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Analysis on Steady States of a Competition System with Nonlinear Diffusion Terms

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Abstract Competition is a fundamental force shaping population size and structure as a result of limited availability of resources. In biomathematics, the biological models with competitive interactions exist widely. Furthermore, the nonlinear-diffusion (including self- and cross-diffusions) terms are incorporated to the biological models to better simulate the actual movement of species. Therefore, better compatibility with reality can be achieved by introducing nonlinear-diffusion into biological models with competitive interactions. As a result, a competition system with nonlinear-diffusion and nonlinear functional response is proposed and analyzed in this paper. We first briefly discuss the stability of trivial and semi-trivial solutions by spectrum analysis. Then the boundedness and the non-existence of steady states are studied. Based on the boundedness of the solutions, the existence of the steady states is also investigated by the fixed point index theory in a positive cone. The result shows that the two species can coexist when their diffusion and inter-specific competition pressures are controlled in a certain range.

Keywords Steady states · Competition model · Nonlinear-diffusion · Boundedness · Existence

Mathematics Subject Classification (2000) 35K57 · 92D25 · 93C20

1 Introduction

Competition models, having enormous impacts on various fields including biological, ecological and biochemical processes, enrich modern research to a large extent [1–3]. In biology, competition models are widely regarded as the crucial tools to understand the mecha-

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nisms leading to biodiversity. During the past years, the competition models derived from interactions of several species have been extensively studied. Among those models, the Lotka-Volterra competition model is considered as the basis of a model reflecting competitive interactions between species [4–9]. However, there is a limitation in classical Lotka-Volterra model that has the competitive interaction of two populations: with the increase of one competitor's density, its competitive capacity will increase and tend to infinity. But in reality, this capacity between different species should be upper-bounded. To overcome this deficiency, several types of models have been proposed. For example, this deficiency can be remedied by the following model with nonlinear functional response:

$$\begin{cases} \frac{\partial u}{\partial t} = u(a_1 - b_1 u - \frac{c_1 v}{1+v}), \\ \frac{\partial v}{\partial t} = v(a_2 - b_2 v - \frac{c_2 u}{1+u}), \end{cases} \tag{1.1}$$

where u and v stand for the densities of two competing species, a_1 and a_2 refer to their intrinsic growth rates, b_1 and b_2 account for their logistic growth rates, c_1 and c_2 are their maximum inter-specific interaction coefficients. Here, all parameters are positive constants. The terms $\frac{c_1 uv}{1+v}$ and $\frac{c_2 uv}{1+u}$ represent the functional response, their limits $\lim_{v\to\infty} \frac{c_1 uv}{1+v} = c_1 u$ and $\lim_{u\to\infty} \frac{c_2 uv}{1+u} = c_2 v$ imply that the competitive capacity of species cannot increase at an infinitely great rate when the density of its competitor increases. In the past few years, the models based on (1.1) have been well studied, and some valuable results have been obtained, see [10, 11] for examples.

In the field of population dynamics, the diffusion phenomenon of different species in the environment is a very universal survival and life style. Therefore, a large number of models of the multi-species interacting populations are described by reaction-diffusion systems. For example, Barabanova [12] studied a reaction-diffusion system with exponential nonlinearity. He discussed the global existence of nonnegative solutions and the asymptotic behavior of global solutions for system. Jia [13] considered a reaction-diffusion population model with predator-prey-dependent functional response. He investigated the conditions which ensure the model has a unique positive constant solution, and studied the dynamical properties of the model, including the large time behaviors of the nonconstant solutions and the local and global asymptotic stability of the positive constant solution. For more detailed backgrounds of reaction-diffusion systems, one can see [4, 14–16] and the references therein.

In recent decades, there has been considerable interest in being able to reveal the dynamics of reaction-diffusion models with nonlinear diffusion. Just because of this, the cross-diffusion terms are introduced into a system of reaction-diffusion equations to model the situation that one species influence the movement of another species, which was proposed firstly by Kerner [17] and applied firstly to biological models by Shigesada et al. [18]. Recently, many researchers have devoted to the study of the population models with cross-diffusion from various mathematical viewpoints. For example, in [19], the authors presented a general instability analysis on cross-diffusion system with two species. They showed that cross-diffusion can destabilize a uniform equilibrium which is stable for the kinetic and self-diffusion-reaction systems; On the other hand, cross-diffusion can also stabilize a uniform equilibrium which is stable for the kinetic system but unstable for the self-diffusion-reaction system. Bendahmane [20] discussed a predator-prey model with cross-diffusion. He established the existence of weak and classical solutions for model by means of an approximation system, the Faedo-Galerkin method, and the compactness method. Paper [21] studied three species food chain model with a Holling type-II functional response involving cross-diffusions. The authors presented the equilibrium solutions of the model and proved the stability of positive coexistence equilibrium, and conducted Turing instability induced by cross-diffusion. In [22], the authors investigated the Shigesada-Kawasaki-Teramoto model for two competing species with triangular cross-diffusion. By using the scalar maximum principle and the Hopf boundary point lemma, they determined explicit parameter ranges within which the model exclusively possesses constant steady state solutions. Moreover, there have many valuable surveys on the mathematical developments of cross-diffusion equations arising from various research fields, one can see [6, 23–31] and the references therein.

Based on model (1.1), in [28], Li et al. proposed the following model with cross-diffusion

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta(d_{11}u + d_{12}v) = u(a_1 - b_1 u - \frac{c_1 v}{1+v}), & (x,t) \in \Omega \times (0,\infty), \\ \frac{\partial v}{\partial t} - \Delta(d_{21}u + d_{22}v) = v(a_2 - b_2 v - \frac{c_2 u}{1+u}), & (x,t) \in \Omega \times (0,\infty), \\ \partial_{\mathbf{n}} u = \partial_{\mathbf{n}} v = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) > 0, v(x,0) = v_0(x) > 0, & x \in \Omega, t = 0, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$, $\mathbf{n}$ is the unit outward normal vector on the boundary $\partial\Omega$, a_i, b_i, c_i, $i = 1, 2$ have the same biological meaning as in model (1.1), d_{11} and d_{22} are the self-diffusion coefficients of two species, d_{12} and d_{21} denote their cross-diffusion pressures. Here, $d_{11}, d_{12}, d_{21}, d_{22}$ are positive parameters. In [28], the authors studied the existence and stability of positive equilibrium, and gave the Turing bifurcation critical value and the condition for the occurrence of Turing pattern to model (1.2).

In [18], Shigesada et al. described that the movement of two species in the actual ecological environment is affected by nonlinear diffusion forces. So, it is more realistic to consider the nonlinear diffusion effects to model (1.2). With what in mind, by making appropriate modifications, model (1.2) can be revised as the following form with nonlinear diffusion effects and Dirichlet boundary conditions

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta[(\alpha_1 + \beta_{11}u + \beta_{12}v)u] = u(a_1 - b_1 u - \frac{c_1 v}{1+v}), & (x,t) \in \Omega \times (0,\infty), \\ \frac{\partial u}{\partial t} - \Delta[(\alpha_2 + \beta_{21}u + \beta_{22}v)v] = v(a_2 - b_2 v - \frac{c_2 u}{1+u}), & (x,t) \in \Omega \times (0,\infty), \\ u = v = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, \not\equiv 0, v(x,0) = v_0(x) \geq 0, \not\equiv 0, & x \in \Omega, t = 0, \end{cases} \tag{1.3}$$

where all parameters are positive constants, the parameters $a_i, b_i, c_i, i = 1, 2$ have the same biological meaning as in model (1.1), α_1 and α_2 are the diffusion rates of two species, β_{11} and β_{22} are their self-diffusion pressures, β_{12} and β_{21} are their cross-diffusion pressures, the nonlinear terms $\Delta(\alpha_1 u)$ and $\Delta(\alpha_2 v)$ model the situation that the two species move in random ways, and the nonlinear terms $\Delta[(\beta_{11}u + \beta_{12}v)u]$ and $\Delta[(\beta_{21}u + \beta_{22}v)v]$ describe that the movement of two species is under influence of population pressure caused by intra- and inter-species interferences, u_0 and v_0 are continuous functions. Obviously, compared with model (1.2), the model (1.3) is more logical and close to real situations.

In this paper, we focus on the existence of steady state solutions of model (1.3), that is, the existence of classical positive solutions of the following elliptic system

$$\begin{cases} -\Delta[(\alpha_1 + \beta_{11}u + \beta_{12}v)u] = u(a_1 - b_1 u - \frac{c_1 v}{1+v}), & x \in \Omega, \\ -\Delta[(\alpha_2 + \beta_{21}u + \beta_{22}v)v] = v(a_2 - b_2 v - \frac{c_2 u}{1+u}), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

Since the cross-diffusion terms are introduced, one interesting problem is that whether their increase will affect the possibility of existence of positive solutions for model (1.4) or not.

Therefore, the main purpose of this paper is to consider the effects of cross-diffusion pressures on the positive solutions of model (1.4). Our approach to the proof is the fixed point index theory [31]. It should be pointed out that we extend the conclusions in [31] in analyzing the existence of positive solution since the standard conclusion frameworks in [31] are not comprehensive for our model (it only can obtain that system (1.4) admits a positive solution if the signs of the first eigenvalues of suitable operators are the same). More specifically, we not only prove that (1.4) has a positive solution when the signs of the first eigenvalues of suitable operators are the same, but also show that (1.4) has a positive solution when one of these first eigenvalues is less than 0 and one is equal to 0. Due to the cross-diffusion pressures are related to these first eigenvalues, and therefore they affect the existence of positive solution of model. Moreover, we also show that the inter-specific competition pressures are also related to the existence of positive solution of model (1.4).

The rest of this paper is organized as follows. Section 2 states some known results about eigenvalue problem, a scalar equation and the fixed point index theory. In Sect. 3, we briefly discuss the stability of trivial and semi-trivial solutions of (1.4) by spectrum analysis. In Sect. 4, we first give the boundedness of positive solution of (1.4), and then present the sufficient conditions which ensure (1.4) having no positive solution. Using the fixed point index theory, the existence of positive solutions of (1.4) is investigated in Sect. 5. Section 6 gives the conclusion to end the investigation.

2 Preliminaries

In this section, we first consider a certain eigenvalue problem and a scalar equation, and then give some known results for fixed point index theory.

