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Exponential Stability for a Thermoelastic Porous System with Microtemperatures Effects

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Abstract
In this article, we consider a one-dimensional thermoelastic porous system with microtemperatures. Based on the energy method we show in the case of zero thermal conductivity that the dissipation given only by the microtemperatures is strong enough to produce an exponential stability irrespective of the wave speeds of the system or any other condition on the coefficients. The result of this paper is new and improves previous results in the literature.

Keywords Exponential decay · Porous system · Microtemperatures effects · Energy method · Lyapunov functional

Mathematics Subject Classification (2000) 35L70 · 35B40 · 93D20

1 Introduction

In this paper, we consider the following one-dimensional thermoelastic porous system with microtemperatures

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \gamma\theta_x, \text{ in } \quad (0,1)\times(0,\infty), \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta, \text{ in } \quad (0,1)\times(0,\infty), \\ c\theta_t = -\gamma u_{tx} - m\varphi_t - k_1 w_x, \text{ in } \quad (0,1)\times(0,\infty), \\ \alpha w_t = k_2 w_{xx} - k_3 w - k_1\theta_x - d\varphi_{tx}, \text{ in } \quad (0,1)\times(0,\infty), \end{cases} \tag{1}$$

under the boundary conditions

$$\begin{aligned} &u(0,t) = u(1,t) = \varphi_x(0,t) = \varphi_x(1,t) = 0, \ t > 0, \\ &\theta_x(0,t) = \theta_x(1,t) = w(0,t) = w(1,t) = 0, \ t > 0, \end{aligned} \tag{2}$$

and the initial conditions

$$\begin{aligned} &u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ \varphi(x,0) = \varphi_0(x), \ x \in (0,\ 1), \\ &\varphi_t(x,0) = \varphi_1(x), \ w(x,0) = w_0(x), \ \theta(x,0) = \theta_0(x), \ x \in (0,\ 1), \end{aligned} \tag{3}$$

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where the functions u, φ, θ, w represent, respectively, the displacement of the solid elastic material, the volume fraction, the temperature difference and the microtemperature vector. The parameters ρ and J which are assumed to be strictly positive constants, are the mass density and product of the mass density by the equilibrated inertia respectively. The coefficients $c, \mu, \delta, \gamma, \xi, m, d, k_1, k_2, k_3, \alpha$ are positive constants in which their physical meaning is well known such that

$$\mu\xi > b^2, \tag{4}$$

where b is a real number different from zero and the initial data u_0, u_1, φ_0, φ_1, w_0, θ_0 belongs to the suitable functional space.

The system (1) was constructed by considering the following basic evolution equations of the one-dimensional porous materials theory with temperature and microtemperature

$$\rho u_{tt} = T_x,\ J\varphi_{tt} = H_x + G,\ \rho\eta_t = q_x,\ \rho E_t = P_x - Q, \tag{5}$$

where T is the stress tensor, H is the equilibrated stress vector, G is the equilibrated body force, q is the heat flux vector, η is the entropy, P is the first heat flux moment, Q is the mean heat flux and E is the first moment of energy. The constitutive equations T, H, G, E, η, P and Q take the following forms

$$\begin{cases} T = \mu u_x + b\varphi - \gamma\theta,\ H = \delta\varphi_x - dw, \\ G = -bu_x - \xi\varphi + m\theta, \\ \rho\eta = \gamma u_x + c\theta + m\varphi,\ q = -k_1 w, \\ \rho E = -\alpha w - d\varphi_x,\ P = -k_2 w_x,\ Q = -k_3 w - k_1\theta_x, \end{cases} \tag{6}$$

and by substituting Eq. (6) into Eq. (5), we obtain the system (1).

In 1972, Goodman and Cowin [10] have given an extension of the classical elasticity theory to porous media by introducing the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition, Nunziato and Cowin [8] have presented a nonlinear theory for the behavior of porous solids in which the skeletal or matrix material is elastic and the interstices are void of material. In this theory the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. Furthermore, this representation introduces an additional degree of kinematic freedom. The intended applications of the theory of elastic materials with voids are to geological materials like rocks and soils and to manufactured porous materials. In [11], Grot has developed a theory of thermodynamics of elastic materials with inner structure whose microelements, in addition to microdeformations of the string, possess microtemperatures which represent the variation of the temperature within a microvolume. Later, many works has been released in this direction (for example [12–14] and the references therein).

