upsilon_2(t, r) dr.$$

By the properties of the Riemann-Stieltjes integrals, one has for each $i = 1, 2$ that

$$\int_{-\infty}^{0} \phi(s) d\Big[\int_{-\infty}^{s} \Upsilon_i(t, r) dr\Big] = \int_{-\infty}^{0} \phi(s)\Upsilon_i(t, s) ds, \quad t \in \mathbb{R}_+,$$

for any $\phi(\cdot) \in BC((-\infty, 0]; \mathbb{R}^n)$. Then for any $t \in \mathbb{R}_+$, $\phi(\cdot) \in BC((-\infty, 0]; \mathbb{R}^n)$,

$$\int_{-\infty}^{0} \phi(s)d[\eta_1(t,s)] = \sum_{i=1}^{n} \gamma_i(t)\phi(-h_i(t)) + \int_{-\infty}^{0} \phi(s)\Upsilon_1(t,s)ds,$$

$$\int_{-\infty}^{0} \phi(s)d[\eta_2(t,s)] = \sum_{i=1}^{n} \zeta_i(t)\phi(-h_i(t)) + \int_{-\infty}^{0} \phi(s)\Upsilon_2(t,s)ds.$$

Therefore, (4.10)–(4.11) imply that (4.5)–(4.6) hold and (4.12) ensures that (4.7) holds. By the Theorem 4.1 we can obtain our desired results. The proof is complete. $\qquad\square$

5 Comparison with Existing Results and Some Examples

Now, we state some comparisons with existing results to illustrate the advantage of our results.

Consider the following stochastic differential equation with distributed delay of the form

$$dx(t) = F\Big(t, x(t), \int_{-\infty}^{0} k_1(s)x(t+s)ds, \ldots, \int_{-\infty}^{0} k_r(s)x(t+s)ds\Big)dt$$
$$+ G\Big(t, x(t), \int_{-\infty}^{0} k_1(s)x(t+s)ds, \ldots, \int_{-\infty}^{0} k_r(s)x(t+s)ds\Big)dw(t) \tag{5.1}$$

on $t \geq 0$ with initial data $x_0 = \xi \in BC((-\infty, 0]; \mathbb{R}^n)$, where $k_i(s) \in L((-\infty, 0]; \mathbb{R}_+)$ for $i = 1, 2, \ldots, r$ and $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{n \times r} \to \mathbb{R}^n$, $G : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{n \times r} \to \mathbb{R}^{n \times m}$ satisfy the local Lipschitz condition and the linear growth condition and $F(t, 0, \ldots, 0) \equiv 0$, $G(t, 0, \ldots, 0) \equiv 0$.

By using the general Razumikhin-type theorem, Yang et al. [21] established the following criterion for the exponential stability in mean square of (5.1) (see the Corollary 4.2 in [21]).

Proposition 5.1 *Assume that there are* $\lambda > 0$ *and nonnegative constants* α_i, β_i, $i = 0, 1, 2, \ldots, r$ *such that*

$$\int_{-\infty}^{0} k_i(s)ds = 1, \qquad \bar{k}_i := \int_{-\infty}^{0} e^{-\lambda s}k_i(s)ds < \infty, \tag{5.2}$$

$$\varphi^T(0)F(t, \varphi(0), 0, \ldots, 0) \leq -\lambda|\varphi(0)|^2, \tag{5.3}$$

$$\left|F(t, \psi(0), 0, \ldots, 0) - F\Big(t, \varphi(0), \int_{-\infty}^{0} k_1(s)\varphi(s))ds, \ldots, \int_{-\infty}^{0} k_r(s)\varphi(s))ds\Big)\right|$$
$$\leq \alpha_0|\varphi(0) - \psi(0)| + \sum_{i=1}^{r} \alpha_i \left|\int_{-\infty}^{0} k_i(s)\varphi(s)ds\right|, \tag{5.4}$$

and

$$\left|G\Big(t, \varphi(0), \int_{-\infty}^{0} k_1(s)\varphi(s))ds, \ldots, \int_{-\infty}^{0} k_r(s)\varphi(s))ds\Big)\right|^2$$
$$\leq \beta_0|\varphi(0)|^2 + \sum_{i=1}^{r} \beta_i \left|\int_{-\infty}^{0} k_i(s)\varphi(s)ds\right|^2, \tag{5.5}$$