2.1 Eigenvalue Problem

For $a(x) > 0$ in $C^2(\overline{\Omega})$ and $b(x) \in L^\infty(\Omega)$, consider the eigenvalue problem

$$\begin{cases} \Delta[a(x)u] + b(x)u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is the same as Ω in (1.2). By [31], we obtain that the problem (2.1) has an infinite sequence of eigenvalues $\{\lambda_i(\Delta a(x) + b(x))\}$ such that $\lambda_i(\Delta a(x) + b(x)) \geq \lambda_{i+1}(\Delta a(x) + b(x))$ with corresponding eigenfunctions $\phi_i, \phi_{i+1}, \ldots, i = 1, 2, \ldots$, where $\lim_{i\to\infty} \lambda_i(\Delta a(x) + b(x)) = -\infty, i \geq 1$.

Denote by $\|\cdot\|_{L^2}$ the usual L^2-norm in $L^2(\Omega)$. From [31] we have

$$\lambda_1(\Delta a(x) + b(x)) = \sup_{u \in W^{1,2}(\Omega)} \frac{\int_\Omega (-|\nabla[a(x)u]|^2 + a(x)b(x)u^2)\mathrm{d}x}{\|\sqrt{a(x)}u\|_{L^2}^2}. \tag{2.2}$$

Clearly, $\lambda_1(\Delta a(x) + b(x))$ is increasing in $b(x)$. The following Lemma 2.1 and Lemma 2.2 can also be obtained from [31].

Lemma 2.1 *Let* $a(x) > 0$ *in* $C^2(\overline{\Omega})$, $b(x) \in L^\infty(\Omega)$, *and* $u \geq 0, \not\equiv 0$ *in* Ω *with* $u = 0$ *on* $\partial\Omega$. *Then the following conclusions hold*:

(i) *If* $(\Delta a(x) + b(x))u \geq, \not\equiv 0$, *then* $\lambda_1(\Delta a(x) + b(x)) > 0$;

(ii) *If* $(\Delta a(x)+b(x))u \leq, \not\equiv 0$, *then* $\lambda_1(\Delta a(x)+b(x))<0$;
(iii) *If* $(\Delta a(x)+b(x))u \equiv 0$, *then* $\lambda_1(\Delta a(x)+b(x))=0$.

Lemma 2.2 *Assume that* $b_1(x)/a_1(x) > b_2(x)/a_2(x)$, *where* $a_i(x)>0$ *in* $C^2(\overline{\Omega})$, $b_i(x) \in L^\infty(\Omega)$ *for* $i=1,2$.

(i) *If* $\lambda_1(\Delta a_1(x)+b_1(x)) \leq 0$, *then* $\lambda_1(\Delta a_2(x)+b_2(x))<0$;
(ii) *If* $\lambda_1(\Delta a_2(x)+b_2(x)) \geq 0$, *then* $\lambda_1(\Delta a_1(x)+b_1(x))>0$.

Let $T: E \to E$ be a linear operator on a Banach space E and denote by $r(T)$ the spectral radius of T. Then we have the following statements for $r(T)$, which can be shown by the similar manner in [32, Lemma 2].

Lemma 2.3 *Let* $a(x)>0$ *in* $C^2(\overline{\Omega})$, $b(x) \in L^\infty(\Omega)$, *and* M *be a positive constant such that* $b(x)+Ma(x)>0$ *for all* $x \in \overline{\Omega}$. *Then we have*

(i) *If* $\lambda_1(\Delta a(x)+b(x))>0$, *then* $r[\frac{1}{a(x)}(-\Delta+M)^{-1}(b(x)+Ma(x))]>1$;
(ii) *If* $\lambda_1(\Delta a(x)+b(x))<0$, *then* $r[\frac{1}{a(x)}(-\Delta+M)^{-1}(b(x)+Ma(x))]<1$;
(iii) *If* $\lambda_1(\Delta a(x)+b(x))=0$, *then* $r[\frac{1}{a(x)}(-\Delta+M)^{-1}(b(x)+Ma(x))]=1$.

2.2 A Scalar Equation

In this subsection, we consider the scalar equation

$$\begin{cases} -\Delta[\varphi(u)u]=uf(u), & x \in \Omega, \\ u=0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$. The functions $\varphi:[0,\infty) \to [0,\infty)$ and $f:[0,\infty) \to \mathbb{R}$ are assumed to satisfy the following hypotheses:

(H2.1) $\varphi(0)>0$ and $\varphi(u)$ is C^2-function in u with $\varphi'(u) \geq 0$ for all $u \geq 0$;
(H2.2) $f(u)$ is C^1-function in u with $f'(u)<0$ for all $u \geq 0$;
(H2.3) $f(0)>0$ and $f(u)<0$ on (C_0,∞) for some positive constant C_0.

Now we give the existence and uniqueness theorem of positive solutions of (2.3), which can be proved by the similar technique in [31, Theorem 2.11].

Theorem 2.1 *Consider the scalar equation* (2.3) *with hypotheses* (H2.1)-(H2.3).

(i) *If* $\lambda_1(\varphi(0)\Delta+f(0)) \leq 0$, *then* (2.3) *has no positive solution*;
(ii) *If* $\lambda_1(\varphi(0)\Delta+f(0))>0$, *then* (2.3) *has a unique positive solution*.

2.3 Fixed Point Index Theory in Banach Space

Let E be a real Banach space and $W \subset E$ a closed convex set. W is called a total wedge if $\alpha W \subset W$ for all $\alpha \geq 0$ and $\overline{W-W}=E$. A wedge is said to be a cone if $W \cap (-W)=\{0\}$. For $y \in W$, define $W_y=\{x \in E: y+\gamma x \in W \text{ for some } \gamma>0\}$, $S_y=\{x \in \overline{W}_y: -x \in \overline{W}_y\}$. Then $\overline{W}_y$ is a wedge containing W, y, $-y$, and S_y is a closed subspace of E containing y.

Let T be a compact linear operator on E which satisfies $T(\overline{W}_y) \subset \overline{W}_y$. We say that T has property α on $\overline{W}_y$ if there is $t \in (0,1)$ and $w \in \overline{W}_y \setminus S_y$ such that $w-tTw \in S_y$. Let $F: W \to W$ is a compact operator with a fixed point $y \in W$ and F is Fréchet differentiable at y. Let $L = F'(y)$ be the Fréchet derivative of F at y. Then L maps $\overline{W}_y$ into itself. For an open subset $U \subset W$, define $\text{index}_W(F, U) = \text{index}(F, U, W) = \text{index}_W(I - F, U, 0)$, where I is the identity map. If y is an isolated fixed point of F, then the fixed point index of F at y in W is defined by $\text{index}_W(F, y) = \text{index}(F, y, W) = \text{index}_W(F, U(y), W)$, where $U(y)$ is a small open neighborhood of y in W.

Next, we represent two results that will be useful in proving the existence of positive solutions of system (1.4). Theorem 2.2 is due to Li [33], and Theorem 2.3 is due to Amann [34].

Theorem 2.2 *Assume that $I - L$ is invertible on $\overline{W}_y$.*

(i) *If L has property α on $\overline{W}_y$, then $\text{index}_W(F, y) = 0$.*
(ii) *If L does not have property α on $\overline{W}_y$, then $\text{index}_W(F, y) = (-1)^\gamma$, where γ is the sum of multiplicities of all the eigenvalues of L which are greater than* 1.

Suppose that B is an open unit ball of E, V is a real vector space, P is a nonempty subset of V. Denote by (E, P) an arbitrary ordered Banach space. For every $\rho > 0$, denote $P_\rho = \rho B \cap P$. Then we have the following statements.

Theorem 2.3 *Let $F : \bar{P}_\rho \to P$ be a compact map such that $F(0) = 0$. Suppose that F has a right derivative $F'_+(0)$ at zero such that* 1 *is not an eigenvalue of $F'_+(0)$ to a positive eigenvector. Then there exists a constant $\sigma_0 \in (0, \rho]$ such that for every $\sigma \in (0, \sigma_0]$,*

(i) *if $F'_+(0)$ has no positive eigenvector to an eigenvalue greater than one, then $\text{index}_W(F, P_\sigma) = 1$.*
(ii) *if $F'_+(0)$ possesses a positive eigenvector to an eigenvalue greater than one, then $\text{index}_W(F, P_\sigma) = 0$.*

3 Stability of Trivial and Semi-Trivial Solutions

This section focuses on the stability of trivial and semi-trivial solutions of model (1.4). The arguments are based on the spectrum analysis of the linearized operators.

Clearly, model (1.4) has a trivial solution $(0, 0)$. With Theorem 2.1, we know that the problem

$$\begin{cases} -\Delta[(\alpha_1 + \beta_{11} u)u] = u(a_1 - b_1 u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases}$$

has a unique positive solution u^* for $\lambda_1(\alpha_1\Delta + a_1) > 0$. Thus (1.4) has semi-trivial solution $(u^*, 0)$ when $\lambda_1(\alpha_1\Delta + a_1) > 0$.

Similarly, (1.4) has semi-trivial solution $(0, v^*)$ when $\lambda_1(\alpha_2\Delta + a_2) > 0$.

Theorem 3.1 *The solution $(0, 0)$ is asymptotically stable if $\lambda_1(\alpha_1\Delta + a_1) < 0$ and $\lambda_1(\alpha_2\Delta + a_2) < 0$, whereas it is unstable if $\lambda_1(\alpha_1\Delta + a_1) > 0$ or $\lambda_1(\alpha_2\Delta + a_2) > 0$.*

Proof The linearized operator of (1.4) at $(0, 0)$ is

$$G_1 = \begin{bmatrix} \alpha_1\Delta + a_1 & 0 \\ 0 & \alpha_2\Delta + a_2 \end{bmatrix}.$$

In view of [35], we know that all eigenvalues of G_1 are $\{\lambda_i(\alpha_1\Delta + a_1)\} \cup \{\lambda_i(\alpha_2\Delta + a_2)\}, i = 1, 2, \ldots$. Thus the conclusion can be obtained directly by spectral analysis. □

Theorem 3.2 (i) *The solution* $(u^*, 0)$ *is asymptotically stable if* $\lambda_1(\Delta(\alpha_2 + \beta_{21}u^*) + a_2 - \frac{c_2u^*}{1+u^*}) < 0$, *whereas it is unstable if* $\lambda_1(\Delta(\alpha_2 + \beta_{21}u^*) + a_2 - \frac{c_2u^*}{1+u^*}) > 0$.

