Boundedness for a Fully Parabolic Keller–Segel Model with Sublinear Segregation and Superlinear Aggregation

Silvia Frassu[1] · Giuseppe Viglialoro[1] ⓘ

Abstract This work deals with a fully parabolic chemotaxis model with nonlinear production and chemoattractant. The problem is formulated on a bounded domain and, depending on a specific interplay between the coefficients associated to such production and chemoattractant, we establish that the related initial-boundary value problem has a unique classical solution which is uniformly bounded in time. To be precise, we study this zero-flux problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot (f(u)\nabla v) & \text{in } \Omega \times (0, T_{max}), \\ v_t = \Delta v - v + g(u) & \text{in } \Omega \times (0, T_{max}), \end{cases} \tag{$\Diamond$}$$

where Ω is a bounded and smooth domain of $\mathbb{R}^n$, for $n \geq 2$, and $f(u)$ and $g(u)$ are reasonably regular functions generalizing, respectively, the prototypes $f(u) = u^\alpha$ and $g(u) = u^l$, with proper $\alpha, l > 0$. After having shown that any sufficiently smooth $u(x, 0) = u_0(x) \geq 0$ and $v(x, 0) = v_0(x) \geq 0$ produce a unique classical and nonnegative solution (u, v) to problem ($\Diamond$), which is defined on $\Omega \times (0, T_{max})$ with T_{max} denoting the maximum time of existence, we establish that for any $l \in (0, \frac{2}{n})$ and $\frac{2}{n} \leq \alpha < 1 + \frac{1}{n} - \frac{l}{2}$, $T_{max} = \infty$ and u and v are actually uniformly bounded in time.

The paper is in line with the contribution by Horstmann and Winkler (J. Differ. Equ. 215(1):52–107, 2005) and, moreover, extends the result by Liu and Tao (Appl. Math. J. Chin. Univ. Ser. B 31(4):379–388, 2016). Indeed, in the first work it is proved that for $g(u) = u$ the value $\alpha = \frac{2}{n}$ represents the critical blow-up exponent to the model, whereas in the second, for $f(u) = u$, corresponding to $\alpha = 1$, boundedness of solutions is shown under the assumption $0 < l < \frac{2}{n}$.

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1 Introduction and Motivations

Most of this article is dedicated to the following Cauchy boundary problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot (f(u)\nabla v) & \text{in } \Omega \times (0, T_{max}), \\ v_t = \Delta v - v + g(u) & \text{in } \Omega \times (0, T_{max}), \\ u_\nu = v_\nu = 0 & \text{on } \partial\Omega \times (0, T_{max}), \\ u(x, 0) = u_0(x), \ v(x, 0) = v_0(x) & x \in \bar{\Omega}, \end{cases} \tag{1}$$

defined in a bounded and smooth domain Ω of $\mathbb{R}^n$, with $n \geq 2$, and formulated through some functions $f = f(s)$ and $g = g(s)$, sufficiently regular in their argument $s \geq 0$, and further regular initial data $u_0(x) \geq 0$ and $v_0(x) \geq 0$. Additionally, the subscript ν in $(\cdot)_\nu$ indicates the outward normal derivative on $\partial\Omega$, whereas T_{max} the maximum time up to which solutions to the system are defined.

The two partial differential equations appearing above generalize

$$u_t = \Delta u - \nabla \cdot (u\nabla v) \quad \text{and} \quad v_t = \Delta v - v + u, \quad \text{in } \Omega \times (0, T_{max}), \tag{2}$$

proposed in the pioneer papers by Keller and Segel ([10, 11]) to model the dynamics of populations (as for instance cells or bacteria), arising in mathematical biology. Precisely, by indicating with $u = u(x, t)$ a certain particle density at the position x and at the time t, the equations describe how the aggregation impact from the coupled cross term $u\nabla v$, related to the chemical signal $v = v(x, t)$ (initially distributed accordingly to the law $v_0(x) = v(x, 0)$, as in (1)), may contrast the natural diffusion (associated to the Laplacian operator, Δu) of the cells, organized at the initial time through the configuration $u_0(x) = u(x, 0)$. In particular, such an attractive impact might influence the motion of the cells so strongly even to lead the system to its chemotactic collapse (blow-up at finite time with appearance of δ-formations for the particle density). In the literature there are many contributions dedicated to the comprehension of this phenomenon. In this regard, in [9, 18] the reader can find an extensive theory dealing with the existence and properties of global, uniformly bounded or blow-up (local) solutions to the Cauchy problem associated to (2), and endowed with homogeneous Neumann boundary conditions (exactly as in (1), and biologically modeling an impermeable domain), especially in terms of the initial mass of the particle distribution, i.e., $m = \int_\Omega u_0(x)dx$. Indeed, the mass of the bacteria, preserved in time for this model, appears as a critical parameter (see, for instance, [5, 18, 25]); more exactly, for $n \geq 2$, the value $m_c = 4\pi$ establishes that when $m < m_c$ global in time solutions are expected, whereas when $m > m_c$ blow-up solutions may be detected.

The size of the initial distribution is not the only factor capable to influence the chemotactic behavior of the cells toward their self-organization. Other elements may also take sensitively part in this process; the impacts of the diffusion and/or of the chemoattractant, weaker or stronger (for instance, if in (2) the cross-diffusion term $u\nabla v$ is replaced by $\chi u\nabla v$, for some $\chi > 0$, then even for initial distribution u_0 with subcritical mass, the system exhibits blow-up at finite time whenever χ increases), the presence of external sources affecting the cells' density, or the law of the signal production from the chemical, dictated by the cells themselves: to a high (low) segregation corresponds a high disorganization (organization) in the motion of the particles. Herein we are interested in the analysis concerning the *mutual interplay between the actions from the chemoattractant and the segregation rates*. In such sense problem (1) is an example of chemotaxis model combining these aspects, exactly as specified: the chemosensitivity function $f(u)$ describes how the population aggregates,

through the interaction with the chemical, and directs its movement in the direction of the gradient of v. In our problem $f(u)$ generalizes u^α, for some α even covering superlinear powers; as said, the larger α the higher is the attraction between each cell, leading the system to undesired instabilities. On the other hand, the second equation indicates that the chemical signal is produced according to the law of $g(u)$, which as well has as prototype u^l, with l smaller than 1. Naturally, this has a segregation impact on the model weaker than the case with $g(u) = u$, especially at large particle densities (see [6, 16, 17] an related references therein); as conceivable, the gathering phenomena of the original model are dampened and more smoothness to the system is supplied.

Before giving our precise objectives in respect of the analysis above developed, let us mention the result which, mainly, inspires and justifies this investigation: For $g(u) = u$ and $f(u) \cong u^\alpha$, it is shown that the value $\alpha = \frac{2}{n}$ decides whether model (1) manifests or not blow-up scenarios. Specifically, with $n \geq 2$, for $\alpha \in (0, \frac{2}{n})$ all solutions are global and uniformly bounded, whereas the same does not apply for $\alpha > \frac{2}{n}$. In fact, (a) for $\alpha > 2$ and any $n \geq 2$, (b) for $\alpha \in (1, 2)$, $n \in \{2, 3\}$ and technical assumptions on f, (c) for $\alpha \in (\frac{2}{n}, 1)$ and $n \in \{2, 3\}$ or (d) for $\alpha \in (\frac{2}{n}, \frac{2}{n-2})$ and $n \geq 4$ (also in this case combined with further assumptions on f), there are initial data (u_0, v_0) leading to unbounded solutions (see [7]).

By continuing within the confines of Keller–Segel models with linear production, when the diffusion is not linear, i.e. $\Delta u = \nabla \cdot \nabla u$ reads $\nabla \cdot (D(u)\nabla u)$, for $n \geq 2$ the asymptotic behavior of the ratio $\frac{f(u)}{D(u)} \cong u^\alpha$ for large values of u indicates that if $\alpha \in (0, \frac{2}{n})$ any (u_0, v_0) produces uniformly bounded classical solutions to problem (1) (see [21]), whilst for $\alpha > \frac{2}{n}$ blow-up solutions either in finite or infinite time can be constructed, even for arbitrarily small initial data (see [26]). Moreover, the insight about the quantitative role of the diffusion of the cells on their evolution reads as follows: for $D(u) \cong u^{m-1}$ and $f(u) \cong u^\alpha$, $m, \alpha \in \mathbb{R}$, it is established in [1, 2] (for the fully parabolic case) and in [28] (for the simplified parabolic-elliptic one; $v_t = 0$ in the second equation) that $\alpha < m + \frac{2}{n} - 1$ is condition sufficient and necessary in order to ensure global existence and boundedness of solutions. (For completeness, we also refer to [14], where an estimate for the blow-up time of unbounded solutions to the simplified model is derived.) Unlike the case where $D(u) = 1$ and $f(u) = u$ where the critical mass m_c is n-independent, the above criterion implies that the size of the initial mass may have no crucial role on the existence of global or local-in-time solutions to nonlinear diffusion chemotaxis-systems. Conversely, the key factor is given by some specific interplay between the coefficients m, α and the dimension n; this is especially observed at high dimensions, for which a magnification of the diffusion parameter to compensate instability effects is required. A similar consideration, appropriately reinterpreted in that context, will be given below when the exponent associated to the nonlinear signal production is introduced.