for all $t \geq 0$, $\varphi, \psi \in BC((-\infty, 0]; \mathbb{R}^n)$. If

$$\lambda > \frac{1}{2}\beta_0 + \frac{1}{2}\sum_{i=1}^{r}\alpha_i + \frac{1}{2}\sum_{i=1}^{r}(\alpha_i + \beta_i)\bar{k}_i, \tag{5.6}$$

then the trivial solution of (5.1) *is exponentially stable in mean square.*

According to (5.3) and (5.4), we know that for all $t \geq 0$, $\varphi \in BC((-\infty, 0]; \mathbb{R}^n)$

$$2\varphi^T(0)F\left(t, \varphi(0), \int_{-\infty}^{0} k_1(s)\varphi(s))ds, \ldots, \int_{-\infty}^{0} k_r(s)\varphi(s))ds\right)$$

$$= 2\varphi^T(0)F(t, \varphi(0), 0, \ldots, 0) + 2\varphi^T(0)\Big(-F(t, \varphi(0), 0, \ldots, 0)$$

$$+ F\left(t, \varphi(0), \int_{-\infty}^{0} k_1(s)\varphi(s))ds, \ldots, \int_{-\infty}^{0} k_r(s)\varphi(s))ds\right)\Big) \tag{5.7}$$

$$\leq -2\lambda|\varphi(0)|^2 + 2|\varphi^T(0)|\sum_{i=1}^{r}\alpha_i\left|\int_{-\infty}^{0} k_i(s)\varphi(s)ds\right|$$

$$\leq -2\lambda|\varphi(0)|^2 + \left(\sum_{i=1}^{r}\alpha_i\int_{-\infty}^{0} k_i(s)ds\right)|\varphi(0)|^2 + \sum_{i=1}^{r}\alpha_i\int_{-\infty}^{0} k_i(s)|\varphi(s)|^2 ds.$$

On the other hand, from (5.5) and the Hölder's inequality, we have for all $t \geq 0$, $\varphi \in BC((-\infty, 0]; \mathbb{R}^n)$

$$\left|G\left(t, \varphi(0), \int_{-\infty}^{0} k_1(s)\varphi(s))ds, \ldots, \int_{-\infty}^{0} k_r(s)\varphi(s))ds\right)\right|^2$$

$$\leq \beta_0|\varphi(0)|^2 + \sum_{i=1}^{r}\beta_i\int_{-\infty}^{0} k_i(s)ds \cdot \int_{-\infty}^{0} k_i(s)|\varphi(s)|^2 ds, \tag{5.8}$$

which means that (3.7) holds with $h_i(t) = 0$, $\gamma_i(t) = \sum_{i=1}^{r}\alpha_i\int_{-\infty}^{0} k_i(s)ds, -2\lambda$, $\Upsilon_1(t,s) = \sum_{i=1}^{r}\alpha_i k_i(s)$ and (3.8) holds with $h_i(t) = 0$, $\zeta_i(t) = \beta_0$, $\Upsilon_2(t,s) = \sum_{i=1}^{r}\beta_i\left(\int_{-\infty}^{0} k_i(s)ds\right)k_i(s)$.

Thus, we deduce that (5.3)–(5.5) are indeed stronger than (3.7) and (3.8). Moreover, by the Corollary 3.1, we deduce that the trivial solution of (5.1) is exponentially stable in mean square if (5.3)–(5.5) are satisfied and there exists a constant $\beta > 0$ such that

$$-2\lambda + \sum_{i=1}^{r}\alpha_i\int_{-\infty}^{0} k_i(s)ds + \sum_{i=1}^{r}\alpha_i\int_{-\infty}^{0} k_i(s)e^{-\beta s}ds + \beta_0$$

$$+ \sum_{i=1}^{r}\beta_i\int_{-\infty}^{0} k_i(s)ds \cdot \int_{-\infty}^{0} k_i(s)e^{-\beta s}ds < -\beta. \tag{5.9}$$