(ii) *The solution* $(0, v^*)$ *is asymptotically stable if* $\lambda_1(\Delta(\alpha_1 + \beta_{12}v^*) + a_1 - \frac{c_1v^*}{1+v^*}) < 0$, *whereas it is unstable if* $\lambda_1(\Delta(\alpha_1 + \beta_{12}v^*) + a_1 - \frac{c_1v^*}{1+v^*}) > 0$.

Proof We only prove the (i) since we can make a similar argument for (ii). The linearized operator of (1.4) at $(u^*, 0)$ is

$$G_2 = \begin{bmatrix} \Delta(\alpha_1 + 2\beta_{11}u^*) + a_1 - 2b_1u^* & \beta_{12}\Delta u^* - c_1u^* \\ 0 & \Delta(\alpha_2 + \beta_{21}u^*) + a_2 - \frac{c_2u^*}{1+u^*} \end{bmatrix}.$$

Similarly, by [35] we derive that all eigenvalues of G_2 are $\{\lambda_i(\Delta(\alpha_1 + 2\beta_{11}u^*) + a_1 - 2b_1u^*)\} \cup \{\lambda_i(\Delta(\alpha_2 + \beta_{21}u^*) + a_2 - \frac{c_2u^*}{1+u^*})\}, i = 1, 2, \ldots$. Clearly, $\lambda_i(\Delta(\alpha_1 + 2\beta_{11}u^*) + a_1 - 2b_1u^*) \le \lambda_1(\Delta(\alpha_1 + 2\beta_{11}u^*) + a_1 - 2b_1u^*) < 0$ for any $i \ge 1$. Combining the spectral analysis, we can obtain the conclusion directly. □

4 Boundedness and Non-existence of Positive Solutions

This section deals with the boundedness and non-existence of positive solutions of (1.4), which play a critical role in proving the existence result in Sect. 5.

Denote

$$\varphi(u, v) = \alpha_1 + \beta_{11}u + \beta_{12}v, \quad \psi(u, v) = \alpha_2 + \beta_{21}u + \beta_{22}v,$$
$$f(u, v) = a_1 - b_1u - \frac{c_1v}{1+v}, \quad g(u, v) = a_2 - b_2v - \frac{c_2u}{1+u}.$$

Then model (1.4) becomes

$$\begin{cases} -\Delta[\varphi(u, v)u] = uf(u, v), & x \in \Omega, \\ -\Delta[\psi(u, v)v] = vg(u, v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

The following lemma is useful in the calculation of the priori upper bound of positive solution to model (1.4), which is an immediate result of the proof of Lemma 3.2 in [31].

Lemma 4.1 *Let* (u, v) *be a positive solution of* (4.1). *If* $\varphi(u, v)u$ *and* $\psi(u, v)v$ *attain their maximum at* $x = x_0$ *and* $x = x_1$ *over* $\overline{\Omega}$, *respectively, then* $f(u(x_0), v(x_0)) \ge 0$ *and* $g(u(x_1), v(x_1)) \ge 0$.

Theorem 4.1 *Suppose* $c_1 \ge a_1$, $c_2 \ge a_2$. *Then there exist constants* $M_1, M_2 > 0$ *such that every positive classical solution* (u, v) *of* (1.4) *satisfies*

$$u(x) \le M_1, \quad v(x) \le M_2.$$

Proof Assume that (u, v) is a positive solution of (1.4). Let x_0 be a point such that $\varphi(u(x_0), v(x_0))u(x_0) = \max_{x\in\overline{\Omega}}\{\varphi(u, v)u\}$. Applying Lemma 4.1 to the first equation of model (4.1), we have $f(u(x_0), v(x_0)) \geq 0$. Since $f_u, f_v < 0$, we have

$$f(u(x_0), 0)) \geq f(u(x_0), v(x_0)) \geq 0 \quad \text{and} \quad f(0, v(x_0))) \geq f(u(x_0), v(x_0)) \geq 0.$$

This means that $u(x_0) \leq \frac{a_1}{b_1}$ and $v(x_0) \leq \frac{a_1}{c_1-a_1}$ by given condition $c_1 \geq a_1$. With $\varphi_u, \varphi_v > 0$, we have

$$\max_{x\in\overline{\Omega}}\{\varphi(u, v)u\} \leq \varphi(\frac{a_1}{b_1}, \frac{a_1}{c_1 - a_1})\frac{a_1}{b_1} = \frac{a_1(c_1 - a_1)(\alpha_1 b_1 + a_1\beta_{11}) + a_1^2 b_1\beta_{12}}{b_1^2(c_1 - a_1)},$$

and so

$$u(x) \leq \frac{1}{\alpha_1}\varphi(u, v)u \leq \frac{1}{\alpha_1}\max_{x\in\overline{\Omega}}\{\varphi(u, v)u\} \leq \frac{a_1(c_1 - a_1)(\alpha_1 b_1 + a_1\beta_{11}) + a_1^2 b_1\beta_{12}}{\alpha_1 b_1^2(c_1 - a_1)} \triangleq M_1$$

for all $x \in \overline{\Omega}$.

By the similar reason, we can show that there exists a positive constant M_2 such that $v(x) \leq M_2$ for all $x \in \overline{\Omega}$ when $c_2 \geq a_2$, where

$$M_2 = \frac{a_2(c_2 - a_2)(\alpha_2 b_2 + a_2\beta_{22}) + a_2^2 b_2\beta_{21}}{\alpha_2 b_2^2(c_2 - a_2)}.$$

Thus, the proof is finished. □

In the following, we give the non-existence of positive solutions for (1.4). By Sect. 2.1, we know that when $a(x) = 1$ and $b(x) = 0$ in (2.1), $\lambda_i(\Delta a(x) + b(x))$ is the eigenvalue of Δ, denoted as λ_i. In this case, the principle eigenvalue of Δ in Ω with the homogeneous Dirichlet boundary condition is λ_1, which will be used many times in the later. In addition, for any $\varphi \in L^1(\Omega)$, we let $\overline{\varphi} = \frac{1}{|\Omega|}\int_\Omega \varphi \mathrm{d}x$.

Theorem 4.2 *Suppose that $A_i, D_i, i = 1, 2$ are given positive constants. Then there exist positive constants D_1^0, D_2^0 such that model (1.4) has no positive solution if*

$$\alpha_1 > D_1^0,\ \alpha_2 > D_2^0,\ D_1 \leq \beta_{12} < A_1,\ D_2 \leq \beta_{21} < A_2.$$

Proof Assume, on the contrary, that (u, v) is a positive solution of (1.4). Multiply the first equation of (1.4) by $(u - \overline{u})$ and the second equation by $(v - \overline{v})$, then integrate over Ω by parts, and then add them together to yield

$$\begin{aligned}
&-\int_\Omega \Delta[(\alpha_1 + \beta_{11}u + \beta_{12}v)u](u - \overline{u})\mathrm{d}x - \int_\Omega \Delta[(\alpha_2 + \beta_{21}u + \beta_{22}v)v](v - \overline{v})\mathrm{d}x \\
&= \int_\Omega \{2\beta_{11}u|\nabla u|^2 + \beta_{12}u\nabla u\nabla v + \alpha_1|\nabla u|^2 + \beta_{12}v|\nabla u|^2 + 2\beta_{22}v|\nabla v|^2 + \beta_{21}v\nabla u\nabla v \\
&\quad + \alpha_2|\nabla v|^2 + \beta_{21}u|\nabla v|^2\}\mathrm{d}x \\
&= \int_\Omega u(a_1 - b_1 u - \frac{c_1 v}{1+v})(u - \bar{u})\mathrm{d}x + \int_\Omega v(a_2 - b_2 v - \frac{c_2 u}{1+u})(v - \overline{v})\mathrm{d}x
\end{aligned} \tag{4.2}$$

$$= \int_{\Omega} \{[a_1 - b_1(u+\overline{u}) - \frac{c_1\overline{v} + c_1 v\overline{v}}{(1+v)(1+\overline{v})}](u-\overline{u})^2 - \frac{c_1 u(v-\overline{v})(u-\overline{u})}{(1+v)(1+\overline{v})}\}\mathrm{d}x$$
$$+ \int_{\Omega} \{[a_2 - b_2(v+\overline{v}) - \frac{c_2\overline{u} + c_2 u\overline{u}}{(1+u)(1+\overline{u})}](v-\overline{v})^2 - \frac{c_2 v(u-\overline{u})(v-\overline{v})}{(1+u)(1+\overline{u})}\}\mathrm{d}x.$$

With (4.2), Young's inequality, Theorem 4.1 and the given conditions $\beta_{12} < A_1, \beta_{21} < A_2$, we have

