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Keller-Segel Chemotaxis Models: A Review

Gurusamy Arumugam[1] · **Jagmohan Tyagi**[1]

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Abstract We recount and discuss some of the most important methods and blow-up criteria for analyzing solutions of Keller-Segel chemotaxis models. First, we discuss the results concerning the global existence, boundedness and blow-up of solutions to parabolic-elliptic type models. Thereafter we describe the global existence, boundedness and blow-up of solutions to parabolic-parabolic models. The numerical analysis of these models is still at a rather early stage only. We recollect quite a few of the known results on numerical methods also and direct the attention to a number of open problems in this domain.

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1 Introduction

Response to the environmental changes is an essential and basic property of the living cells. Through evolution, both unicellular and multicellular organisms develop various mechanisms that help them to regulate their cellular function in response to environmental changes. In general, whole organisms or cells cannot move by random manner, but they sense their

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✉ G. Arumugam
guru.poy@gmail.com

J. Tyagi
jtyagi@iitgn.ac.in

1 Discipline of Mathematics, Indian Institute of Technology Gandhinagar, Gandhinagar-382355, Gujarat, India

environment and respond to it. Movement of the cells is mainly decided by some external stimulants / signals, which determine the direction and distance of the cell movement. In the context of individual cells, migration can be regulated by various environmental factors such as light, chemicals, temperature, electric field, environmental gravity and many more. Of these, chemotaxis is an important sensory phenomenon of the cells by which cells translate chemical signals into motile behavior. Prokaryotic response to chemotaxis have been studied extensively in Escherichia coli, where a quite simple signaling cascade supports the clockwise or anticlockwise rotation of flagella to produce either forward motion or headlong, respectively. In case of eukaryotic organisms, chemotaxis mechanisms have been studied widely in amoeboid Dictyostelium discoideum and mammalian neutrophils in which chemoattractants induce intricate signaling cascades contributing to diverse cellular processes including the establishment of cellular polarity and extension of the cell membrane. Chemotaxis regulates various biological processes such as sperm motivation during fertilization, wound healing, tissue morphogenesis, axon guidance, immune reaction, neuron migration and lymphocyte migration during later phases of development.

Chemotaxis has attracted the significant interest due to its critical role in a wide range of biological phenomena (see [1, 22, 45]). Famous examples of biological species experiencing chemotaxis are the flagellated bacteria Salmonella typhimurium and Escherichia Coli, the slime mold amoebae Dictyostelium discoideum and the human endothelial cells etc. The mathematical model of chemotaxis has provided a cornerstone for much of this work, its success being a consequence of its intuitive simplicity, analytical tractability and capacity to replicate key behaviour of chemotactic populations. Among all properties of chemotaxis behaviour, the important one is the ability to display cell aggregation, which has led to its importance as a mechanism for self-organization of biological systems. This phenomenon has shown to lead to finite time blow-up under certain formulations of the model and a wide spectrum of work has been devoted to determine succinctnessing when blow-up occurs or whether global solutions exist. To illustrate the breadth of this field, we describe some of areas that have benefited from the use of Keller-Segel (KS) models, apologizing to those whose works have been omitted for conciseness.

In general, organism or cell moves from a lower concentration towards a higher concentration of the chemo attractant, which is known as positive chemotaxis. In the same way, the opposite movement of the organisms is known as negative chemotaxis. In particular, microorganisms use chemotaxis to position themselves within the optimal portion of their habitats by monitoring the environmental concentration gradients of specific chemical attractant (positive chemotaxis) and repellent ligands (negative chemotaxis). Let us consider a positive chemotactic response of unicellular organisms such as bacteria. Let us first assume that a chemical gradient has been produced externally and the bacteria are merely responding to it. Organic or inorganic substances which cause bacterial motility by inducing chemotaxis are called chemo attractants or chemo repellents. Bacterial motion involves a sequence of runs and tumbles. At the run stage, the bacterium moves a certain distance in a combination of motion in certain direction with random diffusion. At the tumble stage, the bacterium reorients itself to get ready for a fresh run. Rather than actively changing their direction of motion in response to the stimulus, bacteria seems to turn more frequently when the chemical concentration is low. The easiest model is that the average velocity with which the bacteria responds to the gradient is proportional to the gradient. Therefore the flux should be proportional to the product of the gradient and the density of bacteria. Let u and v be denote the bacteria density and the chemical concentration, respectively. Now, we have

$$J_{\text{chemo}} = \chi u \nabla v,$$

where χ is a proportionality constant and is also called the chemotactic constant or chemotactic sensitivity. The flux J_{chemo} is nothing but the chemotactic flux. This is the most widely used model for chemotactic flux. For negative chemotaxis, χ is negative. Since the chemical concentration v also diffuses, the resulting diffusion equation becomes

$$\partial_t v = \nabla \cdot (d_2 \nabla v) + g(u, v),$$

where $g(u, v) = \alpha u - \beta v$ and $d_2 > 0$. If the bacteria themselves produce the chemical then things become more complicated.

1.1 The Keller-Segel Model

The general form of Keller-Segel model for chemotaxis is

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla(\phi(u, v)\nabla u - \psi(u, v)\nabla v) + f(u, v), \\ \tau \dfrac{\partial v}{\partial t} = d\Delta v + g(u, v) - h(u, v)v, \end{cases} \tag{1.1}$$

where u represents the cell (or organism) density on a given domain $\Omega \subset \mathbb{R}^n$ and v denotes the concentration of the chemical signal. The motility function $\phi(u, v)$ describes the diffusivity of the cells and $\psi(u, v)$ represents the chemotactic sensitivity. Both ϕ and ψ may be functions of u and v. The function $f(u, v)$ describes cell growth and death while the functions $g(u, v)$ and $h(u, v)$ are kinetic functions that describe production and degradation of the chemical signal, respectively. The general form of chemotactic term is of the form $\psi(u, v, |\nabla v|)$ (for example, see [125]). The important properties of (1.1) are self aggregation phenomenon and spatial pattern formation. Theoretical and mathematical modeling of chemotaxis phenomenon dates back to the pioneering works of Patlak in the 1950s in [129] and Keller-Segel in 1970s [86–88]. For the application perspectives of chemotaxis phenomenon, cf. [128]. Since the model derivations have already been available in [70, 73, 119], so we omit the details here.

The Keller-Segel models have fascinated the applied researchers for few decades. In particular, it has attracted to applied mathematicians due to its qualitative behavior like blow-up, pattern formation, stabilization etc. This model mainly describes the aggregation of bacteria or cells induced by concentration of chemical and the aggregation is balanced by diffusion of cells. We interpret this aggregation or concentration of cells as blow-up in mathematics. Since the size of cells is finite so this blow-up is not very realistic from the biological point of view. Hence, avoiding blow-up in theory and numerics is a challenging problem. There are some possible ways to avoid blow-up such as bounded chemotaxis sensibilities, additional cross-diffusion term in the second equation of the classical model, degenerate cell diffusion, logistic sources. For instance, the reader may consult the recent survey [100] for available blow-up techniques and regularity issues for classical Keller-Segel model. In the present review, we carefully discuss quite a few Keller-Segel models from literature and its related issues.

1.2 Keller-Segel-Navier-Stokes Models

Understanding the chemotaxis phenomena in fluid environment is an interesting topic of research in mathematical biology. This is governed by Keller-Segel model together with

Navier-Stokes equation. The Keller-Segel-Navier-Stokes model is presented by the following model:

$$\begin{cases} \partial_t n + u\cdot\nabla n = d_1\Delta n - \nabla\cdot(n\chi(c)\nabla c) + f(u), \\ \partial_t c + u\cdot\nabla c = d_2\Delta c - nf(c), \\ \partial_t u + (u\cdot\nabla)u = \nu\Delta u - \nabla P - n\nabla\Phi, \\ \nabla\cdot u = 0, \end{cases} \tag{1.2}$$

where n, c, u and p represent the cell density, chemical concentration, velocity of the fluids, and pressure of the fluid, respectively. Here $\chi(c)$ denotes the chemotactic sensitivity, $f(c)$ represents the consumption rate of the chemical and Φ is the potential function. The positive constants d_1, d_2, and ν denote the diffusion coefficients of the cells, chemical and fluids, respectively.

Let there is no fluid in the above model, that is, $u = 0$ in the Keller-Segel-Navier-Stokes model (1.2), then it will reduce to Keller-Segel model (which is the centre of the other models). Similarly, there are a few models available such as chemotaxis-haptotaxis, attraction-repulsion models which are generalization of Keller-Segel models, cf. [127]. In order to present the available theory for these models, we need some basic results around Keller-Segel models. Using the same, we can extend the available results of Keller-Segel models to chemotaxis-haptotaxis, attraction-repulsion models, etc. Since the literature related to these models is vast, so in this review, we mainly focus on Keller-Segel models and its variants.

1.3 The Model Variations

A number of variations have been described based on additional biological realism. We record some of the available variations of Keller-Segel chemotaxis system and precisely mention the existing results in the literature. Depending on the properties of the chemoattractant, the system can be simplified to a parabolic-elliptic system (for chemoattractant with 'fast' diffusion) or to a parabolic-ODE system (for non-diffusive chemoattractants or with slow diffusion). We list a few models, below, which have been discussed by many researchers. The minimal model ($\gamma = 1, \alpha = 1$) [70]:

$$\begin{cases} \partial_t u = d\Delta u - \nabla\cdot(\chi u\nabla v), \\ \partial_t v = \Delta v + u - \gamma v. \end{cases} \tag{1.3}$$

The classical Keller-Segel model [199]:

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot(u\nabla v), \\ \partial_t v = \Delta v + u - v. \end{cases} \tag{1.4}$$

The non-local model [70]:

$$\begin{cases} \partial_t u = \nabla\cdot(d\nabla u - \chi u\mathring{\nabla}_\rho v), \\ \partial_t v = \Delta v + u - v, \end{cases} \tag{1.5}$$

the non-local gradient $\mathring{\nabla}_\rho v$ is defined as follows

$$\mathring{\nabla}_\rho v(x,t) = \frac{n}{\omega\rho}\int_{S^{n-1}} \sigma v(x+\rho\sigma, t)\mathrm{d}\sigma, \tag{1.6}$$

where $\omega = |S^{n-1}|$ and S^{n-1} denotes the $(n-1)$ dimensional unit sphere in $\mathbb{R}^n$ and is chosen such that the monomial model follows for $\rho \to 0$. Keller-Segel model like a drift-diffusion type [124]:

$$\begin{cases} \partial_t u = \operatorname{div}(\nabla u^m - u \cdot \nabla \varphi), \\ -\Delta \varphi = u - < u >, \end{cases} \tag{1.7}$$

where $m \geqslant 1$. The nonlinear-diffusion model [70]:

$$\begin{cases} \partial_t u = \nabla \cdot (du^n \nabla u - \chi u \nabla v), \\ \partial_t v = \Delta v + u - v, \end{cases} \tag{1.8}$$

where the minimal model corresponds to the limit of $n \to 0$. The Keller-Segel model in [132]:

$$\begin{cases} \partial_t u = \nabla \cdot (\phi(v) \nabla u) - \nabla(u \chi(v) \nabla v), \\ \partial_t v = D \Delta v + u f(v) - k(v) v. \end{cases} \tag{1.9}$$

The nonlinear gradient model [70]:

$$\begin{cases} \partial_t u = \nabla \cdot (d \nabla u - \chi u \mathbf{F_c}(\nabla v)), \\ \partial_t v = \Delta v + u - v, \end{cases} \tag{1.10}$$

where the vector valued function $\mathbf{F_c} : \mathbb{R}^n \to \mathbb{R}^n$ is defined as follows

$$\mathbf{F_c}(\nabla v) = \frac{1}{c} \left(\tanh\left(\frac{c v_{x_1}}{1+c} \right), \ldots, \tanh\left(\frac{c v_{x_n}}{1+c} \right) \right).$$

Signal-dependent sensitivity models [70]:

Here are the two versions of signal-dependent sensitivity; the receptor model

$$\begin{cases} \partial_t u = \nabla \cdot (d \nabla u - \dfrac{\chi u}{(1+\alpha v)^2} \nabla v), \\ \partial_t v = \Delta v + u - v, \end{cases} \tag{1.11}$$

where for $\alpha \to 0$, the minimal model is obtained and the logistic model:

$$\begin{cases} \partial_t u = \nabla \cdot \left(d \nabla u - \chi u \dfrac{1+\beta}{v+\beta} \nabla v \right), \\ \partial_t v = \Delta v + u - v, \end{cases} \tag{1.12}$$

when $\beta \to \infty$, the minimal model follows and for $\beta \to 0$, we obtain the classical form of $\chi(v) = 1/v$.

Density dependent sensitivity models [70]:

There are two models with density dependent sensitivity, which are described as follows:

$$\begin{cases} \partial_t u = \nabla \cdot \left(d \nabla u - \chi u \left(1 - \dfrac{u}{\gamma} \right) \nabla v \right), \\ \partial_t v = \Delta v + u - v, \end{cases} \tag{1.13}$$

where the limit of $\gamma \to \infty$ leads to the minimal model and the volume-filling model is

$$\begin{cases} \partial_t u = \nabla \cdot \left(d \nabla u - \chi \dfrac{u}{1+\epsilon u} \nabla v \right), \\ \partial_t v = \Delta v + u - v, \end{cases} \tag{1.14}$$

where $\epsilon \to 0$ leads to the minimal model. The derivation of Keller-Segel models and an extensive literature survey on the above mentioned variations of Keller-Segel models can be found in [10, 70, 73]. Here, we choose some variants of Keller-Segel model below, to review already existing recent results on existence, uniqueness, blow-up, boundedness of solutions and its numerical analysis. Singular sensitivity model [209]:

$$\begin{cases} \partial_t u = \Delta u^m - \chi \nabla \cdot \left(\frac{u}{v} \nabla v \right), \\ \partial_t v = \Delta v - uv. \end{cases} \tag{1.15}$$

Keller-Segel model with signal absorption and logistic growth term (when $k = 2$ [97] and when $k > 1$ [216])

$$\begin{cases} \partial_t u = \Delta u - \chi \nabla \cdot \left(\frac{u}{v} \nabla v \right) + ru - \mu u^k, \\ \partial_t v = \Delta v - uv. \end{cases} \tag{1.16}$$

Quasilinear Keller-Segel model [67]:

$$\begin{cases} \partial_t u = \nabla \cdot (\nabla u^m - u^{q-1} \nabla v), \\ \tau \partial_t v = \Delta v - v + u. \end{cases} \tag{1.17}$$

The following model [155] describes a typical chemotaxis process:

$$\begin{cases} \partial_t u = \nabla(\phi(u) \nabla u) - \nabla \cdot (\psi(u) \nabla v), \\ \partial_t v = \Delta v - v + u. \end{cases} \tag{1.18}$$

The Keller-Segel model with special case [11]:

$$\begin{cases} \partial_t u = \nabla \cdot (u^\alpha \nabla u) - \nabla \cdot (u^{1+\alpha} \nabla v) + f(u, v), \\ \partial_t v = v^\beta \Delta v - v^{\beta+1} + uv^\beta. \end{cases} \tag{1.19}$$

Keller-Segel model [114]:

$$\begin{cases} \partial_t u = \nabla \cdot (\nabla u - u^m \nabla v), \\ \Gamma \partial_t v = \Delta v - \lambda v + u. \end{cases} \tag{1.20}$$

The macroscopic model for cell populations [59]:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \chi_0(v) \nabla v) + f(u), \\ \tau \partial_t v = \Delta v - v + u. \end{cases} \tag{1.21}$$

The following model with $m > 1$, $\alpha, \gamma, \chi > 0$ and $N \geqslant 1$ [150]:

$$\begin{cases} \partial_t u = \nabla \cdot (\nabla u^m - \chi u \nabla v), \\ 0 = \Delta v - \gamma v + \alpha u. \end{cases} \tag{1.22}$$

Keller-Segel model [39]:

$$\begin{cases} \partial_t u = \nabla \cdot (\phi(u) \nabla u) - \nabla \cdot (u \nabla v), \\ 0 = \Delta v + u - M. \end{cases} \tag{1.23}$$

Keller-Segel model [21]:

$$\begin{cases} \partial_t u = \operatorname{div}(\nabla u^m - u\nabla v), \\ -\Delta v = u. \end{cases} \tag{1.24}$$

The Keller-Segel model with non-diffusive chemical [153]:

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot\left(u\frac{\nabla v}{v}\right), \\ \partial_t v = uv^{\lambda}. \end{cases} \tag{1.25}$$

The Keller-Segel model with rotation [166]:

$$\begin{cases} \partial_t u = \nabla\cdot(\phi(u)\nabla u - uS(u, v, x)\nabla v), \\ \partial_t v = \Delta v - uf(v). \end{cases} \tag{1.26}$$

The Keller-Segel model with nonlinear diffusion [210]:

$$\begin{cases} \partial_t u - \Delta\phi(u) + \nabla\cdot(\psi(u, v)u\nabla v) = f(u, v), \\ -\Delta v = u - v. \end{cases} \tag{1.27}$$

The Keller-Segel model with nonlinear diffusion [136]:

$$\begin{cases} \partial_t u - (u_x - (\phi(v))_x u)_x = 0, \\ -v_{xx} + v = u. \end{cases} \tag{1.28}$$

Keller-Segel model with nonlinear secretion [217]:

$$\begin{cases} \partial_t u = \nabla\cdot(\phi(u)\nabla u) - \nabla\cdot(\chi u\nabla v) + au - bu^r, \\ 0 = \Delta v + u^k - v. \end{cases} \tag{1.29}$$

Keller-Segel model with nonlinear diffusion and secretion [61]:

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot(\chi u^m\nabla v) + \mu u(1 - u^{\alpha}), \\ 0 = \Delta v + u^{\gamma} - v. \end{cases} \tag{1.30}$$

Simplest form of Keller-Segel model [19]:

$$\begin{cases} \partial_t u = \Delta u - \chi\nabla\cdot(u\nabla v), \\ 0 = \Delta v + u. \end{cases} \tag{1.31}$$

Keller-Segel model in [144]:

$$\begin{cases} \partial_t u = \nabla\cdot(\phi(u)\nabla u) - \nabla(\psi(u)\nabla v), \\ \partial_t v = d\Delta v + \alpha u - \beta v. \end{cases} \tag{1.32}$$

p-Laplacian Keller-Segel model [42]:

$$\begin{cases} \partial_t u = \nabla\cdot(|\nabla u|^{p-2}\nabla u) - \nabla\cdot(u\nabla v), \\ -\Delta v = u. \end{cases} \tag{1.33}$$

Keller-Segel model with logistic source [139]:

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot(\chi u\nabla v) + u(a - bu), \\ 0 = \Delta v + u - v. \end{cases} \tag{1.34}$$

Keller-Segel model with Fokker-Planck diffusion [211]:

$$\begin{cases} \partial_t u = \Delta(\gamma(v)u), \\ \partial_t v = \epsilon \Delta v + bu - av. \end{cases} \tag{1.35}$$

Keller-Segel model with consumption of chemoattractant [82]:

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot (u\chi(v)\nabla v), \\ \partial_t v = \Delta v - uf(v). \end{cases} \tag{1.36}$$

Keller-Segel model with signal-dependent motility [167]; Model 1:

$$\begin{cases} \partial_t u = \Delta(\gamma(v)u) + au - \mu u^2, \\ \partial_t v = \Delta v - v + u. \end{cases} \tag{1.37}$$

Keller-Segel model with density-suppressed motility model [83]; Model 2:

$$\begin{cases} \partial_t u = \Delta(\gamma(v)u) + a(u - u^2), \\ \partial_t v = \Delta v - v + u. \end{cases} \tag{1.38}$$

Keller-Segel model with flux limitation [8]:

$$\begin{cases} \partial_t u = \nabla\cdot \left(\dfrac{u\nabla u}{\sqrt{u^2 + |\nabla u|^2}} \right) - \chi \nabla\cdot \left(\dfrac{u\nabla v}{\sqrt{1 + |\nabla v|^2}} \right), \\ 0 = \Delta v - \mu + u. \end{cases} \tag{1.39}$$

Quasilinear Keller-Segel model with flux limitation [37]:

$$\begin{cases} \partial_t u = \nabla\cdot \left(\dfrac{u^p\nabla u}{\sqrt{u^2 + |\nabla u|^2}} \right) - \chi \nabla\cdot \left(\dfrac{u^q\nabla v}{\sqrt{1 + |\nabla v|^2}} \right), \\ 0 = \Delta v - \mu + u. \end{cases} \tag{1.40}$$

Keller-Segel model with flux limitation [37]:

$$\begin{cases} \partial_t u = \nabla\cdot \left(\dfrac{\phi(u,v)u\nabla u}{\sqrt{u^2 + |\nabla u|^2}} \right) - \chi \nabla\cdot \left(\dfrac{\psi(u,v)u\nabla v}{\sqrt{1 + |\nabla v|^2}} \right) + f(u,v), \\ \partial_t v = d\Delta v + g(u,v). \end{cases} \tag{1.41}$$

Quasilinear Keller-Segel model with logistic source [109]:

$$\begin{cases} \partial_t u = \nabla\cdot (\phi(u)\nabla u) - \chi \nabla\cdot (u\nabla v) + f(u), \\ 0 = \Delta v - \mu(t) + u. \end{cases} \tag{1.42}$$

Keller-Segel model [20]:

$$\begin{cases} \partial_t u = \Delta u - \chi \nabla\cdot (u\nabla v), \\ v = -\dfrac{1}{d\pi} \displaystyle\int_{\mathbb{R}^d} \log|x - y| u \mathrm{d}y. \end{cases} \tag{1.43}$$

Keller-Segel model with additional cross-diffusion [71]:

$$\begin{cases} \partial_t u = \operatorname{div}(\nabla u - u\nabla v), \\ \tau \partial_t v = \Delta v + \delta \Delta u + \mu u - v, \end{cases} \tag{1.44}$$

where $\delta > 0$. Keller-Segel model with additional cross-diffusion [6]:

$$\begin{cases} \partial_t u = \nabla \cdot (d_1 \nabla u - \phi(u, v) \nabla v) + h(u, v), \\ \partial_t v = d_2 \Delta v + \delta \Delta u + g(u, v), \end{cases} \tag{1.45}$$

where $\delta > 0$. Keller-Segel model with additional cross-diffusion [66]:

$$\begin{cases} \partial_t u - \nabla \cdot (d_1 \nabla u - \chi \phi(u, v) \nabla v) = f_u, \\ \partial_t v - d_2 \Delta v - \mu \Delta u - \alpha u + \beta v = f_v. \end{cases} \tag{1.46}$$

Keller-Segel model with additional cross-diffusion [30]:

$$\begin{cases} \partial_t u = \operatorname{div}(\nabla (u^m) - u \nabla v), \\ \alpha \partial_t v = \Delta v + \delta \Delta (u^m) + u - v, \end{cases} \tag{1.47}$$

where $\delta > 0$. The full Keller-Segel model [76]:

$$\begin{cases} \partial_t u - \operatorname{div}(\kappa(u, v) \nabla u) = \operatorname{div}(\sigma(u, v) \nabla v) \\ \partial_t v - k_v \Delta v = -r_1 v p + r_{-1} w + u f(v), \\ \partial_t p - k_p \Delta p = -r_1 v p + (r_{-1} + r_2) w + u g(v, p), \\ \partial_t w - k_w \Delta w = r_1 v p - (r_{-1} + r_2) w. \end{cases} \tag{1.48}$$

Keller-Segel model with singular sensitivity [95]:

$$\begin{cases} \partial_t u = \Delta u - \chi \nabla \cdot \left(\dfrac{u}{v} \nabla v \right), \\ \partial_t v = \Delta v + u - v. \end{cases} \tag{1.49}$$

Keller-Segel model with signal-dependent sensitivity [60]:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \chi(v) \nabla v), \\ 0 = \Delta v - v + u. \end{cases} \tag{1.50}$$

Quasilinear Keller-Segel model with logistic source:

$$\begin{cases} \partial_t u = \nabla \cdot ((1+u)^{m-1} \nabla u) - \chi \nabla \cdot (u(1+u)^{q-1} \nabla v) + au - bu^r, \\ 0 = \Delta v - v + u, \end{cases} \tag{1.51}$$

where $m \geqslant 1, r > 1, a \geqslant 0, b, a, \chi > 0$.
Keller-Segel model with logistic source [169]:

$$\begin{cases} \partial_t u = \nabla \cdot (\phi(u) \nabla u) - \chi \nabla \cdot (\psi(u) \nabla v) + f(u), \\ \partial_t v = \Delta v - v + u. \end{cases} \tag{1.52}$$

Keller-Segel model with logistic source [104]:

$$\begin{cases} \partial_t u = \nabla \cdot (\phi(u) \nabla u) - \nabla \cdot (u \nabla v), \\ 0 = \Delta v - \mu(t) + f(u), \mu(t) = \dfrac{1}{|\Omega|} \displaystyle\int_\Omega f(u(\cdot, t)). \end{cases} \tag{1.53}$$

Keller-Segel model with logistic source [161]:

$$\begin{cases} \partial_t u = \nabla \cdot (\phi(u) \nabla u) - \nabla \cdot (\psi(u) \nabla v) + f(u), \\ \partial_t v = \Delta v - v + g(u). \end{cases} \tag{1.54}$$

Keller-Segel model [200]:

$$\begin{cases} \partial_t u = \nabla\cdot(\phi(u)\nabla u) - \nabla\cdot(\psi(u)\nabla v), \\ 0 = \Delta v - M + u. \end{cases} \tag{1.55}$$

Keller-Segel model [219]:

$$\begin{cases} \partial_t u = \Delta u - \chi\nabla\cdot(\psi(u)\nabla v) + f(u), \\ 0 = \Delta v - v + g(u). \end{cases} \tag{1.56}$$

Keller-Segel model [125]:

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot(\chi u|\nabla u|^{p-2}\nabla v), \\ 0 = \Delta v - M + u, \end{cases} \tag{1.57}$$

where χ is positive constant and $p > 1$.
Keller-Segel model with logistic source [205]:

$$\begin{cases} \partial_t u = \nabla\cdot(d_1\nabla u - \chi u\nabla v) + f(u), \\ \partial_t v = d_2\Delta v - \beta v + \alpha u. \end{cases} \tag{1.58}$$

Keller-Segel model [2]:

$$\begin{cases} \partial_t u = \Delta u - \chi\nabla\cdot(u\nabla \log v), \\ 0 = \eta\Delta v - \beta v + \alpha u. \end{cases} \tag{1.59}$$

Keller-Segel model [188]:

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot(u\nabla v) + \lambda u - \mu u^{\kappa}, \\ 0 = \Delta v - \beta v + \alpha u. \end{cases} \tag{1.60}$$

Keller-Segel model with logistic source [102]:

$$\begin{cases} \partial_t u = \Delta u - \chi\nabla\cdot(u\nabla v) + f(u), \\ \tau v_t = \Delta v - v + u. \end{cases} \tag{1.61}$$

Keller-Segel model with nonlinear secretion [102]:

$$\begin{cases} \partial_t u = \Delta u - \chi\nabla\cdot(\psi(u)\nabla v) + f(u), \\ \tau v_t = \Delta v - v + g(u). \end{cases} \tag{1.62}$$

Keller-Segel model with signal-dependent [165]:

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot(u\psi(v)\nabla v), \\ 0 = \Delta v - v + g(u). \end{cases} \tag{1.63}$$

Keller-Segel system with an environmental dependent logistic source [54]:

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot(u\nabla v) + \kappa(|x|)u - \mu(|x|)u^p, \\ 0 = \Delta v - \dfrac{m(t)}{|\Omega|} + u, m(t) := \displaystyle\int_\Omega u(\cdot, t). \end{cases} \tag{1.64}$$

Keller-Segel system with logistic source [207]:

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot(u\nabla v) + f(u), \\ \tau\partial_t v = \Delta v - v + u. \end{cases} \tag{1.65}$$

Keller-Segel system [96]:

$$\begin{cases} \partial_t u = \nabla(\phi(u)\nabla u) - \nabla\cdot(\frac{u}{v}\nabla v), \\ \partial_t v = \Delta v - uv. \end{cases} \tag{1.66}$$

Keller-Segel system [159]:

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot(u\nabla v), \\ 0 = \Delta v - uv - \mu v + r(x,t). \end{cases} \tag{1.67}$$

Keller-Segel system [41]:

$$\begin{cases} \partial_t u = \nabla\cdot(\phi(u)\nabla u) - \nabla\cdot(\psi(u)\nabla v), \\ \partial_t v = \Delta v - v + u. \end{cases} \tag{1.68}$$

Keller-Segel system [201]:

$$\begin{cases} \partial_t u = d\Delta u - d\chi\nabla\cdot(\frac{u}{v}\nabla v), \\ \partial_t v = d\Delta v - v + u. \end{cases} \tag{1.69}$$

Keller-Segel system [84]:

$$\begin{cases} \partial_t u_\delta = \mathrm{div}(\nabla u_\delta - u_\delta\nabla v_\delta), \\ \epsilon\partial_t v_\delta = \Delta v_\delta - v_\delta + u_\delta. \end{cases} \tag{1.70}$$

Keller-Segel system [112]:

$$\begin{cases} \partial_t u = \nabla\cdot(\nabla u^m - \chi u\nabla v), \\ \epsilon\partial_t v = \Delta v - v + u. \end{cases} \tag{1.71}$$

Keller-Segel system [185]:

$$\begin{cases} \partial_t u = \Delta u - \chi\nabla\cdot(u\nabla v) + au - \mu u^2, \\ \partial_t v = \Delta v - v + u. \end{cases} \tag{1.72}$$

Keller-Segel system [17]:

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot(u\nabla v), \\ \partial_t v = \Delta v - v + u + f(x,t). \end{cases} \tag{1.73}$$

Keller-Segel system [190]:

$$\begin{cases} \partial_t u = \Delta u - \chi\nabla\cdot(\frac{u}{v}\nabla v) - uv + B_1(x,t), \\ \partial_t v = \Delta v + uv - v + B_2(x,t). \end{cases} \tag{1.74}$$

Keller-Segel system [101]:

$$\begin{cases} \partial_t u = \nabla\cdot(\nabla u^{q+1} - u\nabla v), \\ \partial_t v = \Delta v - \alpha v + u. \end{cases} \tag{1.75}$$

Keller-Segel system [193]:

$$\begin{cases} \partial_t u = \Delta - \nabla\cdot(u\nabla v), \\ 0 = \Delta v - \mu + u, \mu := \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u. \end{cases} \tag{1.76}$$

Keller-Segel system [55]:

$$\begin{cases} \partial_t u = \nabla\cdot(\phi(u,v)\nabla u - \phi(u,v)\nabla v), \\ \partial_t v = \Delta v - v + u. \end{cases} \tag{1.77}$$

Keller-Segel system [145]:

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot(u\nabla v), \\ 0 = \Delta v - \mu + u, \mu := \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u_0. \end{cases} \tag{1.78}$$

Keller-Segel system [183]:

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot(\dfrac{u}{v}\nabla v), \\ \partial_t v = \Delta v - uv. \end{cases} \tag{1.79}$$

Keller-Segel system [215]:

$$\begin{cases} \partial_t u + \Lambda^\alpha u + \nabla\cdot(u\nabla v) = 0, \\ \Delta v = u. \end{cases} \tag{1.80}$$

Keller-Segel system [171]:

$$\begin{cases} \partial_t u = \Delta u^m - \nabla\cdot(u\nabla v), \\ \partial_t v = \Delta v - v + u. \end{cases} \tag{1.81}$$

One Species and two stimuli:

One can found a particular example for chemotactic movement caused by two stimuli in [23]. In their experiment, oxygen and glucose are two attractants for the considered E. Coli population. In a very general formulation such an experimental observed movement of the E. Coli population can be described by the following class of coupled chemotaxis systems:

$$\begin{cases} \partial_t u = \nabla\cdot(d_1(u,v_1,v_2)\nabla u + \phi_1(u,v_1,v_2)\nabla v_1) + \nabla(\phi_2(u,v_1,v_2)\nabla v_2) \\ = f(u,v_1,v_2) \text{ in } Q_T, \\ \partial_t v_1 = \nabla\cdot(d_2(u,v_1,v_2)\nabla v_1) + h(u,v_1,v_2) \quad \text{in } Q_T, \\ \partial_t v_2 = \nabla\cdot(d_3(u,v_1,v_2)\nabla v_2) + g(u,v_1,v_2) \quad \text{in } Q_T, \\ \dfrac{\partial u}{\partial n} = 0, \dfrac{\partial v_1}{\partial n} = 0, \dfrac{\partial v_2}{\partial n} = 0 \text{ in } \Gamma, \\ u(x,0) = u_0(x), v_1(x,0) = v_{10}(x), v_2(x,0) = v_{20}(x) \quad \text{in } \Omega. \end{cases} \tag{1.82}$$

The following special cases for the values of d_1, ϕ_2, d_2, d_3, f, h, g which have been considered and presented in [23] are, given below:

$$d_1(u, v_1, v_2) = 1,\ \phi_1(u, v_1, v_2) = -u\frac{\chi_1^2 v_1^2}{1+\chi_1^2 v_1^2},\ \phi_2(u, v_1, v_2) = -u\frac{\chi_2^2 v_2^2}{1+\chi_2^2 v_2^2},$$

$$d_2(u, v_1, v_2) = a_1,\ d_3(u, v_1, v_2) = a_2,$$

$$h(u, v_1, v_2) = -\gamma_1\frac{\alpha_1 s_1}{1+\alpha_1 s_1}(1+\beta_2^4 s_2^4)^{-1} - \gamma_1\delta_1\frac{\alpha_2 s_1}{1+\alpha_2 s_1}\frac{\alpha_3 s_2}{1+\alpha_3 s_2},$$

and

$$g(u, v_1, v_2) = -\gamma_2\frac{\alpha_2 s_3}{1+\alpha_3 s_2}(1+\beta_1^4 s_1^4)^{-1} - \gamma_2\delta_2\frac{\alpha_4 s_2}{1+\alpha_4 s_2}\frac{\alpha_1 s_1}{1+\alpha_1 s_1}.$$

It should be noted that they formulated the above system in several dimensions but they provided some numerical simulations in one dimension for the experimental observed formation of bacterial bands.