Complementary, as far as nonlinear segregation chemotaxis models are concerned, when in problem (1) the case $f(u) = u$ is considered, uniformly boundedness of all its solutions is proved in [13] for $g(u) \cong u^l$, with $0 < l < \frac{2}{n}$. (We also mention [19] for an analysis of a related model with logistic-type terms.) Moreover, by resorting to a simplified parabolic-elliptic version in spatially radial contexts, for $f(u) = u$ and the second equation reduced to $0 = \Delta v - \mu(t) + g(u)$, with $g(u) \cong u^l$ and $\mu(t) = \frac{1}{|\Omega|}\int_\Omega g(u(\cdot, t))$, it is known (see [27]) that the same conclusion on the boundedness continues valid for any $n \geq 1$ and $0 < l < \frac{2}{n}$, whereas for $l > \frac{2}{n}$ blow-up phenomena may appear. (See also [23] for analyses concerning a chemotaxis model with signal-dependent sensitivity and sublinear production.)

2 Presentation of the Main Result and Comparison with a Simplified Model. Plan of the Paper

2.1 Claim of the Main Result

In accordance to what discussed above, we wish to contribute to completion of a picture yet fragmentary in the literature by addressing situations concerning system (1) that, to our knowledge, are not yet studied. To this aim, from now on these assumptions, respectively identifying the actions associated to the chemoattractant and to the segregation of the chemical signal, are fixed:

$$f \in C^2([0, \infty)), \quad f(0) = 0 \quad \text{and} \quad f(s) \le K s^\alpha, \quad \text{for some} \quad K, \alpha > 0 \quad \text{and all } s \ge 0, \tag{3}$$

and

$$g \in C^1([0, \infty)) \quad \text{and} \quad 0 \le g(s) \le K_0 s^l, \quad \text{for some} \quad K_0, l > 0 \quad \text{and all } s \ge 0. \tag{4}$$

In particular, with a specific view to what analyzed in the frame of [7], linear productions of the chemical may be sufficient to generate blow-up solutions when the impact from the chemoattractant, favoring gatherings in the motion of the species, is superquadratic, in any dimension, superlinear and subquadratic, in low dimensions, and sublinear in higher. Thus the following question seems meaningful:

∘ May a sublinear signal segregation of the chemical enforce globability of solutions for superlinear chemosensitivitiy even in high dimensions?

Our result positively addresses this issue in the sense that *independently of the initial data, by weakening in an inversely proportional way to the dimension the impact associated to the production rate of the chemical, the uniform-in-time boundedness of solutions to model* (1) *is ensured, even for superlinear thrusts from the chemoattractant.*

What said is formally claimed in this

Theorem 2.1 *Let Ω be a bounded and smooth domain of $\mathbb{R}^n$, with $n \ge 2$. Moreover, let f and g fulfill* (3) *and* (4), *respectively, with $l \in (0, \frac{2}{n})$ and α satisfying*

$$\frac{2}{n} \le \alpha < 1 + \frac{1}{n} - \frac{l}{2}. \tag{5}$$

Then, for any nontrivial $(u_0, v_0) \in C^0(\bar{\Omega}) \times C^1(\bar{\Omega})$, with $u_0 \ge 0$ and $v_0 \ge 0$ on $\bar{\Omega}$, there exists a unique pair of nonnegative functions $(u, v) \in (C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)))^2$ which solve problem (1) *and satisfy for some $C > 0$*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } t > 0.$$

*Remark 1* We make these considerations:

- For $\alpha = 1$ assumption (5) is simplified into $l \in (0, \frac{2}{n})$. Subsequently, our analysis is an extension of that developed in [13], in the sense that Theorem 2.1 recovers [13, Theorem 1.1] when $f(u) = u$ in problem (1).
- From $l \in (0, \frac{2}{n})$, the comparison with the limit linear signal production model for system (1), makes sense only in two-dimensional settings; for $l = 1$ the upper bound in assumption (5) reads $\alpha < 1$, and Theorem 2.1 is consistent with [7, Theorem 4.1].
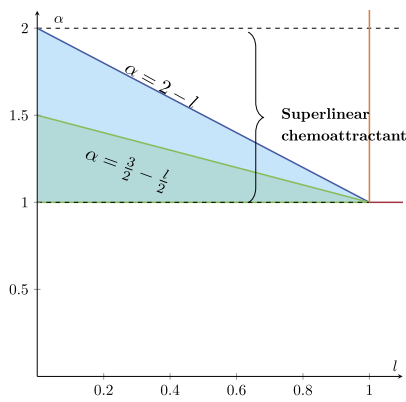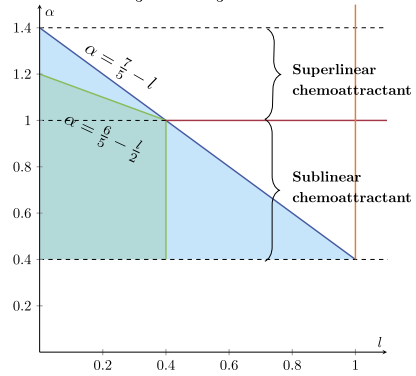
(A) Case $n = 2$. PP: $l \in (0, 1)$, $1 \le \alpha < \frac{3}{2} - \frac{l}{2}$.
PE: $l \in (0, 1)$, $1 \le \alpha < 2 - l$.
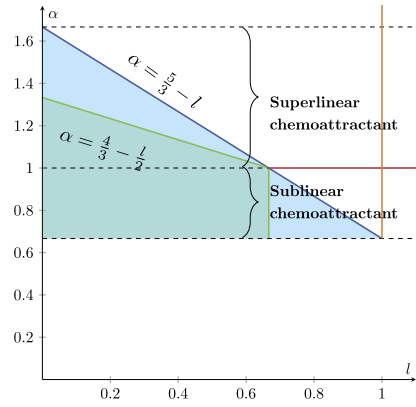
(B) Case $n = 3$. PP: $l \in (0, \frac{2}{3})$, $\frac{2}{3} \le \alpha < \frac{4}{3} - \frac{l}{2}$.
PE: $l \in (0, 1)$, $\frac{2}{3} \le \alpha < \frac{5}{3} - l$.
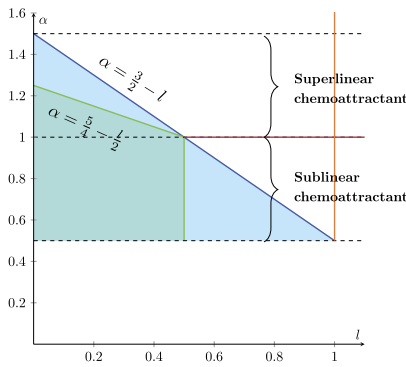
(C) Case $n = 4$. PP: $l \in (0, \frac{1}{2})$, $\frac{1}{2} \le \alpha < \frac{5}{4} - \frac{l}{2}$.
PE: $l \in (0, 1)$, $\frac{1}{2} \le \alpha < \frac{3}{2} - l$.

(D) Case $n = 5$. PP: $l \in (0, \frac{2}{5})$, $\frac{2}{5} \le \alpha < \frac{6}{5} - \frac{l}{2}$.
PE: $l \in (0, 1)$, $\frac{2}{5} \le \alpha < \frac{7}{5} - l$.

Fig. 1 Illustration comparing for some values of the dimension n the regions in the $l\alpha$-plane where both parabolic-parabolic (PP, green sector) and parabolic-elliptic (PE, cyan sector) models from problem (1) possess uniformly bounded solutions. The superlinear ($\alpha > 1$) and sublinear ($\alpha < 1$) chemoattractant zones are also marked. The vertical orange lines (at $l = 1$) indicate the values of $\alpha > \frac{2}{n}$ for which the PP version of the system admits blowing up solutions (see, again, [7]). For the PE case, it seems conceivable (see, again, [27]) that in the horizontal red lines ($\alpha = 1$ and $l > \frac{2}{n}$) the same conclusion may be true. (Color figure online)

- Since [7, Theorem 4.1] is applicable for any $n \ge 2$ whenever $\alpha < \frac{2}{n}$ and $l = 1$, a fortiori it holds true for $l \in (0, \frac{2}{n})$; this is the sole reason why we consider in our analysis $\alpha \ge \frac{2}{n}$.
- Considering that for linear production and nonlinear diffusion (with parameter m) models we discussed that the condition for boundedness reads $\alpha < m + \frac{2}{n} - 1$, from assumption (5) one can observe that the parameter l associated to the nonlinear segregation plays an opposite role with respect m: given α, for high values of n, smaller (larger) values of l (m) are needed to ensure globality and boundedness.