The first investigation concerning the study of temporal asymptotic behavior of the solutions for a one-dimensional porous-elastic system was started by the work of Quintanilla [19], in which he considered a damping through porous-viscosity and he proved that the system is not decay exponentially with this complementary control. In [3, 4], Apalara showed that the same system considered in [19] is exponentially stable for the case of equal speeds of wave propagation, i.e.

$$\frac{\mu}{\rho} = \frac{\delta}{J}.$$

In [6], Casas and Quintanilla considered the following one-dimensional porous system in the presence of the usual thermal effect and the microtemperature damping

$$\begin{cases} \rho u_{tt} = \mu^* u_{xx} + b\varphi_x - \beta\theta_x, \text{ in } \quad (0,\pi)\times(0,\infty), \\ J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta, \text{ in } \quad (0,\pi)\times(0,\infty), \\ c\theta_t = \kappa\theta_{xx} - \gamma u_{tx} - l\varphi_t - k_1 w_x, \text{ in } \quad (0,\pi)\times(0,\infty), \\ \delta w_t = k_4^* w_{xx} - k_2 w - k_3\theta_x - d\varphi_{tx}, \text{ in } \quad (0,\pi)\times(0,\infty). \end{cases}$$

They used the semi-group approach to prove the exponential stability of the solutions regardless to the speeds of wave propagations. In [7], the same authors proved that the combination of porous-viscosity and thermal effects (temperature and microtemperatures) provokes exponential stability of solutions. In [17], Magańa and Quintanilla showed that viscoelasticity damping and temperature produced slow decay in time and when the viscoelasticity is coupled with porous damping or with microtemperatures, the system decays in an exponential way. In [5], Apalara proved that the unique dissipation given by the finite memory is strong enough to stabilize exponentially the system for the case of equal speeds of wave propagation. In [1], Apalara showed that the memory term together with the heat effect are strong enough to stabilize exponentially the system irrespective of the wave speeds.

Interestingly, Apalara [2] proved that the dissipation given only with the microtemperatures is sufficient to get an exponential stability for the case of equal speeds of wave propagation. Furthermore, if the speeds of wave propagation are non-equal, he showed that the system is polynomially stable. In [9], Dridi and Djebabla studied the porous thermoelastic system in case of zero thermal conductivity with temperatures and microtemperatures effects

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \gamma\theta_x, \text{ in } \quad (0,1)\times(0,\infty), \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta - \tau\varphi_t, \text{ in } \quad (0,1)\times(0,\infty), \\ c\theta_t = -\gamma u_{tx} - m\varphi_t - k_1 w_x, \text{ in } \quad (0,1)\times(0,\infty), \\ \alpha w_t = k_2 w_{xx} - k_3 w - k_1\theta_x - d\varphi_{tx}, \text{ in } \quad (0,1)\times(0,\infty), \end{cases}$$

with the following Dirichlet (on φ, θ)-Neumann (on u, w) boundary conditions, and prove the exponential stability without any condition on the coefficients of the system.

In [21], Saci and Djebabla studied a porous-elastic system with dissipation only due to microtemperatures effect

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \gamma\theta_x, \text{ in } \quad (0,1)\times(0,\infty), \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta, \text{ in } \quad (0,1)\times(0,\infty), \\ c\theta_t = -\gamma u_{tx} - m\varphi_t - k_1 w_x, \text{ in } \quad (0,1)\times(0,\infty), \\ \alpha w_t = k_2 w_{xx} - k_3 w - k_1\theta_x - d\varphi_{tx}, \text{ in } \quad (0,1)\times(0,\infty), \end{cases} \tag{7}$$

with the Dirichlet (on φ, θ)-Neumann (on u, w) boundary conditions. They introduced a new stability number and proved that the unique dissipation due to the microtemperatures is strong enough to drive the system to the equilibrium state in an exponential manner.

In [20], Saci et al. investigated the porous-elastic system where two kinds of dissipation processes were considered: the frictional damping acting on the elasticity equation and the microtemperatures dissipation. The authors showed that these both dissipation terms guarantees an exponential stability of the solutions. In [16] Liu et al. considered a one-dimensional porous-elastic system with finite memory term acting on the porous equation. They showed a general decay of the solutions under the assumptions of non-equal wave speeds propagations and positive semidefinite energy.

Recently, Lacheheb et al. in [15] studied a porous-elastic system with thermoelasticity of type III and based on the energy method, they obtained an exponential decay result for the case of equal wave speeds. In the opposite one, they proved a polynomial decay result. Moreover, they used some numerical approximations to validate the theoretical result. In [22] the authors showed the existence of global and exponential attractors for a nonlinear porous-elastic system subjected to a delay-type damping.

In this paper, we consider the same porous-elastic system (7) with temperature and microtemperatures, but with different boundary conditions, i.e., the Dirichlet (on u, w)-Neumann (on φ, θ) boundary conditions. Based on the energy method, we show in case of zero thermal conductivity that the dissipation given only by the microtemperatures is strong enough to produce an exponential stability irrespective of the wave speeds of the system or any other condition on the coefficients.

In view of the boundary conditions, our system can have solutions (uniform in the variable x), which do not decay. To avoid such case and also to be able to use Poincaré's inequality, we perform the following transformation:

By using $(1)_2$, $(1)_3$, and the boundary conditions, we observe that

$$\begin{cases} J\dfrac{d^2}{dt^2}\int_0^1 \varphi(x,t)\,dx + \xi\int_0^1 \varphi(x,t)\,dx = m\int_0^1 \theta(x,t)\,dx, \\ c\dfrac{d}{dt}\int_0^1 \theta(x,t)\,dx = -m\dfrac{d}{dt}\int_0^1 \varphi(x,t)\,dx. \end{cases} \tag{8}$$

The system (8) is equivalent to

$$\begin{cases} J\dfrac{d^2}{dt^2}\int_0^1 \varphi(x,t)\,dx + \tau_2\int_0^1 \varphi(x,t)\,dx - \tau_1 = 0,\ \ \tau_2 > 0, \\ \int_0^1 \theta(x,t)\,dx = -\dfrac{m}{c}\int_0^1 \varphi(x,t)\,dx + \dfrac{\tau_1}{m}, \end{cases} \tag{9}$$

where $\tau_2 = \xi + \frac{m^2}{c}$ and $\tau_1 = m\int_0^1 \theta_0(x)\,dx + \dfrac{m^2}{c}\int_0^1 \varphi_0(x)\,dx$.