Keller-Segel with Lotka-Volterra competitive source [63]:

$$\begin{cases} \partial_t u_1 = d_1\Delta u_1 - \chi_1\nabla(u_1\nabla v) + \epsilon_1 u_1(1-u_1-a_1u_2) \text{ in } Q_T, \\ \partial_t u_2 = d_2\Delta u_2 - \chi_2\nabla(u_2\nabla v) + \epsilon_2 u_2(1-a_2u_1-u_2) \text{ in } Q_T, \\ \tau\partial_t v = d_3\Delta v + u_1 + u_2 - \epsilon_3 v \quad \text{in } Q_T, \\ \dfrac{\partial u_1}{\partial n} = 0,\ \dfrac{\partial u_2}{\partial n} = 0,\ \dfrac{\partial v}{\partial n} = 0 \text{ in } \Gamma, \\ u(x,0) = u_{10}(x), u_2(x,0) = u_{20}(x), v(x,0) = v_0(x) \quad \text{in } \Omega. \end{cases} \tag{1.83}$$

Keller-Segel with Lotka-Volterra competitive source [126]:

$$\begin{cases} \partial_t u_1 = d_1\Delta u_1 - \chi_1\nabla(u_1\cdot\nabla v_1) + g_1(u_1, u_2), \\ \partial_t u_2 = d_2\Delta u_2 - \chi_2\nabla(u_2\cdot\nabla v_2) + g_2(u_1, u_2), \\ -d_3\Delta v_1 + \alpha_1 v_1 = \beta_1 u_2, \\ -d_4\Delta v_2 + \alpha_2 v_2 = \beta_2 u_1. \end{cases} \tag{1.84}$$

Keller-Segel system coupled to Navier-Stokes equation [197]:

$$\begin{cases} \partial_t n + u\cdot\nabla n = \Delta n + \nabla\cdot(n\nabla c), \\ \partial_t c + u\cdot\nabla c = \Delta c - c + n, \\ \partial_t u + u\cdot\nabla u = \Delta u + \nabla P + n\nabla\Phi, \nabla\cdot u = 0. \end{cases} \tag{1.85}$$

Keller-Segel system coupled to Navier-Stokes equation [198]:

$$\begin{cases} \partial_t n + u\cdot\nabla n = \Delta n + \nabla\cdot(n\nabla c), \\ u\cdot\nabla c = \Delta c - c + n, \\ \partial_t u + u\cdot\nabla u = \Delta u + \nabla P + n\nabla\Phi, \nabla\cdot u = 0. \end{cases} \tag{1.86}$$

Keller-Segel system coupled to Navier-Stokes equation:

$$\begin{cases} \partial_t n + u\cdot\nabla n = \nabla\cdot(D(n)\nabla n) - \nabla\cdot(nS(n)\nabla c),, \\ \partial_t c + u\cdot\nabla c = \Delta c - c + n, \\ \partial_t u + \kappa(u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\Phi, \nabla\cdot u = 0. \end{cases} \tag{1.87}$$

Keller-Segel system coupled to Navier-Stokes equation [156]:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) + rn - \mu n^2, \\ \partial_t c + u \cdot \nabla c = \Delta c - c + n, \\ \partial_t u = \Delta u - \nabla P + n \nabla \Phi + g(x,t), \nabla \cdot u = 0. \end{cases} \tag{1.88}$$

Keller-Segel system coupled to Navier-Stokes equation [85]:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) - \mu n^2, \\ \partial_t c + u \cdot \nabla c = \Delta c - c + n, \\ \partial_t u + \kappa (u \cdot \nabla) u = \Delta u - \nabla P - n \nabla \Phi + g(x,t), \nabla \cdot u = 0. \end{cases} \tag{1.89}$$

Keller-Segel system coupled to Navier-Stokes equation [196]:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \chi \nabla \cdot (n \nabla c) + \rho n - \mu n^2, \\ \partial_t c + u \cdot \nabla c = \Delta c - c + n, \\ \partial_t u + (u \cdot \nabla) u = \Delta u + \nabla P + n \nabla \Phi + g(x,t), \nabla \cdot u = 0. \end{cases} \tag{1.90}$$

Keller-Segel system coupled to Navier-Stokes equation [92]:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) - \nabla \cdot (n \nabla v), \\ \partial_t c + u \cdot \nabla c = \Delta c - cn, \\ \partial_t u + (u \cdot \nabla) u = \Delta u - \nabla \pi - nf, \\ \partial_t v + (u \cdot \nabla) v = \Delta v - \gamma v + n, \nabla \cdot u = 0. \end{cases} \tag{1.91}$$

Keller-Segel system coupled to Navier-Stokes equation [184]:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (n \chi(c) \nabla c), \\ \partial_t c + u \cdot \nabla c = \Delta c - nf(c), \\ \partial_t u + (u \cdot \nabla) u = \Delta u + \nabla P + n \nabla \Phi, \nabla \cdot u = 0. \end{cases} \tag{1.92}$$

Keller-Segel system coupled to Navier-Stokes equation [18]:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \chi \nabla \cdot (\frac{n}{c} \nabla c), \\ \partial_t c + u \cdot \nabla c = \Delta c - c + n, \\ \partial_t u = \Delta u + \nabla P + n \nabla \Phi, \nabla \cdot u = 0. \end{cases} \tag{1.93}$$

Keller-Segel system coupled to Navier-Stokes equation [187]:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (n S(n) \nabla c), \\ \partial_t c + u \cdot \nabla c = \Delta c - c + n, \\ \partial_t u + \nabla P = \Delta u + n \nabla \Phi + f(x,t), \nabla \cdot u = 0. \end{cases} \tag{1.94}$$

Keller-Segel system coupled to Navier-Stokes equation [172]:

$$\begin{cases} \partial_t n_\epsilon + u_\epsilon \cdot \nabla n_\epsilon = \Delta n_\epsilon - \nabla \cdot (n_\epsilon S(x, n_\epsilon, c_\epsilon) \nabla c_\epsilon) + f(x, n_\epsilon, c_\epsilon), \\ \partial_t c_\epsilon + u_\epsilon \cdot \nabla c_\epsilon = \Delta c_\epsilon - c_\epsilon + n_\epsilon, \\ \partial_t u_\epsilon + \kappa (u_\epsilon \cdot \nabla u_\epsilon) - \nabla P_\epsilon = \Delta u_\epsilon + n_\epsilon \nabla \Phi, \nabla \cdot u_\epsilon = 0. \end{cases} \tag{1.95}$$

Keller-Segel system coupled to Navier-Stokes equation [189, 192]:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (n S(x,n,c) \nabla c), \\ \partial_t c + u \cdot \nabla c = \Delta c - nf(c), \\ \partial_t u - \nabla P = \Delta u + n \nabla \Phi, \nabla \cdot u = 0. \end{cases} \tag{1.96}$$

Keller-Segel system coupled to Navier-Stokes equation [170]:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (nS(x, n, c)\nabla c), \\ \partial_t c + u \cdot \nabla c = \Delta c - c + n, \\ \partial_t u + (u \cdot \nabla)u = \Delta u - \nabla P + n\nabla\Phi, \nabla \cdot u = 0. \end{cases} \tag{1.97}$$

Keller-Segel system coupled to Navier-Stokes equation [51]:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (n\nabla c) - nm, \\ \partial_t c + u \cdot \nabla c = \Delta c - c + m, \\ \partial_t m + u \cdot \nabla m = \Delta m - nm, \\ \partial_t u + (u \cdot \nabla)u = \Delta u - \nabla P + (n+m)\nabla\Phi, \nabla \cdot u = 0. \end{cases} \tag{1.98}$$

The following Table 1 and Table 2, show the simplified version of most of the models which are reviewed in this section.

1.4 Outline of the Article

We organize this paper as follows. Section 2 deals with definitions of solution to Keller-Segel models. In Sect. 3, we provide the mathematical approach for determining global existence and blow-up of solutions to parabolic-elliptic models. In Sect. 4, we discuss about the boundedness of solutions to parabolic-elliptic models. The global existence and blow-up of solutions to parabolic-parabolic models are discussed in Sect. 5, and boundedness results to parabolic-parabolic models discussed in Sect. 6. The results on numerical analysis of different types of Keller-Segel models are a part of Sect. 7. Finally, in Sect. 8 we discuss the recent findings and mention some of the related issues to attempt in future.

2 Basic Definitions

We recall basic definitions and solution spaces for a few models, which are listed in the previous section.

Definition 2.1 [121]. Let T_{max} be the maximal existence time of the solutions (u, v) of (1.3) with $d = 1$. A point $x_0 \in \overline{\Omega}$ is said to be a blow-up point of u if there exist $\{t_k\}_{k=1}^{\infty} \subset (0, T_{max})$ and $\{x_k\}_{k=1}^{\infty} \subset \overline{\Omega}$ satisfying

$$u(x_k, t_k) \to \infty, t_k \to T_{max}, x_k \to x_0 \text{ as } k \to \infty.$$

Definition 2.2 [72] We say that the solution of (1.3) blows up, provided there is a time $T_{max} \leqslant \infty$ such that

$$\limsup_{t \to T_{max}} \|u(x, t)\|_{L^\infty(\Omega)} = \infty \text{ or } \limsup_{t \to T_{max}} \|v(x, t)\|_{L^\infty(\Omega)} = \infty.$$

If $T_{max} < \infty$, we say that the solution of (1.3) blows up in finite time and if $T_{max} = \infty$, we will call it blow-up in infinite time.

Definition 2.3 (Type I and II blow-up) The blow-up is called type I if $\|u(t)\|_{L^\infty} \leqslant C(T - t)^{-1}$ for all $t \in [0, T)$ with some constant $C > 0$ and type II otherwise.

Now, we define the weak solutions of the Keller-Segel model (1.32).

Table 1 Summary of the listed models

Model	$\phi(u,v)$	$\psi(u,v)$	$f(u,v)$	$g(u,v)$	$h(u,v)$.
(1.3), $\tau=d=1$	d	χu	0	αu	γ
(1.5), $\tau=d=1$	d	χu	0	u	1
(1.9), $\tau=1, d=D$	$\phi(v)$	$u\chi(v)$	0	$uf(v)$	k(v)
(1.8), $\tau=d=1$	du^n	χu	0	u	1
(1.10), $\tau=d=1$	d	χu	0	u	1
(1.11), $\tau=d=1$	d	$\frac{\chi u}{(1+\alpha v)^2}$	0	u	1
(1.12), $\tau=d=1$	d	$\chi u\frac{1+\beta}{(v+\beta)}$	0	u	1
(1.13), $\tau=d=1$	d	$\chi u(1-\frac{u}{\gamma})$	0	u	1
(1.14), $\tau=d=1$	d	$\chi\frac{u}{1+\epsilon u}$	0	u	1
(1.18), $\tau=d=1$	$\phi(u)$	$\psi(u)$	0	u	1
(1.19), $\tau=d=v^\beta$	u^α	$u^{1+\alpha}$	$f(u,v)$	uv^β	v^β
(1.20), $\tau=\Gamma, d=1$	1	u^m	0	u	λ
(1.21), $d=1$	1	$u\chi(v_0)$	f(u)	u	1
(1.23), $\tau=0, d=1$	$\phi(u)$	u	0	u	Mv^{-1}
(1.9), $\tau=1, d=D$	$\phi(v)$	$u\chi(v)$	0	$uf(v)$	$k(v)$
(1.25), $\tau=1, d=0$	1	$\frac{u}{v}$	0	uv^λ	0
(1.26), $\tau=d=1$	$\phi(u)$	$uS(u,v,x)$	0	$-uf(v)$	0
(1.29), $\tau=0, d=1$	$\phi(u)$	χu	$au-bu^r$	u^k	1
(1.30), $\tau=0, d=1$	1	χu^m	$\mu u(1-u^\alpha)$	u^γ	1
(1.31), $\tau=0, d=1$	1	χu	0	u	0
(1.32), $\tau=1$	$\phi(u)$	$\psi(u)$	0	αu	β
(1.33), $\tau=0, d=1$	$\lvert\nabla u\rvert^{p-2}$	u	0	u	0
(1.34), $\tau=0, d=1$	1	χu	$u(a-bu)$	u	1
(1.36), $\tau=d=1$	1	$u\chi(v)$	0	$-uf(v)$	0
(1.36), $\tau=d=1$	1	$u\chi(v)$	0	$-uf(v)$	0
(1.39), $\tau=0, d=1$	$\frac{u}{\sqrt{u^2+\lvert\nabla u\rvert^2}}$	$\chi\frac{u}{\sqrt{1+\lvert\nabla v\rvert^2}}$	0	u	μv^{-1}
(1.42), $\tau=0, d=1$	$\phi(u)$	χu	$f(u)$	u	$\mu(t)v^{-1}$
(1.49), $\tau=d=1$	1	$\chi\frac{u}{v}$	0	u	1
(1.50), $\tau=0, d=1$	1	$u\chi(v)$	0	u	1
(1.51), $\tau=0, d=1$	$(1+u)^{m-1}$	$\chi u(1+u)^{q-1}$	$au-bu^r$	u	1
(1.52), $\tau=d=1$	$\phi(u)$	$\chi\psi(u)$	$f(u)$	$g(u)$	1
(1.54), $\tau=d=1$	$\phi(u)$	$\psi(u)$	$f(u)$	$g(u)$	1
(1.56), $\tau=0, d=1$	1	$\chi\psi(u)$	$f(u)$	$g(u)$	1
(1.58), $\tau=1, d=d_2$	d_1	χu	$f(u)$	αu	β
(1.60), $\tau=0, d=1$	1	u	$\lambda u-\mu u^\kappa$	αu	β

Definition 2.4 ([144], weak solution) A function (u,v) is a weak solution of the system (1.32) if the following conditions hold

$$u\in C([0,T];L^2(\Omega))\cap L^\infty(Q_T)\cap L^2(0,T;H_0^1(\Omega)),\phi(u)\in L^2(0,T;H_0^1(\Omega))$$
$$v\in C([0,T];L^2(\Omega))\cap L^\infty(Q_T)\cap L^p(0,T;W^{2,p}(\Omega)),$$

Table 2 Summary of the listed models

Model	$\phi(u,v)$	$\psi(u,v)$	$f(u,v)$	$g(u,v)$	$h(u,v)$.
(1.64),$\tau=0, d=1$	1	u	$\kappa(\lvert x\rvert)u-\mu(\lvert x\rvert)u^p$	u	$\frac{m(t)}{\lvert\Omega\rvert v}$
(1.65)$d=1$	1	u	$f(u)$	u	1
(1.66)$\tau=d=1$	$\phi(u)$	$\frac{u}{v}$	0	0	u
(1.67)$\tau=0, d=1$	1	u	0	$r(x,t)$	$u+1$
(1.68)$\tau=d=1$	$\phi(u)$	$\psi(u)$	0	u	1
(1.69)	d	$\frac{d\chi u}{v}$	0	u	1
(1.70)$d=1$	1	u_δ	0	u	1
(1.72)$\tau=d=1$	1	χu	$au-\mu u^2$	u	1
(1.73)$\tau=1$	1	u	0	$u+f(x,t)$	1
(1.74)$\tau=d=1$	1	$\frac{\chi u}{v}$	$-uv+B_1(x,t)$	$uv+B_2(x,t)$	1
(1.76)$\tau=0, d=1$	1	u	0	u	$\frac{\mu}{v}$
(1.77)$\tau=d=1$	$\phi(u,v)$	$\psi(u,v)$	0	u	1
(1.78)$\tau=0, d=1$	1	u	0	u	$\frac{\mu}{v}$
(1.79)$\tau=d=1$	1	$\frac{u}{v}$	0	0	u

and for any $\varphi_1, \varphi_2 \in L^2(0,T; H_0^1(\Omega)) \cap C^1(Q_T)$, with $\varphi_1(\cdot,T)=\varphi_2(\cdot,T)=0$, we have

$$-\int_\Omega u_0(x)\varphi_1(x)\mathrm{d}x - \int_{Q_T} u\varphi_{1t}\mathrm{d}x\mathrm{d}t + \int_{Q_T}\phi(u)\nabla u\nabla\varphi_1\mathrm{d}x\mathrm{d}t - \int_{Q_T}\psi(u)\nabla v\nabla\varphi_1\mathrm{d}x\mathrm{d}t$$

$$-\int_\Omega v_0(x)\varphi_2(x)\mathrm{d}x - \int_{Q_T} v\varphi_{2t}\mathrm{d}x\mathrm{d}t + d\int_{Q_T}\nabla v\nabla\varphi_2\mathrm{d}x\mathrm{d}t = \int_{Q_T}(\alpha u-\beta v)\varphi_2\mathrm{d}x\mathrm{d}t.$$

Now, let us define a energy solution, maximal existence time and blow-up for (1.17) in a ball $\Omega = B_R := \{x \in \mathbb{R}^N \,\lvert\, \lvert x\rvert < R\}$ with $N \geqslant 2$, $R > 0$, $m \geqslant 1$, $q \geqslant 2$ and initial conditions $u_0 \geqslant 0$, $u_0 \in L^\infty(\Omega)$ with $\nabla u_0^m \in L^2(\Omega)$, $v_0 \geqslant 0$, $v_0 \in W^{1,\infty}(\Omega)$.

Definition 2.5 ([124]) Suppose $0 < T \leqslant \infty$. A pair (u,φ) of functions $u : \Omega \times [0,T) \longrightarrow [0,\infty)$, $\varphi : \Omega \times [0,T) \longrightarrow \mathbb{R}$ is a weak solution of (1.7) in $\Omega \times [0,T)$ if

(i) $u \in L^\infty((0,T); L^\infty(\Omega))$; $u^m \in L^2((0,T); H^1(\Omega))$ and $\langle u\rangle = M$, where

$$\langle u\rangle = \frac{1}{\lvert\Omega\rvert}\int_\Omega u(t,x)dx.$$

(ii) $\varphi \in L^2((0,T); H^1(\Omega))$ and $\langle\varphi\rangle = 0$.
(iii) (u,φ) satisfies the equation in the sense of distributions; that is,

$$-\int_0^T\int_\Omega(\nabla u^m\cdot\nabla\psi - u\nabla\varphi\cdot\nabla\psi - u\partial_t\psi)\mathrm{d}x\mathrm{d}t = \int_\Omega u_0(x)\psi(0,x)\mathrm{d}x,$$

$$\int_0^T\int_\Omega\nabla\varphi\cdot\nabla\psi\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega(u-M)\psi\mathrm{d}x\mathrm{d}t,$$

for any continuously differentiable function $\psi \in C^1([0,T]\times\overline{\Omega})$ with $\psi(T)=0$ and $T>0$.

Definition 2.6 (Energy solution). [67] Let $T\in(0,\infty]$. A pair (u,v) of nonnegative functions defined on $\Omega\times(0,T)$ is called an energy solution of (1.17) on $\Omega\times[0,T)$ if

(i) $u\in L^\infty(0,T;L^\infty(\Omega))$, $\nabla u^m\in L^\infty(0,T;L^2(\Omega))$, $(u^{\frac{m+1}{2}})_t\in L^2(0,t;L^2(\Omega))$ for all $t<T$,
(ii) $v\in L^\infty(0,T;W^{1,\infty}(\Omega))$, $v_t\in L^2(0,T;L^2(\Omega))$,
(iii) (u,v) satisfies (1.17) in the following sense: for all $\varphi\in L^1(0,T;H^1(\Omega))\cap W^{1,1}(0,T;L^2(\Omega))$ with compact support $\varphi(x)\subset[0,T)(a.a. x\in\Omega)$,

$$\int_0^T\int_\Omega(\nabla u^m\cdot\nabla\varphi-u^{q-1}\nabla v\cdot\nabla\varphi-u\varphi_t)\mathrm{d}x\mathrm{d}t=\int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x, \tag{2.1}$$

$$\int_0^T\int_\Omega(\nabla v\cdot\nabla\varphi+v\varphi-u\varphi-v\varphi_t)\mathrm{d}x\mathrm{d}t=\int_\Omega v_0(x)\varphi(x,0)\mathrm{d}x. \tag{2.2}$$

(iv) (u,v) satisfies the following energy estimate for a.a. $t\in(0,T)$:

$$\frac{2e^{-2t}}{(m+1)^2}\int_0^t\int_\Omega\left|\frac{\partial}{\partial s}u^{\frac{m+1}{2}}\right|^2\mathrm{d}x\mathrm{d}s+\frac{1}{2m}\int_\Omega|\nabla(u^m(t))|^2\mathrm{d}x\leqslant K,$$

where $K>0$ is a constant depending on $\|u_0\|_{L^2(\Omega)}$, $\|\nabla u_0^m\|_{L^2(\Omega)}$, $\|v_0\|_{W^{1,\infty}(\Omega)}$, $\|u\|_{L^\infty(0,T;L^\infty(\Omega))}, m, q, N, |\Omega|$.

The following definition is a weak power-λ solution of (1.49).

Definition 2.7 [146] Let $\lambda\in(0,1)$, $u_0\in L^\lambda(\Omega)$, $v_0\in L^\lambda(\Omega)$ and $T>0$. The pair of nonnegative functions (u,v) is said to be a weak power-λ solution of (1.49) if $u\in L^\lambda_{loc}(\overline{\Omega}\times[0,T))$, $v\in L^\lambda_{loc}(\overline{\Omega}\times[0,T))$ such that

$$(u+1)^{\lambda-2}|\nabla u|^2,\ (v+1)^{\lambda-2}|\nabla v|^2,\ u^\lambda v^{-2}|\nabla v|^2$$
$$\text{and } u(v+1)^{\lambda-1} \text{ belong to } L^1_{loc}(\overline{\Omega}\times[0,T)),$$

that satisfies the following identities

$$-\frac{1}{\lambda}\int_0^T\int_\Omega(u+1)^\lambda\varphi_t+(\lambda-1)\int_0^T\int_\Omega(u+1)^{\lambda-2}|\nabla u|^2\varphi+\int_0^T\int_\Omega(u+1)^{\lambda-1}\nabla u\cdot\nabla\varphi$$
$$-\chi(\lambda-1)\int_0^T\int_\Omega(u+1)^{\lambda-2}\frac{u}{v}\nabla u\cdot\nabla v\varphi$$
$$-\chi\int_0^T\int_\Omega(u+1)^{\lambda-1}\frac{u}{v}\nabla v\cdot\nabla\varphi=\frac{1}{\lambda}\int_\Omega(u_0+1)^\lambda\varphi(\cdot,0)$$

and

$$-\frac{1}{\lambda}\int_0^T\int_\Omega (v+1)^\lambda\varphi_t+(\lambda-1)\int_0^T\int_\Omega (v+1)^{\lambda-2}|\nabla v|^2\varphi+\int_0^T\int_\Omega (v+1)^{\lambda-1}\nabla v\cdot\nabla\varphi$$
$$+\int_0^T\int_\Omega (v+1)^{\lambda-1}v\varphi-\int_0^T\int_\Omega u(v+1)^{\lambda-1}\varphi=\frac{1}{\lambda}\int_\Omega (v_0+1)^\lambda\varphi(\cdot,0),$$

for all $\varphi\in C_0^\infty(\overline{\Omega}\times[0,T))$.

Definition 2.8 [99] A pair (u,v) of functions $u\in L^1_{loc}(\overline{\Omega}\times[0,T))$, $v\in L^1_{loc}([0,\infty);W^{1,1}(\Omega))$ is called a global weak solution of

$$\begin{cases}\partial_t v=\Delta v+u-v,\ x\in\Omega,t>0,\\ \frac{\partial v}{\partial\nu}=0,\ x\in\partial\Omega,t>0,\\ v(x,0)=v_0(x),\ x\in\Omega,\end{cases}\tag{2.3}$$

if

$$-\int_0^\infty\int_\Omega v\varphi_t-\int_\Omega v_0\varphi(\cdot,0)=-\int_0^\infty\int_\Omega\nabla v\cdot\nabla\varphi-\int_0^\infty\int_\Omega v\varphi+\int_0^\infty\int_\Omega u\varphi\tag{2.4}$$

for all $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$.

Definition 2.9 [162] A pair of functions (u,v) is said to be a nonnegative weak solution of (1.56) with $\phi(u)=1$, $\psi(u)=g(u)=u$, $k=1$ if

$$u\in L^1((0,T);W^{1,1}(\Omega)),v\in L^1((0,T);W^{1,1}(\Omega))$$

such that $u\nabla v\in L^1((0,T);L^1(\Omega))$, $f(u)\in L^1((0,T);L^1(\Omega))$ and the following identities hold

$$\int_0^T\int_\Omega u\varphi_t+\int_0^T\int_\Omega\nabla u\cdot\nabla\varphi-\chi\int_0^T\int_\Omega u\nabla v\cdot\nabla\varphi=\int_\Omega u_0\varphi(0)+\int_0^T\int_\Omega f(u)\varphi,\tag{2.5}$$

$$\int_0^T\int_\Omega\nabla v\cdot\nabla\psi+\int_0^T\int_\Omega v\psi=\int_0^T\int_\Omega u\psi,\tag{2.6}$$

for all $\varphi\in C_0^\infty(\overline{\Omega}\times[0,T))$ and $\psi\in C_0^\infty(\overline{\Omega}\times[0,T))$.

Next, we define the *renormalized* solutions to (1.15) with $m=1$.

Definition 2.10 [186] Let $n\geqslant 1$, $\Omega\subset\mathbb{R}^n$ is a bounded domain and that $u_0\in L^1(\Omega)$ and $v_0\in L^1(\Omega)$ be nonnegative. Then, a pair (u,v) of functions $u\in L^1_{loc}(\overline{\Omega}\times[0,\infty))$, $v_0\in L^\infty_{loc}(\overline{\Omega}\times[0,\infty))$, satisfying $u\geqslant 0$ and $v>0$ a.e. in $\Omega\times(0,\infty)$, is called a global renormalized solution of (1.15) with $m=1$ if

$$\begin{cases} \chi_{\{u<M\}} \nabla u \in L^2_{loc}(\overline{\Omega} \times [0,\infty)) \text{ for all } M>0 \quad \text{and} \\ \frac{\nabla v}{v} \in L^2_{loc}(\overline{\Omega} \times [0,\infty)), \end{cases} \tag{2.7}$$

and for all $\phi \in C^\infty([0,\infty))$ with $\phi' \in C_0^\infty([0,\infty))$, we have

$$-\int_0^\infty\int_\Omega \phi(u)\varphi_t - \int_\Omega \phi(u_0)\varphi(\cdot,0) = -\int_0^\infty\int_\Omega \phi''(u)|\nabla u|^2\varphi - \int_0^\infty\int_\Omega \phi'(u)\nabla u\cdot\nabla\varphi$$
$$+\int_0^\infty\int_\Omega u\phi''(u)\left(\nabla u\cdot\frac{\nabla v}{v}\right)\varphi + \int_0^\infty\int_\Omega u\phi'(u)\frac{\nabla v}{v}\cdot\nabla\varphi,$$

for all $\varphi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$ and if moreover the identity

$$\int_0^\infty\int_\Omega v\varphi_t + \int_\Omega v_0\varphi(\cdot,0) = \int_0^\infty\int_\Omega \nabla v\cdot\nabla\varphi + \int_0^\infty\int_\Omega uv\varphi,$$

holds for any $\varphi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$.

Definition 2.11 (Very weak subsolution [97]). A pair (u,v) of functions is called very weak subsolution to the model (1.16) with $k=2$ iff u and v are nonnegative and positive almost everywhere, respectively, and $u \in L^2_{loc}([0,\infty); L^2(\Omega))$, $v \in L^2_{loc}([0,\infty); W^{1,2}(\Omega))$ and $\nabla\log(v) \in L^2_{loc}(\Omega\times[0,\infty))$ hold and further

$$\int_0^\infty\int_\Omega \phi_t u - \int_\Omega u_0\phi(\cdot,0) \leqslant \int_0^\infty\int_\Omega u\Delta\phi + \chi\int_0^\infty\int_\Omega u\nabla\phi\cdot\nabla\log(v) + r\int_0^\infty\int_\Omega u\phi$$
$$-\mu\int_0^\infty\int_\Omega \phi u^2 \tag{2.8}$$

is satisfied for every nonnegative test function $\phi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$ with $\partial_n\phi = 0$ on $\partial\Omega\times(0,\infty)$ and

$$-\int_0^\infty\int_\Omega \psi_t v - \int_\Omega v_0\psi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla v\cdot\nabla\psi - \int_0^\infty\int_\Omega \psi uv \tag{2.9}$$

is fulfilled for every $\psi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$.

Definition 2.12 (Weak logarithmic supersolution [97]). A pair of functions (u,v) is called weak logarithmic supersolution of (1.16) with $k=2$ iff u is nonnegative and v is positive almost everywhere, $u \in L^1_{loc}([0,\infty); L^2(\Omega))$, $v \in L^\infty_{loc}(\Omega\times[0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega))$, $\nabla\log(u+1) \in L^2_{loc}(\Omega\times[0,\infty))$, $\nabla\log(v) \in L^2_{loc}(\Omega\times[0,\infty))$ and

$$-\int_0^\infty\int_\Omega \log(u+1)\phi_t - \int_\Omega \log(u_0+1)\phi(\cdot,0) \geqslant -\int_0^\infty\int_\Omega \nabla\log(u+1)\cdot\nabla\phi$$

$$+\int_0^\infty\int_\Omega \phi|\nabla \log(u+1)|^2+\chi\int_0^\infty\int_\Omega \frac{u}{u+1}\nabla \log(v)\cdot\nabla\phi$$

$$-\chi\int_0^\infty\int_\Omega \frac{u}{u+1}\phi\nabla\log(v)\cdot\nabla\log(u+1)+r\int_0^\infty\int_\Omega \frac{u}{u+1}\phi-\mu\int_0^\infty \Omega\frac{u^2}{u+1}\phi \quad (2.10)$$

is satisfied for every nonnegative test function $\phi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$ and (2.9) holds $\forall\ \psi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$.

Definition 2.13 (Generalized solution [97]). A pair (u, v) of functions is called generalized solution to (1.16) with $k=2$ iff (u, v) is a very weak subsolution and a weak logarithmic super solution to (1.16) with $k=2$.

Definition 2.14 (Mild solution [92]) Let $n \geqslant 2$ and $\{n_0, c_0, v_0, u_0, f\}$ be satisfy

(i) For $n \geqslant 3$ the initial data $\{n_0, c_0, v_0, u_0\}$ satisfies $n_0 \in L_w^{\frac{n}{2}}(\mathbb{R}^n)$, $c_0 \in L^\infty(\mathbb{R}^n)$ with $\nabla c_0 \in L_w^n(\mathbb{R}^n)$, $v_0 \in \mathcal{S}'$ with $\nabla v_0 \in L_w^n(\mathbb{R}^n)$, $u_0 \in PL_w^n(\mathbb{R}^n)$.
(ii) For $n=2$, we replace $n_0 \in L_w^1(\mathbb{R}^2)$ by $n_0 \in L^1(\mathbb{R}^2.)$ The external force f satisfies $f \in L_w^n(\mathbb{R}^n)$. A pair $\{n, c, u, v\}$ of measurable functions on $\mathbb{R}^n\times(0,\infty)$ is called a mild solution of (1.91) on $(0,\infty)$ if $n, c, u, v \in L_{loc}^q(0,\infty; L^r(\mathbb{R}^n))$ for some $1 \leqslant q, r \leqslant \infty$, and if the identities

$$n(t) = e^{t\Delta}n_0 - \int_0^t e^{(t-\tau)\Delta}(u\cdot\nabla n)(\tau)\mathrm{d}\tau - \int_0^t \nabla\cdot e^{(t-\tau)\Delta}(n\nabla c+n\nabla v)(\tau)\mathrm{d}\tau,$$

$$c(t) = e^{t\Delta}c_0 - \int_0^t e^{(t-\tau)\Delta}(u\cdot\nabla c+nc)(\tau)\mathrm{d}\tau,$$

$$v(t) = e^{-\gamma t}e^{t\Delta}v_0 - \int_0^t e^{-\gamma(t-\tau)\Delta}(u\cdot\nabla v-n)(\tau)\mathrm{d}\tau,$$

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}P(u\cdot\nabla u+nf)(\tau)\mathrm{d}\tau,$$

hold for $0<t<\infty$, where $e^{t\Delta}$ denotes the heat semi-group defined by $(e^{t\Delta}g)(x) = \int_{\mathbb{R}^n} G(x-y,t)g(y)\mathrm{d}y$ with $G(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$.

In the next section, we discuss about existence and blow-up of solutions of parabolic-elliptic models.

3 Existence and Blow-up of Solutions of Parabolic-Elliptic Type Models

The conjecture introduced by Childress and Percus [36] and Nanjundiah [123] as either the solution of the complete Keller-Segel system globally exists or it blows-up in finite time, the so called chemotactic collapse. This conjecture was validated in [19] for the simplest Keller-Segel model (1.31). Under the assumptions of the initial data $u_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\mathrm{d}x)$, $u_0 \log u_0 \in L^1(\mathbb{R}^2, \mathrm{d}x)$, the authors [19] proved the following two cases:

- solutions of (1.31) blow-up in finite time when $M > 8\pi/\chi$,
- there exits a global in time solution to (1.31) when $M < 8\pi/\chi$.

Their idea is based on the free energy defined by

$$F[u] := \int_{\mathbb{R}^2} u \log u \,\mathrm{d}x - \frac{\chi}{2}\int_{\mathbb{R}^2} uv \,\mathrm{d}x.$$

Based on the bounds of this free energy, they have classified global existence of solutions and decaying properties. The first term in the above free energy F is called the *entropy* and second is called *a potential energy*. The global existence and blow-up of solutions in two dimension to the minimal model (1.3) with $d = 1$, $\gamma = 1$, $\alpha = 1$ was established in [121]. Nagai has indicated the blow up behavior of solutions at isolated blow-up points and proved the global existence of solutions and decay properties of bounded solutions at $t \to \infty$. In order to establish his results, he assumed that the initial conditions $(u_0, v_0) \in L^1(\Omega) \cap L^\infty(\Omega)$.

He summarized the main results as follows

Theorem 3.1 *Let (u, v) be a finite-time blow-up nonnegative solution to (1.3) with $d = 1$, $\gamma = 1$, $\alpha = 1$. If $x_0 \in \overline{\Omega}$ is a isolated blow-up point of u, then there exists a constant m satisfying*

$$m \geqslant \begin{cases} 8\pi \ if \ x_0 \in \Omega, \\ 4\pi \ if \ x_0 \in \partial\Omega, \end{cases} \tag{3.1}$$

and a nonnegative function $f \in L^1(\Omega(x_0, \epsilon))$ for a positive number ϵ such that

$$\lim_{t \to T_{max}} u(\cdot, t) = m\delta_{x_0} + f \quad weak -^{*} \ star \ in \ \mathcal{M}(\Omega(x_0, \epsilon)),$$

where δ_{x_0} is the delta function at x_0 and $\mathcal{M}(\Omega(x_0, \epsilon))$ is the space of radon measure on $\Omega(x_0, \epsilon)$.

For the proof of Theorem 3.1, we refer to Theorem 1 [121].

Theorem 3.2 *If $\int_{\mathbb{R}^2} u_0 \mathrm{d}x < 4\pi$, then the nonnegative solution (u, v) of (1.3) with $d = 1$, $\gamma = 1$, $\alpha = 1$ with initial function (u_0, v_0) exists globally in time.*

For the proof of Theorem 3.2, we refer to Theorem 2 [121]. For the model (1.17) with $m = 1$, $q = 2$ and $\tau = 0$, Nagai proved that there exist global solutions in time when the space dimension $N = 1$ and showed the blow-up of solutions under some appropriate conditions on the data when $N \geqslant 3$. In the case of space dimension $N = 2$, he also proved that either there exists global solution or blow-up of solutions in a finite time depending on the size of L^1-norm of the initial data u_0.

Sugiyama [149] considered the model (1.17) with $\tau = 0$ or $\tau = 1$. For this model, let us mention the challanges which were encountered by the author. Firstly, the degeneracy of the model (1.17) in case when $m > 1$, and therefore one can not expect classical solutions.