2.2 A View to the Parabolic-Elliptic Case

When the parabolic-parabolic problem (1) is simplified into parabolic-elliptic, with equation for the chemical replaced by $0 = \Delta v - v + g(u)$, assumption (5) becomes sharper; precisely

$\frac{2}{n} \leq \alpha < 1 + \frac{2}{n} - l$, which requires for compatibility only the restriction $l \in (0, 1)$. We highlights this aspect in Fig. 1, where we overlap the regions defined by the interplay between α and l in both models; to be more precise, we also distinguish the zones with superlinear chemoattractant ($\alpha > 1$) and sublinear chemoattractant ($\alpha < 1$). (In the same figure we also spend some words for the blow-up cases.) We understand that the observed gap between the range of parameters is not only justified by some technical reasons (see Remark 3 at the end of the paper, where few mathematical indications are given) but also by biological ones. Indeed, the fact that in the simplified version the values of $v(\cdot, t)$ only depend on the values of $u(\cdot, t)$ at the same time, is a strong modeling assumption. It corresponds to the situation where the signal responses to the concentration of the particles much faster than the organisms do to the signal; in particular, such difference in the relative adjustment of the bacteria and the chemoattractant makes that the last one reaches its equilibrium instantaneously.

2.3 Organization of the Paper

The rest of the paper is structured in this way. Section 3 is concerned with the local existence question of classical solutions to (1) and some of their properties. Some general inequalities are included in Sect. 4. They are mainly devoted to establish how to ensure globality and boundedness of local solutions using their boundedness in some proper Sobolev spaces; a key cornerstone in this direction is the procedure to fix the corresponding exponents of theses spaces (Sect. 4.2). Finally, the mentioned bound is derived in Sect. 5, which also includes the proof of Theorem 2.1.

3 Existence of Local-in-Time Solutions and Main Properties

Let us dedicate to the existence of classical solutions to system (1). It is shown that such solutions are at least local and, additionally, satisfy some crucial estimates.

Lemma 3.1 (Local existence) *Let Ω be a bounded and smooth domain of $\mathbb{R}^n$, with $n \geq 2$. Moreover, let f and g fulfill (3) and (4), respectively, with $l \in (0, \frac{2}{n})$ and α satisfying (5). Then, for any nontrivial $(u_0, v_0) \in C^0(\bar{\Omega}) \times C^1(\bar{\Omega})$, with $u_0 \geq 0$ and $v_0 \geq 0$ on $\bar{\Omega}$, there exist $T_{max} \in (0, \infty]$ and a unique pair of nonnegative functions $(u, v) \in (C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})))^2$, such that this dichotomy criterion holds true:*

$$\text{either } T_{max} = \infty \text{ or } \limsup_{t \to T_{max}}(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty. \tag{6}$$

In addition, the u-component obeys the mass conservation property, i.e.

$$\int_\Omega u(x, t)dx = \int_\Omega u_0(x)dx = m > 0 \quad \text{for all } t \in (0, T_{max}), \tag{7}$$

whilst for some $c_0 > 0$ the v-component is such that

$$\|v(\cdot, t)\|_{W^{1,n}(\Omega)} \leq c_0 \quad \text{on } (0, T_{max}). \tag{8}$$

Proof We just mention that the conclusions concerning the local-in-time well-posedness as well as the dichotomy criterion (6), can be established by straightforward adaptations of widely used methods involving an appropriate fixed point framework and standard parabolic

regularity theory; we can adequately cite [7, Theorem 3.1], for the case $g(u) = u$, and [23, Lemma 3.1], for g as in our hypotheses. Moreover, comparison arguments apply to yield both $u, v \geq 0$ in $\Omega \times (0, T_{max})$.

On the other hand, the mass conservation property easily comes by integrating over Ω the first equation of (1), in conjunction with the boundary and initial conditions.

Finally, the last claim is derived as follows. From the assumption $0 < l < \frac{2}{n}$, we can first of all fix $\frac{n}{2} < \gamma < n$ complying with $\gamma \leq \frac{1}{l}$. In this way, through the Hölder inequality, taking in mind (4) and the mass conservation property (7), we have

$$\int_\Omega g(u)^\gamma \leq K_0^\gamma \int_\Omega u^{\gamma l} \leq K_0^\gamma m^{\gamma l} |\Omega|^{1-\gamma l} \quad \text{for all } t < T_{max}. \tag{9}$$

Henceforth, we can also pick $\frac{1}{2} < \rho < 1$ such that $\zeta = 1 - \rho - \frac{n}{2}(\frac{1}{\gamma} - \frac{1}{n}) > 0$. Subsequently, since by means of the representation formula for v we have

$$v(\cdot, t) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}g(u(\cdot, s))ds \quad \text{for all } t \in (0, T_{max}),$$

with the aid of smoothing properties related to the Neumann heat semigroup $(e^{t\Delta})_{t \geq 0}$ (see Sect. 2 of [7] and Lemma 1.3 of [25]), we obtain for some $\lambda_1 > 0$ and $C_S > 0$

$$\|v(\cdot, t)\|_{W^{1,n}(\Omega)} \leq e^{-t}\|e^{t\Delta}v_0\|_{W^{1,n}(\Omega)} + \int_0^t \|e^{(t-s)(\Delta-1)}g(u(\cdot, s))\|_{W^{1,n}(\Omega)}ds$$

$$\leq C_S\|v_0\|_{W^{1,n}(\Omega)} + C_S \int_0^t \|(-\Delta + 1)^\rho e^{(t-s)(\Delta-1)}g(u(\cdot, s))\|_{L^n(\Omega)}ds$$

$$\leq C_S\|v_0\|_{W^{1,n}(\Omega)} + C_S \int_0^t (t-s)^{-\rho-\frac{n}{2}(\frac{1}{\gamma}-\frac{1}{n})}e^{-\lambda_1(t-s)}\|g(u(\cdot, s))\|_{L^\gamma(\Omega)}ds. \tag{10}$$

As a consequence, the introduction of the Gamma function Γ infers

$$\int_0^t (t-s)^{-\rho-\frac{n}{2}(\frac{1}{\gamma}-\frac{1}{n})}e^{-\lambda_1(t-s)}ds \leq \lambda_1^{-\zeta}\Gamma(\zeta),$$

which combined with bounds (9) and (10) conclude the proof.   □

In the sequel of the paper with (u, v) we will refer to the gained local classical solution to problem (1), and we might tacitly avoid to mention that such solution is produced by the initial data (u_0, v_0).

4 Preliminaries: Inequalities and Parameters

With the local solution (u, v) to problem (1) at disposal, its uniform boundedness on $(0, T_{max})$ is achieved when uniform-in-time bound for u in some L^p-norm and for $|\nabla v|^2$ in some L^q-norm, with proper p and q, is derived. This will be obtained by constructing an absorption inequality satisfied by the functional

$$y(t) := \frac{1}{p(p-1)} \int_\Omega (u+1)^p + \frac{1}{q} \int_\Omega |\nabla v|^{2q} \quad \text{for all } t \in (0, T_{max}). \tag{11}$$

In particular, the entire procedures requires, first, to adequately manipulate inequalities resulting by differentiating $y(t)$ with respect to the time and, secondly, to figure out how to choose the parameters p and q. In view of its decisive role, this second part will be discussed with some details after this subsection.

4.1 Some Algebraic and Functional Inequalities

This three coming lemmas will be used in the next logical steps. We start by considering a suitable version of the Gagliardo–Nirenberg interpolation inequality, commonly used to treat nonlinearities appearing in the diffusion and/or chemosensitivity terms (see [20, 22, 24]), and successively by recalling a particular boundary integral employed to deal with terms defined in non-convex domains.