$$\int_{\Omega} \{\alpha_1|\nabla u|^2 + \alpha_2|\nabla v|^2\}\mathrm{d}x$$
$$= \int_{\Omega} \{2\beta_{11}u|\nabla u|^2 + \beta_{12}u\nabla u\nabla v + \alpha_1|\nabla u|^2 + \beta_{12}v|\nabla u|^2 + 2\beta_{22}v|\nabla v|^2 + \beta_{21}v\nabla u\nabla v$$
$$+ \alpha_2|\nabla v|^2 + \beta_{21}u|\nabla v|^2\}\mathrm{d}x - \int_{\Omega} \{2\beta_{11}u|\nabla u|^2 + \beta_{12}u\nabla u\nabla v + \beta_{12}v|\nabla u|^2 + 2\beta_{22}v|\nabla v|^2$$
$$+ \beta_{21}v\nabla u\nabla v + \beta_{21}u|\nabla v|^2\}\mathrm{d}x$$
$$\leq \int_{\Omega} \{2\beta_{11}u|\nabla u|^2 + \beta_{12}u\nabla u\nabla v + \alpha_1|\nabla u|^2 + \beta_{12}v|\nabla u|^2 + 2\beta_{22}v|\nabla v|^2 + \beta_{21}v\nabla u\nabla v$$
$$+ \alpha_2|\nabla v|^2 + \beta_{21}u|\nabla v|^2\}\mathrm{d}x - \int_{\Omega} \{\beta_{12}u\nabla u\nabla v + \beta_{21}v\nabla u\nabla v\}\mathrm{d}x$$
$$= \int_{\Omega} \{[a_1 - b_1(u+\overline{u}) - \frac{c_1\overline{v} + c_1 v\overline{v}}{(1+v)(1+\overline{v})}](u-\overline{u})^2 - \frac{c_1 u(v-\overline{v})(u-\overline{u})}{(1+v)(1+\overline{v})}\}\mathrm{d}x$$
$$+ \int_{\Omega} \{[a_2 - b_2(v+\overline{v}) - \frac{c_2\overline{u} + c_2 u\overline{u}}{(1+u)(1+\overline{u})}](v-\overline{v})^2 - \frac{c_2 v(u-\overline{u})(v-\overline{v})}{(1+u)(1+\overline{u})}\}\mathrm{d}x$$
$$- \int_{\Omega} \{\beta_{12}u\nabla u\nabla v + \beta_{21}v\nabla u\nabla v\}\mathrm{d}x$$
$$\leq \int_{\Omega} [a_1 + (\frac{c_1 u}{(1+v)(1+\overline{v})})^2/(4\varepsilon_1) + \varepsilon_2](u-\overline{u})^2\mathrm{d}x$$
$$+ \int_{\Omega} [a_2 + (\frac{c_2 v}{(1+u)(1+\overline{u})})^2/(4\varepsilon_2) + \varepsilon_1] \cdot (v-\overline{v})^2\mathrm{d}x$$
$$+ \int_{\Omega} [\frac{\beta_{12}^2 u^2}{4\varrho_1}|\nabla u|^2 + \varrho_1|\nabla v|^2]\mathrm{d}x + \int_{\Omega} [\varrho_2|\nabla u|^2 + \frac{\beta_{21}^2 v^2}{4\varrho_2}|\nabla v|^2]\mathrm{d}x$$
$$\leq \int_{\Omega} [(a_1 + C(\varepsilon_1) + \varepsilon_2)(u-\overline{u})^2 + (a_2 + C(\varepsilon_2) + \varepsilon_1)(v-\overline{v})^2 + (\frac{A_1^2 M_1^2}{4\varrho_1} + \varrho_2)|\nabla u|^2$$
$$+ (\varrho_1 + \frac{A_2^2 M_2^2}{4\varrho_2})|\nabla v|^2]\mathrm{d}x,$$

where, $C(\varepsilon_i) = C(\varepsilon_i)(M_1, M_2), \varepsilon_i > 0, \varrho_i > 0, i = 1, 2, u \leq M_1, v \leq M_2,\ M_1, M_2$ are the same as defined in Theorem 4.1.

It follows from the Poincaré inequality that

$$\int_{\Omega} (\alpha_1|\nabla u|^2 + \alpha_2|\nabla v|^2)\mathrm{d}x \leq \int_{\Omega} [(-\frac{a_1 + C(\varepsilon_1) + \varepsilon_2}{\lambda_1} + \frac{A_1^2 M_1^2}{4\varrho_1} + \varrho_2)|\nabla u|^2$$
$$+ (-\frac{a_2 + C(\varepsilon_2) + \varepsilon_1}{\lambda_1} + \varrho_1 + \frac{A_2^2 M_2^2}{4\varrho_2})|\nabla v|^2]\mathrm{d}x. \tag{4.3}$$

We choose D_1^0 and D_2^0 satisfying

$$D_1^0 \geq -\frac{a_1 + C(\varepsilon_1) + \varepsilon_2}{\lambda_1} + \frac{A_1^2 M_1^2}{4\varrho_1} + \varrho_2 \quad \text{and} \quad D_2^0 \geq -\frac{a_2 + C(\varepsilon_2) + \varepsilon_1}{\lambda_1} + \varrho_1 + \frac{A_2^2 M_2^2}{4\varrho_2},$$

respectively, then (4.3) is a contradiction with assumptions $\alpha_1 > D_1^0, \alpha_2 > D_2^0$. Thus, when $\alpha_1 > D_1^0, \alpha_2 > D_2^0$, $D_1 \leq \beta_{12} < A_1$, $D_2 \leq \beta_{21} < A_2$, model (1.4) has no positive solution. □

5 Existence of the Positive Solutions

In this section, we investigate the existence of positive solutions of (1.4) by calculating the fixed point's index. We assume that the following hypothesis always hold.

(H5.1) $\lambda_1(\alpha_1\Delta + a_1) > 0$ and $\lambda_1(\alpha_2\Delta + a_2) > 0$.

Obviously, the hypothesis (H5.1) shows that the model (1.4) has semi-trivial solutions $(u^*, 0)$ and $(0, v^*)$.

Let $G(u, v) = (\varphi(u, v)u, \psi(u, v)v)$, $S = (S_1, S_2)$, where

$$S_1(u, v) = u(a_1 - b_1 u - \frac{c_1 v}{1+v} + M(\alpha_1 + \beta_{11}u + \beta_{12}v)),$$

$$S_2(u, v) = v(a_2 - b_2 v - \frac{c_2 u}{1+u} + M(\alpha_2 + \beta_{21}u + \beta_{22}v))$$

with M being a sufficiently large positive constant so that S_1 is monotone increasing with respect to u and S_2 is monotone increasing with respect to v for all $(u, v) \in [0, M_1] \times [0, M_2]$. The existence of M follows from $\alpha_1 > 0$ and $\alpha_2 > 0$.

Since the Jacobian determinant $J = \frac{\partial G(u,v)}{\partial(u,v)}$ satisfies

$$J = \frac{\partial G(u, v)}{\partial(u, v)} = \begin{vmatrix} \alpha_1 + 2\beta_{11}u + \beta_{12}v & \beta_{12}u \\ \beta_{21}v & \alpha_2 + \beta_{21}u + 2\beta_{22}v \end{vmatrix}$$
$$= (\alpha_1 + 2\beta_{11}u + \beta_{12}v)(\alpha_2 + \beta_{21}u + 2\beta_{22}v) - \beta_{12}\beta_{21}uv > 0,$$

G is invertible and denote the inverse of G by G^{-1}. Define operator $H : C(\overline{\Omega}) \times C(\overline{\Omega}) \to C(\overline{\Omega}) \times C(\overline{\Omega})$ by $H(u, v) = ((-\Delta + M)^{-1}S_1(u, v), (-\Delta + M)^{-1}S_2(u, v))$. Then H is compact. Simple calculation gives that (u, v) is a solution of (1.4) is equivalent to (u, v) satisfies $(u, v) = (G^{-1} \circ H)(u, v)$. Denote $F = G^{-1} \circ H$ throughout this section.

We introduce the following notations:

$C_0(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$, $\quad E := C_0(\overline{\Omega}) \oplus C_0(\overline{\Omega})$,
$D := \{(u, v) \in C_0(\overline{\Omega}) \oplus C_0(\overline{\Omega}) : u \leq M_1 + 1, v \leq M_2 + 1\}$,
$K := \{u \in C_0(\overline{\Omega}) : 0 \leq u(x), x \in \overline{\Omega}\}$, $\quad W := K \oplus K$,
$Q_{\rho'} := \{(u, v) \in W : u \leq \rho', v \leq \rho', \rho' = \max\{M_1, M_2\} + \varepsilon, \varepsilon > 0\}$,
$D' := \{(\text{int}D) \cap W\}$ for $\rho' > 0$.

Note that D' is open in W and every positive solution of (1.4) is a fixed point of F in D'. To show that model (1.4) has a strictly positive solution (u, v), we prove that F has a nontrivial fixed point in D'.

Let

$$h_1 = \frac{1}{\alpha_1}(a_1 - \frac{c_1 v^*}{1+v^*}) - \frac{a_1 - b_1 u^*}{\alpha_1 + \beta_{11}u^*}, \quad h_2 = \frac{1}{\alpha_2}(a_2 - \frac{c_2 u^*}{1+u^*}) - \frac{a_2 - b_2 v^*}{\alpha_2 + \beta_{22}v^*}.$$

Now we state the existence theorem of positive solutions to model (1.4), which will be proved in the later.

Theorem 5.1 (i) *If the first eigenvalues* $\lambda_1(\Delta(\alpha_1+\beta_{12}v^*)+a_1-\frac{c_1v^*}{1+v^*})$ *and* $\lambda_1(\Delta(\alpha_2+\beta_{21}u^*)+a_2-\frac{c_2u^*}{1+u^*})$ *have the same signs* (*i.e., if both of them are either positive or negative or zero*), *or one of these first eigenvalues is less than* 0 *and one is equal to* 0, *then model* (1.4) *has a positive solution.*

(ii) *If* $\beta_{12}=\beta_{21}=0$ *and both* h_1, h_2 *have the same constant sign, or one of* h_1 *and* h_2 *is less than* 0 *and one is equal to* 0 *on* $(0,\overline{M})$, *where* $\overline{M}=\max\{\frac{a_1}{b_1},\frac{a_2}{b_2}\}$, *then the conditions in* (i) *are necessary and sufficient for the existence of positive solutions to model* (1.4).

Remark 5.1 By the formula of the first eigenvalue (i.e., (2.2)), we know that β_{12},β_{21} influence the signs of $\lambda_1(\Delta(\alpha_1+\beta_{12}v^*)+a_1-\frac{c_1v^*}{1+v^*})$ and $\lambda_1(\Delta(\alpha_2+\beta_{21}u^*)+a_2-\frac{c_2u^*}{1+u^*})$. So the changes of β_{12},β_{21} can lead to all the cases in Theorem 5.1 arising, and therefore the cross-diffusion pressures affect the existence of positive solution of model (1.4). In Theorem 5.1(i), we give a sufficient condition for the existence of positive solutions of model (1.4). Biologically, this implies that the two competition species u and v can coexist, and the cross-diffusion pressures have an important effect on the coexistence of the two species. Theorem 5.1(ii) shows a necessary and sufficient condition for the existence of positive solutions of model (1.4) under $\beta_{12}=\beta_{21}=0$. Biologically, this means that the coexistence of two competition species u and v is not affected by the cross-diffusion pressures when their self-diffusion, inter-specific competition pressures and growth rates meet certain conditions.

In order to complete the proof of Theorem 5.1, we first give the following Lemmas 5.1-5.4.

Lemma 5.1 *Index*$_W(F,D')=1$.