Secondly, the comparison principle doesn't hold for this model and we can not use any representation formula, since there is no explicit fundamental solution for the first equation. Even if we had these difficulties, in both cases (i) $m > 2$ for large initial data, and (ii) $1 < m \leqslant 2 - \frac{2}{N}$ for a small initial data, Sugiyama [149] proved the global existence of solutions and also proved it without assuming the small initial data in case (i). Also, Sugiyama has established uniform bound in both space and time for the solution in both cases (i) and (ii). The decay properties of the model (1.17) are established in cases (ii) and $\tau = 0$. Moreover, she assumed the initial data (u_0, v_0) is nonnegative and belongs to $L^1 \cap L^\infty(\mathbb{R}^N) \times L^1 \cap H^1 \cap W^{1,\infty}(\mathbb{R}^N)$, $u_0^m \in H^1(\mathbb{R}^N)$. Note that the results in [149] were improved in [77] and [78] and the details will be discussed after Theorem 5.7.

Sugiyama [150] has also considered the degenerate parabolic-elliptic system of type (1.22) with $m > 1$, $\alpha, \gamma, \chi > 0$ and $N \geqslant 1$ and proved the global existence and finite time blow-up in sub-critical and super-critical cases, respectively. Without any restriction on the initial data, she proved global solvability of the model when $m > 2 - \frac{2}{N}$ and showed the blow-up of solutions when $1 < m \leqslant 2 - \frac{2}{N}$ for some large initial data. From this it is clear that the existence and non-existence of solutions strongly depend on the exponent $m = 2 - \frac{2}{N}$ which generalized the Fujita exponent of the model (1.22). In [28], Calvez and Carrillo have derived a priori estimates for the classical chemotaxis model of Patlak, Keller-Segel when a nonlinear diffusion or a nonlinear chemotactic sensitivity is considered accounting for the finite size of the cells. They also obtained entropy estimates which give natural conditions on nonlinearities implying the absence of blow-up of solutions. The globally bounded in time, point wise estimate of solutions to simplified model of (1.1), that is, when $\phi(u, v) = a + b|u|^m$, $\psi(u, v) = a + b|u|^m$, $\tau = 0$, $g(u, v) = \gamma$, $h(u, v) = \alpha$ and $f(u, v) = 0$ are derived in [208]. The authors considered their problem in two dimension. Winkler introduced a concept of very weak solutions for the parabolic-elliptic Keller-Segel system and proved boundedness properties in [173]. He considered the chemotaxis system by restricting to $\psi(u, v) = 1$, $\psi(u, v) = \chi u$, $\tau = 0$, $g(u, v) = u$, $h(u, v) = 1$, the logistic function $f(u, v) = Au - b^\alpha$ with $\alpha > 0$, $A \geqslant 0$ and $b > 0$. To prove global existence of very weak solutions for any nonnegative initial data $u_0 \in L^1(\Omega)$ under the assumption that $\alpha > 2 - \frac{1}{n}$, he also assumed the logistic function g to belong to $C^1([0, \infty))$ and to satisfy $g(0) \geqslant 0$. In addition, he also assumed the following assumptions for various $\alpha > 1$:

(1) $g(s) \leqslant a - bs^\alpha$ for all $s \geqslant 0$ with some $a \geqslant 0$ and $b > 0$, and
(2) $g(s) \geqslant -c_0(s + s^\alpha)$ for all $s \geqslant 0$ with some $c_0 > 0$.

The main results of this paper are

Theorem 3.3 [173] *Let $\chi > 0$ and suppose that g satisfies the above assumptions* 1 *and* 2 *with some $\alpha > 2 - \frac{1}{n}$. Then for each nonnegative $u_0 \in L^1(\Omega)$, the parabolic-elliptic system possesses at least one global very weak solution (u, v). This solution can be obtained as the limit of an approximate sequence $((u_\epsilon, v_\epsilon))_{\epsilon=\epsilon_j \searrow 0}$ of global bounded classical solutions of the following problem*

$$\begin{cases} \partial_{\epsilon t} u = \Delta u_\epsilon - \chi \nabla \cdot (u_\epsilon \nabla v_\epsilon) + g(u_\epsilon) - \epsilon u_\epsilon^\beta, \ x \in \Omega, \quad t > 0, \\ 0 = \Delta v_\epsilon + \mu u_\epsilon - v_\epsilon, \quad x \in \Omega, \quad t > 0, \frac{\partial u_\epsilon}{\partial \nu} = \frac{\partial v_\epsilon}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > 0, \\ u_\epsilon(x, 0) = u_{0\epsilon}(x), \quad x \in \Omega, \end{cases} \tag{3.2}$$

where $(u_{0\epsilon})_{\epsilon \in (0,1)} \subset C^0(\overline{\Omega})$ is such that $u_{0\epsilon} > 0$ in Ω and $\|u_{0\epsilon} - u_0\|_{L^1(\Omega)} \leqslant \epsilon, \epsilon \in (0, 1)$, in the sense that $u_\epsilon \to u$ a.e in $\Omega \times (0, \infty)$, $u_\epsilon^{\frac{\gamma}{2}} \rightharpoonup u^{\frac{\gamma}{2}}$ in $L^2_{loc}([0, \infty); W^{1,2}(\Omega))$, $u_\epsilon \rightharpoonup u$ in

$L^{\alpha}_{loc}(\overline{\Omega} \times [0,\infty))$ and $v_\epsilon \rightharpoonup v$ in $L^{\alpha}_{loc}([0,\infty); W^{2,\alpha}(\Omega))$ as $\epsilon = \epsilon_j \searrow 0$ for any $\gamma \in (0,1)$ satisfying $\gamma \leqslant \alpha - 1$.

For the proof of Theorem 3.3, one can refer to [173]. The next theorem states the existence of global bounded solutions with small-data.

Theorem 3.4 *Assume that g satisfies assumptions 1 and 2 with some $\alpha > 1$. Then there exists $\delta > 0$ with the property that if $\frac{a}{b} < \delta$, then for all $\gamma > \max\{1, \frac{n}{2}\}$, one can find $\lambda > 0$ such that whenever $u_0 \in Ł^{\infty}(\Omega)$ satisfies $\|u_0\|_{L^{\gamma}(\Omega)} < \lambda$, the parabolic-elliptic system*

$$\begin{cases} \partial_t u = \Delta u - \chi \nabla \cdot (u \nabla v) + g(u), \ x \in \Omega, \quad t > 0, \\ 0 = \Delta v + u - v, \quad x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geqslant 0, \quad x \in \Omega, \end{cases} \tag{3.3}$$

possesses global bounded very weak solution (u, v).

For the proof of Theorem 3.4, one can refer to [173]. Fujie et al. [59] proved a global existence results to the model (1.21) in a bounded domain $\Omega \subset \mathbb{R}^2$ with initial condition $u_0 \in C^0(\overline{\Omega})$ nonnegative with $u_0 \neq 0$. They have assumed that $\chi_0(v) = \frac{\chi}{v}$, where $\chi > 0$ and $f(u) = ru - \mu u^2$, $r \in \mathbb{R}$, $\mu > 0$. They also noticed that there is a way to prevent the blow-up either by decaying sensitivities or by logistic source functions. In [39], they have considered the model (1.23) with a critical exponent of the nonlinearity in the diffusion which measures the strength of diffusion at points of high densities and it distinguishes between finite-time blow-up and global-in-time existence of uniformly bounded solutions. They summarised their results as follows

- If $\phi(s) \geqslant c(1+s)^{-p}$ for all $s \geqslant 0$ holds with some $c > 0$ and $p < \frac{2}{n} - 1$, then all solutions of (1.7) are global and bounded.
- If, however $\phi(s) \leqslant c(1+s)^{-p}$ for all $s \geqslant 0$ and some $c > 0$ and $p > \frac{2}{n} - 1$, then there exists initial data u_0 such that $\limsup_{t \nearrow T} \|u(\cdot,t)\|_{L^{\infty}(\Omega)} = \infty$ for some finite $T > 0$.

For the proof of the above results, one can refer to Theorems 2.4 and 3.2 [39]. Nasreddine [124] proved the global existence of solutions to (1.7) with homogeneous Neumann boundary conditions under a smallness of initial data. By using some higher regularity condition on solutions, he also established uniqueness of solutions. For the prevention of blow-up related to model (1.7), one can consult the paper [28]. Blanchet et al. [21] considered the model (1.24) in the space of dimension $d \geqslant 3$. They have stated that the qualitative behaviour of solutions is decided by the initial conditions of the model. In addition to this, they proved that there exists a sharp critical mass M_c such that if $M \in (0, M_c]$, the global solution exists in time and otherwise there are blowing up solutions exist. They showed the existence of self similar solutions within the rage $(0, M_c)$. They assumed that the initial data belongs to $u_0 \in L^1(\mathbb{R}^d; (1+|x|^2)\mathrm{d}x) \cap L^{\infty}(\mathbb{R}^d)$, $\nabla u_0^m \in L^2(\mathbb{R}^d)$ and $u_0 \geqslant 0$. For the model (1.25) in the whole space $\mathbb{R}^n$, $n \geqslant 3$, the global existence of solutions with initial data which belongs to $L^1 \cap L^p$, $n < p \leqslant \infty$ are proved in [153]. Sugiyama et al. [153] have introduced the following transformation

$$z = \frac{v^{1-\lambda}}{1-\lambda} \text{ with } \theta = \frac{1}{1-\lambda} \in \mathbb{R}, \tag{3.4}$$

and using the same, the system (1.25) can be re-written as

$$\begin{cases} \partial_t u = \Delta u - \theta \nabla \cdot \left(u \dfrac{\nabla z}{z} \right), \\ \partial_t z = u. \end{cases} \tag{3.5}$$

This system together with initial conditions $u(0,x) = a(x)$, $z(0,x) = \theta b(x)^{1-\lambda}$ is investigated. Further, the system (3.5) can be converted into integral equation of the form

$$u(t) = e^{t\Delta} a - \theta B[u](t), z(t) = 1 + \int_0^t u(\tau) \mathrm{d}\tau, \tag{3.6}$$

where

$$B[u](t) = \int_0^t e^{(t-\tau)\Delta} \nabla \cdot \left(u(\tau) \frac{\nabla z(\tau)}{z(\tau)} \right) \mathrm{d}\tau.$$

They summarized their results as follows:

Theorem 3.5 (*Small data*: *global existence*) *Let* $n \geqslant 3, n/(n-1) < q < n < r < p \leqslant \infty$ *and* $\theta \in \mathbb{R}$. *There exists* $\delta = \delta(n, p, q, r, |\theta|) > 0$ *such that if* $\|a\|_{L^1 \cap L^p} \leqslant \delta$, *then there exists a global solution* $u \in L^1(0, \infty; L^\infty(\mathbb{R}^n))$ *of* (3.6) *satisfying* $\|u\|_{X^1_\infty} + \|u\|_{X^2_\infty} + \|u\|_{X^3_\infty} \precsim \|a\|_{L^1 \cap L^p}$, *where*

$$\|u\|_{X^1_t} = \int_0^t \|u(\tau)\|_{L^\infty} \mathrm{d}\tau,$$

$$\|u\|_{X^2_t} = \sup_{\tau < t} \left\| \int_0^\tau \nabla u(\sigma) \mathrm{d}\sigma \right\|_{L^r} \text{ and}$$

$$\|u\|_{X^3_t} = \sup_{\tau < t} \left\| \int_0^\tau \nabla u(\sigma) \mathrm{d}\sigma \right\|_{L^q}.$$

Moreover, with $U_\infty(t) = \int_0^t \|u(\tau)\|_{L^\infty} \mathrm{d}\tau$ *and* $U(x,t) = \int_0^t u(\tau, x) \mathrm{d}\tau$, *one has* $U_\infty \in C([0,\infty))$ *and* $\nabla U \in C([0,\infty); L^r \cap L^q)$.

For the proof of Theorem 3.5, we refer to Theorem 1.1 [153].

Theorem 3.6 (*Large data*: *local existence*). *Let* $n \geqslant 1, 0 < s < 1$ *and* $\theta \in \mathbb{R}$. *Let*

(i) *There exists a small constant* $C = C(n, s, |\theta|) \in (0, 1)$ *such that for any* $a \in L^\infty$, (3.6) *admits a local existence of solutions with* $T = \frac{C}{\|a\|_{L^\infty}}$ *satisfying*

$$\|u\|_{Y^1_T} + \|u\|_{Y^2_T} + \|u\|_{Y^3_T} + \|u\|_{Y^4_T} \precsim \|a\|_{L^\infty}, \text{ where for } t > 0,$$

$$\|u\|_{Y^1_t} = \sup_{\tau < t} \|u(\tau)\|_{L^\infty}, \ \|u\|_{Y^2_t} = \sup_{\tau < t} t^{s/2} \|u(t)\|_{\dot{B}^s_{\infty,\infty}},$$

$$\|u\|_{Y^3_t} = \sup_{\tau < t} \tau^{1/2} \|\nabla u(\tau)\|_{L^\infty} \text{ and } \|u\|_{Y^4_t} = \sup_{\tau < t} \tau^{(1+s)/2} \|\nabla u(\tau)\|_{\dot{B}^s_{\infty,\infty}}.$$

(ii) *In addition, if the initial data also belongs to* $L^1(\mathbb{R}^n)$, *then the solution* u *constructed in* (i), *is in* $C([0,T];L^1)$, *where* $T=\frac{C}{\|a\|_{L^1\cap L^\infty}}$ *and* C *depending on* n, s *and* $|\theta|$, *and fulfills*

$$\|u\|_{Z_t}=\sup_{\tau<t}\|u(\tau)\|_{L^1}\lesssim\|a\|_{L^1}.$$

For the proof of Theorem 3.5, we refer to Theorem 1.2 [153]. For the model (1.27) when $n\leqslant 3$, the existence of solutions proved in [210]. They considered the nonlinear diffusion and superlinear growth term $f(u,v)=|u|^{\alpha-1}u$ and $f(u,v)$ is Lipschitz continuous. They also assumed that the functions ϕ and ψ satisfy $\phi\in C^1(\mathbb{R})$, $\phi(0)=0$, $\psi\in C^2(\mathbb{R}^2)$, $\psi(0,0)=0$, $0<\phi_0\leqslant\phi'(r)\leqslant\phi_\infty<\infty$ for all $r\in\mathbb{R}$, $(r_1,r_2)\mapsto\psi(r_1,r_2)r_1$ is Lipschitz continuous on $\mathbb{R}^2$. It would be easy to establish the numerical error estimates when the nonlinear diffusion and the growth functions are assumed to be Lipschitz continuous. For more literature regarding existence of solutions of parabolic-elliptic type models, one can refer [7, 14, 110]. It would also be interesting to check that whether (1.27) has a local solution when f is non-Lipschitz continuous.

The model (1.30) describes the movement of cells towards a higher concentration of a chemical signal. In [61], the authors considered the model (1.30) in a bounded and regular domain in $\mathbb{R}^N$, $N\in\mathbb{N}$. They have investigated that if the parameters either $\alpha>m+\gamma-1$ or $\alpha=m+\gamma-1$ and $\mu>\frac{N\alpha-2}{2(m-1)+N\alpha}\chi$. Moreover, they also proved for $\mu>2\chi$ and $u_0\in W^{1,p}(\Omega)$ for some $p>N$ and there exists $\underline{u_0}>0$ such that $u_0\geqslant\underline{u_0}$ and $\frac{\partial u_0}{\partial\nu}=0$ in $\partial\Omega$, the solution satisfies $\|u-1\|_{L^\infty(\Omega)}+\|v-1\|_{L^\infty(\Omega)}\to 0$, as $t\to\infty$. The model (1.33) was considered in [42] and they have proved that the existence of a uniform in time L^∞ bounded weak solution of the model with super critical diffusion exponent $1<p<\frac{3d}{d+1}$ under the assumption that norm of the initial data in $L^{\frac{d(3-p)}{p}}-$ is smaller than the universal constant. For general initial data in $L^1\cap L^\infty$, they have also established the local existence of weak solutions and a blow-up criterion. Salako and Shen [139] considered the model (1.34) in $\mathbb{R}^N$ with $\chi>0$, $a\geqslant 0$, $b>0$. They proved the local existence and uniqueness of classical solutions with given initial conditions (u_0,v_0) with various initial conditions u_0. Moreover, they have also proved the global existence and boundedness of classical solutions. Under the assumptions of strictly positive initial conditions or nonnegative compactly supported initial conditions, they have also established the asymptotic behavior of the global solutions, where the initial condition u_0 belongs to

- $C^b_{\text{unif}}(\mathbb{R}^N)=\left\{u\in C(\mathbb{R}^N)|u(x)\text{ is uniformly continuous in }x\in\mathbb{R}^N\text{ and }\limsup_{x\in\mathbb{R}^N}|u(x)|<\infty\right\}$.
- For given $p\geqslant 1$ and $\alpha\in(0,1)$, let X^α be the fractional power space of $I-\Delta$ on $X=L^p(\mathbb{R}^N)$.
- $L^p(\mathbb{R}^N)$ for every $p>N$ with $p\geqslant 2$.

They have summarised their local existence and uniqueness results as follows:

Theorem 3.7 *For any* $u_0\in C^b_{\text{unif}}(\mathbb{R}^N)$ *with* $u_0\geqslant 0$, *there exits* $T^\infty_{\max}(u_0)\in(0,\infty]$ *such that* (1.34) *has a unique non-negative classical solution* $(u(x,t;u_0),v(x,t;u_0))$ *on* $[0,T^\infty_{\max}(u_0))$ *satisfying that* $\lim_{t\to 0^+}u(\cdot,t;u_0)=u_0$ *in the* $C^b_{\text{unif}}(\mathbb{R}^N)$*- norm,*

$$u(\cdot,\cdot;u_0)\in C([0,T^\infty_{\max}(u_0)),C^b_{\text{unif}}(\mathbb{R}^N))\cap C^1((0,T^\infty_{\max}(u_0)),C^b_{\text{unif}}(\mathbb{R}^N))$$

and

$$u(\cdot,\cdot;u_0),\ \partial_{x_i}u(\cdot,\cdot),\ \partial^2_{x_ix_j}u(\cdot,\cdot),\ \partial_t u(\cdot,\cdot;u_0)\in C^\theta((0,T^\infty_{\max}(u_0)),C^\nu_{\mathrm{unif}}(\mathbb{R}^N))$$

for all $i,j=1,2,\dots,N$, $0<\theta\ll 1$, *and* $0<\nu\ll 1$. *Moreover, if* $T^\infty_{\max}(u_0)<\infty$, *then* $\limsup_{t\to T^\infty_{\max}(u_0)}\|u(\cdot,t;u_0)\|_\infty=\infty$.

For the proof of Theorem 3.7, we refer to Theorem 1.1 [139].

Theorem 3.8 *Assume that* $p>N$ *and* $\alpha\in(\frac{1}{2},1)$. *For every nonnegative* $u_0\in X^\alpha$, *there is a positive number* $T^\alpha_{\max}(u_0)\in(0,\infty]$ *such that* (1.34) *has a unique nonnegative classical solution* $(u(x,t;u_0),v(x,t;u_0))$ *on* $\mathbb{R}^N\times[0,T^\alpha_{\max}(u_0))$ *satisfying that* $\lim_{t\to0^+}u(\cdot,t;u_0)=u_0$ *in the* X^α*-norm,*

$$u(\cdot,\cdot;u_0)\in C([0,T^\alpha_{\max}(u_0)),X^\alpha)\cap C^1((0,T^\alpha_{\max}(u_0)),L^p(\mathbb{R}^N)),$$

$$u(\cdot,\cdot;u_0)\in C((0,T^\alpha_{\max}(u_0)),X^\beta)\cap C^1((0,T^\alpha_{\max}(u_0)),C^b_{\mathrm{unif}}(\mathbb{R}^N)),$$

and

$$u(\cdot,\cdot;u_0),\ \partial_{x_i}u(\cdot,\cdot;u_0),\ \partial^2_{x_ix_j}u(\cdot,\cdot;u_0),\ \partial_t u(\cdot,\cdot;u_0)\in C^\theta((0,T^\alpha_{\max}(u_0)),C^\nu_{\mathrm{unif}}(\mathbb{R}^N))$$

for all $0\leqslant\beta<1$, $i,j=1,2,\dots,0<\theta\ll1$, *and* $0<\nu\ll1$. *Moreover, if* $T^\alpha_{\max}(u_0)<+\infty$, *then* $\lim_{t\to T^\alpha_{\max}(u_0)}\|u(\cdot,t;u_0)\|_{X^\alpha}=\infty$.

For the proof of Theorem 3.8, we refer to Theorem 1.2 [139].

Theorem 3.9 *For every* $p>N$ *with* $p\geqslant2$ *and* $u_0\in L^p(\mathbb{R}^N)$ *with* $u_0\geqslant0$, *there is a positive number* $T^p_{\max}(u_0)\in(0,\infty]$ *such that* (1.34) *has a unique non-negative solution* $(u(x,t;u_0),v(x,t;u_0))$ *on* $[0,T^p_{\max}(u_0))$ *satisfying that* $\lim_{t\to0^+}u(\cdot,t;u_0)=u_0(\cdot)$ *in the* $L^p(\mathbb{R}^N)$*-norm,*

$$u(\cdot,\cdot;u_0)\in C([0,T^p_{\max}(u_0)),L^p(\mathbb{R}^N))\cap C^1((0,T^p_{\max}(u_0)),L^p(\mathbb{R}^N)),$$

$$u(\cdot,\cdot;u_0)\in C((0,T^p_{\max}(u_0)),X^\beta)\cap C^1((0,T^p_{\max}(u_0)),C^b_{\mathrm{unif}}(\mathbb{R}^N)),$$

and

$$u(\cdot,\cdot;u_0),\ \partial_{x_i}u(\cdot,\cdot),\ \partial^2_{x_ix_j}u(\cdot,\cdot;u_0),\ \partial_t u(\cdot,\cdot;u_0)\in C^\theta((0,T^p_{\max}(u_0)),C^\nu_{\mathrm{unif}}(\mathbb{R}^N))$$

for all $0\leqslant\beta<1$, $i,j=1,2,\dots,N$, $0<\theta\ll1$, *and* $0<\nu\ll1$. *Moreover, if* $T^p_{\max}(u_0)<+\infty$, *then* $\lim_{t\to T^p_{\max}(u_0)}\|u(\cdot,t;u_0)\|_{L^p(\mathbb{R}^N)}=\infty$.

For the proof of Theorem 3.9, we refer to Theorem 1.3 [139]. For the global existence of classical solutions, we refer to Theorems 1.5, 1.6 and 1.7 [139] and for the asymptotic behavior of global classical solutions of (1.34), we refer to Theorems 1.8 and 1.9.

In recent years, the researchers have introduced chemotaxis models with gradient dependent chemotactic coefficient, that is, $\chi(\nabla v)$ rather than constant (for example, see [8, 9, 125]). The main novelty of the paper [9] is the introduction of new type of Keller-Segel model (1.39) with flux delimiter features. In [9], the authors considered the model (1.39) in a ball $\Omega = B_R(0) \subset \mathbb{R}^n, n \geqslant 1$. The authors established the existence of a unique classical solution which is extendable in time up to a maximal $T_{max} \in (0, \infty]$ and satisfying the condition if $T_{max} < \infty$, then $\lim_{t \nearrow T_{max}} \sup \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$. They assumed that the initial condition satisfy

$$u_0 \in C^3(\overline{\Omega}) \text{ is radially symmetric and positive in } \overline{\Omega}\ with\ \frac{\partial u_0}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{3.7}$$

They summarized their main results as follows:

Theorem 3.10 *Let $u_0 \in C^3(\overline{\Omega})$. Then there exist $T_{max} \in (0, T]$ and a uniquely determined pair (u, v) of positive radially symmetric functions $u \in C^{2,1}(\overline{\Omega} \times [0, T_{max}))$ and $v \in C^{2,0}(\overline{\Omega} \times [0, T_{max}))$ which solve (1.39) classically in $\Omega \times (0, T_{max})$, and which are such that if*

$$T_{max} < \infty, \quad then \lim_{t \nearrow T_{max}} \sup \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{3.8}$$

For the proof of Theorem 3.10, we refer to Theorem 1.1 [9]. The above Theorem 3.10 provides the extensibility criteria (3.8), which has more crucial importance for deriving global existence and to characterise the asymptotic behavior near blow-up time of non-global solutions. It is to be noted that the extensible criteria eliminates the occurrence of gradient blow-up in [9]. The next theorem states the global existence of solutions.

Theorem 3.11 *Assume that u_0 satisfies (3.7) and that either $n \geqslant 2$ and $\chi < 1$, or $n = 1$, $\chi > 0$ and $\int_\Omega u_0 < m_c$, where in the case $n = 1$, we have set*

$$m_c := \begin{cases} \frac{1}{\sqrt{\chi^2 - 1}}\ if\ \chi > 1, \\ +\infty\ if\ \chi \leqslant 1. \end{cases} \tag{3.9}$$

Then the problem (1.39) possesses a unique global classical solution $(u, v) \in C^{2,1}(\overline{\Omega} \times [0, \infty)) \times C^{2,0}(\overline{\Omega} \times [0, \infty))$ which is radially symmetric and such that for some $C > 0$, we have $\|u(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C$ and $\|v(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C$, for all $t > 0$.

For the proof of Theorem 3.11, we refer to Theorem 1.2 [9]. In [217], the authors studied a unique global bounded classical solution and obtained the large time behavior of the solution for specific logistic source term for the model (1.29). In particular, they considered the model in the space dimension $\mathbb{R}^n$, $n \geqslant 2$ under homogeneous Neumann boundary conditions with $\chi > 0$, $a, b > 0$, $r > 1, k \geqslant 1$ and $\phi(u)$ is smooth and satisfying $\phi(u) > c_D u^{m-1}$ with some $c_D > 0$ and $m \geqslant 1$. It is already known that for any $b > 0$, if any one of the following assumptions holds

- $k \in (1, 2]$ and $m < \frac{2}{n} - 1$;
- $\max\{k, 1 + \frac{2}{n} - m\} > 2$,

then the model (1.42) has a unique nonnegative classical solution (u, v) which is globally bounded. But very recently, for the model (1.42), the blow-up result is shown in [109] in a bounded domain $\Omega \subset \mathbb{R}^n, n \geqslant 2$. The authors established the global boundedness of solutions of (1.42) and blow-up of solutions. In order to establish their studies, they assumed that the logistic source function $f \in C([0, \infty)) \cap C^1((0, \infty))$ and satisfies

$$f(u) \leqslant a - bu^k \text{ for all } u \geqslant 0 \text{ and some } a \geqslant 0, b > 0 \text{ and } k > 1, \tag{3.10}$$

and

$$\phi \in C^2([0,\infty)), \phi(u) > 0, u \geqslant 0 \text{ and } \phi(u) \geqslant \phi_0 u^{-m} \text{for all } u > 0 \text{ with some } \phi_0 > 0 \text{ and } m \in \mathbb{R}. \quad (3.11)$$

In [109], in particular, they extended the results $k > 2$ to more general cases. The main results are summarized as follows:

Theorem 3.12 *Let $\Omega \subset \mathbb{R}^n (n \geqslant 2)$ be an arbitrary bounded domain with smooth boundary, and let $\chi > 0$. Suppose that f and ϕ satisfy (3.10) and (3.11) with some $\phi_0, m \in \mathbb{R}$, $a \geqslant 0, b > 0$ and $k > 2$, respectively. If $k > m + 3 - \frac{4}{n+2}$, then for any nonnegative initial data $u_0 \in C^0(\overline{\Omega})$, the system (1.42) possesses a unique global bounded classical solution.*

For the proof of Theorem 3.12, we refer to Theorem 1.1 [109]. Negreanu and Tello [125] considered the model (1.57) with the range of p:

$$\left\{ p \in (1,\infty), \text{ if } \quad N = 1 \text{ and } p \in \left(1, \frac{N}{N-1}\right), if\ N \geqslant 2, \right. \quad (3.12)$$

and $u_0 \in C^{2,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$. The main result is provided as follows:

Theorem 3.13 *Under the assumptions of (3.12),$u_0 \in C^{2,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$ and $\frac{\partial u_0}{\partial \nu} = 0$, $x \in \partial\Omega$, for any $T < \infty$, there exists a constant $c(u_0, \chi, p, \Omega)$, independent of T, such that $\|u\|_{L^\infty(\Omega)} \leqslant c$.*

For the proof the above Theorem 3.13, we refer to Theorem 1.1 [125]. In addition, he also proved the existence of infinitely many solutions to the steady sate case of (3.12) in a bounded domain in one dimensional setting for the range $p \in (1, 2)$. Winkler [188] investigated the finite time blow-up of radially symmetric solutions to the model (1.60) in a ball $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geqslant 3$, $R > 0$, $\lambda \in \mathbb{R}$, $\mu > 0$. He proved it by using the conditions $\kappa > 1$ such that

$$\kappa < \begin{cases} \frac{7}{6} & \text{if } n \in \{3, 4\}, \\ 1 + \frac{1}{2(n-1)} & \text{if } n \geqslant 5, \end{cases} \quad (3.13)$$

and the initial condition $u_0 \in C^0(\overline{\Omega})$ is nonnegative and radially symmetric. Moreover, he presented his findings in the following

Theorem 3.14 *Let $\Omega = B_R(0) \subset \mathbb{R}^n, n \geqslant 3$ and $R > 0$, and let $\lambda \in \mathbb{R}$, $\mu > 0$ and $\kappa > 1$ be such that (1.60). Then for all $L > 0$, $m > 0$ and $m_0 \in (0, m)$ one can find $r_0(R, \lambda, \mu, \kappa, L, m, m_0) \in (0, R)$ with the property that whenever $u_0 \in C^0(\overline{\Omega})$ and is $u_0(x) \leqslant L|x|^{-n(n-1)}$ for all $x \in \Omega$ as well as $\int_\Omega u_0 \leqslant m$ but $\int_{B_{r_0}(0)} u_0 \geqslant m_0$, there exists $T_{max} \in (0, \infty)$ and a classical solution (u, v) of (1.60), uniquely determined by the inclusions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max}), \\ v \in \cap_{q>n} L^\infty_{loc}([0, T_{max}); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0, T_{max})), \end{cases} \quad (3.14)$$

The following corollary states that the blow-up of solutions to (1.60) for wide class of initial data.

Corollary 3.1 *Let $\Omega = B_R(0) \subset \mathbb{R}^n, n \geqslant 3$ and $R > 0$, and let $\lambda \in \mathbb{R}$, $\mu > 0$ and $\kappa > 1$ be such that (3.13). Then for any positive $u_0 \in C^0(\overline{\Omega})$, there exist initial data $u_{0k}, k \in \mathbb{N}$, which belongs to $C^0(\overline{\Omega})$ and radially symmetric as well as $u_{0k} \to u_0$ in $L^1(\Omega)$ as $k \to \infty$, and which are such that for each $k \in \mathbb{N}$, (1.60) has a classical solution (u_k, v_k) with $u_k|_{t=0} = u_{0k}$ and blowing up in finite time in the sense of the above Theorem 3.14.*

For the proof of Theorem 3.14 and Corollary, we refer Theorem 1.1 and Corollary 1.2 [188], respectively. The Keller-Segel model with fractional diffusion (1.80) studied by J. Zhao in [215]. In this case $1 < \alpha \leqslant 2$, the author established the local existence and uniqueness for any initial data and to prove global well-posedness, the author assumed that the initial data are small and belong to critical Besov spaces $\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n)$, $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$. Further, the global existence and analyticity of solutions also established to (1.80) under for small initial data which is in $\dot{B}_{\infty,1}^{-\alpha}(\mathbb{R}^n)$ and the result includes the limiting case $\alpha = 1$ as well. Biler et al. [16], proved the existence of radial global-in-time solutions for (1.31) with $\chi = 1$ under the criteria in Morrey space norms in dimension $n \geqslant 3$. However, for the same model (1.31) with $\chi = 1$, Biler and Zienkiewicz [15] derived the blowup criteria for radially symmetric solutions under the Morrey spaces norms in space dimension $n \geqslant 2$. This criteria is the Very recently, Li [104] studied the finite time blow-up of solutions to the model (1.53) in a bounded domain $\mathbb{R}^n$, $n \geqslant 1$. To prove his results, he assumed that $\phi(u) \in C^2([0,\infty))$ to be positive function, in particular, $\phi(u) \geqslant C_0(1+u)^{-m}$ for all $u \geqslant 0$, $C_0 > 0$, $m \in \mathbb{R}$ and $f(u) = K(1+u)^\kappa$ for all $u \geqslant 0$, $K > 0$, $\kappa > 0$. His main interest was on finding the interaction between the nonlinear functions $\phi(u)$ and $f(u)$ and to prove finite time blow-up to the model (1.53). In addition, he proved that the exponent $\frac{2}{n}$ is critical. He formulated his main results as follows:

Theorem 3.15 *Let $\Omega \subset \mathbb{R}^n, n \geqslant 1$ be smooth bounded domain. The function $\phi \in C^2([0,\infty))$ and $f \in \cup_{\theta\in(0,1)} C^\theta_{loc}([0,\infty)) \cap C^1((0,\infty))$, $f \geqslant 0$, $f' \geqslant 0$. $m \in \mathbb{R}$ and $K, \kappa > 0$ are given parameters. The initial data $u_0 \in \cup_{\theta\in(0,1)} C^\theta(\overline{\Omega})$ is radially symmetric, $u_0 \geqslant 0, u_0 \not\equiv 0$, $\frac{\partial u_0}{\partial \nu} = 0$ on $\partial\Omega$.*

- *Suppose that $\phi(u) \geqslant C_0(1+u)^{-m}$, $C_0 > 0$ and $f(u) \leqslant K(1+u)^\kappa$ for all $u \geqslant 0$. If*
$$m + \kappa < \frac{2}{n},$$
then the corresponding solution of (1.53) is global and uniformly bounded.
- *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ with $n \geqslant 1$ be a ball, $R > 0$. If $\phi(u) \leqslant C_0(1+u)^{-m}$ with $C_0 > 0$ and $f(u) \geqslant K(1+u)^\kappa$ or all $u \geqslant 0$. Assume that*
$$m + \kappa > \frac{2}{n},$$
then for all $M > 0$ with $\int_\Omega u_0 = M$, there exist $\epsilon = \epsilon(K,\kappa,M,R) \in (0, M)$ and $r^* = r^*(K,\kappa,M,R) \in (0,R)$ such that $\int_{B_{r^*}(0)} u_0 \geqslant M - \epsilon$, the he corresponding solution of (1.53) blow-up in finite time.

For the proof of Theorem 3.15, we refer to Theorem 1.1 [104]. Also we note that it is possible to prove the boundedness of solution to (1.53) as in Theorem 3.15 with nonnegative $u_0 \in C^0(\bar{(\Omega)})$ which need not be radially symmetric. Ahn et al. [2] investigated the global existence of weak solutions to (1.59) under the condition that $\chi < \chi_N := \frac{(4+\sqrt{N})}{(4+\sqrt{N})^2-4}$, $N \geqslant 3$.