Lemma 4.1 (Gagliardo–Nirenberg inequality) *Let Ω be a bounded and smooth domain of $\mathbb{R}^n$, with $n \geq 1$, and $0 < \mathfrak{q} \leq \mathfrak{p} \leq \infty$ satisfying $\frac{1}{2} \leq \frac{1}{n} + \frac{1}{\mathfrak{p}}$. Then, for $a = \frac{\frac{1}{\mathfrak{q}} - \frac{1}{\mathfrak{p}}}{\frac{1}{\mathfrak{q}} + \frac{1}{n} - \frac{1}{2}}$, there exists $C_{GN} = C_{GN}(\mathfrak{p}, \mathfrak{q}, \Omega) > 0$ such that*

$$\|w\|_{L^{\mathfrak{p}}(\Omega)} \leq C_{GN}(\|\nabla w\|_{L^2(\Omega)}^a \|w\|_{L^{\mathfrak{q}}(\Omega)}^{1-a} + \|w\|_{L^{\mathfrak{q}}(\Omega)}) \quad \text{for all } w \in W^{1,2}(\Omega) \cap L^{\mathfrak{q}}(\Omega).$$

Proof See [12, Lemma 2.3].   □

Lemma 4.2 *Let Ω be a bounded and smooth domain of $\mathbb{R}^n$, with $n \geq 1$, and $q \in [1, \infty)$. Then for any $\eta > 0$ there is $C_\eta > 0$ such that for any $w \in C^2(\overline{\Omega})$ satisfying $\frac{\partial w}{\partial \nu} = 0$ on $\partial \Omega$, as well as $\int_\Omega |\nabla w| \leq L_0$, for some $L_0 > 0$, this inequality holds:*

$$\int_{\partial \Omega} |\nabla w|^{2q-2} \frac{\partial |\nabla w|^2}{\partial \nu} \leq \eta \int_\Omega |\nabla |\nabla w|^q|^2 + C_\eta.$$

Proof A more general proof can be found in [8, Propostion 3.2]. More precisely, by identically retracing the steps between expressions (3.7) and (3.9) in [8, Propostion 3.2], when the value of s therein (see (ii) in (3.2)) is as in our assumptions 1, we arrive at (3.10), so concluding by invoking Young's inequality. (When Ω is convex, the left hand side of the inequality is nonpositive; see [3, Appendix] and [21, Lemma 3.2].)   □

Thanks to the next results (variants of Young's inequality), products of powers will be estimated by suitable sums involving their bases and powers of sums controlled by sums of powers.

Lemma 4.3 *Let $a, b \geq 0$ and $d_1, d_2 > 0$ such that $d_1 + d_2 < 1$. Then for all $\epsilon > 0$ there exists $c > 0$ such that*

$$a^{d_1} b^{d_2} \leq \epsilon(a + b) + c.$$

Moreover, for further $d_3, d_4 > 0$, it is possible to find positive d_5 and d such that

$$a^{d_3} + b^{d_4} \geq 2^{-d_5}(a + b)^{d_5} - d.$$

Proof We show the first inequality, the proof of the second being similar. By applying Young's inequality with conjugate exponents $\frac{1}{d_1}$ and $\frac{1}{1-d_1}$, we obtain for any $\epsilon_1 > 0$ and some $c_1(\epsilon_1) > 0$ that

$$a^{d_1} b^{d_2} = a^{d_1} (b^{\frac{d_2}{1-d_1}})^{1-d_1} \leq \epsilon_1 a + c_1(\epsilon_1) b^{\frac{d_2}{1-d_1}}.$$

Moreover, due to $\frac{d_2}{1-d_1} < 1$, a further application of the same inequality to the latter power provides for every positive ϵ_2 and proper $c_2(\epsilon_2) > 0$ this relation: $c_1(\epsilon_1)b^{\frac{d_2}{1-d_1}} \le \epsilon_2 b + c_2(\epsilon_2)$. By putting together the two inequalities and choosing $\epsilon_1 = \epsilon_2$, the first part of the lemma is concluded. All the details of the second inequality can be found in [15, Lemma 3.3]. $\square$

4.2 The Right Procedure in Fixing the Parameters p and q

In this sequence of lemmas, we will verify that the mutual relation between the parameters α in (3) and l in (4), i.e. relation (5), is such that the mentioned parameters p and q may be chosen in the appropriate way to make sure our general machinery work.

Lemma 4.4 *For any $n \in \mathbb{N}$, with $n \ge 2$, let $l \in (0, \frac{2}{n})$ and α comply with assumption (5). Then there exist $1 < \theta < \frac{n}{n-2}$ and $\mu > \frac{n}{2}$ such that*

$$\frac{l(2\mu - 1)}{4\mu - n} < \frac{n(\theta + 1 - 2\alpha\theta) + 2\theta}{2n\theta + n^2 - n^2\theta}. \tag{12}$$

Proof We first point out that for $n = 2$, $1 < \theta < \frac{n}{n-2}$ indicates that θ might also be fixed large as we want; despite that, this is not the case, and we will take θ always sufficiently close to 1.

Precisely, for $n \ge 2$, $1 < \theta < \frac{n}{n-2}$ implies that $2n\theta + n^2 - n^2\theta > 0$, and we can consider $\theta > 1$ small enough so to have $n(\theta + 1 - 2\alpha\theta) + 2\theta > 0$. Hence the function

$$h(\theta, \mu) := \frac{l(2\mu - 1)}{4\mu - n} - \frac{n(\theta + 1 - 2\alpha\theta) + 2\theta}{2n\theta + n^2 - n^2\theta},$$

is the difference of two positive terms. Now, from our assumptions

$$\lim_{\mu \to +\infty} h(1, \mu) = \left(\frac{l}{2} - 1 + \alpha - \frac{1}{n}\right) < 0,$$

and the claim is proved by means of continuity arguments. $\square$

Lemma 4.5 *Let the hypotheses of Lemma 4.4 be satisfied, and $1 < \theta < \frac{n}{n-2}$ and $\mu > \frac{n}{2}$ be therein fixed. Then there is $q_r \in [1, \infty)$ such that for all $q > q_r$ one has this compatibility relation:*

$$-\frac{2l(nq + \mu(4 + n^2 - 2n(2 + q)))}{n(4\mu - n)} = f_1(q) < f_2(q)$$

$$= \frac{q(2n(\theta + 1 - 2\alpha\theta) + 4\theta) + 2n\theta(\alpha - 1)(n - 2)}{2n\theta + n^2 - n^2\theta}. \tag{13}$$

Proof Some easy computations show that for any $n \ge 2$ the claim follows once it is established that for $1 < \theta < \frac{n}{n-2}$ and $\mu > \frac{n}{2}$ as in the hypotheses, and for

$$\mathcal{A} = \mathcal{A}(\theta, \mu) = \frac{2l(2\mu - 1)}{4\mu - n}, \qquad \mathcal{C} = \mathcal{C}(\theta, \mu) = \frac{2n(\theta + 1 - 2\alpha\theta) + 4\theta}{2n\theta + n^2 - n^2\theta},$$

$$\mathcal{B} = \mathcal{B}(\theta, \mu) = \frac{2l\mu(n - 2)^2}{n(4\mu - n)}, \qquad \mathcal{D} = \mathcal{D}(\theta, \mu) = \frac{2n\theta(1 - \alpha)(n - 2)}{2n\theta + n^2 - n^2\theta},$$
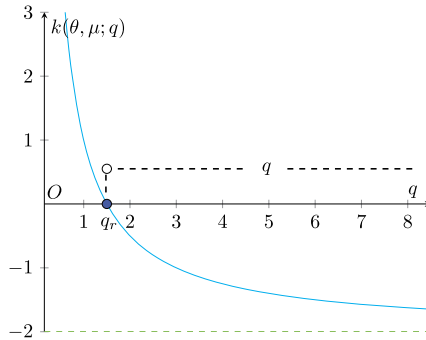
Fig. 2 By setting $k(\theta, \mu; q) := \mathcal{A} - \mathcal{C} - \frac{\mathcal{B}}{q} + \frac{\mathcal{D}}{q}$, from inequality (14) we intend to find q_r such that for some θ and μ we have that $k(\theta, \mu; q) < 0$ for all $q > q_r$. Let θ close to 1 and μ sufficiently large be taken from Lemma 4.4. As a consequence, inequality (12) leads to $\mathcal{A} - \mathcal{C} < 0$, whereas $\mathcal{B} - \mathcal{D} \in \mathbb{R}$. In particular, given that $\frac{\partial k(\cdot, \cdot; q)}{\partial q} = \frac{\mathcal{B} - \mathcal{D}}{q^2}$ and $\lim_{q \to +\infty} k(\cdot, \cdot; q) = \mathcal{A} - \mathcal{C}$, the illustration shows the qualitative behavior of the function $k(\theta, \mu; q)$ for these values of θ and μ, assuming the nontrivial situation $\mathcal{B} - \mathcal{D} < 0$. (If, indeed, θ and μ infer $\mathcal{B} - \mathcal{D} \geq 0$, k is negative for all q.) Then, by indicating with $q_r = \frac{\mathcal{B} - \mathcal{D}}{\mathcal{A} - \mathcal{C}}$ the root of k, any $q \in (q_r, \infty)$ satisfies relation (14). (In order to clarify the role of μ and θ, we observe that for $n = 2$ the chain of inequality in (13) is more manageable; in fact, it reads $lq < q(2\theta - 2\alpha\theta + 1)$, directly coming from $l < 2\theta - 2\alpha\theta + 1$, corresponding to (12) when $n = 2$, and it is μ-independent and true for some θ approaching 1, once $\alpha < \frac{3-l}{2}$ from (5) is considered). (Color figure online)

there are $q \geq 1$ entailing

$$\mathcal{A} - \frac{\mathcal{B}}{q} < \mathcal{C} - \frac{\mathcal{D}}{q}. \tag{14}$$