By introducing the following change of variable

$$z(t) = \tau_2 \int_0^1 \varphi(x,t)\,dx - \tau_1, \tag{10}$$

the differential equation $(9)_1$ becomes

$$z''(t) + \frac{\tau_2}{J}z(t) = 0. \tag{11}$$

So, by solving (11) and using the initial data, we obtain

$$\begin{aligned} z(t) = {}& \left(\tau_2\left(\int_0^1 \varphi_0(x)\,dx\right) - \tau_1\right)\cos\left(\sqrt{\frac{\tau_2}{J}}t\right) \\ & + \sqrt{J\tau_2}\left(\int_0^1 \varphi_1(x)\,dx\right)\sin\left(\sqrt{\frac{\tau_2}{J}}t\right). \end{aligned}$$

We deduce from (10), $(9)_2$ that

$$\int_0^1 \varphi(x,t)\,dx = \left(\left(\int_0^1 \varphi_0(x)\,dx\right) - \frac{\tau_1}{\tau_2}\right)\cos\left(\sqrt{\frac{\tau_2}{J}}t\right)$$

$$
+\sqrt{\frac{J}{\tau_2}}\left(\int_0^1 \varphi_1(x)\,dx\right)\sin\left(\sqrt{\frac{\tau_2}{J}}t\right)+\frac{\tau_1}{\tau_2},
$$

$$
\int_0^1 \theta(x,t)\,dx = -\frac{m}{c}\left(\left(\int_0^1 \varphi_0(x)\,dx\right)-\frac{\tau_1}{\tau_2}\right)\cos\left(\sqrt{\frac{\tau_2}{J}}t\right)
$$

$$
-\frac{m}{c}\sqrt{\frac{J}{\tau_2}}\left(\int_0^1 \varphi_1(x)\,dx\right)\sin\left(\sqrt{\frac{\tau_2}{J}}t\right)+\tau_1\left(\frac{1}{m}-\frac{m}{c\tau_2}\right).
$$

Consequently, if we let

$$
\bar{\varphi}(x,t) = \varphi(x,t) - \left(\left(\int_0^1 \varphi_0(x)\,dx\right)-\frac{\tau_1}{\tau_2}\right)\cos\left(\sqrt{\frac{\tau_2}{J}}t\right)
$$

$$
-\sqrt{\frac{J}{\tau_2}}\left(\int_0^1 \varphi_1(x)\,dx\right)\sin\left(\sqrt{\frac{\tau_2}{J}}t\right)-\frac{\tau_1}{\tau_2},
$$

and

$$
\bar{\theta}(x,t) = \theta(x,t) + \frac{m}{c}\left(\left(\int_0^1 \varphi_0(x)\,dx\right)-\frac{\tau_1}{\tau_2}\right)\cos\left(\sqrt{\frac{\tau_2}{J}}t\right)
$$

$$
+\frac{m}{c}\sqrt{\frac{J}{\tau_2}}\left(\int_0^1 \varphi_1(x)\,dx\right)\sin\left(\sqrt{\frac{\tau_2}{J}}t\right)-\tau_1\left(\frac{1}{m}-\frac{m}{c\tau_2}\right),
$$

we obtain

$$
\int_0^1 \bar{\varphi}(x,t)\,dx = \int_0^1 \bar{\theta}(x,t)\,dx = 0. \tag{12}
$$

Henceforth, we work with $\bar{\varphi}$, $\bar{\theta}$ instead of $\bar{\varphi}$, $\bar{\theta}$ but write φ and θ for simplicity of notation.

2 Well-Posedness

In this section, we give the existence and uniqueness of solutions for the system (1)-(3) using semigroup theory. Introducing the vector function $U = (u, v, \varphi, \psi, \theta, w)^T$, where $v = u_t$, and $\psi = \varphi_t$, the system (1) can be rewritten as follows:

$$
\begin{cases} U_t + \mathcal{A}U = 0, \ t > 0, \\ U(x,0) = U_0(x) = (u_0, u_1, \varphi_0, \varphi_1, \theta_0, w_0)^T, \end{cases}
$$

where the operator $\mathcal{A}$ is defined by

$$
\mathcal{A}U = \begin{pmatrix} -v \\ -\frac{\mu}{\rho}u_{xx} - \frac{b}{\rho}\varphi_x + \frac{\gamma}{\rho}\theta_x \\ -\psi \\ -\frac{\delta}{J}\varphi_{xx} + \frac{b}{J}u_x + \frac{\xi}{J}\varphi + \frac{d}{J}w_x - \frac{m}{J}\theta \\ \frac{\gamma}{c}v_x + \frac{m}{c}\psi + \frac{k_1}{c}w_x \\ -\frac{k_2}{\alpha}w_{xx} + \frac{k_3}{\alpha}w + \frac{k_1}{\alpha}\theta_x + \frac{d}{\alpha}\psi_x \end{pmatrix}. \tag{13}
$$