In addition, they have also proved that the stabilization of bounded solutions in general domains. Mizukami et al. [117] proved the global existence of solutions to the model (1.40) under the condition $p > q + 1 - \frac{1}{n}$. From this later condition, we may expect the blow-up of solutions when q is large. Fortunately, Chiyoda et al. [37] studied the finite-time blow-up of solutions for the model (1.40). The authors considered the model in a ball $B_R(0) \subset \mathbb{R}^n (n \in \mathbb{N})$, $R > 0$, $\chi > 0$, $p, q \geqslant 1$, $\mu := \frac{1}{|\Omega|} \int_\Omega u_0$. Souplet and Winkler [145] investigated the asymptotic behaviour of radially decreasing solutions of the model (1.78) in a ball $\Omega = B_R \mathbb{R}^n$ and $\Omega = \mathbb{R}^n, n \geqslant 3$, that is, they studied the solution behaviour at blow-up time. To prove the results, the authors assumed that the initial condition $u_0 \in L^\infty(\Omega), u_0 \geqslant 0$ u_0 is radially symmetric and nonincreasing with respect to $|x|$ with $u_0 \not\equiv const$.

4 Boundedness of Solutions of Parabolic-Elliptic Models

During the past few decades, the main issue on the Keller-Segel model was whether the solutions of KS models are globally bounded or blows up in finite time. The global existence and boundedness of solutions to the singular Keller-Segel model is always a challenging problem, though the global existence of weak solutions to (1.50) with $\chi(v) = \frac{\chi_0}{v}$ proved by Biler [12]. However, the author had left the boundedness as an open problem. Tello and Winkler [162] proved the existence of global and bounded solutions to (1.56) under the assumptions that $\psi(u) = g(u) = u$. Moreover, they summarized their results as follows

- *If either* $n \leqslant 2$ *or* $n \geqslant 3$ *and* $b > \frac{(n-2)\chi}{n}$, *then for arbitrary initial data,* (1.56) $\psi(u) = g(u) = u$ *has a global bounded classical solution and which is unique.*
- *For all* $n \geqslant 1, b > 0$, *and for any initial conditions, there exist at least one global weak solution* $(u, v) \in (L^1((0, T); W^{1,2}(\Omega)))^2$ *if* $f(s) \geqslant -c_0(s^2 + 1), \forall s > 0$ *and* $c_0 > 0$.

Theorem 4.1 *If g meets the conditions* $f(u) \leqslant c_1 - bu^2, \forall u \geqslant 0$, $f(u) > 0$ *if* $0 < u < 1$ *and* $f(u) < 0$ *if* $u > 1$, *where* c_1 *and* b *are positive constants and satisfying* $b > \frac{n-2}{n}\chi$, *then for any nonnegative* $u_0 \in C^0(\overline{\Omega})$, *the model* (1.56) $\psi(u) = g(u) = u$ *has a unique and uniformly bounded global classical solution* (u, v). *More precisely, there exists* $c = c(\|u_0\|_{L^\infty(\Omega)})$ *such that* $\|u(t)\|_{L^\infty(\Omega)} \leqslant c(\|u_0\|_{L^\infty(\Omega)})$, $\forall t \in (0, \infty)$ *holds.*

For the proof of Theorem 4.1, we refer to Theorem 2.5 [162].

In [200], the authors studied the boundedness and finite-time collapse to the model (1.55). They assumed that $\phi(u) = (u + 1)^{-p}$ and $\psi(u) = u(u + 1)^{q-1}$, $p \geqslant 0, q \in \mathbb{R}$. They summarized their results as follows:

- *If* $p + q < \frac{2}{n}$, *then there exist solutions, which are global and bounded.*
- *If* $p + q > \frac{2}{n}$, $a > 0$ *and* Ω *is a ball, then there exist unbounded solutions in finite time.*

Theorem 4.2 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with, smooth boundary, and* $\phi, \psi \in C^1_{loc}([0, \infty))$ *for some* $\theta > 0$ *and satisfy* $\phi > 0$ *and* $\psi \geqslant 0$ *in* $[0, \infty)$. *Furthermore, assume that the nonnegative function* $u_0 \in C^\alpha(\overline{\Omega})$ *for some* $\alpha > 0$ *and that* $\frac{1}{|\Omega|} \int_\Omega u_0 = M$. *Then there exists a unique classical solution* (u, v) *of* (1.55) *that can be extended up to its maximal existence time* $T_{max} \in (0, \infty]$. *Here, either* $T_{max} = \infty$ *or* $\lim_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$.

For the proof of Theorem 4.2, we refer to Theorem 2.1 [200]. Fujie et al. [60] proved the existence of a unique global classical solution for (1.50), which is uniformly bounded under the condition that $\chi_0 < \frac{2}{n}(k=1)$, $\chi_0 < \frac{2}{n}\frac{k^k}{(k-1)^{k-1}}\gamma^{k-1}(k>1)$, where $\gamma > 0$. They established their results in a bounded domain $\Omega \subset \mathbb{R}^n, n \geqslant 2$ and $u_0 \in C^0(\overline{\Omega}), u_0 \geqslant 0$. Moreover, they assumed that $\chi(v)$ satisfy $0 < \chi(v) \leqslant \frac{\chi_0}{v^k}$, $k \geqslant 1$, $\chi_0 > 0$. In [219], the authors have discussed the boundedness of solutions of the model (1.56).

In heterogeneous environment the growth or death rates shall depend on spatial characteristics. This fact has been incorporated into many mathematical models of population. For the space and time dependent logistic source one can refer the series of papers [140–142].

As we know, the classical Keller-Segel system exhibits a critical-mass phenomena. Now Winkler ensured that a novel type of critical-mass phenomenon, in the radially symmetric setting, exists for (1.76) in [193]. From his findings, there is an interesting fact is that if the mass levels $m := \int_\Omega u_0$ increasing above the critical mass level m_c, then the trajectories collapses in finite time, that is, we are getting extreme unstable solutions. This results are proved in a ball $\Omega \subset \mathbb{R}^n, n \geqslant 2$. Moreover, he summarised his results as follows: 1) For $R > 0$ and $n \geqslant 2$, the model (1.76) exhibits blow-up phenomena in finite time provided whenever $m > m_c(n, R)$. 2) In contrast to 1), if $m < m_c$, there exist infinite number of nonnegative radial u_0, which are concentrated than $u = \frac{m}{|\Omega|}$, nevertheless, it allows global bounded classical solutions to (1.76).

Very recently, the boundedness of solution to the model (1.53) studied in [104] (see Theorem 3.15 in this article). Viglialoro and Woolley [165] considered the model (1.63) in a bounded domain in $\mathbb{R}^2$ and proved its solvability under the suitable assumptions on the production function $g(u)$ belongs to $C^1([0, \infty))$ and it satisfies $\lambda_1 \leqslant g(u) \leqslant \lambda_2(1 + u)^\beta, u \geqslant 0, 0 \leqslant \beta < 1$ and $0 < \lambda_1 \leqslant \lambda_2$ and the general chemotactic sensitivity $\psi(v)$ belongs to $C^1((0, \infty))$. The main result is summarized as follows:

Theorem 4.3 *Let Ω be a smooth and bounded domain in $\mathbb{R}^2, 0 < \chi \in C^(0, \infty)$ and $g \in C^1((0, \infty))$ a function satisfying the above assumption. Then for any nonnegative initial data $0 \not\equiv u_0 \in C^0(\overline{\Omega})$, the model (1.63) has a unique global classical solution (u, v). Moreover, both u and v are bounded in $\Omega \times (0, \infty)$.*

For the proof of Theorem 4.3, we refer to Theorem 1 [165]. Currently, the chemotaxis models with Lotka-Voltera type logistic source are attracted by many researchers see [126]. Under suitable assumptions on the coefficients of the system (1.84), the authors studied the existence of global and bounded solutions in [126]. In [159], the authors considered the model (1.67) in $\Omega \subset \mathbb{R}^n, n \geqslant 1, \mu \geqslant 0$ and a nonnegative function $r \in C^1(\overline{\Omega} \times [0, \infty))$. The authors proved the global classical solutions to (1.67) for any positive initial data $u_0 \in W^{1,\infty}(\Omega)$. Moreover, the stabilization of the solution u is derived provided if r satisfies

$$\int_t^{t+1}\int_\Omega |\nabla\sqrt{r}|^2 \to 0 \text{ as } t \to \infty.$$

Further, the solution u is uniformly bounded if $\sup_{t>0} \|r(\cdot, t)\|_{L^q(\Omega)} < \infty, q \geqslant 1$ and $q > \frac{n}{2}$. The main results are stated as follows

Theorem 4.4 *Let $n \geqslant 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $\mu \geqslant 0$ and $r \in C^1(\overline{\Omega} \times [0, \infty))$ be nonnegative. Then for any choice of $u_0 \in W^{1,\infty}(\Omega)$ such that $u_0 > 0$ in $\overline{\Omega}$, the problem (1.67) admits a global classical solution (u, v), uniquely determined by the inclusions $u \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$ and $v \in C^{2,0}(\overline{\Omega} \times (0, \infty))$ for which furthermore $u > 0, v \geqslant 0$ in $\overline{\Omega} \times (0, \infty)$.*

For the proof of Theorem 4.4, we refer to Theorem 1.1 [159]. To prove stabilization of u, we need decay of gradient $\nabla\sqrt{r}$ instead of the function r itself. The stabilization result reads as

Theorem 4.5 *Let $\mu \geqslant 0$ and suppose that $r \in C^1(\overline{\Omega} \times (0,\infty))$ is a nonnegative function fulfilling $\sqrt{r} \in L^2_{loc}([0,\infty); W^{1,2}(\Omega))$, and $\int_t^{t+1}\int_\Omega |\nabla\sqrt{r}|^2 \to 0$ as $t \to \infty$. Then for any $u_0 \in W^{1,\infty}(\Omega)$ which is such that $u_0 > 0$ in $\overline{\Omega}$ the solution (u, v) of (1.67) satisfies $u(\cdot,t) \to \overline{u}_0$ in $L^1(\Omega)$ as $t \to \infty$ and that*

$$\int_t^{t+1} \|\nabla v^{\frac{1}{4}}(\cdot,s)\|^4_{L^2(\Omega)} ds \to 0 \quad as\ t \to \infty.$$

If furthermore there exists $q \geqslant 1$ such that $q > \frac{n}{2}$ and $\sup_{t>0}\|r(\cdot,t)\|_{L^q(\Omega)} < \infty$, then u belongs to $L^\infty(\Omega \times (0,\infty))$ and we have $u(\cdot,t) \to \overline{u}_0$ in $L^p(\Omega)$ for all $p \in [1,\infty)$ and $u(\cdot,t) \to^\star \overline{u}_0$ in $L^\infty(\Omega)$ as $t \to \infty$.

For the proof of Theorem 4.5, we refer to Theorem 1.2 [159] In [54], Fuest considered the Keller-Segel system with environmental dependent logistic source function (1.64) in a ball $B_R(0) \subset \mathbb{R}^2$, $R > 0$ and $p \geqslant 1$. He assumed that the functions $\kappa, \mu : [0, R] \to [0,\infty)$ are sufficiently smooth. He summarised the main result as follows

Theorem 4.6 *Let $p \geqslant 1, \alpha \geqslant 2(p-1), \mu_1 > 0$ and suppose that $\kappa, \mu \in C^0([0,R]) \cap C^1((0,R))$ satisfy*

$$\kappa, -\kappa', \mu, \mu' \geqslant 0 \quad in\ (0,R)$$

and $\mu(s) \leqslant \mu_1 s^\alpha$ for all $s \in [0,R]$. For any $m_0 > 8\pi$ there exist $r_1 \in (0,R)$ and $c \in (0,m_0)$ such that if $0 \leqslant u_0 \in C^0(\overline{\Omega})$ is radially symmetric and radially decreasing with $\int_\Omega u_0 = m_0$ and $\int_{B_{r_1}(0)} u_0 \geqslant c$, then there exists a classical solution (u,v) to (1.64) with u_0 blowing up in finite time, that is, there exists $T_{\max} \in (0,\infty)$ such that

$$\lim_{t \to T_{\max}} \sup \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

For the proof of Theorem 4.6, we refer to Theorem 1.1 [54]. The additional cross-diffusion term has been introduced in the chemical concentration equation of Keller-Segel models to suppress blow up of solutions. Jüngel, Leingang and Wang [84] answered for the question *how the solutions of added cross-diffusion model approximate those solutions from the main Keller-Segel model?* The authors provided this results by using vanishing cross-diffusion limit and error estimate as well.

In Table 3, we review the existence, blow-up and boundedness results to a variety of parabolic-elliptic models.

5 Existence and Blow-up of Solutions of Parabolic-Parabolic Type Models

We recall that the results of this section are useful to understand under what conditions there exist global existence and blow-up of solutions of variations of Keller-Segel model. Now, we turn to the parabolic-parabolic Keller-Segel models. The global existence and regularity of solutions to (1.3) in a bounded domain in $\mathbb{R}^2$ derived in [120]. The authors assumed that $\chi, d = 1, \gamma, \alpha$ are positive constant and u_0, v_0 are nonnegative functions for the model (1.3). He summarized the results as follows:

Table 3 Summary of the existing results for parabolic-elliptic models

Model	Results	Reference
(1.31)	Blow-up & global existence	[19]
(1.3) $d=\gamma=\alpha=1$	Blow-up & global existence	[121]
(1.31) $m=1$, $q=2$, $\tau=0$	Blow-up & global existence	[19]
(1.17)	Global existence & decay	[149]
(1.22)	Global existence & blow-up	[150]
(1.1)(particular case)	Globally bounded solutions	[208]
(1.1)(particular case)	boundedness	[173]
(1.21)	Global existence	[59]
(1.23)	Finite-time blow-up & global existence	[39]
(1.7)	Global existence	[124]
(1.7)	Blow-up	[28]
(1.24)	Global existence & blow-up	[21]
(1.25)	Global existence	[153]
(1.27)	Existence of solutions	[210]
(1.30)	Existence	[61]
(1.33)	Existence, blow-up criteria	[42]
(1.34)	Global existence & boundedness	[139]
(1.34)	Asymptotic behavior	[139]
(1.39)	Global existence & blow-up	[9]
(1.29)	Unique global bounded & asymptotic solutions	[217]
(1.42)	Blow-up & global boundedness	[109]
(1.60)	Blow-up	[188]
(1.60)	Existence	[16]
(1.31)	Blow-up	[15]
(1.53)	Blow-up	[104]
(1.59)	Global existence & boundedness	[2]
(1.50)	Global existence & boundedness	[12]
(1.56), $\psi(u)=g(u)=u$	Global and bounded solutions	[162]
(1.55)	Boundedness and finite-time collapse	[200]
(1.50)	Global existence & boundedness	[60]
(1.56)	Boundedness	[219]
(1.84)	Global and boundedness	[126]
(1.40)	Global and boundedness	[117]
(1.40)	Blow-up	[37]
(1.40)	Global existence	[117]
(1.63)	Global existence & boundedness	[165]
(1.64)	Blow-up	[54]
(1.76)	Blow-up	[193]
(1.78)	Qualitative behaviour	[145]
(1.80)	Well-posedness & analyticity	[215]

Theorem 5.1 *Let Ω be a bounded domain in $\mathbb{R}^2$. Assume $u_0, v_0 \in H^{1+\epsilon_0}(\Omega)$ for some $0 < \epsilon_0 \leqslant 1$, and $u_0 \geqslant 0$, $v_0 \geqslant 0$ on Ω.*

(i) *If $\int_\Omega u_0(x)\mathrm{d}x < 4\pi/(\alpha\chi)$, then (1.3) admits a unique classical solution (u, v) on $\overline{\Omega} \times (0, \infty)$ satisfying*

$$\sup_{t \geqslant 0}\{\|u(t)\|_{L^\infty} + \|v(t)\|_{L^\infty}\} < \infty.$$

(ii) *Let $\Omega = \{x \in \mathbb{R}^2; |x| < L\}$ and (u_0, v_0) be radial in x. Then the same assertion as (i) holds under the condition $\int_\Omega u_0(x)\mathrm{d}x < 8\pi/(\alpha\chi)$.*

For the proof of Theorem 5.1, we refer to Theorem 1.1 [120].

In [69], Hillen and Painter considered the version of Keller-Segel model with chemotactic sensitivity function depending on both the local population cell density and the external signal. They also assumed that the chemotactic response is switched off at high threshold cell densities. Therefore this mechanism prevents overcrowding and they established the local and global existence of classical solutions. In addition to this they also performed numerical simulations through which they have observed pattern formation and stable aggregates. In [29], the authors derived a critical mass threshold below which they have ensured the global existence of the model related to (1.17). Their main idea was to tackle the global existence issues for the parabolic-parabolic model (1.17) with $\tau > 0$, in space dimension $d = 2$ and in the whole space $\mathbb{R}^2$. They have obtained the optimal critical mass value for the global existence by using the energy method and ad hoc functional inequalities on $\mathbb{R}^2$. Before them, there was no optimal critical mass value for the global existence of solutions to the parabolic - parabolic Keller-Segel model. The author [202], proved the existence of global in time solution to a chemotaxis model with volume filling under no-flux or Dirichlet boundary conditions. In addition, he showed the existence of global attractor in the space $W^{1,p}(\Omega)$, $p > n$, $\Omega \subset \mathbb{R}^n$. Kowalcyk and Szymanska [89] proved aggregation model of Keller-Segel type with nonlinear, degenerate diffusion. They shown the existence, uniform in-time boundedness and uniqueness of solutions. The uniqueness proved by assuming some higher regularity conditions on solutions is known a priori. The crucial assumption for the diffusion coefficient $f(n) \geqslant \delta n^p$ for all $n > 0$, where $\delta > 0$ is a constant and $p > 0$ in case $d = 1$ and $p > 2 - 4/d$ in the case $d \geqslant 2$. For initial conditions $u_0 \in L^{n/2}_w(\mathbb{R}^n)$ and $v_0 \in \mathbf{BMO}$, the authors in [90] proved the global existence of strong solutions to (1.3) with $d = 1$ in $\mathbb{R}^n$, $n \geqslant 3$. Their method is based on the perturbation of linearisation together with the $L^p - L^q$ estimates of the heat semigroup and the fractional powers of Laplace operators. They also proved the existence of global self-similar solutions $\{u, v\}$ of the following system

$$\begin{cases} \partial_t u = \Delta u - \nabla(u\nabla v), & \text{in } x \in \mathbb{R}^n, t \in (0, \infty), \\ 0 = \Delta v + \mu u - v. \end{cases} \tag{5.1}$$

He has chosen the weak L^p- space since it contains homogeneous functions. To solve the original system, they converted it as the following integral equations

$$\begin{cases} u(t) = e^{t\Delta}u_0 - \int\limits_0^t \mathrm{div} e^{(t-\tau)\Delta}(u\nabla v)(\tau)\mathrm{d}\tau, \\ v(t) = e^{-t(-\Delta+\gamma)}v_0 + \int\limits_0^t e^{-(t-\tau)(-\Delta+\gamma)}u(\tau)\mathrm{d}\tau. \end{cases} \tag{5.2}$$

Theorem 5.2 *Let $n \geqslant 3$ and let $u_0 \in L_w^{n/2}$, and $v_0 \in BMO$. Suppose that $n/2 < q < n$ and $n/q - 1 < \alpha < 1$.*

(i) (*global existence*) *There is a constant $\epsilon_0 = \epsilon_0(n, q, \alpha)$ such that if the initial data $\{u_0, v_0\}$ satisfies*

$$\|u_0\|_{L_w^{n/2}} + \|v_0\|_{BMO} \leqslant \epsilon_0, \tag{5.3}$$

then there exists a solution $\{u, v\}$ of (5.2) on $(0, \infty)$ in the class

$$u \in C_w([0, \infty); L_w^{n/2}) \cap C((0, \infty); H^{\alpha,q}), \tag{5.4}$$

$$v \in C_w([0, \infty); BMO), \quad \nabla v \in C_w((0, \infty); L^\infty), \tag{5.5}$$

with the property

$$\sup_{0<t<\infty} \|u(t)\|_{L_w^{n/2}} + \sup_{0<t<\infty} t^{(n/2)(2/n-1/q)+l/2} \|(-\Delta)^{l/2} u(t)\|_{L^q} < \infty, \tag{5.6}$$

$$\sup_{0<t<\infty} \|u(t)\|_{BMO} + \sup_{0<t<\infty} t^{1/2} \|\nabla v(t)\|_{L^\infty} < \infty, \tag{5.7}$$

for $0 \leqslant l \leqslant \alpha$. Here, $C_w([0, \infty); X)$ denotes the set of weakly-star continuous functions on $[0, \infty)$ with values in X, and $H^{\alpha,q}$ is the space of Bessel potentials defined by $H^{\alpha,q} = \{f \in S'; \|(1 - \Delta)^{\alpha/2} f\|_{L^q} < \infty\}$.

(ii) (*uniqueness*) *There is a constant $\delta_0 = \delta_0(n, q, \alpha) > 0$ such that if the solution $\{u, v\}$ of (5.2) in the class (5.4)-(5.5) with the property (5.6)-(5.7) satisfies*

$$\lim_{t \to +0} \sup t^{(n/2)(2/n-1/q)} \|u(t)\|_{L^q} + \lim_{t \to +0} t^{1/2} \|\nabla v(t)\|_{L^\infty} \leqslant \delta_0, \tag{5.8}$$

then $\{u, v\}$ is the unique solution of (5.2).

For the proof of Theorem 5.2, we refer to [90].

As a consequence of Theorem 5.2, the existence of a forward self-similar solution to

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla v), & \text{in } x \in \mathbb{R}^n, t \in (0, \infty), \\ \partial_t v = \Delta v + \mu u, & \text{in } x \in \mathbb{R}^n, t \in (0, \infty), \\ u|_{t=0} = u_0, v|_{t=0} = v_0, & \text{in } x \in \mathbb{R}^n, \end{cases} \tag{5.9}$$

is given as corollary in [90]. For the regularity of the solutions of (5.2), we refer to Theorem 1.3 [90].

There is a natural question arise as whether the model (1.1) has global existence of solutions? The global existence of strong solution to the semi-linear Keller-Segel system of type (1.3) with small data in scale invariant spaces studied in [91]. They considered the system in $\mathbb{R}^n$, $n \geqslant 3$ and chosen initial condition $u_0 \in H^{\frac{n}{r}-2,r}(\mathbb{R}^n)$ and $v_0 \in H^{\frac{n}{r},r}(\mathbb{R}^n)$ for $\max\{1, n/4\} < r < n/2$. Their method is based on the perturbation of linearisation together with $L^p - L^q$ estimates of the heat semigroup and the fractional powers of the Laplace operator. Below, we presented a result of Kozono and Sugiyama, who proved the following main existence theorem.

Theorem 5.3 *Let $n \geqslant 3$, and let $\max\{1, n/4\} < r < n/2$. There is a constant $\epsilon_0(n, r) > 0$ such that if $u_0 \in H^{\frac{n}{r}-2,r}(\mathbb{R}^n)$ and $v_0 \in H^{\frac{n}{r},r}(\mathbb{R}^n)$ satisfy*

$$\|(-\Delta)^{\frac{n}{2r}-1} u_0\|_r + \|(-\Delta)^{\frac{n}{2r}} v_0\|_r \leqslant \epsilon_0, \tag{5.10}$$

then there exists a unique solution $\{u, v\}$ *of* (1.3) *in*

$$u \in C([0,\infty); H^{\frac{n}{r}-2,r}(\mathbb{R}^n)) \cap C((0,\infty); H^{2,r}(\mathbb{R}^n)) \cap C^1((0,\infty); L^r(\mathbb{R}^n)), \tag{5.11}$$

$$v \in C([0,\infty); H^{\frac{n}{r},r}(\mathbb{R}^n)) \cap C^1((0,\infty); L^r(\mathbb{R}^n)). \tag{5.12}$$

Moreover, such a solutions $\{u, v\}$ *has the following decay property*:

$$\|(-\Delta)^\sigma u(t)\|_r = O(t^{\frac{n}{2r}-1-\sigma}), \quad \textit{for}\ \frac{n}{2r} - 1 \leqslant \sigma < 1, \tag{5.13}$$

$$\|(-\Delta)^\zeta v(t)\|_r = O(t^{\frac{n}{2r}-\zeta}), \quad \textit{for}\ \frac{n}{2r} \leqslant \zeta \leqslant \frac{n}{4r} + 1, \tag{5.14}$$

as $t \to \infty$.

For the proof of Theorem 5.3, we refer to [91]. The mass conservation property is also hold if we assume additionally that $u_0, v_0 \in L^1(\mathbb{R}^n)$ (see Theorem 2 in [91]. Payne and Straughan considered the model (1.9) in [132]. They have concentrated mainly on deriving conditions which ensure the solution will decay to a constant state. These sufficient conditions precludes the formation of instabilities in space. Payne have obtained other results in this direction, for example, the interested readers may refer to [130, 131].

The author in [174] considered the model (1.3) with $d = 1$ in space dimension $n \geqslant 3$. For each $q > \frac{n}{2}$, $p > n$ and for bounded initial data (u_0, v_0) such that $\|u_0\|_{L^q(\Omega)} < \epsilon$ and $\|\nabla v_0\|_{L^p(\Omega)} < \epsilon$, he proved the solution global in time and bounded and (u, v) approaches the steady state (m, m) as $t \to \infty$, where m is the total mass $m := \int_\Omega u_0$ of the population. Winkler observed that there were no results available about either global existence of bounded solutions or the occurrence of blow-up solutions for parabolic-parabolic system (1.3) with $d = 1$ in space dimension $n \geqslant 3$. He also was noticed that there were no results in the literature regarding whether (1.3) possesses any nonstationary global solutions, when $n \geqslant 3$. He stated the main results as follows:

(1) *if* $n \geqslant 3$, *given any* $q > n/2$ *and* $p > n$ *one can find a bound for* $u_0 \in L^q(\Omega)$ *and for* $\nabla v_0 \in L^p(\Omega)$ *guaranteeing that* (u, v) *is global in time and bounded, see Theorem 2.1 [174]).*

On the other hand,

(2) *if* $n \geqslant 3$ *and* Ω *is a ball then for arbitrarily small mass* $m > 0$ *there exist* u_0 *and* v_0 *having* $\int_\Omega u_0 = m$ *such that* (u, v) *blows up either in finite or infinite time, see Theorem 3.5 [174].*

In addition, he characterized the large-time behavior of small-data solutions as follows:

(1) *If both* $\|u_0\|_{L^q(\Omega)} < \epsilon$ *and* $\|\nabla v_0\|_{L^p(\Omega)} < \epsilon$ *with* $\epsilon > 0$ *sufficiently small, then solutions* (u, v) *of the model* (1.3) *with* $d = 1$ *satisfies*

$$\|u(\cdot,t) - u_H(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C\epsilon^2 e^{-\lambda_1 t}\ \textit{and} \|\nabla v(\cdot,t) - v_H(\cdot,t)\|_{L^p(\Omega)} \leqslant C\epsilon^2 e^{-\lambda_1 t},$$

for all $t > 1$, *with some constant* $C > 0$, *where* λ_1 *denotes the smallest positive eigenvalue of* $-\Delta$ *in* Ω *and* u_H *and* v_H *are the solutions of* $\partial_t u_H = \Delta u_H$ *and* $\partial_t v_H = \Delta v_H - v_H + u_H$ *under the same initial and boundary data (Theorem 2.1 [174]).*

Winkler [175] proved the global existence of solutions to (1.49) actually it was posted an open problem in [70]. If $0 < \chi < \sqrt{\frac{2}{n}}$, then he generalized the global-in-time classical solution of the model (1.49) for $n \geqslant 2$ with initial conditions $u_0 \in C^0(\overline{\Omega}), u_0 \geqslant 0$ and

$v_0 \in W^{1,\infty}(\Omega)$, $v_0 > 0$ in $\overline{\Omega}$. In addition, he also have established the global existence of weak solutions provided $0 < \chi < \sqrt{\frac{n+2}{3n-4}}$.

If $\chi < \sqrt{2/n}$, then (1.49) has a global solution stated in the below theorem.

Theorem 5.4 *Suppose that $\chi < \sqrt{2/n}$. Then for all $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ satisfying $u_0 \geqslant 0$ and $v_0 > 0$ in $\overline{\Omega}$, then (1.49) has a global classical solution.*

For the proof of Theorem 5.4, we refer to Theorem 3.5 [175].

Theorem 5.5 *Assume that $n \geqslant 2$ and $\chi < \sqrt{(n+2)/(3n-4)}$. Then for all $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ satisfying $u_0 \geqslant 0$ and $v_0 > 0$ in $\overline{\Omega}$, then there exists a global weak solution (u, v) of (1.49).*

For the proof of the above theorem, we refer to Theorem 4.6 [175]. First, Winkler [176] has established the blow-up results for degenerate Keller-Segel system of the type (1.1) with $\phi(u, v) = u$, $\psi(u, v) = u$, $f(u, v) = 0$, $\tau = 1$, $h(u, v) = 1$ and $g(u, v) = u$ under the super critical condition and $u_0 \in C(\overline{\Omega})$ and $v_0 \in C^1(\overline{\Omega})$ in a bounded domain $\Omega \subset \mathbb{R}^n$. He summarised main results as follows:

(i) If Ω is a ball in $\mathbb{R}^n$ for some $n \geqslant 2$, satisfies $\frac{\psi(u)}{\phi(u)}$ grows faster than $u^{2/n}$ as $n \to \infty$, in a certain sense then for any $m > 0$, the model possesses unbounded solutions having mass $\int_\Omega u(x,t)\mathrm{d}x = m$.

(ii) The sufficient conditions for the occurrence of blow-up are
- $n = 2$, $\frac{\psi(u)}{\phi(u)} \geqslant c_0\, u \,\ln\, u$ for some $c_0 > 0$ and sufficiently large u,
- $n \geqslant 3$, and $u^{-\alpha}\frac{\psi(u)}{\phi(u)} \geqslant c_0$ as $u \to \infty$ with some $\alpha > 2/n$;
- $n \geqslant 3$ and $\lim_{u\to\infty} \inf \frac{u(\frac{\psi}{\phi})'(u)}{(\frac{\psi}{\phi})(u)} > \frac{2}{n}$.

The proof of the above results were inspired by the arguments in [74, 143]. In [203], Wu and Zheng considered the Keller-Segel system with fractional diffusion as follows:

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha/2} u = \pm\nabla\cdot(u\nabla v), & \text{in } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ \epsilon\partial_t v + (-\Delta)^{\alpha/2} v = u, & \text{in } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ u|_{t=0} = u_0,\, v|_{t=0} = v_0. \end{cases} \tag{5.15}$$

The Cauchy problem to (5.15) for the initial data u_0, v_0 in critical Fourier-Herz spaces $\dot{\mathcal{B}}_q^{2-2\alpha}(\mathbb{R}^n) \times \dot{\mathcal{B}}_q^{2-\alpha}(\mathbb{R}^n)$, is considered with $q \in [2, \infty)$, where $1 < \alpha \leqslant 2$. By utilizing small initial data and few estimates of the linear dissipative equation in the framework of mixed time-space spaces, the Fourier localization method, the Chemin' mono-norm technique' and the theory of Littlewood-Paley, they have obtained a local and a global well-posedness of the problem. They have summarised their main results as follows

Theorem 5.6 *Let $\alpha \in (1, 2]$ and $(u_0, v_0) \in \dot{\mathcal{B}}_q^{2-2\alpha}(\mathbb{R}^n) \times \dot{\mathcal{B}}_q^{2-\alpha}(\mathbb{R}^n)$. Then there exists $T > 0$ such that the Cauchy problem (5.15) has a unique solution $(u, v) \in \mathcal{L}^{\frac{2\alpha}{\alpha-1}}\left(I; \dot{\mathcal{B}}_2^{\frac{3(1-\alpha)}{2}}\right) \times \mathcal{L}^{\frac{2\alpha}{\alpha-1}}\left(I; \dot{\mathcal{B}}_2^{\frac{(3-\alpha)}{2}}\right)$ and $u \in C(I; \dot{\mathcal{B}}_2^{2-2\alpha}) \cap \mathcal{L}^{\frac{\alpha}{\alpha-1}}\left(I; \dot{\mathcal{B}}_2^{1-\alpha}\right)$, $v \in C(I; \dot{\mathcal{B}}_2^{2-\alpha}) \cap \mathcal{L}^{\frac{\alpha}{\alpha-1}}\left(I; \dot{\mathcal{B}}_2^{1}\right)$.*

For the proof the above theorem, we refer to Theorem 1.1 [203]. Let T^* denotes the maximum life span of the solution. We have the following results: there exists constants $c_1, c_2 > 0$ such that if $\|u_0\|_{\dot{\mathcal{B}}_2^{2-2\alpha}} \leqslant c_1$ and $\|v_0\|_{\dot{\mathcal{B}}_2^{2-\alpha}} \leqslant c_2$, then $T^* = +\infty$.

Theorem 5.7 *If $2 < q \leqslant \infty$ and $\alpha = 2$, then the system (5.15) is ill-posed in $\dot{\mathcal{B}}_q^{-2} \times \dot{\mathcal{B}}_q^0$ and $\dot{\mathcal{B}}_{\infty,q}^{-2} \times \dot{\mathcal{B}}_{\infty,q}^0$.*

The above theorem is proved in [203]. For some models from statistical physics with fractional diffusion, we refer to [13]. Even though, they extended the classical Keller-Segel model to the fractional diffusion, they did not mention about the biological significance of their model. It is worth to describe its biological significance. For the model (1.49), the authors considered (1.49) in a ball in $\mathbb{R}^n$, $n \geqslant 2$. Further, in order to handle the singularity of $\chi(v)$ in (1.49), they have introduced weak power-λ solutions and the regularized problem of (1.49). Within this solution context, the global-in time solutions were proved for arbitrary values of $\chi > 0$ and further, this has extended the already proved global existence (weak) [178] for $\chi < \sqrt{\frac{n+2}{3n-4}}$. Moreover, through the regularity on weak power-λ solutions of (1.49), they assured that the global existence of solutions which satisfy $u(\cdot, t) \in L^r(\Omega)$, for some $r > 0$ and for a.e. $t > 0$.