As we justify and explain it in Fig. 2, the above occurs whenever $\mathcal{A} - \mathcal{C} < 0$. □

Lemma 4.6 *Let the hypotheses of Lemma* 4.4 *be satisfied, and* $1 < \theta < \frac{n}{n-2}$ *and* $\mu > \frac{n}{2}$ *be therein fixed. Then there are* $p \in [1, \infty)$ *and* $q \in [1, \infty)$ *such that*

$$a_1 = \frac{\frac{np}{2}\left(1 - \frac{1}{(p+2\alpha-2)\theta}\right)}{1 - \frac{n}{2} + \frac{np}{2}}, \qquad a_2 = \frac{nq\left(\frac{1}{n} - \frac{1}{2\theta'}\right)}{1 - \frac{n}{2} + q},$$

$$a_3 = \frac{\frac{np}{2l}\left(1 - \frac{1}{2\mu}\right)}{1 - \frac{n}{2} + \frac{np}{2l}}, \qquad a_4 = \frac{nq\left(\frac{1}{n} - \frac{1}{2(q-1)\mu'}\right)}{1 - \frac{n}{2} + q},$$

$$\kappa_1 = \frac{\frac{np}{2}\left(1 - \frac{1}{p}\right)}{1 - \frac{n}{2} + \frac{np}{2}}, \qquad \kappa_2 = \frac{q - \frac{n}{2}}{1 - \frac{n}{2} + q},$$

belong to the interval $(0, 1)$ *and, additionally, imply that these other relations hold true:*

$$\beta_1 + \gamma_1 = \frac{p - 2 + 2\alpha}{p} a_1 + \frac{1}{q} a_2 \in (0, 1) \text{ and } \beta_2 + \gamma_2 = \frac{2l}{p} a_3 + \frac{q-1}{q} a_4 \in (0, 1). \tag{15}$$

Proof For θ, μ, l and α as in our hypotheses, the conjugate exponents θ' and μ' satisfy $\theta' > \frac{n}{2}$ and $\mu' < \frac{n}{n-2}$. Now, let $p > \max\{2 + \frac{1}{\theta}, \frac{2(n-2)l\mu}{n}, \frac{2\theta(\alpha-1)(n-2)}{n-\theta(n-2)}\} = p_0$ and q_r from Lemma 4.5, and note that any $q > q_r$ is such that $(f_1(q), f_2(q))$ is not empty thanks to compatibility (13). Henceforth, in view of $f_1'(q) > 0$ and $f_2'(q) > 0$, we might enlarge q so

to have $f_1(q) < p_0 < f_2(q)$ for some $q > q_r$; subsequently, this procedure always allows us to consider p and q fulfilling

$$
\begin{cases}
q > \max\left\{ \frac{(n-2)}{n}\theta', \frac{n}{2\mu'}+1, \frac{2n\theta(\alpha-1)(2-n)}{2n(\theta+1-2\alpha\theta)+4\theta}, q_r \right\} \\
p > \max\left\{ 2+\frac{1}{\theta}, \frac{2(n-2)l\mu}{n}, \frac{2\theta(\alpha-1)(n-2)}{n-\theta(n-2)} \right\}
\end{cases}
\qquad
\begin{array}{l}
\text{and also complying with} \\
p \in (f_1(q), f_2(q)).
\end{array}
\qquad (16)
$$

Our aim is to show that such restrictions suffice to prove the claim. Straightforward reasoning justify that some of the first relations in (16) imply $a_1, a_2, a_3, a_4, \kappa_1, \kappa_2 \in (0,1)$. The remaining two inequalities in (15) are, conversely, less direct. Indeed, if it can be immediately inferred that $\frac{p-2+2\alpha}{p}a_1 + \frac{1}{q}a_2$ and $\frac{2l}{p}a_3 + \frac{q-1}{q}a_4$ are positive, the other bound requires tedious computations associated to $f_1(q)$ and $f_2(q)$. More exactly, algebraic rearrangements give

$$
\frac{p-2+2\alpha}{p}a_1 + \frac{1}{q}a_2 - 1
$$
$$
= \frac{n^2(2(\alpha-1)\theta + p(\theta-1)) + 2n(q(-2\alpha\theta+\theta+1) - \theta(2\alpha+p-2)) + 4q\theta}{\theta(n(p-1)+2)(n-2(q+1))}, \quad (17)
$$

and

$$
\frac{2l}{p}a_3 + \frac{q-1}{q}a_4 - 1 = \frac{np(n-4\mu) - 2l\left(\mu\left(n^2-2n(q+2)+4\right)+nq\right)}{\mu(n-2(q+1))(l(n-2)-np)}. \quad (18)
$$

To see that expression (17) is negative, we notice from the constrains on p, q, θ and μ that the denominator is negative, so by imposing

$$
n^2(2(\alpha-1)\theta + p(\theta-1)) + 2n(q(-2\alpha\theta+\theta+1) - \theta(2\alpha+p-2)) + 4q\theta > 0,
$$

we obtain

$$
p(n^2\theta - n^2 - 2n\theta) > 4\alpha\theta n - 4n\theta - 4q\theta - 2n^2\theta(\alpha-1) - 2nq(\theta+1-2\alpha\theta). \quad (19)
$$

This, taking into account the negativity of $n^2\theta - n^2 - 2n\theta$, is equivalent to find q such that

$$
4\alpha\theta n - 4n\theta - 4q\theta - 2n^2\theta(\alpha-1) - 2nq(\theta+1-2\alpha\theta) < 0 \quad \text{or also}
$$
$$
q > \frac{2n\theta(\alpha-1)(2-n)}{2n(\theta+1-2\alpha\theta)+4\theta},
$$

which is fulfilled by virtue of the choice on q and since the considered θ complies with $2n(\theta+1-2\alpha)+4\theta > 0$. Subsequently, from (19) we have that

$$
p < \frac{q(2n(\theta+1-2\alpha\theta)+4\theta) + 2n\theta(\alpha-1)(n-2)}{n^2 + 2n\theta - n^2\theta} \quad (20)
$$

is satisfied for p and q as in (16).

Let us now turn our attention to (18). Unlike the previous case, we immediately see that the denominator is positive and, again by invoking (16), it holds that

$$
p > -\frac{2l(nq + \mu(4+n^2-2n(2+q)))}{n(4\mu-n)}. \quad (21)
$$

$\square$

Remark 2 Let us spend some words on how to treat the introduced parameters p and q in accordance with our overall purposes. This technical detail makes the analysis of the present work different and in some sense more thorough with respect those presented in many references above mentioned; therein, indeed, no undesired smallness assumption on p, generally, appears. (Let us specify that bounds from above for p may limit the validity of certain results on Keller–Segel-type systems only to low-dimensional settings: for instance, the proof of both [23, Lemma 2.2] and [4, Lemma 4.6] requires $p \in (1, 2)$, and this in turn makes that the relative analyses are exclusively confined to two-dimensional domains.)

(i) Taking the "lower extremes" for q in (q_r, ∞) and for p in $(f_1(q), f_2(q))$, as specified in Lemma 4.6, might not be appropriate when dealing with other computations where they are involved. In particular, as we will perform in the last step toward the proof of Theorem 2.1, it could be necessary to enlarge each one of this values in order to ensure the validity of certain inequalities/inclusions. Despite that, we understand that some care is needed when this procedure has to be adopted; indeed, p cannot be taken large as we want independently by q, but this is possible when the order $f_1(q) < p < f_2(q)$ related to relation (13) is preserved. (This was already imposed in the same Lemma 4.6.)