We consider the following spaces

$$
\begin{aligned}
&H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1),\\
&L_*^2(0,1) = \left\{\phi \in L^2(0,1) : \int_0^1 \phi(x)\,dx = 0\right\},\\
&H_*^2(0,1) = \left\{\Psi \in H^2(0,1) : \Psi_x(0) = \Psi_x(1) = 0\right\}.
\end{aligned}
$$

Let $\mathcal{H}$ be the energy space given by

$$
\mathcal{H}=H_0^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \times L_*^2(0,1) \times L_*^2(0,1) \times L^2(0,1),
$$

and for any $U = (u, v, \varphi, \psi, \theta, w)^T \in \mathcal{H}$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{\psi}, \tilde{\theta}, \tilde{w})^T \in \mathcal{H}$, we equip $\mathcal{H}$ with the inner product

$$
\begin{aligned}
\left\langle U, \tilde{U}\right\rangle_{\mathcal{H}} &= \rho \int_0^1 v\tilde{v}dx + \mu \int_0^1 u_x\tilde{u}_x dx + J \int_0^1 \psi\tilde{\psi}dx + b \int_0^1 (u_x\tilde{\varphi} + \tilde{u}_x\varphi)\,dx\\
&\quad + \xi \int_0^1 \varphi\tilde{\varphi}dx + \delta \int_0^1 \varphi_x\tilde{\varphi}_x dx + \alpha \int_0^1 w\tilde{w}dx + c \int_0^1 \theta\tilde{\theta}dx.
\end{aligned} \tag{14}
$$

It is easy to see that (14) defines an inner product. In fact, from (14), we have

$$
\begin{aligned}
\langle U, U\rangle_{\mathcal{H}} &= \rho \int_0^1 v^2 dx + \mu \int_0^1 u_x^2 dx + J \int_0^1 \psi^2 dx + 2b \int_0^1 u_x\varphi dx\\
&\quad + \xi \int_0^1 \varphi^2 dx + \delta \int_0^1 \varphi_x^2 dx + \alpha \int_0^1 w^2 dx + c \int_0^1 \theta^2 dx.
\end{aligned}
$$

Since $\mu\xi > b^2$, we deduce that

$$
\mu u_x^2 + 2bu_x\varphi + \xi\varphi^2 > \frac{1}{2}\left[\left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\xi - \frac{b^2}{\mu}\right)\varphi^2\right].
$$

Consequently,

$$
\langle U, U\rangle_{\mathcal{H}} > \int_0^1 \left\{\rho v^2 + \mu_1 u_x^2 + J\psi^2 + \xi_1\varphi^2 + +\delta\varphi_x^2 + \alpha w^2 + c\theta^2\right\}dx,
$$

where

$$
\mu_1 = \frac{1}{2}\left(\mu - \frac{b^2}{\xi}\right) > 0,\ \xi_1 = \frac{1}{2}\left(\xi - \frac{b^2}{\mu}\right) > 0. \tag{15}
$$

Hence, we conclude that $\left\langle U, \tilde{U}\right\rangle_{\mathcal{H}}$ defines an inner product on $\mathcal{H}$ and the associated norm $\|.\|_{\mathcal{H}}$ is equivalent to the usual one.

The domain of $\mathcal{A}$ is

$$
\begin{aligned}
D(\mathcal{A}) = \big\{U \in \mathcal{H} \mid\ & u \in H_0^2(0,1) \cap H_0^1(0,1);\ v \in H_0^1(0,1);\\
& \varphi \in H_*^2(0,1) \cap H_*^1(0,1);\ \psi \in H_*^1(0,1);\ \theta \in H_*^1(\Omega);
\end{aligned}
$$

$$w \in H_0^2(0,1) \cap H_0^1(0,1)\Big\}.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$. Moreover, by using the inner product (14), it follows that, for any $U \in D(\mathcal{A})$

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = k_3 \int_0^1 w^2 dx + k_2 \int_0^1 w_x^2 dx \geq 0, \tag{16}$$

which implies that $\mathcal{A}$ is a monotone operator. By using the Lax–Milgram Lemma and classical regularity arguments, it can be proved that $I + \mathcal{A}$ is surjective. Hence, using Lumer–Phillips theorem (see [18]), we deduce that $\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup on $\mathcal{H}$. Consequently, we have the following well-posedness result.

Theorem 1 *Let $U_0 \in \mathcal{H}$, then there exists a unique solution $U \in C(\mathbb{R}_+, \mathcal{H})$ of problem* (1). *Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C(\mathbb{R}_+,\ D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

3 Exponential Stability

In this section, we use the energy method to establish the exponential stability of the system (1). To achieve our goal we state and prove the following lemmas.