In [78], Ishida and Yokota obtained the global existence of solutions to quasilinear degenerate KS model. They have established the global weak solution without any restriction on the size of initial data when $q < m + \frac{2}{N}$. In addition to this, the have also assumed that the initial conditions $u_0 \geqslant 0$, $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $v_0 \geqslant 0$, $v_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\Delta v_0 \in L^{p_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, for every $p_0 \gg 1$. It is to be noticed that we can not apply semilinear theory to the problem in the degenerate case. Even though the analysis is difficult, there are some results available for the cases $m > 1$ and $q > 2$, one can refer to [151, 152]. In [77], Ishida and Yokota have answered for the unsolved case of parabolic-parabolic model under the super-critical case, where $q \geqslant m + \frac{2}{N}$ and also they assumed the initial data u_0, v_0 satisfies $u_0 \geqslant 0$, $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $v_0 \geqslant 0$, $v_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\Delta v_0 \in L^{\frac{N}{2}+1}(\mathbb{R}^N) \cap L^{\frac{N}{2}(q-m)+1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

The global existence and boundedness of solutions of the model (1.18) were studied in a bounded convex domain in $\mathbb{R}^3$ by Tao and Winkler [155]. The convexity assumption in [155] was removed in [81]. After the work of Keller and Segel, the researchers have been focused on finding the unbounded solutions of (3.4) with $d = 1$, $\chi = 1$ or, more generally, to determine conditions on the initial data and on certain parameters which either guarantee or rule out the existence of blow-up solutions of (3.4) with $d = 1$, $\chi = 1$ and also for variations of Keller-Segel models. The model (1.32) with $d = 1$, $\alpha = 1$, $\beta = 1$ in a ball $\Omega = B_R \subset \mathbb{R}^n$, $n \geqslant 3$, $R > 0$ considered in [40]. They assumed that the initial data $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ such that $u_0, v_0 > 0$ in $\overline{\Omega}$. By generalizing the idea of M. Winkler [179], they proved the blow-up of solutions to the non-degenerate quasilinear case of (1.32) with $d = 1$, $\alpha = \beta = 1$ in finite time if $q > m + \frac{2}{N}$. That is, they assumed that $\phi, \psi \in C^2([0, \infty))$ and there is a function $\beta \in C^2([0, \infty))$ satisfying $\phi(s) > 0$, $\psi(s) = s\beta(s)$, and $\beta(s) > 0$, for all $s \in [0, \infty)$. In the case of degenerate diffusion, the blow-up of solutions shown in [79] but whether the blow-up time is finite or infinite was unknown. In [30], Carrillo et al. considered the Keller-Segel model with nonlinear cell diffusion and additional cross-diffusion terms (1.47) in up to three space dimensions. The important property of this model is that it admits a new entropy functional which provides a gradient estimates for the cell density and chemical substances. They proved the global existence of weak solutions for the arbitrary small cross-diffusion coefficients and for the suitable exponents of the nonlinear diffusions. There are four methods available to avoid the blow-up in Keller-Segel models which are described as follows:

(i) The first way is the modification of chemotactic sensitivity.
(ii) The modification of the cell diffusion.

(iii) Considering nonvanishing growth-death models.
(iv) The introduction of additional cross-diffusion term in the chemical concentration equation.

It is to be noted that we can not apply the maximum principle to (1.47) because of the introduction of the additional cross-diffusion term, which leads the diffusion matrix of the system to neither symmetric nor positive definitive. They have tackled this issues by defining the following logarithmic entropy functional

$$E_0(u, v) = \int_\Omega \left(u(\log u - 1) + \alpha \frac{c^2}{2\delta} \right) \mathrm{d}x.$$

The blow-up of solutions of (3.4) with $d = 1$, $\chi = 1$ in three space dimension was proved by M. Winkler in [179]. In order to prove blow-up of solutions he used the idea that any solution of (3.4) with $d = 1$, $\chi = 1$ satisfies the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(u(\cdot,t), v(\cdot,t)) \leqslant -\mathcal{D}(u(\cdot,t), v(\cdot,t)) \quad \text{for all } t \in (0, T_{max}(u_0, v_0)), \tag{5.16}$$

where $T_{max}(u_0, v_0) \in (0, \infty]$ denotes the maximum existence time of (u, v) and where for arbitrary smooth positive functions u and v, the energy is defined by

$$\mathcal{F}(u, v) := \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega v^2 - \int_\Omega uv + \int_\Omega u \ln u, \tag{5.17}$$

and the dissipation rate is given by

$$\mathcal{D}(u, v) := \int_\Omega v_t^2 + \int_\Omega u \cdot \left| \frac{\nabla u}{u} - \nabla v \right|^2. \tag{5.18}$$

He summarized his main results as follows:

Theorem 5.8 *Let $\Omega = \mathbf{B}_R \subset \mathbb{R}^n$ with some $n \geqslant 3$ and $R > 0$, and let $m > 0$ and $A > 0$. Then there exist $T(m, A) > 0$ and $K(m, A) > 0$ with the property that given any (u_0, v_0) from the set*

$$\mathcal{B}(m, A) := \{(u_0, v_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) | u_0 \text{ and } v_0 \text{ are radially symmetric and positive in } \overline{\Omega} \text{ with } \int_\Omega u_0 = m, \|v_0\|_{W^{1,2}(\Omega)} \leqslant A \text{ and } \mathcal{F}(u_0, v_0) \leqslant -K(m, A)\},$$

for the corresponding solution (u, v) of (3.4) with $d = \chi = 1$, we have $T_{max}(u_0, v_0) \leqslant T(m, A) < \infty$; that is, (u, v) blows up before or at time $T(m, A)$.

For the proof of Theorem 5.8, we refer to Theorem 1.1 [179]. It is worth to notice that the following Theorem is the first result regarding blow-up of solutions of (3.4) with $d = 1$, $\chi = 1$ in space dimension $n \geqslant 3$.

Theorem 5.9 *Let Ω be as in Theorem 5.8, and suppose that $p \in (1, \frac{2n}{n+2})$. Then, for each $m > 0$ and $A > 0$, the set $\mathcal{B}(m, A)$ defined in Theorem 5.8 is dense in the space of* *all radially symmetric positive functions in* $C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega)$ *respect to the topology in* $L^p(\Omega) \times W^{1,2}(\Omega.)$ *In particular, for any positive radial* $(u_0, v_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega)$ *and any* $\epsilon > 0$ *one can find some radial positive* $(u_{0\epsilon}, v_{0\epsilon}) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega)$ *such that*

$$\|u_{0\epsilon} - u_0\|_{L^p(\Omega)} + \|v_{0\epsilon} - v_0\|_{W^{1,2}(\Omega)} < \epsilon,$$

and such that the solution (u_ϵ, v_ϵ) *of* (3.4) *with* $d = 1$, $\chi = 1$ *with initial data* $(u_\epsilon, v_\epsilon)|_{t=0} = (u_{0\epsilon}, v_{0\epsilon})$ *blows up in finite time.*

For the proof of Theorem 5.9, we refer to Theorem 1.2 [179]. There are many results available for the global-in time existence of solutions to the non-degenerate case of (1.1) with $\phi(u, v) > 0$. As we mentioned in the above, though Winkler proved the blow-up of solutions for the model (1.1) with $\phi(u, v) = u$, $\psi(u, v) = u$, $f(u, v) = 0$, $\tau = 1$, $h(u, v) = 1$ and $g(u, v) = u$, the proof of blow-up solutions for the degenerate case under the super-critical condition, he left it as an open problem. Ishida et al. [80] made an attempt on this problem in $B := \{x \in \mathbb{R}^n; |x| < 1\}$, $n \geqslant 2$, and proved there exits blow-up of solutions under the following conditions:

$$\phi \in C([0, \infty)) \cap C^1((0, \infty)), \phi(0) = 0, \quad \phi > 0 \text{ on } (0, \infty),$$

$$\psi \in C^1([0, \infty)), \psi(0) = 0, \quad \psi > 0 \text{ on } (0, \infty),$$

and there exist $r_0 > 1$, $\epsilon_0 \in (0, 1)$, $K > 0$, $k > 0$ such that

$$\int_{r_0}^{r} \frac{\sigma f(\sigma)}{g(\sigma)} d\sigma \leqslant \begin{cases} K \frac{r}{\log r} (r \geqslant r_0) & \text{if } n = 2, \\ \dfrac{n - 2 - \epsilon_0}{n} \displaystyle\int_{r_0}^{r} \int_{r_0}^{\sigma} \frac{f(\xi)}{g(\xi)} d\xi d\sigma + K r (r \geqslant r_0) & \text{if } n \geqslant 3, \end{cases} \tag{5.19}$$

$$\int_{r_0}^{r} \int_{r_0}^{\sigma} \frac{f(\xi)}{g(\xi)} d\xi d\sigma \leqslant \begin{cases} k r (\log r)^{\theta} \ (r \geqslant r_0) & \text{with some } \theta \in (0, 1) \text{ if } n = 2, \\ k r^{2-\alpha} (r \geqslant r_0) & \text{with some } \alpha > \dfrac{2}{n} \text{ if } n \geqslant 3. \end{cases} \tag{5.20}$$

The conditions (5.19) and (5.20) imply that super-critical condition. In [118], the authors proved the global existence and boundedness of solutions of the model related to (1.11) with $d = 1$ and the logistic source under the assumption in the initial data $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,l}(\Omega)$ for some $l > n$. For the model (1.19) with Dirichlet boundary conditions, the global existence and blow-up of solutions were established in [11]. Due to the degenerate case of their model, they first considered the regularised model and then they have established the existence and blow-up of solutions. They summarized the results as follows:

Theorem 5.10 (*Global existence*) *The system* (1.19) *has a global positive solution* $(u, v) \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty))$.

Theorem 5.11 *We assume that* $u_0, v_0 \in C^1(\overline{\Omega})$ *and* $u_0(x) \geqslant \epsilon_0 \psi(x)$, $v_0(x) \geqslant \epsilon_0 \psi(x)$ *and suppose that, for any* $d > 0$ *and* $u, v > 0$,

$$\left| \frac{f(u, v)}{u^{n+1}} \right| \leqslant \frac{d}{u^{n-\alpha}} + \frac{d}{v^{n-\beta}}.$$

Then the solutions of (1.19) *must blow up in a finite time provided that* $\lambda_1 < d$.

For the proof of Theorem 5.10 and Theorem 5.11, we refer to Theorems 2.1 and 3.1 [11]. The nondegeneracy of blow-up points for the parabolic-parabolic Keller-Segel model (1.20) established in [114]. They considered (1.20) in the dimension $N \geqslant 1$ with $m, \Gamma > 0$ and $\lambda \geqslant 0$. Their results are new even for the standard Keller-Segel model. They summarized the main results as follows:

Theorem 5.12 *Assume either* $1 \leqslant m < 2$ *and* $\Gamma > 0$, *or* $0 < m < 1$ *and* $\Gamma = 1$. *Let* u_0, v_0 *satisfy* $u_0 \in L^\infty(\Omega)$, $v_0 \in W^{1,\infty}(\Omega)$, $u_0, v_0 \geqslant 0$ *and in case* $\Omega = \mathbb{R}^N$, $u_0, v_0 \in L^1(\mathbb{R}^N)$, $\nabla u_0 \in L^r(\mathbb{R}^N)$, *for some* $r \in [1, \infty)$ *if* $m < 1$. *Let* (u, v) *be any solution of* (1.20) *with Neumann boundary conditions and* $u(x,0) = u_0(x)$ *and* $v(x,0) = v_0(x)$ *such that* $T = T_{max}(u, v) < \infty$. *Let* $a \in \overline{\Omega}$, $t_0 \in (0, T)$ *and* $\rho > 0$. *There exists a constant* $\epsilon = \epsilon(N, m, \Gamma) > 0$ *such that if*

$$u(x,t) \leqslant \epsilon (T-t)^{-1/m} \quad \textit{for all } (x,t) \in \Omega_{a,\rho} \times (t_0, T),$$

then a *is not a blow-up point.*

They also have established global-in-space lower blow-up estimate as in the following theorem:

Theorem 5.13 *Let* $m, \Gamma > 0$ *and let* (u, v) *be any solution of* (1.20) *with Neumann boundary conditions and* $u(x,0) = u_0(x)$ *and* $v(x,0) = v_0(x)$ *such that* $T = T_{max}(u, v) < \infty$. *Then*

$$\|u(t)\|_\infty^m + \|u(t)\|_\infty^{2(m-1)} \|\nabla v(t)\|_\infty^2 \geqslant c(T-t)^{-1} \quad \textit{for all } t \in [0, T),$$

where $c = c(N, m, \Gamma) > 0$. (*Here, when* $m < 1$, *we make the convention* $\infty/\infty = \infty$.) *If* $1 \leqslant m < 2$, *then we have in particular*

$$\|u(t)\|_\infty + \|\nabla v(t)\|_\infty^{\frac{2}{2-m}} \geqslant c((T-t)^{-1/m} \quad \textit{for all } t \in \ [0, T).$$

For the proof of Theorem 5.12 and Theorem 5.13, we refer to Theorems 1.1 and 1.2 [114]. For the sake of clearness, due to the vast literature existing on the Keller-Segel system, we shall only mention results that studied for the two and three dimensional setting. In [43], Corrias et al. have focused the issues regarding the system (1.1) with $\phi(u, v) = 1$, $\psi(u, v) = 1$, $f(u, v) = 0$, $d = 1$, $g(u, v) = u$ and $h(u, v) = \alpha$ in $\mathbb{R}^n$, when the initial data belong to scaling invariant Lebesgue spaces. In particular, they have established the global existence of integral solutions, uniqueness, optimal time decay and positivity, together with the uniqueness of self-similar solutions. It is very interesting to see that they have also proved that the solutions behave like self-similar solutions in the absence of the degradation term $\alpha = 0$ and the solution behave like the heat kernel, when the presence of the degradation term $\alpha > 0$. The main results are summarized as follows:

Theorem 5.14 (*Local and global existence*) *Let* $\epsilon > 0$, $\alpha \geqslant 0, u_0 \in L^1(\mathbb{R}^2)$ *and* $v_0 \in \dot{H}^1(\mathbb{R}^2)$. *There exist* $\delta = \delta(\|u_0\|_{L^1(\mathbb{R}^2)}, \epsilon) > 0$ *and* $T = T(\|u_0\|_{L^1(\mathbb{R}^2)}, \epsilon) > 0$ *such that if* $\|\nabla v_0\|_{L^2(\mathbb{R}^2)} < \delta$ *there exist an integral solution* (u, v) *of* (1.1) *with* $\phi(u, v) = 1$, $\psi(u, v) = 1$, $f(u, v) = 0$, $d = 1$, $g(u, v) = u$ *and* $h(u, v) = \alpha$ *in* $\mathbb{R}^n$ *with* $u \in L^\infty((0, T); L^1(\mathbb{R}^2))$ *and* $|\nabla v| \in L^\infty((0, T); L^2(\mathbb{R}^2))$. *Moreover, the total mass* M *is conserved and there exists a constant* $C = C(\epsilon)$ *such that if* $\|u_0\|_{L^1(\mathbb{R}^2)} < C(\epsilon)$, *the solution is global and*

$$t^{(1-\frac{1}{p})} \|u(t)\|_{L^p(\mathbb{R}^2)} \leqslant C(\|u_0\|_{L^1(\mathbb{R}^2)}, \epsilon), \quad t > 0, \tag{5.21}$$

$$t^{(\frac{1}{2}-\frac{1}{r})}\|\nabla v(t)\|_{L^r(\mathbb{R}^2)} \leqslant C(\|u_0\|_{L^1(\mathbb{R}^2)}, \epsilon), \quad t > 0, \tag{5.22}$$

for all $p \in [1, \infty]$ *and* $r \in [2, \infty]$.

For the proof of Theorem 5.14, we refer to Theorem 2.1 [43].

Corollary 5.1 (*Uniqueness and positivity*). *The global solution* (u, v) *given by Theorem 5.14 is unique. Moreover, it is non-negative whenever* u_0 *and* v_0 *are non-negative.*

For the proof of the above Corollary 5.1, we refer to Corollary 2.7 [43].

In order to prove non-negative global integral solutions behave like self-similar solutions for large t, when $\alpha = 0$, they have introduced the following space-time rescaled functions $(\tilde{u}, \tilde{v})$ by

$$u(x,t) = \frac{1}{(t+1)}\tilde{u}\left(\frac{x}{\sqrt{t+1}}, \log(t+1)\right) \quad \text{and} \quad v(x,t) = \tilde{v}\left(\frac{x}{\sqrt{t+1}}, \log(t+1)\right), \tag{5.23}$$

or equivalently

$$\tilde{u}(\xi, s) = e^s u(\xi e^{\frac{s}{2}}, e^s - 1) \quad \text{and} \quad \tilde{v}(\xi, s) = v(\xi e^{\frac{s}{2}}, e^s - 1) \tag{5.24}$$

where $\xi = \frac{x}{\sqrt{t+1}}$ and $s = \log(t+1)$. Then $(\tilde{u}, \tilde{v})$ satisfies the parabolic-parabolic system

$$\tilde{u}_s = \Delta\tilde{u} + \frac{\xi}{2}\cdot\nabla\tilde{u} + \tilde{u} - \nabla\cdot(\tilde{u}\nabla\tilde{v}), \qquad \epsilon\tilde{v}_s = \Delta\tilde{v} + \epsilon\frac{\xi}{2}\cdot\nabla\tilde{v} + \tilde{u}, \tag{5.25}$$

where the differential operators are taken with respect to ξ and $\tilde{u}(\xi, 0) = u_0(\xi)$, $\tilde{v}(\xi, 0) = v_0(\xi)$. For the long time behaviour of solutions, when $\alpha = 0$, we refer to the Proposition 4.2 and Theorem 4.3 [43]. In the presence of the degradation term ($\alpha > 0$), the global solutions behaved like the heat kernel.

Remark 5.1 [43] In the case $\alpha > 0$, the following system

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot(u\nabla v), \\ \epsilon\partial_t v = \Delta v - \alpha v + u, \end{cases} \tag{5.26}$$

is no more invariant under the space time scaling $u_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t)$, $v_\lambda(x,t) = v(\lambda x, \lambda^2 t)$, $\lambda > 0$. Therefore self-similar solutions do not exist. Under this case they proved that the global solutions of (5.26) behaved as the heat kernel $G(t)$ as $t \to \infty$.

To the best of our Knowledge, the paper [144] only considered the Keller-Segel model with Dirichlet boundary conditions. They have established the existence and uniqueness of solutions by using Galerkin's method and non-variational methods. In [94], the global weak solutions are established for (1.65) with $\tau = 1$ and $f(u) = \kappa u - \mu u^2$ and any $\mu > 0$. For sufficiently small κ, the author also proved that the weak solutions become smooth after certain finite time. The global bounded weak solutions of degenerate quasilinear system (1.26) in $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, established in [166]. He assumed that

$$\phi(u) \in C^2([0,\infty)), \phi(u) \geqslant \phi_0 u^{m-1}, \tag{5.27}$$

for all $u \geqslant 0$, and

$$\phi(u) \leqslant \phi_1(u+1)^{k-m}u^{m-1}, \tag{5.28}$$

for all $u \geqslant 0$ with some positive constants ϕ_0, ϕ_1, $m \geqslant 1$ and $k \geqslant 1$. And also assumed that $S(u, v, x) = (s_{ij})_{n\times n}$ is a chemotactic sensitivity matrix with

$$s_{ij} \in C^2([0,\infty)\times[0,\infty)\times\overline{\Omega}), \tag{5.29}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

$$|S(u, v, x)| \leqslant u^{l-2}\tilde{S}(v), \tag{5.30}$$

for all $(u, v, x) \in ([0,\infty)\times[0,\infty)\times\overline{\Omega})$, where $l \geqslant 2$, $\tilde{S}(v)$ is a nondecreasing function on $[0,\infty)$ and $|S|$ denotes the Frobenius norm of matrix and

$$f(u) \in C^1([0,\infty)), \tag{5.31}$$

is a non-negative with $f(0) = 0$. He summarized his results as follows:

Theorem 5.15 *Let $\Omega \subset \mathbb{R}^n$, $(n \geqslant 2)$ be a bounded domain with smooth boundary. Assume that $S(u)$ and $f(u)$ satisfy* (5.29) (5.30) *and* (5.31). *Suppose that $\phi(u)$ satisfies* (5.27) *and* (5.28) *with $m > l - \frac{2}{n}$.*

Then for any choice of the initial data (u_0, v_0) fulfilling $u_0 \geqslant 0, v_0 \geqslant 0, u_0 \in L^\infty(\Omega)$, $W^{1,\infty}(\Omega)$, system (1.26) *possesses at least one non-negative global bounded weak solution (u, v).*

For the proof of Theorem 5.15, we refer to Theorem 1.1 [166]. For suitably arbitrary regular initial data $u_0 \geqslant 0$, $v_0 > 0$, Winkler [183] extended the global existence of generalized solutions of (1.79) from one dimensional to two dimensional setting. His results mainly based on the derivation of a priori estimates of the terms $\nabla \ln(u+1)$ and ∇v in L^2 spaces. Further boundedness and compactness results are derived by using Moser-Trudinger inequality. But Viglialoro [164] provided the definition of very weak solutions to (1.65) with $\tau = 1$ and then he proved that the existence of global solutions under the assumptions of initial condition $(u_0, v_0) \in C^0(\overline{\Omega}) \times C^2(\overline{\Omega})$. It would be interesting to know whether the Keller-Segel model that allows the global existence and the cell aggregation phenomenon at the same time. Very recently, it is investigated by Yoon and Kim. They were introduced a new Keller-Segel model with Fokker-Plank diffusion in [211]. The difference between the Keller-Segel model with Fokker-Planck type and the Keller-Segel type models is in the first equation where the Fokker-Planck type diffusion was introduced. The global existence and the instability of constant steady states were obtained for (1.35) in $\mathbb{R}^n$ in [211]. For the sack of simplicity, they have assumed that a and b are equal to one. Further, they have assumed that the chemotactic sensitivity function $\gamma(v) := c_0 v^{-k}$ and the global existence results were proved for all $k > 0$ with a smallness assumption on $c_0 > 0$. In addition to this, the constant steady states were shown as unstable only if $k > 1$ and $\epsilon > 0$ is small. Moreover, they have found the threshold diffusivity $\epsilon_1 > 0$ and observed that any constant steady state is unstable and an aggregation pattern appears provided if $\epsilon < \epsilon_1$. They have summarized the results as follows:

Theorem 5.16 (*Local existence*) *Let the initial values $f \in C^0(\overline{\Omega})$ and $g \in W^{1,p}(\Omega)(p > n)$ be nonnegative. Then there exists a solution of* (1.35) *in the classical sense, i.e.,*

$$u^d, v^d \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})),$$

where T_{max} denotes the maximal existence time. This solution is unique and nonnegative. Moreover, if $T_{max} < \infty$, then

$$\|u^d(\cdot, t)\|_{L^\infty(\Omega)} + \|v^d(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \to T_{max}.$$

For the proof of Theorem 5.16, we refer to Theorem 2.6 [211].

Theorem 5.17 *Let (u, v) be the solution of* (1.35) *with boundary $\partial_\nu u = \partial_\nu v = 0$ and initial conditions $u(x, 0) = f(x) \geqslant 0$ and $v(x, 0) = g(x) > 0$. The motility function γ is given by $\gamma(v) = \frac{c_0}{v^k}$, $c_0 > 0$, $k > 0$ with a small $c_0 > 0$. Suppose that $t \in (0, T_{max})$ is the maximal time domain for the solution. Then, there exists a constant $C > 0$ such that*

$$\|u\|_{L^\infty(\Omega\times(0,T_{max}))} + \|v\|_{L^\infty(\Omega\times(0,T_{max}))} \leqslant C.$$

Furthermore, the solution is global, i.e, $T_{max} = \infty$.

For the proof of Theorem 5.17, we refer to Theorem 2.9 [211].

For the nonexistence of non-constant steady state solutions, we refer to Theorem 3.3 [211]. The instability conditions for a constant steady state solution is obtained in Theorem 3.4 [211]. Cieslak and Winkler [41] proved the existence of global classical solutions for (1.68) under the assumptions that if ϕ and ψ are smooth enough nonnegative functions which satisfy

$$k_1 e^{-\beta^- s} \leqslant \phi(s) \leqslant k_2 e^{-\beta^+ s} \tag{5.32}$$

for all $s \geqslant 0$ where $k_1 > 0$, $k_2 > 0$, β^+ and $\beta^- \geqslant \beta^+$ then whenever ψ satisfies

$$\frac{\psi(u)}{\phi(u)} \leqslant k_3 s^\alpha \tag{5.33}$$

for all $s \geqslant 0$, with $k_3 > 0$ and $\alpha \in (0, 1)$, and for nonnegative regular initial data, the model (1.68) has a global classical solution and the solution u is bounded. We summarize their main results as

Theorem 5.18 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume that ϕ and ψ satisfy $\phi \in C^{1+\nu}([0, \infty))$ is positive and $\psi \in C^{1+\nu}([0, \infty))$ is nonnegative with $\psi(0) = 0$ and the assumptions* (5.32), (5.33) *hold and $\alpha(0, 1)$. Then for any u_0, v_0 satisfying $u_0 \in W^{1,l}(\Omega)$ for some $l > 2$ with $u_0 > 0$ in $\overline{\Omega}$ and $v_0 \in W^{1,l}(\Omega)$ for some $l > 2$ with $v_0 \geqslant 0$ in Ω, the problem* (1.68) *possesses a uniquely determined global classical solution (u, v) with*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \\ v \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \cap L^\infty_{loc}([0, \infty); W^{1,l}(\Omega)), \end{cases} \tag{5.34}$$

such that both u and v are nonnegative in $\Omega \times (0, \infty)$. Moreover, $\|u(\cdot, t)\|_{L^\infty(\Omega)} \leqslant c$ for all $t > 0$.

For the proof of Theorem 5.18, we refer to Theorem 1.1 [41]. In order to obtain the boundedness of u, they used ODI and the Moser-Trudinger inequality.

In [157], the model (1.35) with $\epsilon = a = b = 1$, is considered in smooth domain when space dimension is greater than or equal to two. As we know already that in the two-dimensional case, the solutions of the classical Keller-Segel model may blow up in finite time under large initial data, in stark contrast to this, the authors in [157] have established the globally bounded classical solution for (1.35) with $\epsilon = a = b = 1$ with suitably regular initial data provided that the uniform positive motility function ϕ belongs to $C^3([0,\infty)) \cap W^{1,\infty}(0,\infty)$. Further, they assumed that the function $k_\phi \leqslant \phi(s) \leqslant K_\phi$, for all $s \geqslant 0$, $|\phi'(s)| \leqslant K_{\phi'}$, for all $s \geqslant 0$ with certain positive constants k_ϕ, K_ϕ and $K_{\phi'}$. Th e initial data are assumed to be in $u_0 \in C^0(\overline{\Omega})$ is non-negative, $u \not\equiv 0$ and $v_0 \in W^{1,\infty}(\Omega)$ is non-negative. As the main results, they summarized as follows:

Theorem 5.19 *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary, and suppose that ϕ satisfies $\phi \in C^3([0,\infty))$, $k_\phi \leqslant \phi(s) \leqslant K_\phi$ and $|\phi'(s)| \leqslant K_{\phi'}$, for all $s \geqslant 0$ with $k_\phi, K_\phi, K_{\phi'} > 0$. Then for all $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ fulfilling $u_0 \in C^0(\overline{\Omega})$ is non-negative, $u \not\equiv 0$ and $v_0 \in W^{1,\infty}(\Omega)$ is non-negative, the problem (1.35) with $\epsilon = a = b = 1$, homogeneous Neumann boundary conditions for u and v, and initial conditions $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$ possesses a global classical solution $(u,v) \in (C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)))^2$ such that both u and v are non-negative in $\Omega \times (0,\infty)$, and such that (u,v) is bounded in the sense that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C \quad \text{for all } t > 0 \tag{5.35}$$

with some constant $C > 0$.

For the proof of Theorem 5.19, we refer to Theorem 1.1 [157]. From the above theorem there is a natural question arises as what about the existence of solutions in the higher dimensional cases? The answer is in the following Theorem:

Theorem 5.20 *Let $n \geqslant 3$, and assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Moreover, suppose that ϕ satisfies $\phi \in C^3([0,\infty))$, $k_\phi \leqslant \phi(s) \leqslant K_\phi$ and $|\phi'(s)| \leqslant K_{\phi'}$, for all $s \geqslant 0$ with $k_\phi, K_\phi, K_{\phi'} > 0$. Then for all $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ fulfilling $u_0 \in C^0(\overline{\Omega})$ is non-negative, $u \not\equiv 0$ and $v_0 \in W^{1,\infty}(\Omega)$ is non-negative, the problem (1.35) with $\epsilon = a = b = 1$, homogeneous Neumann boundary conditions for u and v, and initial conditions $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$ possesses at least one global weak solution in the sense of Definition 5.1 in [157] and this solution can be gained as the limit a.e. in $\Omega \times (0,\infty)$ of solutions (u_ϵ, v_ϵ) to the regularized problems*

$$\begin{cases} \partial_t u_\epsilon = \Delta(\gamma(v_\epsilon)u_\epsilon), \\ \partial_t v_\epsilon = \epsilon\Delta v_\epsilon + f(u_\epsilon) - v_\epsilon, \end{cases}$$

for $\epsilon \in (0,1)$, where $f_\epsilon(s) := \frac{s}{1+s}$, $s \geqslant 0$, along a suitably chosen sequence $(\epsilon_k)_{k\in\mathbb{N}} \subset (0,1)$ such that $\epsilon \searrow 0$ as $k \to \infty$.

For the proof of Theorem 5.20, we refer to Theorem 1.1 [157]. Under the smallness assumptions on the initial data, they also proved the global boundedness of classical solutions. The global-in-time existence of solutions to (1.71) studied in higher dimensions by Mimura in [112]. The author's approach is to rewrite the problem as a gradient flow on the Wasserstein space. Finally, when time step size goes to zero, the author proved the discrete solutions are converge to a weak solution of the main system. The Lyapunov functional corresponding to (1.71) is defined by

$$\phi_m(u,v) := \frac{1}{m-1}\int_\Omega u^m \mathrm{d}x - \chi\int_\Omega uv\mathrm{d}x + \frac{\chi}{2\alpha}|\nabla v|^2 + \gamma v^2\mathrm{d}x.$$

To prove the results, the functional space is $X_M(\Omega) := \{(u,v) \in (L^1(\Omega)\cap L^m(\Omega)) \times H_0^1(\Omega); \|u\|_{L^1} = M, u \geqslant 0, v \geqslant 0\}$ and $\mu_M(\Omega) := \inf_{(u,v)\in X_M(\Omega)} \phi_m(u,v)$, $M_*(\Omega) := \sup\{M \geqslant 0; \mu_M(\Omega) > -\infty\}$. The author has derived theorems based on the properties of M_*. The main global existence result is sated as follows:

Theorem 5.21 *Let $m \geqslant 2 - 2/d$. For any $u_0 \in L^2(\Omega)\cap L^m(\Omega)$ and $v_0 \in H_0^1(\Omega)$ with u_0, $v_0 \geqslant 0$ there exists a weak solution (u,v) of (1.71) with this initial data that exists globally for all $t \geqslant 0$ provided that u_0 satisfies $\int_\Omega u_0 \mathrm{d}x < M_*$.*

For the proof of Theorem 5.21, we refer to Theorem 1.4 [112].

It was unknown that whether the solutions of the model (1.37) will exist globally, when $\mu > 0$ in higher dimensions. Wang and Wang answered for this question in [167]. As we explained earlier, the strong logistic source term in classical Keller-Segel model will prevent blow-up of solutions. The authors were introduced a strong logistic term in the model in order to prevent blow-up of solutions in the higher dimensions. When $\mu > 0$ is large, they proved the global existence and boundedness of the solution to (1.37). For their analysis of the model (1.37), they assumed that

(i) $\gamma \in C^3([0,\infty))$, $\gamma(s) > 0$, $\gamma'(s) < 0$ and $|\gamma'(s)| \leqslant R$ in $[0,\infty)$ with positive constant R and $\lim_{v\to\infty}\gamma(v) = 0$.
(ii) The initial data u_0, v_0 are satisfy $0 \leqslant u_0 \neq$ and $0 \leqslant v_0 \neq 0$, $u_0, v_0 \in W^{1,\infty}(\Omega)$.

They summarized the main results as follows:

Theorem 5.22 *Let $a > 0, n \geqslant 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with a smooth boundary. Suppose that conditions $\gamma \in C^3([0,\infty))$, $\gamma(s) > 0$, $\gamma'(s) < 0$ in $[0,\infty)$, $\lim_{s\to\infty}\gamma(s) = 0$ and $\lim_{s\to\infty}\frac{\gamma'(s)}{\gamma(s)}$ exists and (i) hold. Then there exists $\hat{\mu} > 0$ such that, when $\mu \geqslant \hat{\mu}$, the problem (1.37) admits a unique nonnegative global classical solution $(u,v) \in [C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty))]^2$, which is bounded in the sense that there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leqslant C, \quad t > 0.$$

In [83], the authors considered the model (1.38) in the space of dimension two and assumed that the motility function $\gamma(v) \in C^3([0,\infty))$, $\gamma(v) > 0$, $\gamma'(v) < 0$ for all $v \geqslant 0$, and in $\lim_{v\to\infty}\gamma(v) = 0$ and $\lim_{v\to\infty}\frac{\gamma'(v)}{\gamma(v)}$ exists. This proposed model has very interesting biological properties. This density-suppressed will produce spatio-temporal pattern formation through self-trapping. The major difficulty in the analysis of the model (1.38) is the degeneracy of diffusion. They derived a priori L^∞-bound of v to eliminate the degeneracy and proved the global existence of classical solutions with uniform boundedness in-time by using the function $\gamma(v)$ as weight function and applying method of weighted energy estimates. Moreover, by constructing a Lyapunov functional, they obtained the large time behavior of solutions as well. They stated the main results as follows:

Theorem 5.23 *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary and the assumption $\gamma(v) \in C^3([0,\infty))$, $\gamma(v) > 0$, $\gamma'(v) < 0$ for all $v \geqslant 0$, and in $\lim_{v\to\infty} \gamma(v) = 0$ and $\lim_{v\to\infty} \frac{\gamma'(v)}{\gamma(v)}$ exists, holds. Suppose that $(u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$ with $u_0, v_0 \geqslant 0 (\neq 0)$. Then, the problem (1.38) has a unique nonnegative global solution $(u, v) \in [C^0([0,\infty) \times \overline{\Omega}) \cap C^{2,1}((0,\infty) \times \overline{\Omega}) \cap L^\infty_{loc}([0,\infty); W^{1,\infty}(\Omega))]^2$ satisfying*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leqslant C, \quad \text{for all } t > 0,$$

where $C > 0$ is a constant independent of t. Furthermore, if $\mu > \frac{K_0}{16}$ with $K_0 = \max_{0\leqslant v\leqslant\infty} \frac{|\gamma'(v)|^2}{\gamma(v)}$, the constant steady state (1.38) is globally asymptotically stable in the sense that

$$\lim_{t\to\infty} (\|u(\cdot,t) - 1\|_{L^\infty(\Omega)} + \|v(\cdot,t) - 1\|_{L^\infty(\Omega)}) = 0.$$

For the proof of Theorem 5.23, we refer to Theorem 1.1 [83]. Besides these results by Tao and Winkler and those of Yoon and Kim, and Jin, Kim, Wang, and, Wang and Wang, we are not aware of any other results regarding the model related to (1.35).

It is worth to notice that their resulting model is closely related to the logarithmic model. Even though Winkler has introduced the Lyapunov function to prove the blow-up of solutions, it does not work well for the degenerate parabolic-parabolic model (1.17) with $\tau = 1$. However, the authors in [67] devised the new estimates for the Lyapunov function. The finite time blow-up of solutions of quasilinear degenerate parabolic-parabolic model of the form (1.17) with $\tau = 1$ studied in [67]. They considered the model in a ball of $\mathbb{R}^N$, $N \geqslant 2$, $m \geqslant 1$, $q \geqslant 2$. Further, they have answered for the unsolved problem stated as whether the blow-up time is finite or infinite for quasilinear degenerate model considered in [67]. That is, when $q > m + \frac{2}{N}$, they have established the finite time blow-up of energy solutions.