(ii) In support to the previous item, we point out that even though asymptotically we have

$$\begin{cases} \frac{p-2+2\alpha}{p} a_1 = \frac{n(\theta(2\alpha+p-2)-1)}{\theta(n(p-1)+2)} \nearrow 1 & \text{increasing with } p, \\ \frac{1}{q} a_2 = \frac{n-2\theta'}{\theta'(n-2(q+1))} \nearrow 0 & \text{decreasing with } q, \end{cases} \quad \text{and}$$

$$\begin{cases} \frac{2l}{p} a_3 = \frac{l(1-2\mu)n}{\mu(l(n-2)-np)} \nearrow 0 & \text{decreasing with } p, \\ \frac{q-1}{q} a_4 = \frac{n-2\mu'(q-1)}{\mu'(n-2(q+1))} \nearrow 1 & \text{increasing with } q, \end{cases}$$

this is not sufficient to ensure that there exists a couple (p, q) for which both $\frac{p-2+2\alpha}{p} a_1 + \frac{1}{q} a_2 < 1$ and $\frac{2l}{p} a_3 + \frac{q-1}{q} a_4 < 1$ are satisfied. Surely each one of this inequality holds true for two different couples, let's say (p_0, q_0) and (p_1, q_1), but the identification of a single (p, q) producing simultaneously those inequalities requires the extra condition $p \in (f_1(q), f_2(q))$, intimately linked to the main assumption (5).

5 Deriving Uniform-in-Time $L^p \times L^q$-Bounds for $(u, |\nabla v|^2)$. Proof of the Main Result

The coming lemma provides a uniform-in-time bound on $(0, T_{max})$ for u in $L^p(\Omega)$ and for $|\nabla v|^2$ in $L^q(\Omega)$.

Lemma 5.1 *Under the hypotheses of Lemma* 3.1, *we have the following conclusion*: *For some $p \in (1, \infty)$ and $q \in (1, \infty)$ there exists $L > 0$ such that*

$$\|u(\cdot, t)\|_{L^p(\Omega)} + \|\nabla v(\cdot, t)\|_{L^{2q}(\Omega)} \leq L \quad \text{for all } t < T_{max}.$$

Proof With θ, μ, p and q as in Lemma 4.6, the validity of all the computations along this lemma is justified.

As announced, let us differentiate with respect to the time $y(t)$ defined in (11) and split the resulting derivations in three main steps, altogether yielding the proof.

Estimating $\frac{1}{p(p-1)} \frac{d}{dt} \int_\Omega (u+1)^p$ on $(0, T_{max})$ We take $\frac{(u+1)^{p-1}}{p-1}$ as test function for the first equation in (1), so that by integrating by parts we obtain, also in view of the no-flux boundary conditions, that

$$
\frac{1}{p(p-1)} \frac{d}{dt} \int_\Omega (u+1)^p
$$

$$
= \frac{1}{p-1} \int_\Omega (u+1)^{p-1} \nabla \cdot \nabla u - \frac{1}{p-1} \int_\Omega (u+1)^{p-1} \nabla \cdot (f(u) \nabla v) \tag{22}
$$

$$
= - \int_\Omega (u+1)^{p-2} |\nabla u|^2 + \int_\Omega (u+1)^{p-2} f(u) \nabla u \cdot \nabla v \quad \text{on } (0, T_{max}).
$$

Through an application of Young's inequality and (3), the latter term reads

$$
\int_\Omega (u+1)^{p-2} f(u) \nabla u \cdot \nabla v \leq \frac{1}{2} \int_\Omega (u+1)^{p-2} |\nabla u|^2 + \frac{K^2}{2} \int_\Omega (u+1)^{p+2\alpha-2} |\nabla v|^2 \tag{23}
$$

for all $t \in (0, T_{max})$,

and the second integral at the right-hand side is estimated by the Hölder inequality so to have

$$
\int_\Omega (u+1)^{p+2\alpha-2} |\nabla v|^2 \leq \left(\int_\Omega (u+1)^{(p+2\alpha-2)\theta} \right)^{\frac{1}{\theta}} \left(\int_\Omega |\nabla v|^{2\theta'} \right)^{\frac{1}{\theta'}} \quad \text{on } (0, T_{max}). \tag{24}
$$

Now (recall that $p > 2 - 2\alpha$ by virtue of (16)) we can apply Lemma 4.1 with $\mathfrak{p} = \frac{2(p-2+2\alpha)\theta}{p}$, $\mathfrak{q} = \frac{2}{p}$ and, once the following inequality (used in the sequel without mentioning)

$$
(x+y)^s \leq 2^s (x^s + y^s) \quad \text{for any } x, y \geq 0 \text{ and } s > 0
$$

is also considered, we obtain for every $t \in (0, T_{max})$

$$
\left(\int_\Omega (u+1)^{(p+2\alpha-2)\theta} \right)^{\frac{1}{\theta}} = \| (u+1)^{\frac{p}{2}} \|_{L^{\frac{2(p+2\alpha-2)}{p}\theta}(\Omega)}^{\frac{2(p+2\alpha-2)}{p}}
$$

$$
\leq c_1 \| \nabla (u+1)^{\frac{p}{2}} \|_{L^2(\Omega)}^{\frac{2(p+2\alpha-2)}{p} a_1} \| (u+1)^{\frac{p}{2}} \|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+2\alpha-2)}{p}(1-a_1)} \tag{25}
$$

$$
+ c_1 \| (u+1)^{\frac{p}{2}} \|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+2\alpha-2)}{p}},
$$

where $c_1 > 0$ depends on C_{GN}, and with $a_1 \in (0,1)$ taken from Lemma 4.6. As a consequence, by observing that the mass conservation property (7) implies the boundedness of $(u+1)^{\frac{p}{2}}$ in $L^\infty((0, T_{max}); L^{\frac{2}{p}}(\Omega))$, from (25) we have that for some $c_2 > 0$ and $\beta_1 \in (0,1)$ deduced from Lemma 4.6

$$
\left(\int_\Omega (u+1)^{(p+2\alpha-2)\theta} \right)^{\frac{1}{\theta}} \leq c_2 \left(\int_\Omega |\nabla (u+1)^{\frac{p}{2}}|^2 \right)^{\beta_1} + c_2 \quad \text{for every } t < T_{max}. \tag{26}
$$

In a similar way, we can again invoke the Gagliardo–Nirenberg inequality, with an evident choice of $\mathfrak{p}$ and $\mathfrak{q}$, to have for some $c_3 > 0$ and $a_2 \in (0, 1)$ as in Lemma 4.6

$$\left(\int_\Omega |\nabla v|^{2\theta'}\right)^{\frac{1}{\theta'}} = \||\nabla v|^q\|^{\frac{2}{q}}_{L^{\frac{2\theta'}{q}}(\Omega)} \leq c_3 \|\nabla|\nabla v|^q\|^{\frac{2}{q}a_2}_{L^2(\Omega)} \||\nabla v|^q\|^{\frac{2}{q}(1-a_2)}_{L^{\frac{n}{q}}(\Omega)} + c_3\||\nabla v|^q\|^{\frac{2}{q}}_{L^{\frac{n}{q}}(\Omega)}$$

for all $t < T_{max}$.

In particular, by exploiting (8), we entail that (taking in mind $\gamma_1 \in (0, 1)$ from Lemma 4.6)

$$\left(\int_\Omega |\nabla v|^{2\theta'}\right)^{\frac{1}{\theta'}} \leq c_4 \left(\int_\Omega |\nabla|\nabla v|^q|^2\right)^{\gamma_1} + c_4 \quad \text{for all } t \in (0, T_{max}), \tag{27}$$

with some computable $c_4 > 0$, also in terms of c_0.

Subsequently, by collecting (23), (24) and adjusting the product between (26) and (27) by means of the Young inequality, relation (22) becomes

$$\frac{1}{p(p-1)}\frac{d}{dt}\int_\Omega (u+1)^p + \frac{1}{2}\int_\Omega (u+1)^{p-2}|\nabla u|^2$$

$$\leq c_5 \left(\int_\Omega |\nabla(u+1)^{\frac{p}{2}}|^2\right)^{\beta_1} \left(\int_\Omega |\nabla|\nabla v|^q|^2\right)^{\gamma_1} \tag{28}$$

$$+ \frac{1}{2p^2}\int_\Omega |\nabla(u+1)^{\frac{p}{2}}|^2 + \frac{q-1}{3q^2}\int_\Omega |\nabla|\nabla v|^q|^2 + c_5$$

for all $t \in (0, T_{max})$ and some $c_5 > 0$.