Besides, we note that the condition (5.2) is not required by using our Corollary 3.1. But, if (5.2) is satisfied, then (5.9) reduce to

$$-2\lambda + \sum_{i=1}^{r}\alpha_i + \sum_{i=1}^{r}\alpha_i\int_{-\infty}^{0} k_i(s)e^{-\beta s}ds + \beta_0 + \sum_{i=1}^{r}\beta_i\int_{-\infty}^{0} k_i(s)e^{-\beta s}ds < -\beta. \tag{5.10}$$

We remark that the condition (5.10) is weaker than (5.6). Indeed, if (5.6) holds with $\bar{k}_i := \int_{-\infty}^{0} e^{-\lambda s} k_i(s) ds$, then

$$-2\lambda + \beta_0 + \sum_{i=1}^{r} \alpha_i + \sum_{i=1}^{r} (\alpha_i + \beta_i) \bar{k}_i < 0.$$

Therefore, by the continuity

$$-2\lambda + \sum_{i=1}^{r} \alpha_i + \sum_{i=1}^{r} \alpha_i \int_{-\infty}^{0} k_i(s) e^{-\lambda s + (-\beta + \lambda)s} ds + \beta_0 + \sum_{i=1}^{r} \beta_i \int_{-\infty}^{0} k_i(s) e^{-\lambda s + (-\beta + \lambda)s} ds$$
$$< -\beta \tag{5.11}$$

for some $\beta > 0$ sufficient small such that $-\beta + \lambda > 0$, which implies that (5.10) holds for some $\beta > 0$ sufficient small such that $-\beta + \lambda > 0$.

Now, we consider the following stochastic functional differential equation with time varying infinite delay

$$
\begin{aligned}
dx(t) = &F(t, x(t), x(t - \delta_1(t)), \dots, x(t - \delta_r(t)))dt \\
&+ G(t, x(t), x(t - \delta_1(t)), \dots, x(t - \delta_r(t)))dw(t)
\end{aligned}
\tag{5.12}
$$

on $t \geq 0$ with initial data $x_0 = \xi \in BC((-\infty, 0]; \mathbb{R}^n)$, where $\delta_i(s) : \mathbb{R}_+ \to \mathbb{R}_+$ for $i = 1, 2, \dots, r$ and $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{n \times r} \to \mathbb{R}^n$, $G : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{n \times r} \to \mathbb{R}^{n \times m}$ satisfy the local Lipschitz condition and the linear growth condition and $F(t, 0, \dots, 0) \equiv 0$, $G(t, 0, \dots, 0) \equiv 0$.

To compare with the results of [16], let us introduce the following definitions on both p-th moment and almost surely stability with a certain rate.

Definition 5.1 Let the function $\lambda(t) \in C(\mathbb{R}_+; \mathbb{R}_+)$ be strictly increasing and $\lambda(t) \uparrow \infty$ as $t \to \infty$. Then, the trivial solution of equation (2.1) is said to be p-th moment stable with decay $\lambda(t)$ of order γ if there exists a pair of constants $\gamma > 0$ and $c(\xi) > 0$ such that

$$\mathbb{E}|x(t, \xi)|^p \leq c(\xi) \lambda^{-\gamma}(t), \quad t \geq 0$$

holds for any $\xi \in BC((-\infty, 0]; \mathbb{R}^n)$. The trivial solution of equation (2.1) is said to be almost surely stable with decay $\lambda(t)$ of order γ if

$$\limsup_{t \to \infty} \frac{\ln|x(t, \xi)|}{\ln \lambda(t)} \leq -\gamma, \quad \text{a.s.}$$

By using the general Razumikhin-type theorem, Pavlović and Janković [16] established the following criterion for stability in mean square of (5.12) (see the Corollary 3.1).