When we are exploring the literature, it is exciting to know the existence results of the full Keller-Segel chemotaxis model. In this direction, under the suitable regularity assumptions on the initial data, the authors in [76] have proved that the full Keller-Segel model (1.48) is well-posed on nonsmooth domains. Their idea is based on an abstract theorem by Amann for nonlocal quasilinear equations. Seemingly, this is the first result concerning the full Keller-Segel model. Their main aim was to prove the existence of local-in-time solutions of (1.48) in a general nonsmooth domain, that is, in a Lipschitz domain. In addition to the classical Keller-Segel model, there are multi species Keller-Segel models are also available in the literature. In the following table, we review the existence, blow-up, boundedness of solutions of parabolic-parabolic Keller-Segel models. In [82], the authors have considered the model (1.36) with consumption of chemoattractant in three space dimension and established the blow-up of solutions of (1.36). Yan and Li [209] proved that the existence of generalized solution to the model (1.15) under the condition that $m > 1 + \frac{n-2}{2n}$. To be precise, they stated their main result as follows:

Theorem 5.24 *Let $\Omega \subset \mathbb{R}^n, n \geqslant 2$ be a bounded domain with smooth boundary. Then for all u_0 and v_0 satisfying $u_0 \in C^0(\overline{\Omega}), u_0 \geqslant 0$ in $\Omega, u_0 \not\equiv 0$, $v_0 \in W^{1,\infty}(\Omega)$, $v_0 > 0$ in $\overline{\Omega}$, the model (1.15) for $m > 1 + \frac{n-2}{2n}$ has at least one global generalized solution in the sense of Definition 2.1(see p. 290, [209]).*

For the proof of Theorem 5.24, we refer to Theorem 1.1 [209]. As far as we know, for large initial data, the global existence of classical solutions to (1.15) with $m = 1$ are available in the one dimensional case only (see [106, 160]). However, the same results available [107] to the particular case of (1.15) with $m=1$ without the diffusion term in the second equation in higher dimension case provided for smallness on initial data. Without any restriction on the initial data and space dimension, Winkler [186] proved the existence of global generalized solution to (1.15) with $m=1$ in the case of radially symmetric. He assumed that $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geqslant 2$, $R > 0$ and initial data satisfy

$$\begin{cases} u_0 \in C^0(\overline{\Omega}) \text{ is radially symmetric with } u_0 \geqslant 0 \text{ in } \Omega, \\ v_0 \in W^{1,\infty}(\overline{\Omega}) \text{ is radially symmetric with } v_0 > 0 \text{ in } \overline{\Omega}, \end{cases} \tag{5.36}$$

and he summarized his results as follows:

Theorem 5.25 *Let $n \geqslant 2$, $R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$, and suppose that u_0, v_0 satisfy (5.36). Then there exists at least one pair (u, v) of radially symmetric functions*

$$\begin{cases} u \in L^\infty((0,\infty); L^1(\Omega)) \cap C^0((\overline{\Omega}\backslash\{0\} \times [0,\infty)) \cap C^{2,1}((\overline{\Omega}\backslash\{0\} \times (0,\infty)), \\ v \in L^\infty(\Omega \times (0,\infty)) \cap C^0((\overline{\Omega}\backslash\{0\} \times [0,\infty)) \cap C^{2,1}((\overline{\Omega}\backslash\{0\} \times (0,\infty)), \end{cases} \tag{5.37}$$

such that $u \geqslant 0$ and $v > 0$ in $\overline{\Omega}\backslash\{0\} \times [0,\infty)$, that

$$\frac{\nabla u}{u+1} \in L^2_{loc}(\overline{\Omega} \times [0,\infty)) \quad \text{and} \quad \frac{\nabla v}{v} \in L^2_{loc}(\overline{\Omega} \times [0,\infty)),$$

and that (u, v) is a global renormalized solution of (1.1) with $m=1$ in the sense of Definition 2.9 in Sect. 2. Moreover, (u, v) solves (1.15) with $m=1$ classically in $(\overline{\Omega}\backslash\{0\} \times [0,\infty))$.

For the proof of Theorem 5.25, we refer to Theorem 1.1 [186]. Lankeit and Lankeit studied the global existence of classical solution to the model (1.16) with $k=2$ in [98]. In particular, they have established it under the assumptions of $0 < \chi < \sqrt{\frac{2}{n}}$ and $\mu > \frac{n-2}{n}$, $n \geqslant 2$, $r \geqslant 0$ and the initial data $u_0 \in C^0(\overline{\Omega})$, $v_0 \in W^{1,\infty}(\Omega)$, $u_0 \geqslant 0$, $v_0 > 0$ in $\overline{\Omega}$. When we look for the range of the parameter χ, the immediate question arises as what will happen in other range of the parameter χ? In this direction, Lankeit and Lankeit answered for the following question:

(i) What will happen for small values of $\mu > 0$ when $n \geqslant 3$, and
(ii) what if the assumption $\chi < \sqrt{\frac{2}{n}}$ is removed?

For any value $\mu, r, \chi > 0$, $k=2$ and initial data of the model (1.16) in $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, the generalized global solution is proved by Lankeit and Lankeit in [97]. The main result is presented as follows:

Theorem 5.26 *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$ be a smooth and bounded domain and let $u_0 \in C^0(\overline{\Omega})$ be nonnegative and $v_0 \in W^{1,\infty}(\Omega)$ be positive in $\overline{\Omega}$. Let the chemotactic coefficient $\chi \geqslant 0$, $r \geqslant 0$, $\mu > 0$ be arbitrary. Then the model (1.16) with $k=2$ possesses a global generalized solution in the sense of Definition* 2.12.

For a proof of Theorem 5.26, we refer to Theorem 1.1 [97]. But for the same model (1.16) in $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, Zhao and Zheng [216] showed the global existence of classical solution under the conditions that $k > 1$ for $n=1$ or $k > 1 + \frac{n}{2}$ for $n \geqslant 2$. Moreover, the asymptotic behaviour of solutions was also established under suitable assumptions. The main results are summarized as follows:

Theorem 5.27 *Under the initial data*

$$\begin{cases} u_0(x) \in C^0(\bar(\Omega)), u_0(x) \geqslant 0, & \text{with } u_0(x) \not\equiv 0, x \in \overline{\Omega} \\ v_0(x) \in W^{2,\infty}(\Omega), v_0(x) > 0, & x \in \overline{\Omega}, \text{ and } \frac{\partial v_0}{\partial n} = 0, x \in \partial\Omega, \end{cases} \tag{5.38}$$

the model (1.16) *has a unique positive global classical solution provided* $k > 1$ *with* $n = 1$ *or* $k > 1 + \frac{n}{2}$ *with* $n \geqslant 2$.

Theorem 5.28 *Let* $n = 2$. *If* $k > 2$, *there exists* $\mu_* > 0$ *such that*

$$\left(u, v, \frac{|\nabla v|}{v}\right) \to \left((r/\mu)^{\frac{1}{k-1}}, 0, 0\right) \quad \text{in } L^\infty(\Omega) \text{ as } t \to \infty \tag{5.39}$$

provided $\mu > \mu_*$.

For the proof of Theorem 5.27 and Theorem 5.28, we refer to Theorems 1 and 2 [216]. Li [102] proved the global existence of classical solutions to model (1.61), which is uniformly bounded. Moreover, Li assumed that the model in a bounded domain with smooth boundary in $\mathbb{R}^3$ but without convexity of the domain and the function $|\psi(u)| \leqslant |\chi|u^q$, $u \geqslant 0$ with $\chi \in \mathbb{R}$, $q > 0$. In addition, the logistic source function $f(u)$ satisfies $f(u) \leqslant a - bu^\alpha$, $u \geqslant 0$, where $a \geqslant 0, b > 0$ and $\alpha \geqslant 1$, the production function is $0 \leqslant g(u) \leqslant ku^\gamma$, $u \geqslant 0$ and the initial data $u_0(x) \in C(\overline{\Omega})$, $v_0(x) \in W^{1,\infty}(\Omega)$, $u_0 \not\equiv 0$, $v_0 \not\equiv 0$ in $\overline{\Omega}$.

For the two species Keller-Segel model (1.83), the global existence of nonnegative solutions is proved in [63]. The global existence of classical solutions is established in [191] for (1.1) with $f(u, v) = 0$, $g(u, v) = u$, $h(u, v) = 1$ and $\tau = d = 1$. Under the assumptions that if $\phi(u, v)$ is positive smooth function and decays at most algebraically with respect to u, then it is shown that $\psi(u, v) = \psi(u)$ with $\psi(0) = 0$. Furthermore, if $\Omega \subset \mathbb{R}^n, n \geqslant 2$ is a ball, then infinite time blow-up of solution is shown under some additional assumptions on $\phi(u, v) = (u + 1)^{m-1}$ and $\psi(u, v) = u(u + 1)^{\sigma-1}, u \geqslant 0, v \geqslant$ and $m, \sigma \in \mathbb{R}$ satisfy $m - \frac{n-2}{n} < \sigma \leqslant 0$ and the author assumed that the set of initial data is dense in the set of positive and radially symmetric functions over $\overline{\Omega}$. It is important to observe that the blow-up of solutions of more general quasilinear models depends on the size of ψ relative to ϕ. For the global solvability, he assumed that

$$\begin{cases} \phi \in C^2([0, \infty)^2) \text{ satisfies } \phi > 0 \text{ in } [0, \infty)^2 \\ \psi \in C^2([0, \infty)^2) \text{ is nonnegative and such that } \psi(0, v) = 0 \text{ for all } v \geqslant 0. \end{cases} \tag{5.40}$$

Further, suppose that ϕ satisfies

$$\phi(u, v) \geqslant k(v)(u + 1)^{m-1} \quad \text{for all } u \geqslant 0, v \geqslant 0, \tag{5.41}$$

where $k(v)$ is a nonincreasing positive function and some $m \in \mathbb{R}$. In addition to this assumption, we also need

$$\psi(u, v) \leqslant c, \quad \text{for all } u \geqslant 0, v \geqslant 0, \tag{5.42}$$

where $c > 0$ and

$$\frac{\partial \psi(u, v)}{\partial v} \geqslant -c_\psi u^{-\lambda}(v + 1)^{-\mu}, \quad \text{for all } u \geqslant 0, v \geqslant 0, \tag{5.43}$$

where $c_\psi > 0$, $\lambda > 0$, $\mu \in \mathbb{R}$. He summarized the existence result as follows:

Theorem 5.29 *Let $n \geqslant 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and suppose that ϕ and ψ satisfy (5.40) as well as (5.41)-(5.43) fulfilling $\lambda > \frac{n-2}{n}(1-\mu)_+$. Then for any choice of u_0, v_0 satisfying $u_0 \in W^{1,\infty}(\Omega)$ with $u_0 > 0$ in $\overline{\Omega}$ and $v_0 \in W^{1,\infty}(\Omega)$ with $u_0 > 0$ in $\overline{\Omega}$, the problem (1.1) with $f(u,v)=0$, $g(u,v)=u$, $h(u,v)=1$, $\phi(u,v)=\phi(u)$, $\psi(u,v)=\psi(u)$ and $\tau=d=1$ possesses a global classical solution (u,v) such that*

$$\begin{cases} u \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)), \\ v \in \cap_{q>n} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)), \end{cases} \tag{5.44}$$

and both u and v are positive in $\overline{\Omega}\times(0.\infty)$.

For the proof of Theorem 5.29, we refer to Theorem 1.1 [191]. For the proof of unbounded solutions, we refer to Theorem 1.3 [191]. The global existence of weak solutions to one dimensional case of (1.65) with $f(u)=\tau=0$ is studied in [195]. To prove his results, he assumed that $\mu_0 \in \mathcal{M}(\overline{\Omega})$, $\mu_0 \geqslant 0$, $v_0 \in L^2(\Omega)$, $v_0 \geqslant 0$, where $\mathcal{M}(\overline{\Omega})$ denotes the space of Radon measures over $\overline{\Omega}$ and used the identification $\mathcal{M}(\overline{\Omega}) \equiv (C^0(\overline{\Omega}))^*$. Tao and Winkler considered Keller-Segel type new model (1.38) in [158] in a bounded convex domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, which describes the processes of stripe pattern formation in biological tissues. They established the following results:

(i) obtained the globally bounded classical solutions in two dimension to (1.38) with $a=0$, for all sufficiently regular initial data and the uniformly positive motility function $\gamma \in C^3([0,\infty)) \cap W^{1,\infty}((0.\infty))$.
(ii) proved the global existence of weak solutions in higher dimensional setting and furthermore this solution is classical and bounded in three dimensional only provided the initial data are sufficiently small in $L^2(\Omega) \times W^{1,4}(\Omega)$.

Studying blow-up rate is very important to understand the blow-up phenomena. Very recently, Mizoguchi [115] has proved that each blow-up is of type II without any extra conditions to (1.3) with $d=\chi=\gamma=1$. The author has used a new approach to generalize the case rather than improving [113], where the main ingredient therein is to combine both isolation of blow-up points and backward uniqueness theorem in a half space. Here is the main result:

Theorem 5.30 *Let Ω be a smoothly bounded domain in $\mathbb{R}^2$ or $\Omega=\mathbb{R}^2$. If a solution of (1.3) with $d=\chi=\gamma=1$ blow-up in finite time then the blow-up is type II.*

For the proof of Theorem 5.30, we refer to Theorem 1.1 [115]. The authors have considered the model (1.45) in [6] and established the existence of weak solutions to the considered system by using Schauder's fixed point theorem, a priori energy estimates and the compactness results. They used the crucial assumption $d > \frac{\delta}{2} + \frac{\chi}{2\gamma}\mathcal{K}$ for the regularised system of (1.45) to prove the uniform positive definiteness of the diffusion matrix provided $\gamma \geqslant 1$, χ, d_1, d_2 are positive constants. First, they proved existence of weak solutions to the regularised system and then they shown the existence for the original system (1.45) by letting $\epsilon \to 0$.

In the past few years, there are many findings in the literature related to chemotaxis in fluid environment. In this direction, Winkler [184] considered (1.92) in an open bounded domain in $\mathbb{R}^3$ with the assumptions on $\Phi \in W^{2,\infty}(\Omega)$, $f \in C^1([0,\infty))$ and $\chi \in C^2([0,\infty))$ are nonnegative and $f(0)=0$. Basically this model describes the mutual interaction of populations of swimming aerobic bacteria with the surrounding fluid. He obtained the global weak solutions from derived energy estimates and compactness arguments. Under the assumptions on arbitrary regular initial data and $S(n) \leqslant k_s(n+1)^{-\alpha}$, $\forall n \geqslant 0$ and $\alpha > \frac{1}{3}$, the global existence and boundedness of classical solutions to (1.94) in three dimensional setting are proved by Wang and Xiang in [168]. Winkler [187] investigated the global existence of solutions to (1.94) with $|S(n)| \leqslant k_s(n+1)^{-\alpha}$, $\forall\, n \geqslant 0$ and some $\alpha, k_s > 0$. He has ensured that the assumption $|S(n)| \leqslant k_s(n+1)^{-\alpha}$, $\forall\, n \geqslant 0$ and $\alpha > \frac{1}{3}$ is sufficient to prevent forming singularities in (1.94). In addition to this, he also assumed that $\phi \in C^2(\overline{\Omega})$ and $f \in C^1(\overline{\Omega} \times [0,\infty); \mathbb{R}^3) \cap L^\infty(\Omega \times (0,\infty); \mathbb{R}^3)$. Due to the refined modelling approach, Winkler considered the matrix valued chemotactic sensitivity in [189], in particular, he considered the model (1.95) in a bounded domain in $\mathbb{R}^2$. This modifications brought new challenges in mathematical analysis of this model. For this model, he proved the global mass-preserving generalized weak solutions under smooth enough functions S, f and Φ and in addition S is bounded and f is nonnegative function.

Black et al. [18] proved the global existence of solutions to (1.93) in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ if

$$\chi < \begin{cases} \infty, & n = 2, \\ \frac{5}{3}, & n = 3. \end{cases}$$

Wang et al. considered (1.97) in a bounded convex domain in [170]. From their results, it can also be noted that we could prevent blow-up for (1.97) by choosing chemotactic sensitivity function in such a way that its algebraic saturation is very small. That is, for $c > 0$, and $\alpha > 0$, the function ψ satisfies $|\psi(x,n,c)| \leqslant c(1+n)^{-\alpha}$ for all $x \in \overline{\Omega}, n \geqslant 0, c \geqslant 0$. Authors also assumed that the potential function $\Phi \in W^{2,\infty}(\Omega)$ and $\psi \in C^2(\overline{\Omega} \times [0,\infty)^2; \mathbb{R}^{2\times 2})$ To obtain global-in-time bounded classical solutions, authors also assumed that the potential function $\Phi \in W^{2,\infty}(\Omega)$ and $\psi \in C^2(\overline{\Omega} \times [0,\infty)^2; \mathbb{R}^{2\times 2})$. Furthermore, their analysis were performed by using three energy functionals. Winkler [194] considered the model (1.85) in a smooth bounded planar domains. It is surprising to see his results regarding the model (1.85), that is, in contrast to the classical Keller-Segel system, he proved the global classical existence of solutions to (1.85) for arbitrarily large initial data. He summarized the main result as follows:

Theorem 5.31 *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, that $\Phi \in W^{2,\infty}(\Omega)$ holds and that n_0, c_0 and u_0 satisfy $n_0 \in C^0(\overline{\Omega})$ is nonnegative with $\overline{n_0} > 0$, $c_0 \in W^{1,\infty}(\Omega)$ is nonnegative and $u_0 \in W^{2,2}(\Omega; \mathbb{R}) \cap W^{1,2}_\sigma(\Omega)$, $W^{1,2}_{0,\sigma}(\Omega) := W^{1,2}_0(\Omega; \mathbb{R}^2) \cap L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega; \mathbb{R}^2) | \nabla \cdot \varphi \in \mathcal{D}(\Omega)\}$. Then there exist functions n, c and u uniquely determined by the inclusions*

$$\begin{cases} n \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ c \in \cap_{q>2} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{1,2}(\overline{\Omega} \times (0,\infty)), \\ u \in \cap_{\alpha \in (\frac{1}{2},1)} C^0([0,\infty); \mathcal{D}(A^\alpha)) \cap C^{1,2}(\overline{\Omega} \times (0,\infty); \mathbb{R}^2) \end{cases} \tag{5.45}$$

such that $n > 0$ and $c \geqslant 0$ in $\overline{\Omega} \times (0,\infty)$ and that (1.85) is satisfied in the classical sense with some $P \in C^0(\overline{\Omega} \times [0,\infty))$.

For the proof of Theorem 5.31, we refer to Theorem 1.1 [194], where $A = \mathcal{P}\Delta$ is the realization of the Stokes operator in $L^2(\Omega; \mathbb{R}^2)$ and $\mathcal{D}(\Omega) = W^{2,2}(\Omega; \mathbb{R}^2) \cap W^{1,2}_{0,\sigma}(\Omega)$ and $\mathcal{P}$ is denoting the Helmholtz projection on $L^2(\Omega; \mathbb{R}^2)$. The corresponding chemoattraction system of (1.85) possesses exploding solutions. With some general functions $D(u)$ and $S(u)$ in (1.87), for some relaxing effects of repulsion when compared with attraction, one can see ([53, 93]). The pointwise convergence of solutions to (1.95) and its associated parabolic-elliptic counterpart is addressed in [172]. Furthermore, the authors applied the general result to other two problems. For regular initial data, Winkler [198] also proved the global existence of classical solution (n, c, u, P) to (1.86). Further, he obtained that the cell density n is still uniformly bounded. The summarized result is as follows:

Theorem 5.32 *Let $\Omega \subset \mathbb{R}^2$ a bounded domain with smooth boundary and assume that $\Phi \in W^{1,\infty}(\Omega)$ holds and that n_0 and u_0 comply with $n_0 \in C^0(\overline{\Omega})$ is nonnegative with $\overline{n_0} > 0$ and that $u_0 \in W^{2,2}(\Omega;\mathbb{R}^2) \cap W^{1,2}_{0,\sigma}(\Omega)$. Then there exist functions*

$$\begin{cases} n \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ c \in C^{2,0}(\overline{\Omega} \times (0,\infty)), \\ u \in \bigcap_{\alpha \in (\frac{1}{2},1)} C^0([0,\infty); \mathcal{D}(A^\alpha)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty); \mathbb{R}^2) \quad \textit{and} \\ P \in C^{1,0}(\Omega \times (0,\infty)) \end{cases} \tag{5.46}$$

uniquely determined up to addition of constants to P, such that $n > 0$ and $c > 0$ in $\overline{\Omega} \times (0,\infty)$ that (n, c, u, P) solves (1.86) classically in $\Omega \times (0,\infty)$ and that

$$\sup_{t>0} \|n(\cdot,t)\|_{L^\infty(\Omega)} < \infty.$$

For the proof of Theorem 5.32, we refer to Theorem 1.1 [198]. Kang et al. [85] improved the result of [156] to the whole space $\mathbb{R}^3$ instead of proving in a bounded domain with the restriction $\mu \geqslant 23$. To prove their results, they assumed that the assumptions for $d = 2, 3$ and $n_0 \in (L^1 \cap H^2)(\mathbb{R}^d)$, $c_0 \in (L^q \cap H^3)(\mathbb{R}^d)$, $u_0 \in H^3(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $n_0 \geqslant 0$, $c_0 \geqslant 0$ and for given $q > 2 - \frac{1}{d}$, $d = 2, 3$, a number r satisfies $2 < r < 2q$ if $d = 2$ and $\frac{5}{3} \leqslant r < \frac{3q}{2}$ if $d = 3$ and let $r' := \frac{dr}{d+r}$. The initial data (n_0, c_0, u_0) satisfy

$$\begin{cases} n_0 \in (L^1 \cap H^2)(\mathbb{R}^2), c_0 \in (L^{\frac{r}{2}} \cap W^{1,r} \cap W^{1,r'} \cap H^3)(\mathbb{R}^2), \\ u_0 \in (W^{2,r'} \cap H^3)(\mathbb{R}^2), \text{ if } d = 2, \\ n_0 \in (L^1 \cap H^2)(\mathbb{R}^3), c_0 \in (L^{\frac{3r}{5}} \cap W^{1,r} \cap H^3)(\mathbb{R}^3), \\ u_0 \in (W^{2,r'} \cap H^3)(\mathbb{R}^3) \text{ if } d = 3, \\ n_0(x), c_0(x) \geqslant 0, x \in \mathbb{R}^3. \end{cases} \tag{5.47}$$

The existence of regular solution is summarized as follows:

Theorem 5.33 *Let $\mu > 0$. Suppose that either $q \geqslant 2, (\kappa, d) = (1, 2)$ or $q > 2$, $(\kappa, d) = (0, 3)$. If the initial data (n_0, c_0, u_0) satisfy the above stated assumption, then (1.89) possesses the unique regular solution (n, c, u) of (1.89) satisfying for any $T < \infty$ $(n, c, u) \in L^\infty(0, T; H^2(\mathbb{R}^d) \times H^3((R)^d) \times H^3(\mathbb{R}^d))$, $(\nabla n, \nabla c, \nabla u) \in L^2(0, T; H^2(\mathbb{R}^d) \times H^3((R)^d) \times H^3(\mathbb{R}^d))$.*

For the proof of Theorem 5.33, we refer to Theorem 1.1 [85]. To prove there exists at least one global weak solution to (1.90), Winkler [196] assumed that $\sup_{t>0} \int_t^{t+1} \|g(\cdot,t)\|_{L^{\frac{6}{5}}} \mathrm{d}s$ to be finite and the initial data are sufficiently regular. Moreover, he proved that the global weak solutions are bounded uniformly in the space $L^1(\Omega) \times L^6(\Omega) \times L^2(\Omega; \mathbb{R}^3)$. For more results related to Keller-Segel Navier-Stokes models one can refer [51, 92, 103, 105, 154, 181, 182, 192, 212, 218] and the references therein. As far as we know, the solutions behaviour far from equilibrium seems not well understood for parabolic-parabolic Keller-Segel models and models with high diffusion as well. In this scenario, Winkler developed an approach for parabolic-parabolic Keller-Segel model in [185]. His results will help for the better understanding of the dynamical emergence of structures in models with large diffusion. He provided the results by focusing only on the proliferation parameter of the model. The main result is summarized as follows:

Theorem 5.34 *Let $n \geqslant 3$ and $\Omega = B_R(0) \subset \mathbb{R}^n$ with some $R > 0$, let $a \geqslant 0$ and suppose that $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ are radially symmetric and positive in $\overline{\Omega}$. Then for all $K > 0$ and each $T \in (0, 1)$ there exist sequences $(u_{0k})_{k\in\mathbb{N}} \in C^0(\overline{\Omega})$ and $(v_{0k})_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega)$ of radially symmetric positive functions u_{0k} and v_{0k} on $\overline{\Omega}$ such that $\int_\Omega u_{0k} = \int_\Omega u_0$ for all $k \in \mathbb{N}$, that $u_{0k} \to u_0$ in $L^p(\Omega)$ for all $p \in [1, \frac{2n}{n+2})$ and $v_{0k} \to v_0$ in $W^{1,2}(\Omega)$ as $k \to \infty$ and that for all $k \in \mathbb{N}$, and $\epsilon \in (0, 1)$ one can find $t_{\epsilon,k} \in (0, T)$ with the property that (1.72) with ϵ instead of μ, possesses a classical solution $(u_{\epsilon,k}, v_{\epsilon,k}) \in (C^0(\overline{\Omega} \times [0, t_{\epsilon,k}]) \cap C^{2,1}(\overline{\Omega} \times (0, t_{\epsilon,k})))^2$ which is such that $u_{\epsilon,k}(x_{\epsilon,k}, t_{\epsilon,k}) > \frac{K}{\epsilon}$ for some $x_{\epsilon,k} \in \Omega$.*

For the proof of Theorem 5.34, we refer to Theorem 1.1 [185]. Based on Theorem 5.34, he also added one more result which deals with the effects of large chemotactic sensitivities with fixed logistic source. Urban crime propagation model (1.74) is considered in [190]. The author assumed that the functions $B_1, B_2 \in C^{\nu}_{loc}(\overline{\Omega} \times [0, \infty))$, $\nu \in (0, 1)$ are nonnegative and radially symmetric and initial data $u_0 \in C^{\nu}_{loc}(\overline{\Omega})$ $v_0 \in W^{1,\infty}(\Omega)$ are radially symmetric and nonnegative. Under these assumptions, the global existence of renormalized solution to (1.74) is proved. In [199] Winkler considered the model (1.4) in a ball which is a subset of $\mathbb{R}^n$, $n \geqslant 2$. He asserted that any radial classical solution (which has finite time blow-up) has a unique blow-up profiles and satisfying pointwise upper inequality as well, which can be obtained through pointwise time-independent estimates of radially symmetric solutions. Further, the next result states that the extensibility of non-global solution beyond its blow-up time which is avoiding breakdown of well-posedness of the model (1.4). As we know from the above discussed results, mostly the global existence and blow-up of solutions depend on the initial data. While proving existence of the global solutions to the degenerate Keller-Segel model (1.81), Wang et al. [171] derived a new relationship between the sharp constant of Sobolev inequality and initial data. Thus the range of exponent of diffusion coefficient is $\frac{2n}{2+n} < m < 2 - \frac{2}{n}$. They have shown that the smallness of initial condition is not necessary to prove the global existence of weak solution to (1.81). We would notice that to perform the analysis, they transformed the original model (1.81) into gradient flow structures. Very recently, the existence of nonnegative weak solutions to the time fractional Keller-Segel system is studied in [3].

6 Boundedness of Solutions of Parabolic-Parabolic Models

Horstmann et al. [75] studied the global existence and boundedness of solutions to (1.52) with $\phi(u) = 1$ and $f(u) = 0$. The authors have assumed that $\psi \in C^{1+\theta}([0, \infty))$ and the nonnegative initial data $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in \cup_{q>n} W^{1,q}(\Omega)$. The main result is the following

Theorem 6.1 *If $n \geqslant 1$ and ψ satisfies $\psi(u) \leqslant c_0 u^\alpha, \forall u \in (1, \infty)$ for some $c_0, \alpha > 0, \alpha < \frac{2}{n}$, then all solutions of (1.52) are global in time and uniformly bounded. Moreover, given $\Lambda > 0$ and $\tau \in (0, 1)$, there exist $c(\Lambda, \tau) > 0$, $m > 0$ and $\nu > 0$ such that $\|u_0\|_{L^1(\Omega)} \leqslant \Lambda$ and $\|v_0\|_{L^1(\Omega)} \leqslant \Lambda$ implies*

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} \leqslant c(\Lambda, \tau)(1 + K^m(\tau)e^{-\nu t}) \quad \forall\, t \geqslant \tau.$$

Also

$$\|u(t)\|_{C^\delta(\overline{\Omega})} + \|v(t)\|_{C^{2+\delta}(\overline{\Omega})} \leqslant c(\delta, \Lambda, \tau)(1 + K^m(\tau)e^{-\nu t}) \quad \forall\, t \geqslant \tau,$$

for any $\delta \in (0,1)$ *with* $m = m(\delta)$ *and* $\nu = \nu(\delta)$. *Here, we have set* $K(\tau) := \max_{t\in[\frac{\tau}{4},\tau]}(\|u(t)\|_{L^\infty(\Omega)} + \|\nabla v(t)\|_{L^2(\Omega)})$.

For the proof of Theorem 6.1, we refer to Theorem 4.1 [75]. In [177], the author proved that the model (1.9) with $\phi(v) = 1$, $\chi(v) \leqslant \frac{\chi_0}{(1+\alpha v)^k}$, $\chi_0 > 0$, $\alpha > 0$, $k > 1$, $v \geqslant 0$, $D = 1$, $f(v) = k(v) = 1$ and $u_0 \in C^0(\overline{\Omega})$, $v_0 \in W^{1,r}(\Omega)$, $r > n$ admits a unique global classical solution which is uniformly bounded in time. However, we can not directly apply his method to the singular case. He concluded the main result as follows:

Theorem 6.2 *The solution* (u, v) *of* (1.9) *with* $\phi(v) = 1$, $\chi(v) = \frac{\chi_0}{(1+\alpha v)^2}$, $v \geqslant 0$, $D = 1$, $f(v) = k(v) = 1$ *is global and bounded.*

For the proof of Theorem 6.2, we refer to Theorem 3.2 [177]. It is to be noticed that there was a gap between the *singular case* $0 < \chi(v) \leqslant \frac{\chi_0}{v^k}$, $\chi_0 > 0$, $k > 1$ and the *regular case* $0 < \chi(v) \leqslant \frac{\chi_0}{(1+\alpha v)^k}$, $\alpha > 0$, $\chi_0, k > 1$ (see [177]). This gap was filled by Fujie and Yokota in [58]. Their main result is as follows:

Theorem 6.3 *Suppose that* χ *satisfies* $\chi \in C^{1+\delta}_{loc}((0.\infty))$ *for some* $\delta > 0$ *and* $0 < \chi(v) \leqslant \frac{\chi_0}{v^k}$ *for some* $\chi_0 > 0$, $k > 1$ *and assume that* $u_0 \in C^0(\overline{\Omega})$, $u_0 \geqslant 0$ *in* $\overline{\Omega}$, $u_0 \not\equiv 0$, *and* $v_0 \in W^{1,\infty}(\Omega)$, $v_0 > 0$ *in* $\overline{\Omega}$. *Then the problem* (1.9) *with* $\phi(v) = 1$, $D = 1$, $f(v) = k(v) = 1$ *has a global classical solution* (u, v) *and moreover the solution is bounded in the sense that there exists* $C > 0$ *such that* $\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C$ *for all* $t > 0$.

For the proof of Theorem 6.3, we refer to Theorem 1.1 [58].

Wang et al. [169] proved the global existence and boundedness of solutions to (1.52). They summarized their main result as follows:

Theorem 6.4 *Let* $\Omega \subset \mathbb{R}^n (n \geqslant 2)$ *be a bounded convex domain with smooth boundary, and initial data* u_0 *and* v_0 *be non-negative functions with* $u_0 \in C^0(\overline{\Omega})$ *and* $v_0 \in W^{1,\theta}(\Omega)$ *(with some* $\theta > n$*). Assume that* $f(u) \leqslant a - bu^\gamma$ *for all* $u \geqslant 0$ *with* $a \geqslant 0, b > 0$, *moreover,* ϕ, ψ *satisfy* $\phi \in C^2([0,\infty))$, $\psi \in C^2([0,\infty))$ *with* $\phi(0) = 0$ *and* $\phi(u) \geqslant M_1(u+1)^{-\alpha}$ *for all* $u \geqslant 0$, $\psi(u) \leqslant M_2(u+1)^\beta$ *for all* $u \geqslant 0, \alpha, \beta \in \mathbb{R}$, $M_1, M_2 > 0$ *with* $0 < \alpha + \beta < \frac{2}{n}$. *Then problem* (1.52) *with* $\phi(u) = (u+1)^{-\alpha}$, $\psi(u) = u(u+1)^{\beta-1}$, $\alpha, \beta \in \mathbb{R}$ *possesses a unique global classical solution* (u, v) *for which both* (u, v) *are non-negative and uniformly bounded* $\Omega \times (0,\infty)$.

For the proof of Theorem 6.4, we refer to Theorem 1.1 [169]. Though Winkler [178] first proved the global existence of classical solutions when $\chi_0 < \sqrt{\frac{2}{n}}$ and global existence of weak solutions when $\chi_0 < \sqrt{\frac{n+2}{3n-4}}$ to the Keller-Segel model (1.49), his results could not arrive at boundedness of solutions to the same model.