Lemma 2 *Let (u, φ, θ, w) be a solution of* (1)-(3). *Then, the energy functional $E(t)$, defined by*

$$E(t) = \frac{1}{2}\int_0^1 \big(\rho u_t^2 + J\varphi_t^2 + \mu u_x^2 + \delta\varphi_x^2 + c\theta^2 + \xi\varphi^2 + \alpha w^2 + 2b\varphi u_x\big)\, dx, \tag{17}$$

satisfies

$$E'(t) = -k_3 \int_0^1 w^2 dx - k_2 \int_0^1 w_x^2 dx \leq 0. \tag{18}$$

Proof Multiplying $(1)_1$, $(1)_2$, $(1)_3$, $(1)_4$ by u_t, φ_t, θ, w respectively, integrating over $(0, 1)$ and summing them up, we obtain

$$\frac{d}{2dt}\int_0^1 \big(\rho u_t^2 + J\varphi_t^2 + \mu u_x^2 + \delta\varphi_x^2 + c\theta^2 + \xi\varphi^2 + \alpha w^2 + 2b\varphi u_x\big)\, dx = -k_2 \int_0^1 w_x^2 dx - k_3 \int_0^1 w^2 dx. \quad \square \tag{19}$$

Remark 3 The energy $E(t)$ defined by (17) is non-negative. In fact, as in the second section, we can easily show that

$$E(t) > \frac{1}{2}\int_0^1 \big(\rho u_t^2 + J\varphi_t^2 + \mu_1 u_x^2 + \delta\varphi_x^2 + c\theta^2 + \xi_1\varphi^2 + \alpha w^2\big)\, dx,$$

where μ_1 and ξ_1 are given in (15). Therefore, $E(t)$ is non-negative.

Lemma 4 *Let* (u, φ, θ, w) *be a solution of* (1)-(3). *Then, the functional*

$$I_1(t) = \frac{\gamma}{4}\int_0^1 u_t u dx - c\int_0^1 \theta\left(\int_0^x u_t(y)\,dy\right)dx,\ t \geq 0,$$

satisfies, $\forall t \geq 0$

$$I_1'(t) \leq -\frac{\mu\gamma}{8\rho}\int_0^1 u_x^2 dx - \frac{\gamma}{4}\int_0^1 u_t^2 dx + C_0\int_0^1 \left(\varphi_t^2 + \varphi_x^2 + \theta^2 + w_x^2\right)dx \tag{20}$$

Proof Differentiating $I_1(t)$ and integrating by parts, we get

$$\begin{aligned} I_1'(t) = &-\frac{3}{4}\gamma\int_0^1 u_t^2 dx - \frac{\mu\gamma}{4\rho}\int_0^1 u_x^2 dx - \frac{b\gamma}{4\rho}\int_0^1 \varphi u_x dx + \left(\frac{\gamma^2}{4\rho} - \frac{c\mu}{\rho}\right)\int_0^1 \theta u_x dx \\ &+ \frac{\gamma c}{\rho}\int_0^1 \theta^2 dx - \frac{cb}{\rho}\int_0^1 \varphi\theta dx + m\int_0^1 \varphi_t\left(\int_0^x u_t(y)\,dy\right)dx \\ &+ k_1\int_0^1 w_x\left(\int_0^x u_t(y)\,dy\right)dx. \end{aligned} \tag{21}$$

Using Young's and Cauchy Schwarz inequalities,

$$m\int_0^1 \varphi_t\left(\int_0^x u_t(y)\,dy\right)dx \leq \frac{\gamma}{4}\int_0^1 u_t^2 dx + C_0\int_0^1 \varphi_t^2 dx, \tag{22}$$

$$k_1\int_0^1 w_x\left(\int_0^x u_t(y)\,dy\right)dx \leq \frac{\gamma}{4}\int_0^1 u_t^2 dx + C_0\int_0^1 w_x^2 dx. \tag{23}$$

Using Young's inequality

$$\left(\frac{\gamma^2}{4\rho} - \frac{c\mu}{\rho}\right)\int_0^1 \theta u_x dx \leq \frac{\mu\gamma}{16\rho}\int_0^1 u_x^2 dx + C_0\int_0^1 \theta^2 dx. \tag{24}$$

Using Young's and Poincaré inequalities

$$-\frac{cb}{\rho}\int_0^1 \varphi\theta dx \leq C_0\int_0^1 \left(\theta^2 + \varphi_x^2\right)dx, \tag{25}$$

$$-\frac{b\gamma}{4\rho}\int_0^1 \varphi u_x dx \leq \frac{\mu\gamma}{16\rho}\int_0^1 u_x^2 dx + C_0\int_0^1 \varphi_x^2 dx. \tag{26}$$

By substituting (22)-(26) into (21), we get (20). □

Lemma 5 *Let* (u, φ, θ, w) *be a solution of* (1)-(3). *Then, the functional*

$$I_2(t) = J\int_0^1 \varphi_t\varphi dx - \frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi(y)\,dy\right)dx,\ t \geq 0,$$

satisfies, for any $\varepsilon_1 > 0$,

$$I_2'(t) \leq -\frac{\delta}{2}\int_0^1 \varphi_x^2 dx - 2\xi_1\int_0^1 \varphi^2 dx + \varepsilon_1\int_0^1 u_t^2 dx + C_1\int_0^1 \left(\theta^2 + w_x^2\right)dx$$

$$+ C_1\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 \varphi_t^2 dx, \tag{27}$$

where $\xi_1 = \frac{1}{2}\left(\xi - \frac{b^2}{\mu}\right)$.