Proof Clearly, ∂D contains no fixed points of F. Thus $\text{index}_W(F,D')$ is well-defined. Define an operator F_μ by $G^{-1}\circ H_\mu$ for $\mu\in[0,1]$, where

$$H_\mu(u,v)=((-\Delta+M)^{-1}S_{1,\mu}(u,v),(-\Delta+M)^{-1}S_{2,\mu}(u,v)),$$
$$S_{1,\mu}=u(\mu(a_1-b_1u-\frac{c_1v}{1+v})+M(\alpha_1+\beta_{11}u+\beta_{12}v)),$$
$$S_{2,\mu}=v(\mu(a_2-b_2v-\frac{c_2u}{1+u})+M(\alpha_2+\beta_{21}u+\beta_{22}v)).$$

Then clearly $F=F_1$ and, for each μ, (u,v) is the fixed point of F_μ if and only if (u,v) is the solution of the following problem

$$\begin{cases}-\Delta[(\alpha_1+\beta_{11}u+\beta_{12}v)u]=\mu u(a_1-b_1u-\frac{c_1v}{1+v}), & x\in\Omega,\\ -\Delta[(\alpha_2+\beta_{21}u+\beta_{22}v)v]=\mu v(a_2-b_2v-\frac{c_2u}{1+u}), & x\in\Omega,\\ u=v=0, & x\in\partial\Omega.\end{cases} \tag{5.1}$$

As in Theorem 4.1, we can see that every fixed point of F_μ satisfies $u(x)\le M_1$ and $v(x)\le M_2$ in $\overline{\Omega}$ for each $\mu\in[0,1]$, and so every fixed point of F_μ is in D but not on ∂D. Further, the homotopy invariance property of degree shows that $\text{index}_W(F_\mu,D')$ is independent of μ. So

$$\text{index}_W(F,D')=\text{index}_W(F_1,D')=\text{index}_W(F_0,D').$$

Noting that if $\mu = 0$, then (5.1) has only the trivial solution $(0,0)$, we get $\text{index}_W(F_0, D') = \text{index}_W(F_0, (0,0))$. Moreover, by the definition of λ_1, we know that $\alpha_1\lambda_1 < 0$ and $\alpha_2\lambda_1 < 0$.

For the point $y = (0,0)$, we observe that $\overline{W}_y = K \oplus K$, $S_y = \{0\} \oplus \{0\}$, and $\overline{W}_y \backslash S_y = (K \oplus K)\backslash\{(0,0)\}$. Set $L_1 = F_0'(0,0)$. Assume that $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ is an eigenfunction of L_1 corresponding to some eigenvalue $\lambda \geq 1$. Then we have

$$\begin{cases} (-\Delta + M)^{-1}(M\alpha_1\xi) = \alpha_1(\lambda\xi), \\ (-\Delta + M)^{-1}(M\alpha_2\eta) = \alpha_2(\lambda\eta). \end{cases}$$

If $\eta \not\equiv 0$, then we have $r(\frac{1}{\alpha_2}(-\Delta + M)^{-1}(M\alpha_2)) < 1$ by $\alpha_2\lambda_1 < 0$ and Lemma 2.3(ii), which contradicts $\lambda \geq 1$. So $\eta \equiv 0$. Similarly, we can derive $\xi \equiv 0$ by $\alpha_1\lambda_1 < 0$ and Lemma 2.3(ii). This implies that $I - L_1$ is invertible on $\overline{W}_y$ and L_1 does not have an eigenvalue which is greater than or equal to one.

Now we suppose that L_1 has property α on $\overline{W}_y$. Then there exist $0 < t < 1$ and $(\phi_1^*, \phi_2^*) \in \overline{W}_y \backslash S_y$ such that $(I - tL_1)\begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \in S_y$. So we get

$$\phi_2^* - \frac{t}{\alpha_2}(-\Delta + M)^{-1}(\alpha_2 M)\phi_2^* = 0.$$

Since $\phi_2^* \in K\backslash\{0\}$, we may conclude that $\frac{1}{t} > 1$ is an eigenvalue of $\frac{1}{\alpha_2}(-\Delta + M)^{-1}(M\alpha_2)$, which contradicts the above conclusion. This shows that L_1 does not have property α on $\overline{W}_y$. Then we conclude that $\text{index}_W(F_0, (0,0)) = 1$ by Theorem 2.2(ii). Therefore, $\text{index}_W(F, D') = 1$. □

Lemma 5.2 *$\text{Index}_W(F, (0,0)) = 0$.*

Proof Clearly, $F(0,0) = (0,0)$ and F is compact in $Q_{\rho'}$. Let $L_2 = F'(0,0)$, where $F'(0,0)$ is the Fréchet derivative of F at $(0,0)$. Then by calculation, we have

$$L_2\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha_1}(-\Delta + M)^{-1}[(a_1 + M\alpha_1)\xi] \\ \frac{1}{\alpha_2}(-\Delta + M)^{-1}[(a_2 + M\alpha_2)\eta] \end{pmatrix}$$

for each $(\xi, \eta) \in E$.

We first show that 1 is not an eigenvalue of L_2 corresponding to a positive eigenfunction $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$. Assume that L_2 has an eigenvalue 1, i.e., $L_2\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$. This can be written as follows

$$\begin{cases} -\Delta(\alpha_1\xi) = a_1\xi, & x \in \Omega, \\ -\Delta(\alpha_2\eta) = a_2\eta, & x \in \Omega, \\ \xi = \eta = 0, & x \in \partial\Omega. \end{cases}$$

By Lemma 2.1(iii) we know that if $\xi > 0$ or $\eta > 0$, then $\lambda_1(\alpha_1\Delta + a_1) = 0$ or $\lambda_1(\alpha_2\Delta + a_2) = 0$, which contradicts the hypothesis (H5.1). Thus 1 is not an eigenvalue of L_2 corresponding to a positive eigenfunction.

Next we calculate $\text{index}_W(F, (0,0))$. Since $\lambda_1(\alpha_1\Delta + a_1) > 0$, we get $r(T_1) > 1$ by Lemma 2.3(i), where

$$T_1 := \frac{1}{\alpha_1}(-\Delta + M)^{-1}(a_1 + M\alpha_1).$$

Then using the Krein-Rutman theorem, one can see that $r(T_1)$ is an eigenvalue of T_1 with a positive eigenfunction ϕ. That is, if we consider the pair $\begin{pmatrix}\phi\\0\end{pmatrix}$ and $\lambda = r(T_1) > 1$, then there is an eigenvalue greater than one with a positive eigenfunction. By Theorem 2.3, there exists a $\sigma_0' \in (0, \rho')$ such that $\text{index}_W(F, Q_{\sigma'}) = 0$ for any $0 < \sigma' < \sigma_0'$. On the other hand, since $(0, 0)$ is isolated, there exists $\delta > 0$ such that $(0, 0)$ is the only fixed point of F in Q_δ. If we take $\sigma' < \min\{\sigma_0', \delta\}$, then

$$\text{index}_W(F, (0, 0)) = \text{index}_W(F, Q_{\sigma'}) = 0. \qquad \square$$

Lemma 5.3 (i) *If* $\lambda_1(\Delta(\alpha_2 + \beta_{21}u^*) + a_2 - \frac{c_2u^*}{1+u^*}) > 0$, *then* $\text{index}_W(F, (u^*, 0)) = 0$.
(ii) *If* $\lambda_1(\Delta(\alpha_1 + \beta_{12}v^*) + a_1 - \frac{c_1v^*}{1+v^*}) > 0$, *then* $\text{index}_W(F, (0, v^*)) = 0$.

Proof (i) For the point $y = (u^*, 0)$, we observe that $\overline{W}_y = C_0(\overline{\Omega}) \oplus K$. Set $L_3 = F'(u^*, 0)$. By calculation, we have

$$L_3 = \left((-\Delta + M)\begin{pmatrix}\alpha_1 + 2\beta_{11}u^* & \beta_{12}u^*\\ 0 & \alpha_2 + \beta_{21}u^*\end{pmatrix}\right)^{-1}\begin{pmatrix}\alpha & \beta\\ 0 & \gamma\end{pmatrix},$$

$$\begin{cases}\alpha = a_1 - 2b_1u^* + M(\alpha_1 + 2\beta_{11}u^*),\\ \beta = u^*(-c_1 + M\beta_{12}),\\ \gamma = a_2 - \frac{c_2u^*}{1+u^*} + M(\alpha_2 + \beta_{21}u^*).\end{cases} \tag{5.2}$$

First we prove that $I - L_3$ is invertible on $\overline{W}_y$. Suppose that there is $(\xi, \eta) \in \overline{W}_y$ such that $(I - L_3)\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. Then we have

$$\begin{cases}(-\Delta + M)^{-1}(\alpha\xi + \beta\eta) = (\alpha_1 + 2\beta_{11}u^*)\xi + \beta_{12}u^*\eta,\\ (-\Delta + M)^{-1}[(a_2 - \frac{c_2u^*}{1+u^*} + M(\alpha_2 + \beta_{21}u^*))\eta] = (\alpha_2 + \beta_{21}u^*)\eta.\end{cases} \tag{5.3}$$

The second equation in (5.3) implies

$$\begin{cases}-\Delta[(\alpha_2 + \beta_{21}u^*)\eta] = (a_2 - \frac{c_2u^*}{1+u^*})\eta, & x \in \Omega,\\ \eta = 0, & x \in \partial\Omega,\end{cases}$$

where $\eta \in K$. If $\eta \not\equiv 0$, then we can consider η as a positive eigenfunction of $\Delta(\alpha_2 + \beta_{21}u^*) + (a_2 - \frac{c_2u^*}{1+u^*})$, and so $\lambda_1(\Delta(\alpha_2 + \beta_{21}u^*) + a_2 - \frac{c_2u^*}{1+u^*}) = 0$, which contradicts our assumption. Thus $\eta \equiv 0$. Substituting $\eta = 0$ into the first equation of (5.3), we have

$$\begin{cases}\Delta[(\alpha_1 + 2\beta_{11}u^*)\xi] + (a_1 - 2b_1u^*)\xi = 0, & x \in \Omega,\\ \xi = 0, & x \in \partial\Omega.\end{cases}$$

If $\xi \not\equiv 0$, then 0 is an eigenvalue of $\Delta(\alpha_1 + 2\beta_{11}u^*) + (a_1 - 2b_1u^*)$, and so we have $\lambda_1(\Delta(\alpha_1 + 2\beta_{11}u^*) + a_1 - 2b_1u^*) \geq 0$, which contradicts the fact that $\lambda_1(\Delta(\alpha_1 + 2\beta_{11}u^*) + a_1 - 2b_1u^*) < 0$. Thus $\xi \equiv 0$, i.e., $(\xi, \eta) = (0, 0)$, and so $I - L_3$ is invertible on $\overline{W}_y$.