Later, the uniform-in time boundedness of solutions to the model (1.49) was established by Fujie in [56]. The author solved the open problem proposed by Winkler in [178]. That is, the author derived uniform-in-time boundedness of solutions of (1.49) for $\chi < \sqrt{\frac{2}{n}}$. He summarized his main result as follows:

Theorem 6.5 *Let $n \geqslant 2$. Assume that χ satisfies $0 < \chi < \sqrt{\frac{2}{n}}$, and suppose that $u_0 \in C^0(\overline{\Omega})$, $u_0 \geqslant 0$ in $\overline{\Omega}$, $u_0 \neq 0$ and $v_0 \in W^{1,\infty}(\Omega)$, $v_0 > 0$ in $\overline{\Omega}$. Then the global solution of (1.49) is bounded in the sense that there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C \quad \text{for all } t > 0.$$

For the proof of Theorem 6.5, we refer to [56]. Lankeit [95] introduced a new technique to obtain the boundedness of solutions to parabolic-parabolic chemotaxis model (1.49) with singular sensitivity. He proved the global existence and boundedness of solutions to (1.49) for $\chi \in (0, \chi_0)$, $\chi_0 > 1$, in smooth, convex and bounded domain in two dimension. In order to prove his results, he mainly used the energy functional

$$\mathcal{F}_{a,b}(u,v) = \int_\Omega u \ln u - a \int_\Omega u \ln v + b \int_\Omega |\nabla \sqrt{v}|^2,$$

where $a > 0, b \geqslant 0$. His main result is stated as follows:

Theorem 6.6 [95] *Let $\Omega \subset \mathbb{R}^2$ be a convex, bounded domain with smooth boundary. Let $0 \leqslant u_0 \in C^0(\overline{\Omega})$, $u_0 \neq 0$, $0 < v_0 \in \cup_{q>2} W^{1,q}(\Omega)$. Then there exists $\chi_0 > 1$ such that for any $\chi \in (0, \chi_0)$, the system (1.49) has a global classical solution, which is bounded.*

For the proof of Theorem 6.6, we refer to Theorem 1.1 [95]. At this point, it is important to observe that the results in [177] are affected by the results in [58]. That is, Mizukami and Yokota [116] pointed out that they could not verify the results in [177], that is the global existence and boundedness hold for $\chi_0 > 0$ due to the issue in finding the derivative of $\psi(s) := (1+\alpha s)^{-2k}(1+\beta s)^{k+2}$, where $s \in (0, 2k-2)$, $\alpha, \beta > 0$. Winkler's proof for the boundedness of solutions is based on the large value of β. However, Mizukami and Yokota [116] stated that it is not possible to take large value of β to obtain $\psi'(s) \leqslant 0$. Later on, the author [57] made an attempt to resolve this issue. It was difficult to prove globally bounded solutions for arbitrary $\chi_0 >$. In order to overcome this, the authors [116] introduced an unified treatment to obtain the boundedness of solutions for parabolic-parabolic Keller-Segel model (1.9) with $\phi(v) = 1$, $D = 1$, $f(v) = k(v) = 1$ with singular sensitivity $\chi(v)$. Moreover, they have assumed the general assumption on the sensitivity to fill the gap between the cases $k = 1$ and $k > 1$ as

$$\chi \in C^{1+\lambda}((0,\infty)) \text{and } 0 \leqslant \chi(s) \leqslant \frac{\chi_0}{(a+s)^k}, s > 0, \tag{6.1}$$

with $\lambda > 0$, $a \geqslant 0$, $\chi > 0$ satisfying

$$\chi_0 < k(a+\eta)^{k-1}\sqrt{\frac{2}{n}}, \tag{6.2}$$

where

$$\eta := \sup_{\tau>0}\left(\min\{e^{-2\tau}\min_{x\in\overline{\Omega}} v_0(x), c_0\|u_0\|_{L^1(\Omega)}(1-e^{-\tau})\}\right)$$
$$= c_0\|u_0\|_{L^1(\Omega)}\left(1 - \frac{-c_0\|u_0\|_{L^1(\Omega)} + \sqrt{c_0^2\|u_0\|^2_{L^1(\Omega)} + 4c_0\|u_0\|_{L^1(\Omega)}\min_{x\in\overline{\Omega}} v_0(x)}}{2\min_{x\in\overline{\Omega}} v_0(x)}\right). \tag{6.3}$$

They summarized their main result as follows:

Theorem 6.7 *Let $n \geqslant 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Assume that χ satisfies (6.1) with same $\lambda > 0$, $k \geqslant 1, a \geqslant 0, \chi_0 > 0$ satisfying (6.2). Then for any u_0, v_0 satisfying $0 \leqslant u_0 \in C(\overline{\Omega})\{0\}$ and $0 < v_0 \in W^{1,q}(\Omega)$, $(\exists q > n)$ if $a = 0$, and $0 \leqslant v_0 \in W^{1,q}(\Omega)$, $(\exists q > n)$ if $a > 0$, there exists an exactly one pair (u, v) of solutions*

$$u, v \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)),$$

which solves (1.9) with $\phi(v) = 1$, $D = 1$, $f(v) = k(v) = 1$. Moreover, the solution (u, v) is uniformly bounded, that is, there exists a constant $C > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leqslant C, \ \forall\, t > 0.$$

For the proof of Theorem 6.7, we refer to Theorem 1.1 [116]. They also have assured the connection with the results of [56] through the above unified assumption (6.1) together with $\chi_0 > 0$ satisfying (6.2). In two dimensional case, Black [17] proved that the global existence of solutions to (1.73), which is uniformly bounded in time under the assumptions that f to be constant and initial data are nonnegative such that $\int_\Omega u_0 \mathrm{d}x < 4\pi$. In addition, the author extended the results of Nagai, Senba and Yoshida to the classical Keller-Segel model to (1.73). Furthermore, under suitable assumption, the author extended Winkler's results for the classical Keller-Segel model to (1.73).

The global existence of classical solutions to (1.66) in a smooth bounded domain $\mathbb{R}^n$, $n \geqslant 2$ is studied in [96]. Moreover, the author assumed that the strict positivity for the nonlinear diffusion coefficient $\phi \geqslant \delta u^{m-1}$, where some $\delta > 0$ provided $m > 1 + \frac{N}{4}$. Since the solutions are locally bounded, it does not blow-up in finite time. The main results are summarized as follows:

Theorem 6.8 *Let $n \geqslant 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Then for every $\delta > 0$ and $m \geqslant 1$ satisfying $m > 1 + \frac{n}{4}$, every $\phi \in C^+(\delta, m) := \{d \in C^1([0, \infty)); d(s) \geqslant \delta s^{m-1}$ for all $s \in [0, \infty)$ and $d(0) > 0\}$ and every pair (u_0, v_0) fulfilling $u_0 \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$, $v_0 \in W^{1,\infty}(\Omega)$, $u_0 \geqslant 0$, $v_0 > 0$ in $\overline{\Omega}$, the following problem*

$$\begin{cases} \partial_t u = \nabla \cdot (\phi(u)\nabla u) - \nabla(\frac{u}{v}\nabla v), & in\ \Omega \times (0, T_{max}), \\ \partial_t v = \Delta v - uv, \ in\ \Omega \times (0, T_{max}), \end{cases} \tag{6.4}$$

has a classical solution $(u, v) \in (C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times [0, T_{max})))^2$ which is global in time.

For the proof of Theorem 6.8, we refer to Theorem 1.1 [96]. Without using strict positivity of $\phi(u)$, we prove global existence of weak solutions by an approximation technique. The global existence result is

Theorem 6.9 *Let $n \geqslant 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Then for every $\delta > 0$ and $m > 1 + \frac{n}{4}$, every initial data $u_0 \in L^{\max\{1,m-1\}}(\Omega)$, $v_0 \in W^{1,\infty}(\Omega)$, $u_0 \geqslant 0$, $v_0 > 0$ and every $\phi \in C_{\delta,m}$, the problem (6.4) has a globally bounded weak solution (u, v), which in particular satisfies $\|u\|_{L^\infty(\Omega \times (0,T))} < \infty$ for every $T \in (0, \infty)$.*

For the proof of Theorem 6.9, we refer to Theorem 1.2 [96].

In [206], author obtained the global existence and boundedness of solutions to (1.52) with $\phi(u) = 1$, $\psi(u) = u$ and $f(u) = u - \mu u^2$ in 3D. He also established a global existence of bounded solutions to (1.58) in a nonconvex domains under the condition $\mu > \mu_0\chi = \frac{9}{\sqrt{10}-2}\chi$ in [205]. Moreover, he assumed that the logistic source $f : \mathbb{R} \to \mathbb{R}$ is smooth and $f(0) \geqslant 0$, and $f(u) \leqslant a - \mu u^2$, $\forall u \geqslant 0$ for some $a \geqslant 0$, $\mu > 0$. The logistic source term will affect the Keller-Segel mode's behaviour. The next important question is that how the logistic growth source influence the boundedness of solutions to a general Keller-Segel system? By using the criteria on the boundedness (see [180, 204]), the author in [205] derived lower bounded for logistic damping rate μ. His main results stated as follows:

Theorem 6.10 *Let $\Omega \subset \mathbb{R}^n, n \geqslant 3$ be a bounded smooth domain, the initial data (u_0, v_0) satisfy $u_0 \in C(\overline{\Omega})$ and $v_0 \in W^{1,p_0}(\Omega)$ with some $p_0 > n$ and let f satisfies $f(0) \geqslant 0$, and $f(u) \leqslant a - \mu u^2$, $\forall u \geqslant 0$ for some $a \geqslant 0$, $\mu > 0$ and $d_1, d_2, \alpha, \beta > 0, a \geqslant 0$ and $\chi \in \mathbb{R}$.*

- *For $n = 3$, let the lower logistic damping rate $\mu_0 = \mu_0(3, d_1, d_2, \alpha, \chi)$ of μ be explicitly given by*

$$\mu_0 = \begin{cases} \frac{3}{4d_1}\alpha\chi, & \text{if } d_1 = d_2,\ \chi > 0 \text{ and } \Omega \text{ is convex}, \\ \frac{3}{\sqrt{10}-2}\left(\frac{1}{d_1} + \frac{2}{d_2}\right)\alpha\chi, & \text{otherwise}; \end{cases} \tag{6.5}$$

- *For $n = 4, 5$, the lower logistic damping rate $\mu_0 = \mu_0(n, d_1, d_2, \alpha, \chi)$ of μ be explicitly given by*

$$\mu_0 = \begin{cases} \frac{n}{4d_1}\alpha\chi, & \text{if } d_1 = d_2,\ \chi > 0 \text{ and } \Omega \text{ is convex}, \\ \max\left\{\frac{1}{3}h(n, d_1, d_2), \frac{n}{\sqrt{2n+4}-2}\left(\frac{1}{d_1} + \frac{2}{d_2}\right)\right\}\alpha|\chi|, & \text{otherwise}; \end{cases} \tag{6.6}$$

with

$$h(n, d_1, d_2) = \inf_{0<\epsilon<d-1, 0<\eta<d_2}\left\{\sqrt{\frac{n}{18d_2\epsilon}} + \sqrt{\frac{1}{2\epsilon}\left(\frac{1}{\eta} + \frac{\eta}{2d_2}\right)} + \sqrt{\frac{1}{(d_2-\eta)}\left(\frac{2}{\eta} + \frac{\eta}{2d_2}\right)}\left[\sqrt{2} + \frac{(d_1+d_2)}{2\sqrt{(d_1-\epsilon)(d_2-\eta)}}\right]\right\}.$$

Then, whenever $\mu > \mu_0$, the model (1.58) has a unique global-in-time classical solution (u, v) for which both u and v are positive and uniformly bounded in $\Omega \times (0, \infty)$.

For the proof of Theorem 6.10, we refer to Theorem 1.1 [205]. Xiang [207] studied the blow-up prevention for (1.65) in two dimension. Winkler and Yokota [201] established the stabilization to the model (1.69) in a bounded domain $\Omega \subset \mathbb{R}^n, n \geqslant 2$ under the suitable assumptions on $\chi < \sqrt{\frac{2}{n}}$ and $d > 0$, which reduced to $\frac{\chi^2}{d}$ is sufficiently very small.

Tao et al. [161] studied the boundedness of solutions to (1.54) under suitable assumptions on $\phi(u)$, $\psi(u)$, $f(u)$ and $g(u)$, by which they have improved the existing results. Their goal was to obtain the global existence of bounded solutions to (1.54). Their main results are the following:

Theorem 6.11 *Let $n \geqslant 2$, $f \equiv 0$ and the initial conditions satisfy $u_0 \in C^0(\overline{\Omega})$, $u_0 \not\equiv 0$, $v_0 \in C^1(\overline{\Omega})$ are nonnegative. Suppose that $\phi(u)$, $\psi(u)$ and $g(u)$ satisfy $\phi, \psi \in C^2([0,\infty))$, $\psi(0) = 0$ and*

$$d_0(1+u)^{-\alpha} \leqslant \phi(u) \leqslant d_1(1+u)^{-\alpha_1}, 0 \leqslant \psi(u) \leqslant s_1 u(1+u)^{\beta-1},$$

for all $u \geqslant 0, d_0, d_1, s_1 > 0$ and $\alpha, \alpha_1, \beta \in \mathbb{R}$ and $f \in C^0([0,\infty))$ with $f(0) \geqslant 0$ and $g \in C^1([0,\infty))$ such that $f(u) \leqslant ru - \mu u^k, 0 \leqslant g(u) \leqslant g_1 u^\gamma, \forall u \geqslant 0$, with $r \in \mathbb{R}, \mu, g_1, \gamma > 0$, $k > 1$. If $0 < \gamma \leqslant 1$ and

$$\alpha + \beta + \gamma < 1 + \frac{2}{n},$$

then (1.54) has a nonnegative classical solution (u, v), which is globally bounded.

For the proof of Theorem 6.11, we refer to Theorem 1.1 [161].

Theorem 6.12 *Let $n \geqslant 2$, $f \equiv 0$ and the initial conditions satisfy $u_0 \in C^0(\overline{\Omega})$, $u_0 \not\equiv 0$, $v_0 \in C^1(\overline{\Omega})$ are nonnegative. Suppose that $\phi(u)$, $\psi(u)$ and $g(u)$ satisfy $\phi, \psi \in C^2([0,\infty))$, $\psi(0) = 0$ and*

$$d_0(1+u)^{-\alpha} \leqslant \phi(u) \leqslant d_1(1+u)^{-\alpha_1}, 0 \leqslant \psi(u) \leqslant s_1 u(1+u)^{\beta-1},$$

for all $u \geqslant 0, d_0, d_1, s_1 > 0$ and $\alpha, \alpha_1, \beta \in \mathbb{R}$ and $f \in C^0([0,\infty))$ with $f(0) \geqslant 0$ and $g \in C^1([0,\infty))$ such that $f(u) \leqslant ru - \mu u^k, 0 \leqslant g(u) \leqslant g_1 u^\gamma, \forall u \geqslant 0$, with $r \in \mathbb{R}, \mu, g_1, \gamma > 0$, $k > 1$.

- *If $\beta + \gamma < k$, then (1.54) has a nonnegative classical solution (u, v), which is globally bounded.*
- *Assume $\beta + \gamma = k$. Then there exists $\mu_0 > 0$ such that if $\mu \geqslant \mu_0$, then (1.54) has a nonnegative classical solution (u, v), which is globally bounded.*

For the proof of Theorem 6.12, we refer to Theorem 1.2 [161]. Since the model (1.54) has a logistic source term, we can discuss about the bounded solutions which we have obtained in Theorem 6.12 to stabilize towards the homogeneous steady states. Tao et al. stated this as their conjecture (see, p. 735 [161]). Fuest [55] obtained the blow-up of solutions to (1.77) in a ball $\Omega \subset \mathbb{R}^n, n \geqslant 2$, and derived the blow-up profile as well under the following assumptions on ϕ and ψ such that $\phi(u, v) = (u+1)^{m-1}$ and $\psi(u, v) = u(u+1)^{q-1}$, where $m, q \in \mathbb{R}$. Thus, the blow-up profile $U : \Omega\ \{0\} \to [0, \infty)$ is obtained provided $m > \frac{n-2}{n}$, $m - q > -\frac{1}{n}$ and u, v are nonnegative radially symmetric classical blow-up solutions of (1.77). It should also be noted that for $\alpha > n$ and $\alpha > \frac{n(n-1)}{(m-q)n+1}$, we have $U(x) \leqslant C|x|^{-\alpha}, C > 0, \forall x \in \Omega$. In Table 4 and Table 5, we review the existence, blow-up and boundedness results to a variety of parabolic-parabolic models. Further, we present the existing results of Keller-Segel-Navier-Stokes models in Table 6.

7 Numerical Analysis of Keller-Segel Models

The fundamental idea of any of the numerical methods is to discretize the continuous problem with infinitely many Degrees of Freedom (DoF) to get discrete problem with finitely many DoF. Here, we recall and review some of the numerical methods to Keller-Segel models. Even though theoretical aspects of the Keller-Segel model is well developed, there are

Table 4 Summary of the existing results for parabolic-parabolic models

Model	Results	Reference
(1.3)	Global existence & regularity	[120]
(1.17)	Global existence	[29]
(1.3)	Global existence	[90]
(1.3)	Global existence	[91]
(1.9)	Decay	[132]
(1.3) with $d = 1$	Globally bounded	[174]
(1.49)	Global solutions	[175]
(1.1) (particular case)	Blow-up	[176]
(5.15)	Local & Global solutions	[203]
(1.49)	Global existence	[13]
(1.18)	Global existence, boundedness	[155]
(1.32)	Blow-up(Non-degenerate)	[40]
(1.32)	Blow-up(degenerate)	[79]
(1.47)	Global existence	[30]
(1.3) with $d = 1$, $\chi = 1$	Blow-up	[179]
(1.1) (particular case)	Blow-up	[80]
(1.11) with $d = 1$	Global existence & boundedness	[118]
(1.37)	Global boundedness	[167]
(1.38)	Global existence & boundedness	[83]
(1.48)	Well-posedness	[76]
(1.83)	Global existence	[63]
(1.1)	Global existence	[191]
(1.36)	Blow-up	[82]
(1.15)	Existence	[209]
(1.15),$m = 1$	Global existence	[186]
(1.19)	Global existence and blow-up	[11]
(1.20)	Blow-up	[114]
(1.1)	Global existence	[43]
(1.26)	Global bounded solutions	[166]
(1.35)	Global existence	[211]
(1.35)(particular case)	Blow-up	[157]

very few articles only available for the numerical analysis. Exploration of the literature reveals that there are different types of numerical methods available for solving the following Keller-Segel chemotaxis models.

$$\begin{cases} \partial_t u = k\Delta u - \nabla \cdot (\chi u \nabla v), & \text{in } Q_T, \\ \tau \partial_t v = \Delta v - v + u, & \text{in } Q_T. \end{cases} \tag{7.1}$$

$$\begin{cases} \partial_t u = k\Delta u - \nabla \cdot (\chi u \nabla v), & \text{in } Q_T, \\ \tau \partial_t v = \Delta v - \alpha_v v + \alpha_u u, & \text{in } Q_T. \end{cases} \tag{7.2}$$

Let us mention some of the main ones.

Table 5 Summary of the existing results for parabolic-parabolic models

Model	Results	Reference
(1.8)	Global existence	[150]
(1.21)	Global existence	[59]
(1.7)	Blow-up and global existence	[39]
(1.25)	Global existence	[153]
(1.10)	Global existence	[90]
(1.3)	Global existence for small initial data	[91]
(1.3) with $d = 1$	Global existence and boundedness	[174]
(1.49)	Global existence	[175]
(1.1) related type	Blow-up of solutions	[176]
(1.17), degenerate case	Blow-up of solutions	[67]
(1.52)	Global existence & boundedness	[75]
(1.9)(particular case)	Global & boundedness	[177]
(1.52)	Global & boundedness	[169]
(1.49)	Global existence	[178]
(1.49)	Boundedness	[56]
(1.49) (singular sensitivity)	Boundedness	[95]
(1.9)(singular sensitivity)	Boundedness	[116]
(1.52)(particular case)	Global existence & boundedness	[206]
(1.58)(singular sensitivity)	Global existence & boundedness	[205]
(1.54)	Boundedness	[161]
(1.16)	Global generalized solution	[97]
(1.16)	Global existence	[98]
(1.16)	Global existence	[216]
(1.61)	Global existence	[102]
(1.65)	Global existence	[207]
(1.66)	Global existence	[96]
(1.67)	Global existence & stabilization	[159]
(1.68)	Global existence & boundedness	[41]
(1.69)	Global existence	[201]
(1.70)	Convergence of solutions	[84]
(1.71)	Stabilization	[112]
(1.72)	Existence	[185]
(1.73)	Global existence	[17]
(1.74)	Global existence	[190]
(1.77)	Blow-up	[55]
(1.79)	Global existence	[183]
(1.81)	Global existence	[171]

7.1 Finite Difference Method (FDM)

The main idea in finite difference method is that we are replacing the derivatives in the continuous PDEs by the difference quotients to obtain discrete problem. In this section, let us briefly explain about the finite difference method for Keller-Segel models. It is to be noted that a crucial property of solutions to (7.1) with $k = \chi = 1$, $\tau = 0$, and $v = av$, where a is a positive constant. In [138], Saito and Suzuki made a finite difference scheme to satisfy

Table 6 Summary of the existing results for Keller-Segel-Navier-Stokes models

Model	Results	Reference
(1.85)	Global existence	[197]
(1.86)	Global existence	[198]
(1.88)	Global existence	[156]
(1.89)	Existence	[85]
(1.90)	Global existence	[196]
(1.92)	Global existence	[184]
(1.93)	Global existence	[18]
(1.94)	Global existence	[187]
(1.95)	Qualitative property	[172]
(1.96)	Global mass-preserving weak solutions	[189]
(1.97)	Blow-up prevention	[170]
(1.98)	Global existence &stabilization	[51]

the conservation of a discrete L^1 norm by applying the upwind technique. For the sake of simplicity, we recall the finite difference method in one dimensional case only to (7.1) with $k = \chi = 1$, $\tau = 0$, and $v = av$, where a is a positive constant. Take $\Omega = (0, 1)$ and consider (7.1) with $k = \chi = 1$, $\tau = 0$, and $v = av$ in dimension one as follows:

$$\begin{cases} u_t = u_{xx} - (uv_x)_x, & 0 < x < 1, 0 < t \leqslant T, \\ 0 = v_{xx} - av + u, & 0 < x < 1, 0 < t \leqslant T. \end{cases} \tag{7.3}$$

Choose a positive integer N and $h = 1/N$. Now, we introduce two types of mesh points as $x_i = (i - \frac{1}{2})h$, $i = 0, \ldots, N + 1$ and $\hat{x}_i = ih$, $i = -1, \ldots, N + 1$, and define the intervals $J_i = (\hat{x}_{i-1}, \hat{x}_i)$, $i = 1, \ldots, N$ and $\hat{J} = (x_i, x_{i+1})$, $i = 0, \ldots, N$. In addition, we define

$$V_h = \left\{ \sum_{i=1}^{N} \alpha_i \chi_i \middle| \{\alpha_i\}_{i=1}^{N} \subset \mathbb{R} \right\} \quad \text{and} \quad \hat{V}_h = \left\{ \sum_{i=1}^{N} \beta_i \hat{\chi}_i \middle| \{\beta_i\}_{i=1}^{N} \subset \mathbb{R} \right\},$$

where the characteristic functions of J_i and $\hat{J}_i \cap [0, 1]$ are χ_i and $\hat{\chi}_i$, respectively. We seek the unknowns u_h^n, b_h^n, F_h^n, v_h^n as follows:

$$u_h^n = \sum_{i=1}^{N} u_i^n \chi_i \approx u(x, t_n), \quad u_i^n \in \mathbb{R},$$

$$b_h^n = \sum_{i=1}^{N} b_i^n \chi_i \approx v_x(x, t_n), \quad b_i^n \in \mathbb{R},$$

$$F_h^n = \sum_{i=0}^{N} F_i^n \hat{\chi}_i \approx F^n \equiv (u - uv_x)(x, t_n), \quad F_i^n \in \mathbb{R},$$

$$v_h^n = \sum_{i=1}^{N} v_i^n \hat{\chi}_i \approx v(x, t_n), \quad v_i^n \in \mathbb{R}, \ i = 0, 1, \ldots, m.$$

Now, we describe the proposed FDM for (7.3). Assume that u_h^{n-1} has been calculated. By using the relation $v_h^{n-1} = G_h u_h^{n-1}$, we can compute v_h^n. Utilizing the upwind approximation, the flux can be approximated as follows:

$$F_i^n = \frac{u_{i+1}^n - u_i^n}{h} - b_i^{n-1,+} u_i^n + b_{i+1}^{n-1,-} u_{i+1}^n, \quad i = 1, \dots, N-1,$$

where $b_i^{n-1} = \frac{v_i^{n-1} - v_{i-1}^{n-1}}{h}$, $b_i^{n-1,+} = \max\{0, b_i^{n-1}\}$, $b_i^{n-1,-} = \max\{0, -b_i^{n-1}\}$. The FDM is formulated as follows

$$\begin{aligned}\frac{u_i^n - u_i^{n-1}}{\tau_n} &= \theta[\Delta_h u_i^n - D_h(b_i^{n-1,+} u_i^n) + D_h^+(b_i^{n-1,-} u_i^n)] \\ &\quad + (1-\theta)[\Delta_h u_i^{n-1} - D_h(b_i^{n-1,+} u_i^{n-1}) + D_h^+(b_i^{n-1,-} u_i^{n-1}),\end{aligned} \tag{7.4}$$

with

$$D_h u_1^n - b_0^{n-1,+} u_0^n + b_1^{n-1,-} u_1^n = 0, \qquad D_h u_N^n - b_N^{n-1,+} u_N^n + b_{N+1}^{n-1,-} u_{N+1}^n = 0,$$

where $D_h \phi_i = \frac{\phi_i - \phi_{i-1}}{h}$, $D_h^+ \phi_i = \frac{\phi_{i+1} - \phi_i}{h}$, $\Delta_h = D_h D_h^+ = D_h^+ D_h$, $\theta \in [0, 1]$. This method is called explicit method when $\theta = 0$ and semi-implicit method when $\theta = 1$. As we know already that an important feature of (7.1) is the existence of a Lyapunov function, that is, the following function

$$W(u, v) = \int_\Omega (u \log u - u) \mathrm{d}x - \frac{1}{2} u v \mathrm{d}x, \tag{7.5}$$

exists such that

$$\frac{\mathrm{d}}{\mathrm{d}t} W(u(\cdot, t), v(\cdot, t)) \leqslant 0, \quad t \in [0, T]. \tag{7.6}$$

It is interesting to note that Saito and Suzuki proved the property (7.6) in discrete level by introducing the following time discretization scheme

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\tau_n} = \nabla \cdot (\nabla u^{n+1} - u^{n+1} \nabla v^n), & \text{in } \Omega, \\ \dfrac{\partial u^{n+1}}{\partial \nu} - u^{n+1} \dfrac{\partial v^n}{\partial \nu} = 0, & \text{in } \Omega, \end{cases} \tag{7.7}$$

where $\tau_n > 0$ is the mesh size at the nth time level and u^n and v^n denote the approximate solutions at n-th times step. The conservative property is very important in the numerical analysis. So, we have to design numerical schemes to satisfy conservative property at discrete level. In this direction, Saito [136] introduced the conservative finite difference method for (1.28). It satisfies the conservation of positivity and total mass.

7.2 Finite Volume Methods (FVMs)

Finite volume method is a type of numerical approximation method which is used to approximate the solutions of a wide range of conservation type PDEs. In FVM, the domain Ω is decomposed by control volumes or cells, denoted as Ω_i with center c_i. These control volumes or cells are either constructed from the elements of cell center scheme and cell vertex scheme or the elements of triangulation. Then on each such component Ω_i, the function (or solution) $u_i(\cdot, t)$ is approximated by its mean value all over the cell Ω_i instead of approximating by its value at the center c_i of Ω_i. Hence, $u_i(\cdot, t)$ is treated as a constant inside the control volumes Ω_i, which is given by

$$u_i(t) \cong \frac{1}{|\Omega_i|} \int\limits_{\Omega_i} u(x, t) \mathrm{d}x,$$

where $|\Omega_i|$ denotes the area of Ω_i. It is also a robust numerical scheme. Some of the important features of the FVM are similar to those of the FEM. We shall use this FVM on general domains by using arbitrary meshes such as structured or unstructured. The main feature in this method is that the numerical fluxes are locally conserved. For more details on FVM, one can refer [64].

Let us define an admissible mesh of Ω as a family $\mathcal{T}$ of control volumes and $\mathcal{E}$ is a family of edges and a family of points $(x_K)_{K \in \mathcal{T}}$. This leads to the straight line between two adjacent centers of cells (x_K, x_L) is orthogonal to the edge $\sigma = K|L$. The set of interior edges and boundary edges are denoted by $\mathcal{E}_{int}K$ and $\mathcal{E}_{ext}K$, respectively. The set all edges $\mathcal{E} = \mathcal{E}_{int}K \cup \mathcal{E}_{ext}K$. The distance in $\mathbb{R}^2$ is denoted by d and the Lebesgue measure in $\mathbb{R}^2$ or $\mathbb{R}$ is denoted by m. Further, we assume that the mesh which we have constructed satisfies the regularity property as follows:

$$d(x_K, \sigma) \geqslant \xi d(x_K, x_L), \text{ for } K \in \mathcal{T}, \text{for } \sigma \in \mathcal{E}_{int}K, \sigma = K|L. \tag{7.8}$$

In order to apply discrete Sobolev-type inequalities, the above regularity property is required. The δ denotes the mesh size which is defined as follows

$$\delta = \max_{K \in \mathcal{T}}(\mathrm{diam}(K)), \tag{7.9}$$

and the transmissibility coefficient is defined for all $\sigma \in \mathcal{E}$ as follows:

$$\tau_\sigma = \begin{cases} \frac{m(\sigma)}{d(x_K, x_L)}, & \text{for } \sigma \in \mathcal{E}_{int}, \sigma = K|L, \\ \frac{m(\sigma)}{d(x_K, \sigma)}, & \text{for } \sigma \in \mathcal{E}_{ext}K. \end{cases} \tag{7.10}$$

Let T be a final time and let M_T be the total number of time steps. The step sizes are defied as follows:

$$\Delta t = \frac{T}{M_T}, \quad t^k = k\Delta t, \ 0 \leqslant k \leqslant M_T.$$

The set of all linear space of functions from $\Omega \to \mathbb{R}$ is denoted by $X(\mathcal{T})$, which are constants on each $K \in \mathcal{T}$. Let us define the approximations of initial datum u_0 by L^2 projection as

$$u^0_{\mathcal{T}} = \sum_{K\mathcal{T}} u^0_{\mathcal{T}} \mathbf{1}_K, \quad \text{where } u^0_{\mathcal{T}} = \frac{1}{m(K)} \int\limits_K u_0(x) \mathrm{d}x, \tag{7.11}$$

where $\mathbf{1}_K$ is the characteristic function on K. Filbet [52] applied the finite volume method to solve (1.34) with $a = b = 0$. The approximations of mean value of $u(\cdot, t^k)$ and $v(\cdot, t^k)$ are denoted by u_K^k and v_K^k, respectively. The finite volume scheme is formulated as follows

$$m(K)\frac{u_K^{k+1}-u_K^k}{\Delta t} - \sum_{\sigma\in\mathcal{E}_K} \tau_\sigma D u_{K,\sigma}^{k+1} + \chi \sum_{\sigma\in\mathcal{E}_K,\sigma=K|L} \tau_\sigma \left[\left(Dv_{K,\sigma}^{k+1}\right)^+ u_K^{k+1} - \left(Dv_{K,\sigma}^{k+1}\right)^- u_L^{k+1}\right] = 0, \quad (7.12)$$

and

$$-\sum_{\sigma\in\mathcal{E}_K} \tau_\sigma D v_{K,\sigma}^{k+1} = m(K)\left(u_K^{k+1} - v_K^{k+1}\right), \quad (7.13)$$

for all $K \in \mathcal{T}$ and $0 \leqslant k \leqslant M_T - 1$, and $\eta^+ = \max(\eta, 0)$, $\eta^- = \max(-\eta, 0)$ and

$$Dv_{K,\sigma}^k = \begin{cases} v_L^k - v_K^k, & \text{if } \sigma = K|L \in \mathcal{E}_{int,K}, \\ 0, & \text{if } \sigma = K|L \in \mathcal{E}_{ext,K}, \end{cases} \quad (7.14)$$

for all $K \in \mathcal{T}$ and $0 \leqslant k \leqslant M_T$. For the proposed finite volume scheme, he proved existence and uniqueness of a numerical solution and derived a priori estimates. Andreianov et al. applied FVM for the degenerate system (1.32) with $\phi(u) = a(u)$, $\psi(u) = \chi(u)$ in [5]. To establish the numerical analysis they assumed $\psi : [0, 1] \mapsto \mathbb{R}$ is continuous and $\psi(0) = \psi(1) = 0$; and $\phi : [0, 1] \mapsto \mathbb{R}^+$ is continuous $\phi(0) = \phi(1) = 0$ and $\phi(s) > 0$ for $0 < s < 1$. They have established the existence of discrete solution by using fixed point argument and established the compactness arguments for the discrete solutions. Bessemoulin-Chatard and Jüngel [31] analyzed the finite volume scheme for (1.44) with $\tau = 0$. The main characteristic of the model is that it admits a new entropy functional. They have proved that the existence of solutions for the discrete problem and established convergence of discrete solutions to the continuous solution. Very recently, the authors in [4], they applied the semi-implicit time discretization and an upwind finite volume approximation for (1.34) in two and three space dimensions. In addition, they proved the existence, uniqueness and nonnegativity of the numerical solutions. Zhou and Saito [220] were applied linear FV scheme for (1.34) with $\chi = a = b = 1$ and derived few discrete free energy inequalities. They derived error estimates in L^p norm when $p > 2$ in spatial dimension two under suitable assumptions on admissible meshes, the regularity of solutions and a priori estimates of the numerical solution. In particular they considered the admissible Voronoi mesh which has a dual triangulation and used the idea of [32] to the error analysis. Chertock and Kurganov [34] studied the finite-volume central-upwind method for (7.1) with $k = \tau = 1$. This method preserves the positivity for a second order numerical method. In addition, their numerical simulations show high resolution, robustness and stability of the proposed method.

7.3 Finite Element Method (FEM)

Let us briefly mention about finite element method. It is one of the robust numerical methods for finding approximation solutions to the partial differential equations. It is based on the weak formulation/variational formulation of the given differential equations. It expresses the unknowns in terms of linear combination of constructed basis functions. By which we can reduce the weak formulation/variational formulation of the continuous problem into a finite dimensional approximation problem. Let V be a Hilbert space and let $a : V \times V \to \mathbb{R}$

be the continuous and coercive bilinear form. And let the linear functional $f : V \to \mathbb{R}$ be continuous. Now, we consider the variational problem

$$a(u, v) = f(v) \quad \forall\, v \in V,$$

where u is the solution of the variational problem. Let V_h be a finite dimensional subspace of V. The discrete problem is defined as follows: seek $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \quad \forall\, v_h \in V_h.$$

Since V_h is a subspace of V, the discrete bilinear form a_h is coercive and continuous. According to the Lax-Milgram lemma, we can show that the existence and uniqueness of solution $u_h \in V_h$. The discrete space V_h is constructed by dividing the problem domain into a family of subdomains T and define a space of polynomials on this subdomains T. If V_h is a subspace of V, then the FEM is called a conforming FEM. If V_h is not a subspace of V, then the FEM is called a nonconforming FEM. For more details on FEM, one can refer to the books [24, 25, 33, 38, 62, 163]. Let us recall that on the following grounds, the finite element method performs better than the finite difference method:

- comparatively the complex geometry of the domain, more general boundary conditions and continuously varying material properties can be handled easily.
- the finite element method is more adaptable and explicit so that the software developed can be applied to a wider range of applications.
- the theoretical foundations are strong enough which give it more authenticity and therefore the possibility of expecting sharp error estimates are good enough in finite element solutions.