Proposition 5.2 *Let* $\lambda(t) \in C^1(\mathbb{R}_+; \mathbb{R}_+)$ *be strictly increasing and* $\lambda(t) \uparrow \infty$ *as* $t \to \infty$, $\lambda(0) = 1$, *and* $\lambda(s + t) \leq \lambda(t)\lambda(s)$ *for all* $t, s \geq 0$. *Assume that there exist constants* $\mu > 0$, $\rho > 0$, $\nu \geq 1$ *and* $\alpha_i, \beta_i \geq 0$, $0 \leq i \leq r$ *such that*

$$x^T F(t, x, 0, \dots, 0) \leq -\rho \frac{\lambda'(t)}{\lambda(t)} |x|^2, \tag{5.13}$$

$$|F(t, x, 0, \ldots, 0) - F(t, \bar{x}, y_1, \ldots, y_r)| \leq \frac{\lambda'(t)}{\lambda(t)} \left[\alpha_0 |x - \bar{x}| + \frac{1}{\nu} \sum_{i=1}^{r} \alpha_i \lambda^{-\mu} (\delta_i(t)) |y_i| \right],$$
(5.14)

$$|G(t, x, y_1, \ldots, y_r)| \leq \frac{\lambda'(t)}{\lambda(t)} \left[\beta_0 |x|^2 + \frac{1}{\nu} \sum_{i=1}^{r} \beta_i \lambda^{-\mu} (\delta_i(t)) |y_i|^2 \right]$$
(5.15)

for all $t \geq 0$ and $x, \bar{x}, y_1, \ldots, y_r \in \mathbb{R}^n$. If

$$\rho > \frac{1}{2} \beta_0 + \frac{1}{\nu} \sum_{i=1}^{r} \left(\alpha_i + \frac{1}{2} \beta_i \right),$$
(5.16)

then the trivial solution of equation (5.12) is stable in mean square with the rate function $\lambda(t)$ of order $\gamma = 2\rho - \beta_0 - \frac{2}{\nu} \sum_{i=1}^{r} \left(\alpha_i + \frac{1}{2} \beta_i \right) \wedge \mu$.

By the Definition 2.1 and 5.1, we easily know that if the trivial solution of equation (5.12) is exponentially stable in mean square, then the trivial solution of equation (5.12) is stable in mean square with decay e^t of order λ, and if the trivial solution of equation (5.12) is stable in mean square with decay e^t of order λ, then the trivial solution of equation (5.12) is exponentially stable in mean square. In other word, exponential stability in mean square is equivalent to the stability in mean square with decay e^t. So, the Proposition 5.2 is equivalent to the following Proposition 5.3.

Proposition 5.3 *Assume that there exist constants $\mu > 0$, $\rho > 0$, $\nu \geq 1$ and $\alpha_i, \beta_i \geq 0$, $0 \leq i \leq r$ such that*

$$x^T F(t, x, 0, \ldots, 0) \leq -\rho |x|^2,$$
(5.17)

$$|F(t, x, 0, \ldots, 0) - F(t, \bar{x}, y_1, \ldots, y_r)| \leq \alpha_0 |x - \bar{x}| + \frac{1}{\nu} \sum_{i=1}^{r} \alpha_i e^{-\mu \delta_i(t)} |y_i|,$$
(5.18)

$$|G(t, x, y_1, \ldots, y_r)| \leq \beta_0 |x|^2 + \frac{1}{\nu} \sum_{i=1}^{r} \beta_i e^{-\mu \delta_i(t)} |y_i|^2$$
(5.19)

for all $t \geq 0$ and $x, \bar{x}, y_1, \ldots, y_r \in \mathbb{R}^n$. If

$$\rho > \frac{1}{2} \beta_0 + \frac{1}{\nu} \sum_{i=1}^{r} \left(\alpha_i + \frac{1}{2} \beta_i \right),$$
(5.20)

then the trivial solution of equation (5.12) is exponentially in mean square.