Estimating $\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q}$ on $(0, T_{max})$ First, by applying the identity $\Delta|\nabla v|^2 = 2\nabla v \cdot \nabla\Delta v + 2|D^2v|^2$, we arrive for all $x \in \Omega$ and $t \in (0, T_{max})$ at

$$(|\nabla v|^2)_t = \Delta|\nabla v|^2 - 2|D^2v|^2 - 2|\nabla v|^2 + 2\nabla g(u)\cdot\nabla v = 2\nabla v\cdot\nabla\Delta v - 2|\nabla v|^2 + 2\nabla g(u)\cdot\nabla v.$$

With such a relation in mind, by using $|\nabla v|^{2q-2}$ as test function, a differentiation of the second equation of problem (1) implies that on $(0, T_{max})$ this estimate holds:

$$\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} = -(q-1)\int_\Omega |\nabla v|^{2q-4}|\nabla|\nabla v|^2|^2 + \int_{\partial\Omega} |\nabla v|^{2q-2}\frac{\partial|\nabla v|^2}{\partial \nu}$$

$$- 2\int_\Omega |\nabla v|^{2q-2}|D^2v|^2 - 2\int_\Omega |\nabla v|^{2q} + 2\int_\Omega |\nabla v|^{2q-2}\nabla g(u)\cdot\nabla v.$$

Now, by relying on bound (8), some $L_0 > 0$ providing $\int_\Omega |\nabla v| \leq L_0$ exists; henceforth, an application of Lemma 4.2 allows us to find $C_\eta > 0$ such that for some suitable $\eta > 0$ we have

$$\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} + (q-1)\int_\Omega |\nabla v|^{2q-4}|\nabla|\nabla v|^2|^2 + 2\int_\Omega |\nabla v|^{2q-2}|D^2v|^2 + 2\int_\Omega |\nabla v|^{2q}$$

$$\tag{29}$$

$$\leq \eta\int_\Omega |\nabla|\nabla v|^q|^2 + C_\eta + 2\int_\Omega |\nabla v|^{2q-2}\nabla g(u)\cdot\nabla v \quad \text{on } (0, T_{max}).$$

By integrating by parts the latter integral above and using Young's inequality, we get

$$
2\int_\Omega |\nabla v|^{2q-2}\nabla g(u)\cdot\nabla v
$$

$$
= -2(q-1)\int_\Omega g(u)|\nabla v|^{2q-4}\nabla v\cdot\nabla|\nabla v|^2 - 2\int_\Omega g(u)|\nabla v|^{2q-2}\Delta v
$$

$$
\leq \frac{(q-1)}{2}\int_\Omega |\nabla v|^{2q-4}|\nabla|\nabla v|^2|^2 + 2(q-1)\int_\Omega (g(u))^2|\nabla v|^{2q-2}
$$

$$
+ \frac{2}{n}\int_\Omega |\nabla v|^{2q-2}|\Delta v|^2 + \frac{n}{2}\int_\Omega (g(u))^2|\nabla v|^{2q-2} \quad \text{on } (0, T_{max}),
$$

(30)

where

$$
\frac{2}{n}\int_\Omega |\nabla v|^{2q-2}|\Delta v|^2 \leq 2\int_\Omega |\nabla v|^{2q-2}|D^2 v|^2
$$

due to the pointwise inequality $|\Delta v|^2 \leq n|D^2 v|^2$. Henceforth, by exploiting (30) and recalling assumption (4), we can rephrase (29) as

$$
\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} + \left(\frac{2(q-1)}{q^2} - \eta\right)\int_\Omega |\nabla|\nabla v|^q|^2 \leq c_6\int_\Omega (u+1)^{2l}|\nabla v|^{2q-2} \quad \text{on } (0, T_{max}),
$$

(31)

where c_6 is a positive constant depending also on K_0. Let us now estimate the last integral in the previous bound. By employing the Hölder inequality, we first obtain the following estimate

$$
\int_\Omega (u+1)^{2l}|\nabla v|^{2q-2} \leq \left(\int_\Omega (u+1)^{2l\mu}\right)^{\frac{1}{\mu}}\left(\int_\Omega |\nabla v|^{2(q-1)\mu'}\right)^{\frac{1}{\mu'}} \quad \text{on } (0, T_{max}),
$$

(32)

whereas by relying on Lemma 4.1, we find a constant $c_7 > 0$, depending on C_{GN}, such that for $a_3 \in (0, 1)$ from Lemma 4.6 we arrive at

$$
\left(\int_\Omega (u+1)^{2l\mu}\right)^{\frac{1}{\mu}} = \|(u+1)^{\frac{p}{2}}\|_{L^{\frac{4\mu l}{p}}(\Omega)}^{\frac{4l}{p}}
$$

$$
\leq c_7\|\nabla(u+1)^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{4l}{p}a_3}\|(u+1)^{\frac{p}{2}}\|_{L^{\frac{2l}{p}}(\Omega)}^{\frac{4l}{p}(1-a_3)}
$$

(33)

$$
+ c_7\|(u+1)^{\frac{p}{2}}\|_{L^{\frac{2l}{p}}(\Omega)}^{\frac{4l}{p}} \quad \text{on } (0, T_{max}).
$$

On the other hand, by arguing as before, we infer for some $c_8 > 0$, $\beta_2 \in (0, 1)$ in Lemma 4.6 and by the finiteness of $\|(u+1)^{\frac{p}{2}}\|_{L^{\frac{2l}{p}}(\Omega)}$ (immediately coming from (7) in view of $0 < l < \frac{2}{n} < 1$)

$$
\left(\int_\Omega (u+1)^{2l\mu}\right)^{\frac{1}{\mu}} \leq c_8\left(\int_\Omega |\nabla(u+1)^{\frac{p}{2}}|^2\right)^{\beta_2} + c_8 \quad \text{for all } t \in (0, T_{max}).
$$

(34)

(Let us note that in (33) we have intentionally applied the Gagliardo–Nirenberg inequality with exponent $\mathfrak{q} = \frac{2l}{p}$ only for exhibiting reasons; indeed, the expression of $\mathcal{B}$ in Lemma

4.5 appears more compact than the one that would be obtained by considering the optimal exponent $\mathfrak{q} = \frac{2}{p}$. This does not preclude the sharpness of the assumption because, since μ is taken indefinitely large, the exponent $\mathfrak{p}$ has the control on a_3, and not $\mathfrak{q}$.)

At this point, by making use again of the Lemma 4.1, positive constants c_9 and c_{10} imply

$$\left(\int_\Omega |\nabla v|^{2(q-1)\mu'}\right)^{\frac{1}{\mu'}} = \||\nabla v|^q\|_{L^{\frac{2(q-1)}{q}\mu'}(\Omega)}^{\frac{2(q-1)}{q}}$$

$$\leq c_9 \|\nabla |\nabla v|^q\|_{L^2(\Omega)}^{\frac{2(q-1)}{q}a_4} \||\nabla v|^q\|_{L^{\frac{n}{q}}(\Omega)}^{\frac{2(q-1)}{q}(1-a_4)}$$

$$+ c_9 \||\nabla v|^q\|_{L^{\frac{n}{q}}(\Omega)}^{\frac{2(q-1)}{q}} \qquad \text{on } (0, T_{max}),$$

$$\left(\int_\Omega |\nabla v|^{2(q-1)\mu'}\right)^{\frac{1}{\mu'}} \leq c_{10}\left(\int_\Omega |\nabla |\nabla v|^q|^2\right)^{\gamma_2} + c_{10} \qquad \text{for every } t \in (0, T_{max}), \qquad (35)$$

where, once more through Lemma 4.6, $\gamma_2 \in (0, 1)$ and $a_4 \in (0, 1)$.

Finally, by plugging relations (32), (34) and (35) into bound (31), a further application of Young's inequality gives

$$\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} + \left(\frac{2(q-1)}{q^2} - \eta\right)\int_\Omega |\nabla |\nabla v|^q|^2$$

$$\leq c_{11}\left(\int_\Omega |\nabla (u+1)^{\frac{p}{2}}|^2\right)^{\beta_2}\left(\int_\Omega |\nabla |\nabla v|^q|^2\right)^{\gamma_2} \qquad (36)$$

$$+ \frac{1}{2p^2}\int_\Omega |\nabla (u+1)^{\frac{p}{2}}|^2 + \frac{q-1}{3q^2}\int_\Omega |\nabla |\nabla v|^q|^2 + c_{11} \qquad \text{for all } t < T_{max},$$

c_{11} being a proper positive constant.