Proof By differentiating $I_2(t)$, we obtain

$$I_2'(t) = J\int_0^1 \varphi_{tt}\varphi dx + J\int_0^1 \varphi_t^2 dx - \frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi_t(y)\,dy\right)dx$$
$$- \frac{b\rho}{\mu}\int_0^1 u_{tt}\left(\int_0^x \varphi(y)\,dy\right)dx.$$

Now, by using integration by parts together with the boundary conditions, we get

$$I_2'(t) = -\delta\int_0^1 \varphi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \varphi^2 dx + J\int_0^1 \varphi_t^2 dx$$
$$- d\int_0^1 w_x\varphi dx - \frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi_t(y)\,dy\right)dx$$
$$+ \left(m - \frac{b\gamma}{\mu}\right)\int_0^1 \theta\varphi dx. \tag{28}$$

Using Young's and Cauchy Schwarz inequalities, we get

$$-\frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi_t(y)\,dy\right)dx \le \varepsilon_1\int_0^1 u_t^2 dx + \frac{C_1}{\varepsilon_1}\int_0^1 \varphi_t^2 dx. \tag{29}$$

Young's and Poincaré inequalities leads to

$$-d\int_0^1 w_x\varphi dx \le \frac{\delta_1}{4}\int_0^1 \varphi_x^2 dx + C_1\int_0^1 w_x^2 dx, \tag{30}$$

$$\left(m - \frac{b\gamma}{\mu}\right)\int_0^1 \theta\varphi dx \le \frac{\delta_1}{4}\int_0^1 \varphi_x^2 dx + C_1\int_0^1 \theta^2 dx. \tag{31}$$

Inserting (29)-(31) in (28) and letting $\delta_1 = \frac{\delta}{2}$, we obtain (27). □

Lemma 6 *Let* (u, φ, θ, w) *be a solution of* (1)-(3). *Then, the functional*

$$I_3(t) = c\alpha\int_0^1 \theta\left(\int_0^x w(y)\,dy\right)dx - \alpha\int_0^1 w\left(\int_0^x \varphi_t(y)\,dy\right)dx$$
$$+ \frac{\gamma J\alpha_0}{2\beta_0}\int_0^1 \varphi_t^2 dx + \frac{bc\alpha_0}{\beta_0}\int_0^1 \theta\varphi dx + \frac{\gamma b\alpha_0}{\beta_0}\int_0^1 u_x\varphi dx$$
$$+ \frac{\alpha_0\gamma_0}{2\beta_0}\int_0^1 \varphi^2 dx + \frac{\gamma\delta\alpha_0}{2\beta_0}\int_0^1 \varphi_x^2 dx,$$

where $\alpha_0 = dc + k_1$, $\beta_0 = \gamma m + bc$, $\gamma_0 = \gamma\xi + mb$, *satisfies, for any* $\varepsilon_2, \varepsilon_3, \varepsilon_4 > 0$, *the following estimate*

$$I_3'(t) \leq -\frac{k_1 c}{4}\int_0^1 \theta^2 dx - \frac{d}{4}\int_0^1 \varphi_t^2 dx + \varepsilon_2 \int_0^1 u_t^2 dx + 2\varepsilon_3 \int_0^1 \varphi_x^2 dx + \varepsilon_4 \int_0^1 u_x^2 dx + C_2 \int_0^1 w_x^2 dx + C_2\left(1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_4}\right)\int_0^1 w^2 dx. \tag{32}$$

Proof By differentiating $I_3(t)$, integrating by parts and using (12), we obtain

$$\begin{aligned} I_3'(t) = & -k_1 c \int_0^1 \theta^2 dx - d\int_0^1 \varphi_t^2 dx + \left(\alpha k_1 + \frac{\alpha d}{J}\right)\int_0^1 w^2 dx \\ & + \alpha\gamma \int_0^1 w u_t dx + \left(k_2 - \frac{\gamma d \alpha_0}{\beta_0}\right)\int_0^1 w_x \varphi_t dx \\ & + \frac{\alpha b}{J}\int_0^1 w u dx + k_2 c \int_0^1 w_x \theta dx + \left(\frac{k_1 \alpha_0 b}{\beta_0} - \frac{\alpha\delta}{J}\right)\int_0^1 w\varphi_x dx \\ & - k_3 c \int_0^1 \theta\left(\int_0^x w(y)\,dy\right)dx - \alpha m \int_0^1 \varphi_t \left(\int_0^x w(y)\,dy\right)dx \\ & + k_3 \int_0^1 w\left(\int_0^x \varphi_t(y)\,dy\right)dx - \frac{\alpha m}{J}\int_0^1 w\left(\int_0^x \theta(y)\,dy\right)dx \\ & + \frac{\alpha\xi}{J}\int_0^1 w\left(\int_0^x \varphi(y)\,dy\right)dx. \end{aligned} \tag{33}$$