Next we show that L_3 has property α on $\overline{W}_y$. Observe that $S_y = C_0(\overline{\Omega}) \oplus \{0\}$ and $\overline{W}_y \backslash S_y = C_0(\overline{\Omega}) \oplus \{K\backslash\{0\}\}$. Since $\lambda_1(\Delta(\alpha_2 + \beta_{21}u^*) + a_2 - \frac{c_2u^*}{1+u^*}) > 0$ from the assumption, $r(T_2) > 1$ by Lemma 2.3(i), where

$$T_2 := \frac{1}{\alpha_2 + \beta_{21}u^*}(-\Delta + M)^{-1}[a_2 - \frac{c_2u^*}{1 + u^*} + M(\alpha_2 + \beta_{21}u^*)],$$

and so $r(T_2)$ is an eigenvalue of T_2 with a corresponding positive eigenfunction $\phi_3^* \in K \setminus \{0\}$ by the Krein-Rutman theorem. Set $t = 1/r(T_2)$. Then $t \in (0, 1)$ and $(0, \phi_3^*) \in \overline{W}_y \backslash S_y$. Thus

$$\begin{aligned}(I - tL_3)\begin{pmatrix} 0 \\ \phi_3^* \end{pmatrix} &= \begin{pmatrix} \frac{t\beta_{12}u^*(-\Delta+M)^{-1}((a_2-\frac{c_2u^*}{1+u^*}+M(\alpha_2+\beta_{21}u^*))\phi_3^*)}{(\alpha_1+2\beta_{11}u^*)(\alpha_2+\beta_{21}u^*)} - \frac{t(-\Delta+M)^{-1}((-c_1+M\beta_{12})u^*\phi_3^*)}{\alpha_1+2\beta_{11}u^*} \\ \phi_3^* - \frac{t(-\Delta+M)^{-1}((a_2-\frac{c_2u^*}{1+u^*}+M(\alpha_2+\beta_{21}u^*))\phi_3^*)}{\alpha_2+\beta_{21}u^*} \end{pmatrix} \\ &= \begin{pmatrix} \frac{t\beta_{12}u^*(-\Delta+M)^{-1}((a_2-\frac{c_2u^*}{1+u^*}+M(\alpha_2+\beta_{21}u^*))\phi_3^*)}{(\alpha_1+2\beta_{11}u^*)(\alpha_2+\beta_{21}u^*)} - \frac{t(-\Delta+M)^{-1}((-c_1+M\beta_{12})u^*\phi_3^*)}{\alpha_1+2\beta_{11}u^*} \\ \phi_3^* - \frac{1}{r(T_2)}T_2\phi_3^* \end{pmatrix} \\ &= \begin{pmatrix} \frac{t\beta_{12}u^*(-\Delta+M)^{-1}((a_2-\frac{c_2u^*}{1+u^*}+M(\alpha_2+\beta_{21}u^*))\phi_3^*)}{(\alpha_1+2\beta_{11}u^*)(\alpha_2+\beta_{21}u^*)} - \frac{t(-\Delta+M)^{-1}((-c_1+M\beta_{12})u^*\phi_3^*)}{\alpha_1+2\beta_{11}u^*} \\ 0 \end{pmatrix} \\ &\in S_y,\end{aligned}$$

i.e., L_3 has property α. Therefore $\text{index}_W(F, (u^*, 0)) = 0$ by Theorem 2.2(i).

Using the similar technique, we can prove (ii). □

Lemma 5.4 (i) *If* $\lambda_1(\Delta(\alpha_2 + \beta_{21}u^*) + a_2 - \frac{c_2u^*}{1+u^*}) \le 0$, *then* $\text{index}_W(F, (u^*, 0)) = 1$.

(ii) *If* $\lambda_1(\Delta(\alpha_1 + \beta_{12}v^*) + a_1 - \frac{c_1v^*}{1+v^*}) \le 0$, *then* $\text{index}_W(F, (0, v^*)) = 1$.

Proof We only prove (i) since we can make a similar argument for (ii). Consider the following two cases:

(a) $\lambda_1(\Delta(\alpha_2 + \beta_{21}u^*) + a_2 - \frac{c_2u^*}{1+u^*}) < 0$;
(b) $\lambda_1(\Delta(\alpha_2 + \beta_{21}u^*) + a_2 - \frac{c_2u^*}{1+u^*}) = 0$.

Case (a). Note that $\overline{W}_y = C_0(\overline{\Omega}) \oplus K$, $S_y = C_0(\overline{\Omega}) \oplus \{0\}$, $\overline{W}_y \backslash S_y = C_0(\overline{\Omega}) \oplus \{K \backslash \{0\}\}$. Assume that $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ is an eigenfunction of L_3 corresponding to some eigenvalue $\lambda \ge 1$. Then we have

$$\begin{cases} (-\Delta + M)^{-1}(\alpha\xi + \beta\eta) = (\alpha_1 + 2\beta_{11}u^*)(\lambda\xi) + \beta_{12}u^*(\lambda\eta), \\ (-\Delta + M)^{-1}[(a_2 - \frac{c_2u^*}{1+u^*} + M(\alpha_2 + \beta_{21}u^*))\eta] = (\alpha_2 + \beta_{21}u^*)(\lambda\eta). \end{cases} \tag{5.4}$$

By Lemma 2.3(ii), our assumption (a) implies

$$r(\frac{1}{\alpha_2 + \beta_{21}u^*}(-\Delta + M)^{-1}(a_2 - \frac{c_2u^*}{1 + u^*} + M(\alpha_2 + \beta_{21}u^*))) < 1,$$

and so $\eta \equiv 0$. Substituting $\eta = 0$ into the first equation of (5.4), we can similarly derive $\xi \equiv 0$ by $\lambda_1(\Delta(\alpha_1 + 2\beta_{11}u^*) + a_1 - 2b_1u^*) < 0$ and Lemma 2.3(ii). This implies that $I - L_3$ is invertible on $\overline{W}_y$ and L_3 does not have an eigenvalue which is greater than or equal to one. Further, as in Lemma 5.1, one can easily check that L_3 does not have property α, so $\text{index}_W(F, (u^*, 0)) = 1$.

Case (b). Define $F_{\mu_1} = G^{-1} \circ H_{\mu_1}$ for $\mu_1 \in [0, 1]$, where

$$H_{\mu_1}(u, v) = ((-\Delta + M)^{-1}S_1(u, v), (-\Delta + M)^{-1}S_{2,\mu_1}(u, v)),$$

$$S_1(u,v) = u(a_1 - b_1 u - \frac{c_1 v}{1+v} + M(\alpha_1 + \beta_{11} u + \beta_{12} v)),$$

$$S_{2,\mu_1}(u,v) = v(a_2 - b_2 v - \frac{c_2 u}{1+u} - \mu_1 + M(\alpha_2 + \beta_{21} u + \beta_{22} v)).$$

Clearly, $(u^*, 0)$ is a fixed point of F_{μ_1} for each $\mu_1 \in [0, 1]$ and $F_0 = F$. Also we can easily verify that every fixed point of F_{μ_1} satisfies $u(x) \le M_1$ and $v(x) \le M_2$. Hence F_{μ_1} has no fixed points on $\partial D \times [0, 1]$. By the homotopy invariance property of degree, $\text{index}_W(F, (u^*, 0)) = \text{index}_W(F_{\mu_1}, (u^*, 0))$.

Now we show that $\text{index}_W(F_{\mu_1}, (u^*, 0)) = 1$. Set $L_{\mu_1} = F'_{\mu_1}(u^*, 0)$. Then we have

$$L_{\mu_1} = \left((-\Delta + M)\begin{pmatrix} \alpha_1 + 2\beta_{11}u^* & \beta_{12}u^* \\ 0 & \alpha_2 + \beta_{21}u^* \end{pmatrix}\right)^{-1} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma^* \end{pmatrix},$$

where α, β are defined as in (5.2) and $\gamma^* = a_2 - \frac{c_2 u^*}{1+u^*} - \mu_1 + M(\alpha_2 + \beta_{21}u^*)$. Fix $\mu_1 > 0$. Suppose that $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ is an eigenfunction of L_{μ_1} corresponding to some eigenvalue $\lambda \ge 1$. Then η satisfies $\lambda\eta(\alpha_2 + \beta_{21}u^*) = (-\Delta + M)^{-1}(\gamma^*\eta)$, i.e., $\Delta((\alpha_2 + \beta_{21}u^*)\eta) + B^*\eta = 0$ in Ω and $\eta = 0$ on $\partial\Omega$, where

$$B^* = a_2 - \frac{c_2 u^*}{1+u^*} + \frac{1-\lambda}{\lambda}(a_2 - \frac{c_2 u^*}{1+u^*} + M(\alpha_2 + \beta_{21}u^*)) - \frac{\mu_1}{\lambda},$$

$\eta \in K$. If $\eta \not\equiv 0$, then we can consider η as a positive eigenfunction of $\Delta(\alpha_2 + \beta_{21}u^*) + B^*$. This implies $\lambda_1(\Delta(\alpha_2 + \beta_{21}u^*) + B^*) = 0$. Since $\lambda \ge 1$ and $\mu_1 > 0$, we have $0 = \lambda_1(\Delta(\alpha_2 + \beta_{21}u^*) + B^*) < \lambda_1(a_2 + \beta_{21}u^* + a_2 - \frac{c_2 u^*}{1+u^*})$ by Lemma 2.2(i), which contradicts our assumption (b). So $\eta \equiv 0$. Thus ξ satisfies $\lambda(\alpha_1 + 2\beta_{11}u^*)\xi = (-\Delta + M)^{-1}(\alpha\xi)$, and so

$$\Delta((\alpha_1 + 2\beta_{11}u^*)\xi) + (a_1 - 2b_1 u^* + \frac{1-\lambda}{\lambda}\alpha)\xi = 0$$

in Ω and $\xi = 0$ on $\partial\Omega$. If $\xi \not\equiv 0$, then 0 is an eigenvalue of $\Delta(\alpha_1 + 2\beta_{11}u^*) + (a_1 - 2b_1 u^* + \frac{1-\lambda}{\lambda}\alpha)$, and so $\lambda_1(\Delta(\alpha_1 + 2\beta_{11}u^*) + a_1 - 2b_1 u^* + \frac{1-\lambda}{\lambda}\alpha) \ge 0$. Since $\lambda \ge 1$, we get

$$\lambda_1(\Delta(\alpha_1 + 2\beta_{11}u^*) + a_1 - 2b_1 u^*) \ge 0$$

by Lemma 2.2(ii), which also contradicts the fact that $\lambda_1(\Delta(\alpha_1 + 2\beta_{11}u^*) + a_1 - 2b_1 u^*) < 0$. Hence $I - L_{\mu_1}$ is invertible in $\overline{W}_y$ and L_{μ_1} has no eigenvalue greater than or equal to one. As in Lemma 5.1, one can easily check that L_{μ_1} does not have property α on $\overline{W}_y$. Thus we conclude that $\text{index}_W(F_{\mu_1}, (u^*, 0)) = 1$ by Theorem 2.2(ii). □

Combing Lemma 5.1-Lemma 5.4, we give the proof of Theorem 5.1.