In fact, by (5.17) and (5.18) we obtain for all $t \geq 0$ and $x, y_1, \ldots, y_r \in \mathbb{R}^n$

$$
\begin{aligned}
&2x^T F(t, x, y_1, \ldots, y_r) \\
&= 2x^T F(t, x, 0, \ldots, 0) + 2x^T (F(t, x, y_1, \ldots, y_r) - F(t, x, 0, \ldots, 0)) \\
&\leq -2\rho|x|^2 + 2|x^T| \frac{1}{\nu} \sum_{i=1}^r \alpha_i e^{-\mu \delta_i(t)} |y_i| \\
&\leq -2\rho|x|^2 + \frac{1}{\nu} \sum_{i=1}^r \alpha_i e^{-\mu \delta_i(t)} |x|^2 + \frac{1}{\nu} \sum_{i=1}^r \alpha_i e^{-\mu \delta_i(t)} |y_i|^2.
\end{aligned}
$$

Combining with (5.19), we can deduce that (3.7) and (3.8) hold. Therefore, by our Corollary 3.1 we declare that the trivial solution of equation (5.12) is exponentially stable in mean square if

$$
-2\rho + \beta_0 + \frac{1}{\nu} \sum_{i=1}^r \alpha_i e^{-\mu \delta_i(t)} + \frac{1}{\nu} \sum_{i=1}^r \alpha_i e^{-\mu \delta_i(t)} e^{\beta \delta_i(t)} + \frac{1}{\nu} \sum_{i=1}^r \beta_i e^{-\mu \delta_i(t)} e^{\beta \delta_i(t)} \leq -\beta \quad (5.21)
$$

holds for some $\beta > 0$. We remark that (5.21) is equivalent (5.20). Indeed, by (5.20) we have

$$
-2\rho + \beta_0 + \frac{2}{\nu} \sum_{i=1}^r \alpha_i + \frac{1}{\nu} \sum_{i=1}^r \beta_i < 0. \quad (5.22)
$$

By the continuity and (5.22), we know that

$$
-2\rho + \beta_0 + \frac{1}{\nu} \sum_{i=1}^r \alpha_i + \frac{1}{\nu} \sum_{i=1}^r (\alpha_i + \beta_i) e^{-\mu \delta_i(t)} e^{\beta \delta_i(t)} < -\beta \quad (5.23)
$$

holds for some sufficient small $0 < \beta < \mu$, which means that (5.21) holds for some sufficient small $0 < \beta < \mu$.

Now, we present some examples to illustrate the advantage of our results.

Consider the scalar linear time-varying stochastic differential equation with delay

$$
d[x(t) - k(x(t - h_0(t)))] = (-a(t)x(t) + b(t)x(t - h_1(t)))dt + c(t)x(t - h_2(t))dw(t), \quad (5.24)
$$

where $0 < k < 1$, $a(t), b(t), c(t) : \mathbb{R}_+ \to \mathbb{R}$, $h_0(t), h_1(t), h_2(t) : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions, and $w(t)$ is scalar Brownian motion.

We can deduce that (4.3) holds. Let

$$
f(t, \varphi) := -a(t)\varphi(0) + b(t)\varphi(-h_1(t)), \quad g(t, \varphi) := c(t)\varphi(-h_2(t))
$$

for $t \in \mathbb{R}_+$, $\varphi \in BC((-\infty, 0]; \mathbb{R})$. Then, for all $t \in \mathbb{R}_+$, $\varphi \in BC((-\infty, 0]; \mathbb{R})$ we have

$$
\begin{aligned}
2\varphi(0) f(t, \varphi) &= -2a(t)|\varphi(0)|^2 + 2b(t)\varphi(0)\varphi(-h_1(t))) \\
&\leq -2a(t)|\varphi(0)|^2 + |b(t)|(\varphi^2(0) + \varphi^2(-h_1(t))),
\end{aligned} \quad (5.25)
$$

$$
\begin{aligned}
2G(\varphi) f(t, \varphi) &\leq \sqrt{k}[-a(t)|\varphi(0)| + b(t)\varphi(0)\varphi(-h_1(t))]^2 + \sqrt{k}|\varphi(-h_0(t))|^2 \\
&\leq 2\sqrt{k}[a^2(t)|\varphi(0)|^2 + b^2(t)\varphi^2(-h_1(t))] + \sqrt{k}|\varphi(-h_0(t))|^2,
\end{aligned} \quad (5.26)
$$