The important properties of the solutions of Keller-Segel models are the conservation of mass

$$\int_\Omega |u(t)| \mathrm{d}x = \int_\Omega |u_0| \mathrm{d}x,$$

for all $t \in (0, T)$ which is the consequence of positivity $u, v > 0$, $(x, t) \in \overline{\Omega} \times (0, T)$ and the conservation of total mass

$$\int_\Omega u(x, t) \mathrm{d}x = \int_\Omega u_0(x) \mathrm{d}x,$$

for all $t \in (0, T)$. Let $\mathcal{T}_h$ be a family of triangulation of Ω and the set of all vertices of $\mathcal{T}_h$ be denoted by $\{\mathcal{P}_h\} = \{P_i\}_{i=1}^N$. In order to apply FEM to the model (7.1) with $\tau = 0$, first, we introduce the following variational formulation to the model (7.1) with $\tau = 0$ as

$$(u_t, \varphi) + a(u, \varphi) + \chi b(u, u, \varphi) = 0, \quad \forall\, \varphi \in H^1(\Omega), \forall\, t \in (0, T), \tag{7.15}$$

where $a(u, \varphi) = (k\nabla u, \nabla\varphi)$ and $b(u, u, \varphi) = (\chi\nabla u, \nabla u, \nabla\varphi)$. Next, we define the finite element formulation to (7.15) as follows

$$\begin{cases} (\partial_{\tau_n} u_h^n, \chi_h)_h + (k\nabla u_h^n, \nabla\chi_h) + \lambda b_h(u_h^{n-1}, u_h^n, \chi_h) = 0, \quad \forall\, \chi_h \in X_h, \\ \forall\, n \geqslant 1, u_h^0 = u_{0h}, \end{cases} \tag{7.16}$$

where X_h is the finite element space. Marrocco studied mixed finite element method (MFEM) for the following model

$$\begin{cases} \partial_t u - \text{div}(D_b \text{ grad } u - ku \text{ grad } v) = 0, & \text{in } Q_T, \\ -\text{div}(D_c \text{grad } v) - \alpha u = 0, & \text{in } Q_T, \end{cases} \tag{7.17}$$

in [111]. He introduced a scaling on the variable v via the parameter q, $v = \frac{\tilde{v}}{q}$ and a new variable φ_n by $u = u(\tilde{v}, \varphi_n) = u_0 e^{\frac{\tilde{v}+\varphi_n}{z}}$, where z is a constant and $z = \frac{qD_b}{k}$. After that he has rewritten the above model by using new variables and then discretized by mixed finite element. Also, we mention a paper [27] for mixed finite element discretization of a Keller-Segel model with nonlinear diffusion. Strehl et al. [148] applied an implicit finite element method for the following Keller-Segel model (7.18)

$$\begin{cases} \partial_t u = \nabla \cdot (D(u)\nabla u - A(u)B(v)C(\nabla v)) + q(u), \\ \partial_t v = d\Delta v - s(u)v + g(u)u. \end{cases} \tag{7.18}$$

Their aim was to maintain mass conservation and positivity of solutions in discrete case as well. By solving blow-up problems, they proved the efficiency and robustness of the proposed numerical scheme. Saito [135] applied the upwind finite element scheme for (7.1). It is to be noted that for the acute type triangulation one can prove upwind finite element scheme preserves L^1-norm of solutions. With suitable assumptions on partition of triangle and the regularity of solutions he has established the error estimates as Theorem 7.1. In order to prove error estimates, he assumed the following hypothesis:

(H1) Acuteness. It is assumed that

$$\max\{\cos(\nabla\phi_i^T, \nabla\phi_j^T) | 1 \leqslant i, j \leqslant d+1\} \leqslant 0 \quad \forall T \in \mathscr{T}_h \in \{\mathscr{T}_h\},$$

where $\{\phi_i^T\}_{i=1}^{d+1}$ represent the barycentric coordinates of T with respect to the vertices of T.

(H2) Quasiuniformity assumption. There exists a positive constant γ_2 such that

$$\gamma_2 h \leqslant h_T, \quad \forall T \in \mathscr{T}_h \in \{\mathscr{T}\}.$$

(R) Elliptic regularity. There exists $\mu \in (d, \infty)$ such that the following holds true: For any $p \in (1, \mu)$ and $f \in L^p(\Omega)$, the linear elliptic problem

$$-\Delta v + v = f \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{7.19}$$

admits a unique solution $v \in \mathscr{W}_p$ that satisfies $\|v\|_{2,p} \leqslant C\|f\|_p$, with a constant $C = C(p, \Omega) > 0$.

Theorem 7.1 *Suppose that* (*H*1), (*H*2) *and* (*R*) *are satisfied. Assume that*

$$\begin{cases} (\frac{\mathrm{d}u(t)}{\mathrm{d}t}, \chi) + (k\nabla u(t), \nabla\chi) + \lambda b(u(t), u(t), \chi) = 0 \quad \forall \chi \in H^1, \\ \forall t \in (0, J), u(0) = u_0 \in H^1, \end{cases} \tag{7.20}$$

admits a unique solution u satisfying $u \in C([0, T] : \mathscr{W}_p)$, $u' \in C([0, J] : W^{1,p}) \cap C^\sigma([0, J] : L^p)$ *for some* $p \in (d, \mu)$ *and* $\sigma \in (0, 1]$. *Moreover, let* $u_{0h} \in X_h$ *be chosen as*

$$\|u_0 - u_{0h}\|_p \leqslant \alpha_{0,p} h^{1-d/p},$$

with some $\alpha_{0,p} = \alpha_{0,p}(u_0) > 0$. *Then, there exist positive constants* h_0, τ_0 *depending on* Ω, J, k, λ, p, σ, $\gamma_i' s$ *and* $\alpha_{i,p}' s$ *such that we have the error estimates*

$$\sup_{0\leqslant n\leqslant l} \|u(t_n) - u_h^n\|_p \leqslant C_1(h^{1-d/p} + \tau^\sigma), \tag{7.21}$$

$$\sup_{0\leqslant n\leqslant l} \|Gu(t_n) - G_h u_h^n\|_{1,\infty} \leqslant C_2(h^{1-d/p} + \tau^\sigma), \tag{7.22}$$

for $h \in (0, h_0)$ *and* $\tau \in (0, \tau_0)$, *where* $l = l(\tau, h) = \max\{n \in \mathcal{N} | t_n < J\}$, $\{u_h^n\}_{n\geqslant 0} \subset X_h$ *is the solution of* (7.16). *Furthermore, the constants* C_1 *and* C_2 *can be taken as* $C_i = C(J + 1)(\alpha_{0,p} + \alpha_{1,p}^2 + \alpha_{2,p}^2 + \alpha_{3,p}) \exp[C'(1 + \alpha_{1,p}^2)J]$, $i = 1, 2$, *where* C *and* C' *are positive constants that depend only on* Ω, k, λ, $\gamma_i' s$, h_0 *and* τ_0.

For the proof of Theorem 7.1, we refer to Theorem 2.3 [135]. In [137], the author studied the error analysis of conservative FEM for fully parabolic-parabolic Keller-Segel model. For the efficient and robust finite element solver of parabolic-parabolic Keller-Segel model one can refer [147]. Nakaguchi and Yagi [122] applied Galerkin finite element and Runge-Kutta approximations for (7.17) with $D(u) = a$, $A(u) = u$, $B(v) = 1$, $C(\nabla v) = \nabla B(v)$, $q(u) = c(u)$, $g(u) = f$ and $s(u) = g$. They derived the error estimates and stability for the proposed scheme by using semigroup theory. Moreover, they generalized their obtained results to abstract evolution equations of quasilinear parabolic type. In order to over come the computational difficulty, Zhang et al. developed a new numerical method that is the characteristic splitting mixed finite element method for (3.4) with $d = \gamma = \alpha = 1$ in [214]. Because when we apply the classical mixed FEM, it needs the Ladyzhenskaya-Babuska-Brezzi (LBB) condition which leads to saddle-point problems. In practical computation, the LBB condition is not necessary, so some general piecewise polynomial spaces can be chosen as the corresponding finite element approximate spaces. And the flux equation is separated from the second equation. In addition, they have also used a mass conservative characteristic FEM to solve the first equation to avoid the nonphysical oscillations and maintain mass balance. In [66], Gurusamy and Balachandran studied FEM for (1.46). First, they were introduced a semi-implicit scheme for weak formulation of the problem and then a fixed point formulation is defined for the corresponding scheme. Next, the existence of approximate solutions was established by using Schauder's fixed point theorem. In addition, a priori error estimate for the approximate solutions were also established.

7.4 Discontinuous Galerkin Finite Element Methods (DGFEM)

In standard finite element methods, functions used in finite element spaces for the discretization of second-order PDEs are continuous across interelement boundaries whereas the DGFEMs use completely discontinuous piecewise polynomial space for the test and trial functions. That is the functions in the finite element spaces are totally discontinuous across interelement boundaries which is called discontinuous finite elements. In recent years, there has been increasing interest in discontinuous Galerkin methods for a wide spectrum of partial differential equations. This is due to their attractive features such as local mesh adaptivity and elementwise mass conservation, flexibility to use high-order polynomial and non-polynomial basis functions, ability to easily increase the order of approximation on each mesh element independently and block diagonal mass matrices which are of great computational advantage if an explicit time integration is used and suitability for parallel computations owing to local data communications. The DGFEM is a natural generalization of the finite volume method (FVM) and finite element method (FEM) and it appears to be suitable for problems with solutions containing discontinuous and or steep gradients. These methods have both the advantages of FVM and FEM. Similarly as in the FVM, the DGFEM uses discontinuous piece- wise constant approximations and element boundary fluxes are evaluated with the aid of a numerical flux which allows precise capturing of discontinuities and steep gradient solutions. As in the FEM, it uses higher degree polynomial approximations of solutions which in turn produce an accurate resolution in regions where the solution is smooth. Moreover, a good exposure regarding DG methods for elliptic, parabolic and hyperbolic problems can be found in the recent books, see [44, 68, 134]. For the more details on DGFEM, we refer to [44, 134] and a research report [133].

Since the Keller-Segel model is convection-dominated convection-diffusion equations. So when we apply the standard finite element method, this leads to spurious oscillations in the computational domain. It would be difficult to capture the blow-up of solutions of Keller-Segel chemotaxis model and is really a challenging problem as well. Epshteyn and Kurganov [49] introduced a new interior penalty discontinuous Galerkin FEMs for classical Keller-Segel chemotaxis model (7.1) with $k = \tau = 1$. This method is designed to handle rectangular domains. It is to be noticed that the convective part of (7.1) with $k = \tau = 1$ is of a mixed hyperbolic-eliptic type. In order to apply the DG method, they reformuled the original model (7.1) with $k = \tau = 1$ in the form of convection-diffusion-reaction system with hyperbolic convective part as

$$\begin{cases} \partial_t u + (\chi u \rho)_x + (\chi u \eta)_y = \Delta u, \\ \partial_t v = \Delta v - v + u, \\ \rho = v_x, \eta = v_y. \end{cases} \tag{7.23}$$

These new unknowns satisfy the following boundary conditions $\nabla u \cdot \mathbf{n} = \nabla v \cdot \mathbf{n} = (u, v)^T \cdot \mathbf{n} = 0$, $(x, y) \in \partial\Omega$. The above system (7.23) could be treated as a convection-diffusion-reaction system

$$k\mathbf{Q}_t + \mathbf{F}(\mathbf{Q})_x + \mathbf{G}(\mathbf{Q})_y = k\Delta\mathbf{Q} + \mathbf{R}(\mathbf{Q}), \tag{7.24}$$

where $\mathbf{Q} := (u, v, \rho, \eta)^T$, where $\mathbf{F}(\mathbf{Q}) := (\chi u\rho, 0, -v, 0)^T$ and $\mathbf{G}(\mathbf{Q}) := (\chi u\eta, 0, 0, -v)^T$ and $\mathbf{R}(\mathbf{Q}) := (0, u - v, -\rho, -\eta)$. The DG formulation of the system (7.23) is defined as follows: Seek $(u^{DG}(\cdot, t), v^{DG}(\cdot, t), \rho^{DG}(\cdot, t), \eta^{DG}(\cdot, t)) \in \mathcal{W}^u_{r_u,h} \times \mathcal{W}^v_{r_v,h} \times \mathcal{W}^\rho_{r_\rho,h} \times \mathcal{W}^\eta_{r_\eta,h}$

$$\begin{aligned} &\int_\Omega \partial_t u^{DG}\varphi^u + \sum_{E\in\mathcal{E}_h}\int_E \nabla u^{DG}\nabla\varphi^u - \sum_{e\in\Gamma_h}\int_e \{\nabla u^{DG}\cdot\mathbf{n}_e\}[\varphi^u] + \epsilon\sum_{e\in\Gamma_h}\int_e \{\nabla\varphi^u\cdot\mathbf{n}_e\}[u^{DG}] \\ &+ \sigma_u\sum_{e\in\Gamma_h}\frac{r_u^2}{|e|}\int_e [u^{DG}][\varphi^u] - \sum_{E\mathcal{E}_h}\int_E \chi u^{DG}\rho^{DG}(\varphi^u)_x + \sum_{e\in\gamma_h^{ver}}\int_e (\chi u^{DG}\rho^{DG})^* n_x[\varphi^u] \\ &- \sum_{E\mathcal{E}_h}\int_E \chi u^{DG}\eta^{DG}(\varphi^u)_y + \sum_{e\in\gamma_h^{hor}}\int_e (\chi u^{DG}\eta^{DG})^* n_y[\varphi^u] = 0, \end{aligned} \tag{7.25}$$

$$\begin{aligned} &\int_\Omega \partial_t v^{DG}\varphi^v + \sum_{E\in\mathcal{E}_h}\int_E \nabla v^{DG}\nabla\varphi^v - \sum_{e\in\Gamma_h}\int_e \{\nabla v^{DG}\cdot\mathbf{n}_e\}[\varphi^v] + \epsilon\sum_{e\in\Gamma_h}\int_e \{\nabla\varphi^v\cdot\mathbf{n}_e\}[v^{DG}] \\ &+ \sigma_v\sum_{e\in\Gamma_h}\frac{r_v^2}{|e|}\int_e [v^{DG}][\varphi^v] + \int_\Omega v^{DG}\varphi^v - \int_\Omega u^{DG}\varphi^v = 0, \end{aligned} \tag{7.26}$$

$$\int_{\Omega} \rho^{DG}\varphi^{\rho} + \sum_{E\in\mathcal{E}_h}\int_{E} v^{DG}(\varphi^{\rho})_x + \sum_{e\in\Gamma_h^{ver}}\int_{e}(-v^{DG})^*_{\rho} n_x[\varphi^{\rho}] - \sum_{e\in\partial\Omega_{ver}}\int_{e} v^{DG} n_x\varphi^{\rho}$$

$$+\sigma_{\rho}\sum_{e\in\Gamma_h\cup\partial\Omega_{ver}}\frac{r_{\rho}^2}{|e|}\int_{e}[\rho^{DG}][\varphi^{\rho}] = 0, \tag{7.27}$$

$$\int_{\Omega} \eta^{DG}\varphi^{\eta} + \sum_{E\in\mathcal{E}_h}\int_{E} v^{DG}(\varphi^{\eta})_y + \sum_{e\in\Gamma_h^{hor}}\int_{e}(-v^{DG})^*_{\eta} n_y[\varphi^{\eta}] - \sum_{e\in\partial\Omega_{hor}}\int_{e} v^{DG} n_y\varphi^{\eta}$$

$$+\sigma_{\eta}\sum_{e\in\Gamma_h\cup\partial\Omega_{hor}}\frac{r_{\eta}^2}{|e|}\int_{e}[\eta^{DG}][\varphi^{\eta}] = 0. \tag{7.28}$$

In order to prove the well-posedness, they proved existence of solutions by using Schauder's fixed point theorem. The assumptions for the error analysis of dG solutions are the following:

$$(u, v, \rho, \eta) \in H^{s_1}([0, T]) \cap H^{s_2}(\Omega), \quad s_1 > 3/2, s_2 \geqslant 3, \tag{7.29}$$

which is needed for the h-analysis, or

$$(u, v, \rho, \eta) \in H^{s_1}([0, T]) \cap H^{s_2}(\Omega), \quad s_1 > 3/2, s_2 \geqslant 5, \tag{7.30}$$

which is needed for the r-analysis. Their existence and error estimates are sated as follows.

Theorem 7.2 ($L^2(H^1)$- *and* $L^\infty(L^2)$- *error estimates*) *Let the solution of the Keller-Segel system* (7.23) *satisfy the smoothness assumption* (7.30). *If the penalty parameters* $\sigma_u, \sigma_v, \sigma_\rho$ *and* σ_η *in the DG method* (7.25)-(7.28) *are sufficiently larger and* $r_{min} \geqslant 2$, *then there exist constants* C_u *and* C_v, *independent of* h, r_u, r_v, r_ρ *and* r_η, *such that the following two error estimates hold*:

$$\|u^{DG} - u\|_{L^\infty([0,T];L^2(\Omega))} + \|\nabla(u^{DG} - u)\|_{L^2([0,T];L^2(\Omega))}$$
$$+\left(\int_0^T \sum_{e\in\Gamma_h}\frac{r_u^2}{|e|}\|[u^{DG} - u]\|_{0,e}^2\right)^{\frac{1}{2}} \leqslant C_u E,$$

$$\|v^{DG} - v\|_{L^\infty([0,T];L^2(\Omega))} + \|\nabla(v^{DG} - v)\|_{L^2([0,T];L^2(\Omega))}$$
$$+\left(\int_0^T \sum_{e\in\Gamma_h}\frac{r_v^2}{|e|}\|[v^{DG} - v]\|_{0,e}^2\right)^{\frac{1}{2}} \leqslant C_v E,$$

where $E := \sum_{\alpha\in\{u,v,\rho,\eta\}} \frac{h^{\min(r_\alpha+1, s_\alpha)-1}}{r_\alpha^{s_\alpha-2}}$.

For the proof of Theorem 7.2, we refer to Theorem 5.5 [49]. In this theorem, we can observe that they obtained h-optimal but only sub-optimal for r. Later, this sub-optimal has improved by Li et al. in [108] by using local discontinuous Galerkin method (LDG). It is very important to find the blow-up time of the solutions of Keller-Segel models. The next theorem states that upper bound for the blow-up time of its exact solutions.

Theorem 7.3 *Let us denote by t_b the blow-up time of the exact solution of the Keller-Segel system (7.1) with $k = \tau 1$ and by t_b^{DG}, the blow-up time of the DG solution of (7.25)-(7.28). Then $t_b \leqslant t_b^{DG}$.*

For the proof of Theorem 7.3, we refer to Theorem 5.6 [49]. Epshteyn and Izmirlioglu [48] introduced the fully discrete numerical scheme for the model (7.1) with $k = \tau = 1$. They used the DGFEMs for the spatial discretization and the temporal discretization was performed by either the second order explicit total variation diminishing (TVD) Runge-Kutta or Forward Euler schemes. As we mentioned in the above, Li et al. [108] have improved the results of [49] and obtained optimal order of convergence rate by using suitable finite element spaces but it is before blow up happens. Their LDG method is defined as follows:

Local discontinuous Galerkin method:
Let us define a finite element space $\mathcal{V}_h^k$ as $\mathcal{V}_h^k = \{\phi : \phi|_T \in P^k(T), \forall\ T \in \Omega_h\}$, where h is a diameter of an element $T, h = \max_T h_T$. In order to introduce the LDG method, we need to introduce the axillary variables $\mathbf{p} = \nabla u$ and $\mathbf{r} = \nabla v$, therefore the model (7.1) with $k = \tau = 1$ rewritten as

$$\begin{aligned} u_t =& - \nabla \cdot (\mathbf{r}u) + \nabla \cdot \mathbf{p}, \\ \mathbf{p} =& \nabla u, \\ v_t =& \nabla \cdot \mathbf{r} + u - v, \\ \mathbf{r} =& \nabla v, \end{aligned}$$

where $P^k(T)$ is the set of polynomials of degree up to k in each element T. The proposed LDG method is to seek $u_h \in \mathcal{V}_h^{k_1}$, $\mathbf{p}_h \in \mathcal{V}_h^{k_1}$, $v_h \in \mathcal{V}_h^{k_2}$ and $\mathbf{r}_h \in \mathcal{V}_h^{k_2}$ such that

$$\begin{aligned} (u_{ht}, \phi_u)_T = (\mathbf{r}_h u_h - \mathbf{p}_h, \nabla \phi_u)_T - \langle (\widehat{\mathbf{r}_h u_h} - \widehat{\mathbf{p}}_h) \cdot \mathbf{n}_T, \phi_u \rangle_{\partial T} \\ (\mathbf{p}_h, \phi_p)_T = -(u_h, \nabla \cdot \phi_p)_T + \langle \widehat{u}_h, \phi_p \cdot \mathbf{n}_T \rangle_{\partial T}, \\ (v_{ht}, \phi_v)_T = -(r_h, \nabla \phi_v)_T + \langle \widehat{\mathbf{r}_h} \cdot \mathbf{n}_T, \phi_v \rangle_{\partial T} + (u_h - v_h, \phi_v)_T, \\ (\mathbf{r}_h, \phi_r)_T = -(v_h, \nabla \cdot \phi_r)_T + \langle \widehat{v}_h, \phi_r \cdot \mathbf{n}_T \rangle_{\partial T}, \end{aligned}$$

where $(u, v)_T = \int_T uv \mathrm{d}x\mathrm{d}y$, $(\mathbf{u}, \mathbf{v})_T = \int_T \mathbf{u} \cdot \mathbf{v} \mathrm{d}x\mathrm{d}y$ and $\langle u, v \rangle_{\partial T} = \int_{\partial T} uv \mathrm{d}s$. And also the numerical fluxes defined on $e \in \Gamma_h$ are $\widehat{u}_h$, $\widehat{v}_h$, $\widehat{\mathbf{p}_h}$, $\widehat{\mathbf{r}_h}$ and $\widehat{\mathbf{r}_h u_h}$. Their main results stated as follows:

Theorem 7.4 *Suppose $u, v \in H^{\min\{k_1,k_2\}+1}(\Omega)$, $\mathbf{r}$ is uniformly bounded for $t \leqslant T$. The numerical approximations $u_h \in \mathcal{V}_h^{k_1}$, $\mathbf{p}_h \in \mathcal{V}_h^{k_1}$, $v_h \in \mathcal{V}_h^{k_2}$ and $\mathbf{r}_h \in \mathcal{V}_h^{k_2}$. The initial discretization is given as the standard L^2- projection $(Pu, v)_T = (u, v)_T$, $\forall\ v \in P^k(T)$ and α is chosen to be a bounded constant independent of h. If we take $k_1 \geqslant 1$ and $k_2 \geqslant 2$, then there exists an $H = \frac{1}{2C}$ such that for any $h < H$, if the numerical approximations obtained from*

$$\begin{aligned} (u_{ht}, \phi_u) = \mathcal{L}^c(\mathbf{r}_h, u_h, \phi_u) - \mathcal{L}^d(\mathbf{p}_h, \phi_u), \\ (\mathbf{p}_h, \phi_p) = -\mathcal{Q}(u_h, \phi_p), \\ (v_{ht}, \phi_v) = -\mathcal{L}^d(\mathbf{r}_h, \phi_v) + (u_h - v_h, \phi_v), \\ (\mathbf{r}_h, \phi_r) = -\mathcal{Q}(v_h, \phi_r), \end{aligned}$$

where

$$\mathcal{L}^c(\mathbf{r}, u, \phi) = (\mathbf{r}u, \nabla\phi) - \sum_{T\in\Omega_h} \langle \widehat{\mathbf{r}_h u_h} \cdot \mathbf{n}_T, \phi\rangle_{\partial T},$$

$$\mathcal{L}^d(\mathbf{p}, \phi) = (\mathbf{p}, \nabla\phi) - \sum_{T\in\Omega_h} \langle \widehat{\mathbf{p}} \cdot \mathbf{n}_T, \phi\rangle_{\partial T},$$

$$\mathcal{Q}(u, \phi) = (u, \nabla\cdot\phi) - \sum_{T\in\Omega_h} \langle \hat{u}, \phi\cdot\mathbf{n}_T\rangle_{\partial T},$$

exist for all $t \in [0, T]$, *where* T *is the time that the smooth solution* u *and* v *of the KS model* (3.4) *with* $d = 1$ *exist in* $[0, T]$, *then*

$$\|(u - u_h)(t)\| + \|(v - v_h)(t)\| \leqslant Ch^{\min\{k_1+1, k_2\}}, \quad \forall t \in [0, T], \tag{7.31}$$

where the positive constant C *does not depend on* h.

For the proof of Theorem 7.4, we refer to Theorem 3.1 [108]. For energy dissipative LDG Methods of Keller-Segel chemotaxis model one can refer [65]. Operator splitting combined with positivity-preserving discontinuous Galerkin method for the chemotaxis model (7.1) when $k = \tau = 1$ studied in [213]. Further, the implementation of DG methods for chemotaxis-haptotaxis model was presented by Epshteyn in [47].

7.5 Potential Difference Methods, Hybrid Finite Volume Methods-Finite Difference (FVFD)

Epshteyn introduced DGFEMs to handle rectangular regions. Even though there are several advantages in DGFEMs, there are also some drawbacks with it. These are due to high memory and high computational costs and very involved implementation compare to continuous Galerkin finite element, finite volume, or finite difference methods. Epshteyn [46] developed a novel upwind-difference potentials method for the model (7.1) with $k = \tau = 1$, that can be used to approximate problems in complex geometries. In this method, the unstructured meshes is not required to handle complex geometries and this method can be utilized with poisson solvers. It combines the positivity preserving upwind method for chemotaxis models on cartesian grids and with the flexibility of the Difference Potentials method. Chertock et al. [35] developed a novel higher order hybrid finite-volume-finite-difference methods for (7.2). In order to derive higher order positivity preserving numerical methods for Keller-Segel models (7.2), it would be convenient to re-write the original model as follows:

$$\begin{cases} u_t + (\chi u w_1 - u_x)_x + (\chi u w_2 - u_y)_y = 0, \\ \tau v_t = \Delta v - \alpha_v v + \alpha_u u, \quad w_1 := v_x, \; w_2 := v_y, \end{cases} \tag{7.32}$$

Let Ω be a square domain in $\mathbb{R}^2$ and consider the model (7.32) in Ω. Next, we define the Cartesian mesh using $I_{j,k} := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [y_{k-\frac{1}{2}}, k_{j+\frac{1}{2}}]$ and assumed to satisfy the uniform size $\Delta x \Delta y$, where $\Delta x = x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}$ for every j and $\Delta y = y_{k-\frac{1}{2}}, y_{k+\frac{1}{2}}$ for every k. The semi-discrete hybrid FVFD method for (7.32) is defined on the above mesh $I_{j,k}$ as follows:

$$\begin{cases} \dfrac{\mathrm{d}\overline{u}_{j,k}}{\mathrm{d}t} = -\dfrac{\mathcal{F}_{j+\frac{1}{2}} - \mathcal{F}_{j-\frac{1}{2},k}}{\Delta x} - \dfrac{\mathcal{G}_{j,k+\frac{1}{2}} - \mathcal{G}_{j,k-\frac{1}{2}}}{\Delta y}, \\ \tau \dfrac{\mathrm{d}\overline{v}_{j,k}}{\mathrm{d}t} = \Delta_{j,k} v - \alpha_v v_{j,k} + \alpha_u u_{j,k}, \end{cases} \tag{7.33}$$

Table 7 Summary of the numerical analysis results of Keller-Segel models

Model	Results	Reference
(7.1) with $k=\chi=1$, $\tau=0$, and $v=av$	FDM	[45]
(1.28)	FDM	[136]
(1.34) with $a=b=0$	FVM	[52]
(1.32) with $\phi(u)=a(u)$, $\psi(u)=\chi(u)$	FVM	[5]
(1.34)	FVM	[4]
(1.44) with $\tau=0$	FVM	[31]
(1.34) with $\chi=a=b=1$	FVM	[220]
(7.17)	MFEM	[111]
(7.18)	FEM	[148]
(7.1)	FEM	[135]
(7.17)(particular case)	FEM	[122]
(1.3) with $d=\gamma=\alpha=1$	MFEM	[214]
(1.46)	FEM	[66]
(7.1) with $k=\tau=1$	DGFEM	[49]
(7.1) with $k=\tau=1$	DGFEM	[48]
(7.1) with $k=\tau=1$	LDG	[108]
(7.1) when $k=\tau=1$	DGFEM	[213]
(7.1) with $k=\tau=1$	PUD	[46]
(7.1) with $k=\tau=1$	DMM	[26]
(1.43)	SDS	[20]
(7.2)	HFVFD	[35]

where the cell averages of the density $\overline{u}_{j,k} \approx \frac{1}{\Delta x \Delta y} \iint_{I_{j,k}} u(x,y,t)\mathrm{d}x\mathrm{d}y$ and the chemical concentration point values $v_{j,k} \approx v(x_j, y_k, t)$, both are evolved with respect to time. The numerical fluxes in the x- and y- directions are $\mathcal{F}_{j+\frac{1}{2},k}$ and $\mathcal{G}_{j,k+\frac{1}{2}}$, respectively. The $u_{j,k} \approx u(x_j, y_k, t)$ denotes the approximated point values of the cell density and $\Delta_{j,k}$ denotes the discrete Laplacian.

Budd et al. computed the behaviour of the blow-up of solutions of (7.1) with $k=\tau=1$ by using the dynamic moving-mesh method in [26]. In [20], Blanchet et al. considered a variational steepest descent approximation method for (1.43) with a logarithmic interaction kernel in any dimension. The numerical scheme based on difference potentials method for chemotaxis models in three dimensions with Neumann boundary conditions is introduced in [50]. In Table 7, we review the numerical methods of Keller-Segel models.

8 Discussions and Future Perspectives

We briefly summarize the review on Keller-Segel chemotaxis models during the last few decades. From a mathematical point of view, it is a meaningful question whether solutions blow-up or remain bounded. More precisely, it would be interesting to study the global existence vs. finite-time blow-up of solutions in Keller-Segel chemotaxis models. One of the reasons is that blow-up expresses the aggregation of cells and boundedness implies that the power of diffusion is stronger than that of chemotaxis. When we explore the literature, we got important possible future extensions regarding Keller-Segel models and listed a few open problems. In this review, we observed that how strong the logistic damping source affects the boundedness and global-in-time existence of solutions to a variety of models. Clearly, the lower bound for the logistic damping rate depends on the various parameters such as diffusion, degradation and creation rates and which is independent of degradation rate, the domain Ω, the birth rate, initial conditions and embedding constants. It is interesting to note that the small diffusion or nonlinear diffusion will lead to the occurrence of blow-up of solutions in Keller-Segel models (see [205]). The existence and blow-up of solutions also depend on the value of the chemotactic constant χ. For parabolic-parabolic and parabolic-elliptic models, upto now threshold level for χ is $\frac{N}{N-2}$, where N is the space of the dimension. Regarding the numerical solutions, authors in [49] have used rectangular subdivision for the proposed DGFEM of (7.1) with $k = \tau = 1$ but the DGFEM with triangle subdivision is still open.

Next, we recollect the following problems for future investigation.

- We have already seen that the global existence of classical solutions to (1.16) with $k = 2$ is obtained under the assumptions that $0 < \chi < \sqrt{\frac{2}{n}}$ and $\mu > \frac{n-2}{n}$ sufficiently large. But from the biological point of view, the values of μ is small and positive. This condition $\mu > \frac{n-2}{n}$ is acceptable in dimension $n = 2$, nevertheless, the interesting cases are open in higher dimensions (see [97]).
- The blow-up is not possible for the model (1.61) when the cells are repelled by the stimuli. In this case, the chemotactic coefficient $\chi < 0$. In this situation, the existence and uniqueness are proved for the models with logistic source or repulsive case of the model with nonlinear diffusion coefficient or in the lower dimensional case. However, the same results for higher dimension are open.
- Threshold value for the global existence and blow-up of solutions is available only for classical parabolic-elliptic and parabolic-parabolic Keller-Segel models.
- This optimal critical mass value is still not yet found for all the extended models.
- There has not been any analysis on the model (1.41) for general case of $\phi(u, v)$ and $\psi(u, v)$. We have seen in the review that the blow-up of solutions to (1.40) is proved when $q \geqslant p$. In contrast to this case, the global existence is established when $p > q + 1 - \frac{1}{n}$. Here, we could see that the critical case $p = q$ when $n = 1$. But other than the case $n = 1$, the solution behaviour is still open in the range $q < p \leqslant q + 1 - \frac{1}{n}$.
- The blow-up prevention by introducing additional cross-diffusion is only available up to three space dimension. Hence, this result is open for $n \geqslant 4$, (see [30]).
- To obtain existence, blow-up and numerical analysis for a general class of local-nonlocal Keller-Segel models.
- It would be interesting to study the Keller-Segel models with $p(x)$-Laplacian. To the best of our knowledge, there is no work available for the Keller-Segel models with $p(x)$-Laplacian.
- To extend more case studies of chemotactic sensitivity functions in the Keller-Segel model with Fokker-Plank diffusion. There are a variety of chemotactic sensitivity functions which are available in the literature and require further investigation. One can think of to establish the existence and blow-up of solutions with various initial conditions. For example, initial conditions in the interpolation spaces.
- In literature, there is an introduction of a logistic growth restriction. In view of this, there is a natural question to ask. How strong a logistic damping can prevent blow-up for (1.37), see [167]?
- The available numerical methods are almost for simplified versions of Keller-Segel models only. The same numerical methods for Keller-Segel model with various types of chemotactic coefficient, logistic source and diffusion coefficient are still open.

- For (1.61) with $f(u) = \kappa u - \mu u^p$ and $\tau = 1$,, we know that the term μu^p, $\mu > 0$, $p > 1$ suppress the finite-time blow-up. But the range of exponents μ and p to ensure the global existence is still open.
- For models related to (1.68) with exponentially decreasing diffusion coefficient, the questions on the boundedness of solutions for possibly slow growth of ψ relative to ϕ are still open, (cf. [41]).
- Biologically, the value of μ in logistic growth term is very small. For very small value of μ, proving global existence of solutions is a challenging problem. For instance, authors in [156] proved existence of solutions to (1.88) for the value of $\mu \geqslant 23$. For small range of μ, the existence of solutions to (1.88) is still open and for higher dimension $d \geqslant 4$ as well.
- Even for lower dimension, the existence questions to (1.88) seems unaddressed for $q \in (0, 2)$.
- The literature has left the question "whether blow-up may occur in three-dimensional version of (1.90) in cases when μ is positive but small".
- The additional cross-diffusion term in the chemical concentration equation of Keller-Segel model has been studied in the literature to avoid blow-up upto three space dimension. We would mention that the same results are still open for higher dimensions and for other models as well.
- The global existence of renormalized solution to (1.74) is still open for dimension $n \geqslant 3$. The available result is only for a disk in $\mathbb{R}^2$.
- We would mention that for various types of Keller-Segel-Navier-Stokes system, the knowledge on the convergence of solutions of parabolic-parabolic counter part of Keller-Segel-Navier-Stokes system to its associated parabolic-elliptic counter-part is not fully developed.
- The extensibility of non-global solutions of (1.4) beyond its blow-up time has been restricted to the radial case only. This result to the same model and its variants in the non-radial case seems open.
- From the literature, Winkler left the following question as open, "*how far the statement of Theorem 1.5 [199] is to be valid for general geometric settings*?"

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