Proof of Theorem 5.1 (i) By Theorem 4.1 we obtain that $(0, 0), (u^*, 0), (0, v^*) \in D'$. Suppose that F has no positive fixed point in D'. Then by Lemma 5.1 and the additivity of index, we have

$$\text{index}_W(F, (0,0)) + \text{index}_W(F, (u^*, 0)) + \text{index}_W(F, (0, v^*)) = \text{index}_W(F, D') = 1. \quad (5.5)$$

If $\lambda_1(\Delta(\alpha_1 + \beta_{12}v^*) + a_1 - \frac{c_1 v^*}{1+v^*}) > 0$ and $\lambda_1(\Delta(\alpha_2 + \beta_{21}u^*) + a_2 - \frac{c_2 u^*}{1+u^*}) > 0$, then by Lemmas 5.2 and 5.3 we have

$$\text{index}_W(F, (0,0)) + \text{index}_W(F, (u^*, 0)) + \text{index}_W(F, (0, v^*)) = 0,$$

which contradicts (5.5). By the similar arguments, when the signs of first eigenvalues $\lambda_1(\Delta(\alpha_1+\beta_{12}v^*)+a_1-\frac{c_1v^*}{1+v^*})$ and $\lambda_1(\Delta(\alpha_2+\beta_{21}u^*)+a_2-\frac{c_2u^*}{1+u^*})$ are both negative or zero, or one of these first eigenvalues is less than 0 and one is equal to 0, we can also derive a contradiction by using Lemmas 5.1, 5.2 and 5.4. Therefore model (1.4) must have a positive solution in D'. This concludes the proof of Theorem 5.1(i).

(ii) If $\beta_{12}=\beta_{21}=0$, then model (1.4) becomes

$$\begin{cases} -\Delta[(\alpha_1+\beta_{11}u)u]=u(a_1-b_1u-\frac{c_1v}{1+v}), & x\in\Omega,\\ -\Delta[(\alpha_2+\beta_{21}u)v]=v(a_2-b_2v-\frac{c_2u}{1+u}), & x\in\Omega,\\ u=v=0, & x\in\partial\Omega. \end{cases} \tag{5.6}$$

Similar to the proof of Theorem 5.1(i), we can show the sufficiency of the conditions in Theorem 5.1(ii). We now prove the necessity. Here we assume $h_1<0$ and $h_2<0$. The other cases are proved in the same way.

Let $(\bar{u},\bar{v})$ be a positive solution to system (5.6). Then $\bar{u}$ solves

$$\Delta[(\alpha_1+\beta_{11}\bar{u})\bar{u}]+\bar{u}(a_1-b_1\bar{u}-\frac{c_1\bar{v}}{1+\bar{v}})=0,\quad x\in\Omega,\quad \bar{u}=0,\quad x\in\partial\Omega.$$

Hence $\bar{u}$ is a positive lower solution to

$$\Delta[(\alpha_1+\beta_{11}u)u]+u(a_1-b_1u)=0,\quad x\in\Omega,\quad u=0,\quad x\in\partial\Omega. \tag{5.7}$$

Obviously, $\frac{a_1}{b_1}$ is an upper solution to (5.7). So (5.7) has a positive solution u^*, and hence

$$\lambda_1(\Delta(\alpha_1+\beta_{11}u^*)+a_1-b_1u^*)=0.$$

By the similar technique, we can obtain that

$$\Delta[(\alpha_2+\beta_{22}v)v]+v(a_2-b_2v)=0,\quad x\in\Omega,\quad v=0,\quad x\in\partial\Omega \tag{5.8}$$

has a positive solution v^*. Hence

$$\lambda_1(\Delta(\alpha_2+\beta_{22}v^*)+a_2-b_2v^*)=0.$$

By $h_1<0$, $h_2<0$ and Lemma 2.2, we have

$$\lambda_1(\alpha_1\Delta+a_1-\frac{c_1v^*}{1+v^*})<\lambda_1(\Delta(\alpha_1+\beta_{11}u^*)+a_1-b_1u^*)=0$$

and

$$\lambda_1(\alpha_2\Delta+a_2-\frac{c_2u^*}{1+u^*})<\lambda_1(\Delta(\alpha_2+\beta_{22}v^*)+a_2-b_2v^*)=0.$$

Therefore the signs of $\lambda_1(\alpha_1\Delta+a_1-\frac{c_1v^*}{1+v^*})$ and $\lambda_1(\alpha_2\Delta+a_2-\frac{c_2u^*}{1+u^*})$ are all negative. Similarly, we can also prove that the other cases are true. This completes the proof of (ii). □

Remark 5.2 If the signs of the first eigenvalues $\lambda_1(\Delta(\alpha_1+\beta_{12}v^*)+a_1-\frac{c_1v^*}{1+v^*})$ and $\lambda_1(\Delta(\alpha_2+\beta_{21}u^*)+a_2-\frac{c_2u^*}{1+u^*})$ are opposite, or one of these first eigenvalues is greater than 0 and one is equal to 0, then by Lemmas 5.2-5.4, we have

$$\text{index}_W(F,(0,0))+\text{index}_W(F,(u^*,0))+\text{index}_W(F,(0,v^*))=1.$$

By Lemma 5.1, we get $\text{index}_W(F, D') = 1$, hence it cannot show that whether (1.4) has a positive solution or not. However, if $\alpha_1, \alpha_2, \beta_{12}, \beta_{21}$ also satisfy $\alpha_1 > D_1^0, \alpha_2 > D_2^0, D_1 \leq \beta_{12} < A_1, D_2 \leq \beta_{21} < A_2$, then by Theorem 4.2 we know that model (1.4) has no positive solution. Biologically, this implies that there may be at least one species cannot persist. From the nature point of view, it is also reasonable.

According to Theorem 5.1(i), if we can find some conditions such that the signs of the first eigenvalues $\lambda_1(\Delta(\alpha_1 + \beta_{12}v^*) + a_1 - \frac{c_1 v^*}{1+v^*})$ and $\lambda_1(\Delta(\alpha_2 + \beta_{21}u^*) + a_2 - \frac{c_2 u^*}{1+u^*})$ satisfy any of the situations in Theorem 5.1(i), then the existence of positive solution of (1.4) can be obtained. In fact, the following Corollary 5.1 and Corollaries 5.2-5.3 give some sufficient conditions for the signs of the first eigenvalues $\lambda_1(\Delta(\alpha_1 + \beta_{12}v^*) + a_1 - \frac{c_1 v^*}{1+v^*})$ and $\lambda_1(\Delta(\alpha_2 + \beta_{21}u^*) + a_2 - \frac{c_2 u^*}{1+u^*})$ to be negative and positive, respectively. From Theorem 5.1(i), it follows naturally that those conditions are sufficient for system (1.4) to have a positive solution. We give the specific analysis as follows.

Corollary 5.1 *System* (1.4) *has a positive solution either if*

(i) *the cross-diffusion pressures* β_{12} *and* β_{21} *are sufficiently large for fixed* $\alpha_i, a_i, b_i, c_i, \beta_{ii}$, $i = 1, 2$, *or*
(ii) *the inter-specific competition pressures* c_1 *and* c_2 *are sufficiently large for fixed* α_i, a_i, $b_i, \beta_{ij}, i, j = 1, 2$.

Proof (i) By (2.2), we have

$$\lambda_1(\Delta(\alpha_1 + \beta_{12}v^*) + a_1 - \frac{c_1 v^*}{1+v^*})$$
$$= \sup_{u \in W^{1,2}(\Omega)} \frac{\int_\Omega (-|\nabla[(\alpha_1 + \beta_{12}v^*)u]|^2 + (\alpha_1 + \beta_{12}v^*)(a_1 - \frac{c_1 v^*}{1+v^*})u^2)dx}{\|\sqrt{(\alpha_1 + \beta_{12}v^*)}u\|_{L^2}^2}.$$

Obviously, there exists a constant $m_1 > 0$ such that

$$\lambda_1(\Delta(\alpha_1 + \beta_{12}v^*) + a_1 - \frac{c_1 v^*}{1+v^*}) < 0$$

for all $\beta_{12} > m_1$. Similarly, there also exists a constant $m_2 > 0$ such that

$$\lambda_1(\Delta(\alpha_2 + \beta_{21}u^*) + a_2 - \frac{c_2 u^*}{1+u^*}) < 0$$

for all $\beta_{21} > m_2$. Thus the result follows from Theorem 5.1(i).