Using Young's inequality, we find

$$k_2 c \int_0^1 w_x \theta dx \leq \frac{k_1 c}{4}\int_0^1 \theta^2 dx + C_2 \int_0^1 w_x^2 dx, \tag{34}$$

$$\left(k_2 - \frac{\gamma d\alpha_0}{\beta_0}\right)\int_0^1 w_x \varphi_t dx \leq \frac{d}{4}\int_0^1 \varphi_t^2 dx + C_2 \int_0^1 w_x^2 dx, \tag{35}$$

$$+\alpha\gamma \int_0^1 w u_t dx \leq \varepsilon_2 \int_0^1 u_t^2 dx + \frac{C_2}{\varepsilon_2}\int_0^1 w^2 dx, \tag{36}$$

$$\left(\frac{k_1\alpha_0 b}{\beta_0} - \frac{\alpha\delta}{J}\right)\int_0^1 w\varphi_x dx \leq \varepsilon_3 \int_0^1 \varphi_x^2 dx + \frac{C_2}{\varepsilon_3}\int_0^1 w^2 dx. \tag{37}$$

Using Young's and Poincaré inequalities, we have

$$\frac{\alpha b}{J}\int_0^1 w u dx \leq \varepsilon_4 \int_0^1 u_x^2 dx + \frac{C_2}{\varepsilon_4}\int_0^1 w^2 dx. \tag{38}$$

Using Young's and Cauchy Schwarz inequalities

$$-k_3 c \int_0^1 \theta\left(\int_0^x w(y)\,dy\right)dx \leq \frac{k_1 c}{4}\int_0^1 \theta^2 dx + C_2 \int_0^1 w^2 dx, \tag{39}$$

$$-\frac{\alpha m}{J}\int_0^1 w\left(\int_0^x \theta(y)\,dy\right)dx \le \frac{k_1 c}{4}\int_0^1 \theta^2 dx + C_2\int_0^1 w^2 dx, \tag{40}$$

$$-\alpha m\int_0^1 \varphi_t\left(\int_0^x w(y)\,dy\right)dx \le \frac{d}{4}\int_0^1 \varphi_t^2 dx + C_2\int_0^1 w^2 dx, \tag{41}$$

$$k_3\int_0^1 w\left(\int_0^x \varphi_t(y)\,dy\right)dx \le \frac{d}{4}\int_0^1 \varphi_t^2 dx + C_2\int_0^1 w^2 dx. \tag{42}$$

Using Young's, Poincaré and Cauchy Schwarz inequalities

$$\frac{\alpha\xi}{J}\int_0^1 w\left(\int_0^x \varphi(y)\,dy\right)dx \le \varepsilon_3\int_0^1 \varphi_x^2 dx + \frac{C_2}{\varepsilon_3}\int_0^1 w^2 dx. \tag{43}$$

Estimate (32) follows by substituting (34)-(43) into (33). □

Now, we define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = NE(t) + I_1(t) + N_1 I_2(t) + N_2 I_3(t), \tag{44}$$

where N, N_1, N_2 are positive constants.

Theorem 7 *Let (u, φ, θ, w) be a solution of (1)-(3). Then, there exist two positive constants κ_1 and κ_2 such that the Lyapunov functional (44) satisfies*

$$\kappa_1 E(t) \le \mathcal{L}(t) \le \kappa_2 E(t), \ \forall t \ge 0, \tag{45}$$

and

$$\mathcal{L}'(t) \le -\beta_1 E(t), \ \beta_1 > 0. \tag{46}$$

Proof From (44), we have

$$\begin{aligned}
|\mathcal{L}(t) - NE(t)| \le{}& \frac{\gamma}{4}\int_0^1 |u_t u|\,dx + c\int_0^1 \left|\theta\left(\int_0^x u_t(y)\,dy\right)\right|dx \\
&+ N_1 J\int_0^1 |\varphi_t\varphi|\,dx + N_1\frac{|b|\rho}{\mu}\int_0^1 \left|u_t\left(\int_0^x \varphi(y)\,dy\right)\right|dx \\
&+ N_2 c\alpha\int_0^1 \left|\theta\left(\int_0^x w(y)\,dy\right)\right|dx + N_2\alpha\int_0^1 \left|w\left(\int_0^x \varphi_t(y)\,dy\right)\right|dx \\
&+ N_2\frac{\gamma J\alpha_0}{2|\beta_0|}\int_0^1 \varphi_t^2 dx + N_2\frac{|b|c\alpha_0}{|\beta_0|}\int_0^1 |\theta\varphi|\,dx + N_2\frac{\gamma|b|\alpha_0}{|\beta_0|}\int_0^1 |u_x\varphi|\,dx \\
&+ N_2\frac{\alpha_0|\gamma_0|}{2|\beta_0|}\int_0^1 \varphi^2 dx + N_2\frac{\gamma\delta\alpha_0}{2|\beta_0|}\int_0^1 \varphi_x^2 dx.
\end{aligned}$$