Combining Terms: The Absorptive Inequality on $(0, T_{max})$ Adding the two contributions from (28) and (36) yields for some $c_{12} > 0$

$$\frac{d}{dt}\left(\frac{1}{p(p-1)}\int_\Omega (u+1)^p + \frac{1}{q}\int_\Omega |\nabla v|^{2q}\right) + \frac{1}{p^2}\int_\Omega |\nabla (u+1)^{\frac{p}{2}}|^2$$

$$+ \left(\frac{4(q-1)}{3q^2} - \eta\right)\int_\Omega |\nabla |\nabla v|^q|^2 \qquad (37)$$

$$\leq c_{12}\left(\int_\Omega |\nabla (u+1)^{\frac{p}{2}}|^2\right)^{\beta_1}\left(\int_\Omega |\nabla |\nabla v|^q|^2\right)^{\gamma_1}$$

$$+ c_{12}\left(\int_\Omega |\nabla (u+1)^{\frac{p}{2}}|^2\right)^{\beta_2}\left(\int_\Omega |\nabla |\nabla v|^q|^2\right)^{\gamma_2} + c_{12} \qquad \text{on } (0, T_{max}),$$

where accordingly to Lemma 4.6, the coefficients $\beta_1 + \gamma_1 \in (0, 1)$ and $\beta_2 + \gamma_2 \in (0, 1)$. Therefore we can apply the first inequality of Lemma 4.3 to (37) so to write for any $\epsilon > 0$ and some $c(\epsilon) > 0$ the above two products of powers as

$$\epsilon \int_\Omega |\nabla (u+1)^{\frac{p}{2}}|^2 + \epsilon \int_\Omega |\nabla |\nabla v|^q|^2 + c(\epsilon) \qquad \text{on } (0, T_{max}).$$

If now we fix $\epsilon > 0$ and $\eta > 0$ so small to ensure that $\tilde{c} = \frac{1}{p^2} - \epsilon > 0$ and $\hat{c} = \frac{4(q-1)}{3q^2} - \eta - \epsilon > 0$, a positive constant c_{13} producing

$$\frac{d}{dt}\left(\frac{1}{p(p-1)}\int_\Omega (u+1)^p + \frac{1}{q}\int_\Omega |\nabla v|^{2q}\right) + \tilde{c}\int_\Omega |\nabla (u+1)^{\frac{p}{2}}|^2$$

$$+ \hat{c}\int_\Omega |\nabla|\nabla v|^q|^2 \leq c_{13} \quad \text{on } (0, T_{max}), \tag{38}$$

can be computed. Again by employing twice the Gagliardo–Nirenberg inequality, we have for $\kappa_1 \in (0,1)$ and $\kappa_2 \in (0,1)$ derived in Lemma 4.6, and suitable large $c_{14} > 0$, that these estimates hold true for all $t \in (0, T_{max})$:

$$\int_\Omega (u+1)^p = \|(u+1)^{\frac{p}{2}}\|^2_{L^2(\Omega)} \leq c_{14}\|\nabla(u+1)^{\frac{p}{2}}\|^{2\kappa_1}_{L^2(\Omega)}\|(u+1)^{\frac{p}{2}}\|^{2(1-\kappa_1)}_{L^{\frac{2}{p}}(\Omega)}$$

$$+ c_{14}\|(u+1)^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)},$$

and

$$\int_\Omega |\nabla v|^{2q} = \||\nabla v|^q\|^2_{L^2(\Omega)} \leq c_{14}\|\nabla|\nabla v|^q\|^{2\kappa_2}_{L^2(\Omega)}\||\nabla v|^q\|^{2(1-\kappa_2)}_{L^{\frac{n}{q}}(\Omega)} + c_{14}\||\nabla v|^q\|^2_{L^{\frac{n}{q}}(\Omega)}.$$

The already used mass conservation property and the boundedness of $\|v(\cdot, t)\|_{W^{1,n}(\Omega)}$, provide some positive constant c_{15} such that

$$\int_\Omega (u+1)^p \leq c_{15}\left(\int_\Omega |\nabla(u+1)^{\frac{p}{2}}|^2\right)^{\kappa_1} + c_{15} \quad \text{for all } t \in (0, T_{max}), \tag{39}$$

and

$$\int_\Omega |\nabla v|^{2q} \leq c_{15}\left(\int_\Omega |\nabla|\nabla v|^q|^2\right)^{\kappa_2} + c_{15} \quad \text{on } (0, T_{max}). \tag{40}$$

Consequently, by collecting (39) and (40), we can rewrite (38) in the following way

$$\frac{d}{dt}\left(\frac{1}{p(p-1)}\int_\Omega (u+1)^p + \frac{1}{q}\int_\Omega |\nabla v|^{2q}\right) + c_{16}\left(\int_\Omega (u+1)^p\right)^{\frac{1}{\kappa_1}} + c_{16}\left(\int_\Omega |\nabla v|^{2q}\right)^{\frac{1}{\kappa_2}}$$

$$\leq c_{17} \quad \text{on } (0, T_{max}),$$

with positive constants c_{16}, c_{17}.

From all of the above, we invoke the second inequality in Lemma 4.3, so to see that the function $y = y(t)$ satisfies this initial value problem

$$\begin{cases} y'(t) + c_{18}y^\kappa(t) \leq c_{19} \quad \text{on } (0, T_{max}), \\ y(0) = y_0 = \frac{1}{p(p-1)}\int_\Omega (u_0(x)+1)^p dx + \frac{1}{q}\int_\Omega |\nabla v_0(x)|^{2q} dx, \end{cases}$$

with suitable constants $\kappa, c_{18}, c_{19} > 0$. This leads to the conclusion for appropriate $L > 0$ since standard ODE comparison arguments give

$$y(t) \leq \max\left\{y_0, \left(\frac{c_{19}}{c_{18}}\right)^{\frac{1}{\kappa}}\right\} \quad \text{for every } t < T_{max}. \qquad \square$$

With these gained bounds, we exploit a general boundedness result to quasilinear parabolic equations (see [21]) so to ensure uniform-in-time boundedness of the local solution (u, v) to system (1).

Proof of Theorem 2.1 Let (u, v) be the local classical solution to (1). Upon enlarging p and q accordingly to what said in item (i) of Remark 2, we can obtain that the term $f(u)\nabla v \in L^\infty((0, T_{max}); L^{q_1}(\Omega))$, for some $q_1 > n + 2$. So we conclude thanks to Lemma 5.1, [21, Lemma A.1] and the dichotomy criterion (6). $\square$

Remark 3 (Some hints about the parabolic-elliptic model) Let us consider the equations

$$u_t = \Delta u - \nabla \cdot (f(u)\nabla v) \text{ and } 0 = \Delta v - v + g(u), \quad \text{in } \Omega \times (0, T_{max}), \quad (41)$$

endowed with homogeneous Neumann boundary conditions, nontrivial initial data $u(x, 0) = u_0(x) \geq 0$, where f and g comply with assumptions in Theorem 2.1. Similarly to what already done, we have

$$\frac{1}{p(p-1)} \frac{d}{dt} \int_\Omega (u+1)^p = \frac{1}{p-1} \int_\Omega (u+1)^{p-1} \nabla \cdot \nabla u - \frac{1}{p-1} \int_\Omega (u+1)^{p-1} \nabla \cdot (f(u)\nabla v)$$

$$= -\int_\Omega (u+1)^{p-2} |\nabla u|^2 + \int_\Omega (u+1)^{p-2} f(u)\nabla u \cdot \nabla v$$

$$\leq -\int_\Omega (u+1)^{p-2} |\nabla u|^2$$

$$+ \frac{KK_0}{p+\alpha-1} \int_\Omega (u+1)^{p+\alpha+l-1} \quad \text{on } (0, T_{max}).$$

This estimate is essentially the same than that derived in [27, §4] so that, as therein, in order to take advantage from a combination of the Gagliardo–Nirenberg and Young's inequalities, one has to impose $\alpha - 1 + l < \frac{2}{n}$ (coinciding, exactly as discussed in Sect. 2.2, with the parabolic-elliptic version of assumption (5)); consequently, the integral $\frac{KK_0}{p+\alpha-1} \int_\Omega (u+1)^{p+\alpha+l-1}$ can be suitably treated. Standard procedures, successively, provide that $u \in L^\infty((0, T_{max}); L^p(\Omega))$ for arbitrarily large $p > 1$, and hence also $g \in L^\infty((0, T_{max}); L^p(\Omega))$ for any $l \in (0, 1)$. Finally, elliptic regularity theory applied to the second equation in problem (41) infers uniform bound of ∇v, on $(0, T_{max})$, so that u and v are uniformly bounded for all $t > 0$.

We note that the necessary regularity of ∇v is gained only by differentiating $\int_\Omega (u+1)^p$, by using the initial-boundary value problem (41) and, solely, the mass conservation property; neither an estimate like that in (8) is a priory needed nor the analysis of the term $\int_\Omega |\nabla v|^{2q}$, involving the extra parameter q, has to be developed.

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