Similarly, (ii) can also be proved. □

Corollary 5.2 *If* $\lambda_1 > -\frac{a_1 b_2 - c_1 a_2/(a_2+b_2)}{\alpha_1 b_2 + a_2 \beta_{12}}$, $\lambda_1 > -\frac{a_2 b_1 - c_2 a_1/(a_1+b_1)}{\alpha_2 b_1 + a_1 \beta_{21}}$, $c_1 < a_1, c_2 < a_2$, $\beta_{12} < \frac{\alpha_1(a_1 - c_1)}{a_1}$, $\beta_{21} < \frac{\alpha_2(a_2 - c_2)}{a_2}$, *then model* (1.4) *has a positive solution.*

Proof From the proof of Theorem 5.1(ii), we know that $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are the positive upper solutions of (5.7) and (5.8), respectively. Since u^* and v^* are the unique positive solution of (5.7) and (5.8), respectively, $u^* \leq \frac{a_1}{b_1}$, $v^* \leq \frac{a_2}{b_2}$. From the assumptions $\lambda_1 > -\frac{a_1 b_2 - c_1 a_2/(a_2+b_2)}{\alpha_1 b_2 + a_2 \beta_{12}}$

and $\lambda_1 > -\frac{a_2 b_1 - c_2 a_1/(a_1+b_1)}{\alpha_2 b_1 + a_1 \beta_{21}}$, we get

$$\lambda_1(\Delta(\alpha_1 + \beta_{12}(a_2/b_2)) + a_1 - \frac{c_1(a_2/b_2)}{1 + a_2/b_2}) > 0$$

and

$$\lambda_1(\Delta(\alpha_2 + \beta_{21}(a_1/b_1)) + a_2 - \frac{c_2(a_1/b_1)}{1 + a_1/b_1}) > 0.$$

By assumptions $c_1 < a_1, c_2 < a_2$, $\beta_{12} < \frac{\alpha_1(a_1-c_1)}{a_1}$, $\beta_{21} < \frac{\alpha_2(a_2-c_2)}{a_2}$, we get that $\frac{a_1 - c_1 v/(1+v)}{\alpha_1 + \beta_{12} v}$ and $\frac{a_2 - c_2 u/(1+u)}{\alpha_2 + \beta_{21} u}$ are monotone decreasing in $v \geq 0$ and $u \geq 0$, respectively. So

$$\frac{a_1 - c_1 v^*/(1+v^*)}{\alpha_1 + \beta_{12} v^*} > \frac{a_1 - \frac{c_1(a_2/b_2)}{1+a_2/b_2}}{\alpha_1 + \beta_{12}(a_2/b_2)} \quad \text{and} \quad \frac{a_2 - c_2 u^*/(1+u^*)}{\alpha_2 + \beta_{21} u^*} > \frac{a_2 - \frac{c_2(a_1/b_1)}{1+a_1/b_1}}{\alpha_2 + \beta_{21}(a_1/b_1)},$$

and so $\lambda_1(\Delta(\alpha_1 + \beta_{12} v^*) + a_1 - \frac{c_1 v^*}{1+v^*}) > 0$ and $\lambda_1(\Delta(\alpha_2 + \beta_{21} u^*) + a_2 - \frac{c_2 u^*}{1+u^*}) > 0$ by Lemma 2.2(ii). Therefore model (1.4) has a positive solution by Theorem 5.1(i). □

Corollary 5.3 *If* $\lambda_1 > -\frac{a_2 b_1 - c_2 a_1/(a_1+b_1)}{\alpha_2 b_1 + a_1 \beta_{21}}$, $c_1 < a_1, c_2 < a_2$, $\beta_{12} < \frac{\alpha_1(a_1-c_1)}{a_1}$, $\beta_{21} < \frac{\alpha_2(a_2-c_2)}{a_2}$, $\frac{a_1}{a_2} > \frac{\alpha_1}{\alpha_2} > \max\{\frac{\beta_{12}}{\beta_{22}}, \frac{c_1 \beta_{11}}{b_1 \beta_{22}}\}$ *and* $\frac{b_1}{b_2} < \frac{\beta_{11}}{\beta_{22}}$, *then model* (1.4) *has a positive solution.*

Proof Similar to the proof of Corollary 5.2, we can prove that

$$\lambda_1(\Delta(\alpha_2 + \beta_{21} u^*) + a_2 - \frac{c_2 u^*}{1 + u^*}) > 0$$

when $\lambda_1 > -\frac{a_2 b_1 - c_2 a_1/(a_1+b_1)}{\alpha_2 b_1 + a_1 \beta_{21}}$, $c_2 < a_2$ and $\beta_{21} < \frac{\alpha_2(a_2-c_2)}{a_2}$. Substituting $u = \frac{\alpha_1 \beta_{22}}{\alpha_2 \beta_{11}} v^*$ into (5.7) and using the given conditions $\frac{a_1}{a_2} > \frac{\alpha_1}{\alpha_2}$, $\frac{b_1}{b_2} < \frac{\beta_{11}}{\beta_{22}}$, we get

$$\begin{aligned}
\Delta[(\alpha_1 + \beta_{11} u)u] + u(a_1 - b_1 u) &= \alpha_1 \Delta[(1 + \frac{\beta_{11}}{\alpha_1} u)u] + u(a_1 - b_1 u) \\
&= \alpha_1 \Delta[(1 + \frac{\beta_{22}}{\alpha_2} v^*)\frac{\alpha_1 \beta_{22}}{\alpha_2 \beta_{11}} v^*] + \frac{\alpha_1 \beta_{22}}{\alpha_2 \beta_{11}} v^*(a_1 - \frac{b_1 \alpha_1 \beta_{22}}{\alpha_2 \beta_{11}} v^*) \\
&= \frac{\alpha_1^2 \beta_{22}}{\alpha_2^2 \beta_{11}} v^*(b_2 v^* - a_2) + \frac{\alpha_1 \beta_{22}}{\alpha_2 \beta_{11}} v^*(a_1 - \frac{b_1 \alpha_1 \beta_{22}}{\alpha_2 \beta_{11}} v^*) \\
&= \frac{\alpha_1 \beta_{22}}{\alpha_2 \beta_{11}} v^*[a_1 - \frac{a_2 \alpha_1}{\alpha_2} + (\frac{b_2 \alpha_1}{\alpha_2} - \frac{b_1 \alpha_1 \beta_{22}}{\alpha_2 \beta_{11}})v^*] \\
&> 0.
\end{aligned}$$

So, $\frac{\alpha_1 \beta_{22}}{\alpha_2 \beta_{11}} v^*$ is a positive lower solution to (5.7), and $\frac{\alpha_1 \beta_{22}}{\alpha_2 \beta_{11}} v^* < u^*$.

By assumptions $c_1 < a_1$ and $\beta_{12} < \frac{\alpha_1(a_1-c_1)}{a_1}$, we know that $\frac{a_1 - c_1 v/(1+v)}{\alpha_1 + \beta_{12} v}$ is monotone decreasing with respect to $v \geq 0$, so by Lemma 2.2(ii) and given condition $\frac{\alpha_1}{\alpha_2} >$

$\max\{\frac{\beta_{12}}{\beta_{22}}, \frac{c_1\beta_{11}}{b_1\beta_{22}}\}$, we have

$$\begin{aligned}\lambda_1(\Delta(\alpha_1+\beta_{12}v^*)+a_1-\frac{c_1v^*}{1+v^*}) &> \lambda_1(\Delta(\alpha_1+\frac{\alpha_2\beta_{11}\beta_{12}}{\alpha_1\beta_{22}}u^*)+a_1-\frac{c_1\alpha_2\beta_{11}u^*/(\alpha_1\beta_{22})}{1+\alpha_2\beta_{11}u^*/(\alpha_1\beta_{22})})\\ &> \lambda_1(\Delta(\alpha_1+\frac{\alpha_2\beta_{11}\beta_{12}}{\alpha_1\beta_{22}}u^*)+a_1-c_1\alpha_2\beta_{11}u^*/(\alpha_1\beta_{22}))\\ &> \lambda_1(\Delta(\alpha_1+\beta_{11}u^*)+a_1-b_1u^*)\\ &= 0.\end{aligned}$$

Therefore, by Theorem 5.1(i) we know that model (1.4) has a positive solution. □

6 Conclusion

A rigorous investigation on the dynamics of a competitive model with cross-diffusion and nonlinear functional response subject to the homogeneous Dirichlet boundary condition is discussed. We first prove the stability of trivial and semi-trivial solutions of model (1.4) by spectrum analysis. Then the boundedness and non-existence of positive solution of (1.4) are obtained. By using the fixed point index theory in a positive cone, we prove that the existence of positive solutions of model (1.4) can be characterized by the signs of the first eigenvalues $\lambda_1(\Delta(\alpha_1+\beta_{12}v^*)+a_1-\frac{c_1v^*}{1+v^*})$ and $\lambda_1(\Delta(\alpha_2+\beta_{21}u^*)+a_2-\frac{c_2u^*}{1+u^*})$. Combining with the previous analysis in this paper, we get the following conclusions:

(I) Assume that the cross-diffusion pressures of two competition species u and v are controlled in the certain range. (i) If the signs of the first eigenvalues $\lambda_1(\Delta(\alpha_1+\beta_{12}v^*)+a_1-\frac{c_1v^*}{1+v^*})$ and $\lambda_1(\Delta(\alpha_2+\beta_{21}u^*)+a_2-\frac{c_2u^*}{1+u^*})$ are the same, or one of these first eigenvalues is less than 0 and one is equal to 0, then (1.4) has a positive solution (see Theorem 5.1(i)); (ii) Suppose that the signs of the first eigenvalues above are opposite, or one of these first eigenvalues is greater than 0 and one is equal to 0. If the self- and cross-diffusion pressures also meet additional conditions (i.e., $\alpha_1 > D_1^0, \alpha_2 > D_2^0$, $D_1 \le \beta_{12} < A_1$, $D_2 \le \beta_{21} < A_2$), then model (1.4) has no positive solution (see Remark 5.2). Biologically, the former case implies that the two competition species u and v can coexist, and the later case implies that there may be at least one species cannot persist. From the nature point of view, it is also reasonable.

(II) Assume that the cross-diffusion pressures $\beta_{12}=\beta_{21}=0$, and some certain conditions also hold (i.e. h_1 and h_2 have the same signs on $(0, \overline{M})$). Then the conditions in Theorem 5.1(i) are necessary and sufficient for the existence of positive solutions to system (1.4) (see Theorem 5.1(ii)). Biologically, this means that the two competition species u and v can coexist, and when their self-diffusion, inter-specific competition pressures and growth rates meet certain conditions, their coexistence is not affected by the cross-diffusion pressures.

(III) Either if the cross-diffusion pressures β_{12} and β_{21} or the inter-specific competition pressures c_1 and c_2 are sufficiently large, then system (1.4) has a positive solution (see Corollary 5.1). Biologically, this means that the two competition species u and v can coexist when their pressures of cross-diffusion or inter-specific competition are sufficiently large. Moreover, we also prove that when cross-diffusion pressures are controlled within a certain range, two species can also coexist (see Corollary 5.2 and Corollary 5.3).

The methods and results in the present paper may enrich the research of dynamics in the competition model. Further studies are necessary to analyze the behavior of more complex spatial models such as competition model with time delay or other kinds of cross-diffusion terms and functional responses.

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