By using Young's, Poincaré and Cauchy-Schwarz inequalities, we obtain

$$|\mathcal{L}(t) - NE(t)| \le \tau E(t),$$

which yields

$$(N-\tau)\,E\,(t)\le\mathcal{L}\,(t)\le(N+\tau)\,E\,(t)\,,$$

by choosing N (depending on N_1, N_2) sufficiently large we obtain (45). Now, by differentiating $\mathcal{L}\,(t)$, exploiting (20), (27), (32) and setting $\varepsilon_1=\dfrac{\gamma}{16N_1}$, $\varepsilon_2=\dfrac{\gamma}{16N_2}$, $\varepsilon_3=\dfrac{N_1\delta}{8N_2}$, $\varepsilon_4=\dfrac{\mu\gamma}{16N_2}$, we get

$$\begin{aligned}
\mathcal{L}'\,(t)\le&-\frac{\mu\gamma}{16\rho}\int_0^1 u_x^2dx-\frac{\gamma}{8}\int_0^1 u_t^2dx-2\xi_1N_1\int_0^1\varphi^2dx\\
&-\left(N_2\frac{k_1c}{4}-N_1C_1-C_0\right)\int_0^1\theta^2dx\\
&-\left(\frac{N_1\delta}{4}-C_0\right)\int_0^1\varphi_x^2dx\\
&-\left(N_2\frac{d}{4}-N_1C_1\left(1+\frac{16N_1}{\gamma}\right)-C_0\right)\int_0^1\varphi_t^2dx\\
&-\left(Nk_3-N_2C_2\left(1+\frac{16N_2}{\gamma}+\frac{8N_2}{N_1\delta}+\frac{16N_2}{\mu\gamma}\right)\right)\int_0^1 w^2dx\\
&-(Nk_2-N_2C_2-N_1C_1-C_0)\int_0^1 w_x^2dx.
\end{aligned}\tag{47}$$

Now, we select our parameters appropriately as follows:

First, we choose N_1 large enough so that

$$\frac{N_1\delta}{4}-C_0>0.$$

Next, we select N_2 large enough so that

$$N_2\frac{k_1c}{4}-N_1C_1-C_0>0,$$

and

$$N_2\frac{d}{4}-N_1C_1\left(1+\frac{16N_1}{\gamma}\right)-C_0>0.$$

Finally, we choose N large enough (even larger so that (45) remains valid) such that

$$\begin{cases}Nk_3-N_2C_2\left(1+\dfrac{16N_2}{\gamma}+\dfrac{8N_2}{N_1\delta}+\dfrac{16N_2}{\mu\gamma}\right)>0,\\ \text{and}\\ Nk_2-N_2C_2-N_1C_1-C_0>0.\end{cases}$$

All these choices with the relation (47) leads to

$$\mathcal{L}'(t)\le-\alpha_1\int_0^1\left(u_x^2+u_t^2+\varphi^2+\theta^2+\varphi_x^2+\varphi_t^2+w^2\right)dx,\ \alpha_1>0.\tag{48}$$

On the other hand, from Eq. (17) and by using Young's inequality, we obtain

$$E(t) \le \frac{1}{2}\int_0^1 \left(\rho u_t^2 + J\varphi_t^2 + (\mu + |b|)\, u_x^2 + \delta\varphi_x^2 + c\theta^2 + (\xi + |b|)\,\varphi^2 + \alpha w^2\right) dx$$
$$\le \varrho_1 \left(\int_0^1 \left(u_t^2 + \varphi_t^2 + u_x^2 + \varphi_x^2 + \theta^2 + \varphi^2 + w^2\right) dx\right),\ \varrho_1 > 0,$$

which implies that

$$-\int_0^1 \left(u_t^2 + \varphi_t^2 + u_x^2 + \varphi_x^2 + \theta^2 + \varphi^2 + w^2\right) dx \le -\varrho_2 E(t),\ \varrho_2 > 0. \tag{49}$$

The combination of (48) and (49) gives (46). □

We are now ready to state and prove the following exponential stability result.

Lemma 8 *Let* (u, φ, θ, w) *be a solution of* (1)-(3) *and assume that* (4) *holds. Then, for any* $U_0 \in D(\mathcal{A})$, *there exist two positive constants* λ_1 *and* λ_2 *such that*

$$E(t) \le \lambda_2 e^{-\lambda_1 t},\ \forall t \ge 0. \tag{50}$$

Proof By using the estimation (46), we get

$$\mathcal{L}'(t) \le -\beta_1 E(t),\ t \ge 0,$$

having in mind the equivalence of $E(t)$ and $\mathcal{L}(t)$ we infer that

$$\mathcal{L}'(t) \le -\lambda_1 \mathcal{L}(t),\ t \ge 0, \tag{51}$$

where $\lambda_1 = \dfrac{\beta_1}{\kappa_2} > 0$. A simple integration of (51) gives

$$\mathcal{L}'(t) \le -\mathcal{L}(0)e^{-\lambda_1 t},\ t \ge 0,$$

which yields the serial result (50) and by using the other side of the equivalence relation (45) again. The proof is complete. □

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