and

$$g^2(t, \varphi) = c^2(t)\varphi^2(-h_2(t)). \tag{5.27}$$

Then, by the Corollary 4.1 we deduce that if for any $t \in \mathbb{R}_+$,

$$-2a(t) + |b(t)| + |b(t)|e^{\beta h_1(t)} + 2\sqrt{k}(a^2(t) + b^2(t)e^{\beta h_1(t)}) + \sqrt{k}e^{\beta h_0(t)} + c^2(t)e^{\beta h_2(t)}$$

$$\leq -\beta, \tag{5.28}$$

then the trivial solution of (5.24) is exponentially stable in mean square. In particular, $\mathbb{E}|x(t, \xi)|^2$ exponentially decays with the rate β for any $\xi \in BC((-\infty, 0]; \mathbb{R})$.

For simplicity, we consider the following stochastic scalar equation

$$dx(t) = \left(-a(t)x(t) + \int_{-\infty}^0 x(t+s)d[\eta(s)] \right)dt + b(t)x(t)dw(t), \tag{5.29}$$

for $t \geq 0$, where $\eta(t)$ is a function of bounded variation on $(-\infty, 0]$ and $a(t)$, $b(t)$ are continuous functions and $w(t)$ the one-dimensional Brownian motion.

Let

$$f(t, \varphi) := -a(t)\varphi(0) + \int_{-\infty}^0 \varphi(s)d[\eta(s)], \quad g(t, \varphi) := b(t)\varphi(0),$$

for $t \geq 0$, $\varphi \in BC((-\infty, 0]; \mathbb{R})$. Define $V(s) := Var_{(-\infty, s]}\eta(\cdot)$, $s \in (-\infty, 0]$. Then $V(s)$ is non-decreasing on $(-\infty, 0]$. By the properties of the Riemann-Stieltjes integral, we have

$$\left| \int_{-\infty}^0 \varphi(0)\varphi(s)d[\eta(s)] \right| \leq \int_{-\infty}^0 |\varphi(0)\varphi(s)|d[V(s)].$$

Thus,

$$2\varphi(0)f(t, \varphi) \leq -2a(t)\varphi^2(0) + 2\int_{-\infty}^0 |\varphi(0)\varphi(s)|d[V(s)]$$

$$\leq \left(-2a(t) + \int_{-\infty}^0 d[V(s)] \right)\varphi^2(0) + \int_{-\infty}^0 \varphi^2(s)d[V(s)].$$

By the Corollary 3.1, the trivial solution of (5.29) is exponentially mean-square stable if there exists $\beta > 0$ such that

$$-2a(t) + \int_{-\infty}^0 d[V(s)] + \int_{-\infty}^0 e^{-\beta s}d[V(s)] + b^2(t) \leq -\beta, \tag{5.30}$$

for all $t \geq 0$.

To illustrate further the effectiveness of the obtained result, we consider scalar stochastic functional differential equation

$$dx(t) = -\alpha x(t)dt + \left(\int_{-\infty}^0 e^s x(t+s)ds \right)dt + x(t)dsdw(t), \quad t \geq 0, \tag{5.31}$$

where $\alpha > 0$ stands for a parameter and $w(t)$ is the 1-dimensional Brownian motion.

Clearly, (5.31) is the form of (5.29) with $a(t) = \alpha$, $\eta(t) = e^t$, $b(t) = 1$. So, by the Corollary 3.1, the trivial solution of (5.31) is exponentially mean-square stable if there exists $\beta > 0$ such that

$$-2\alpha + \int_{-\infty}^{0} e^s ds + \int_{-\infty}^{0} e^{-\beta s} e^s ds + 1 \leq -\beta. \tag{5.32}$$

In other word, we can deduce that the zero solution of (5.31) is exponentially stable in mean square if

$$\alpha \geq \frac{3}{2}.$$

But, by Yang et al. [21] we have the zero solution of (5.31) is exponentially stable in mean square if

$$\alpha > 1 + \frac{1}{2} \int_{-\infty}^{0} e^{-\alpha s} e^s ds,$$

which implies that by Yang et al. [21] we can not deduce that the zero solution of (5.31) is exponentially stable in mean square if

$$\frac{3}{2} \leq \alpha < \frac{\sqrt{2}+2}{2}.$$

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