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# Global Dynamics of an SEIR Model with Two Age Structures and a Nonlinear Incidence

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**Abstract** In this paper, we study an SEIR model with both infection and latency ages and also a very general class of nonlinear incidence. We first present some preliminary results on the existence of solutions and on bounds of solutions. Then we study the global dynamics in detail. After proving the existence of a global attractor $\mathcal{A}$, we characterize it in two cases distinguished by the basic reproduction number $R_0$. When $R_0 < 1$, we apply the Fluctuation Lemma to show that the disease-free equilibrium $E_0$ is globally asymptotically stable, which means $\mathcal{A} = \{E_0\}$. When $R_0 > 1$, we show the uniform persistence and get $\mathcal{A} = \{E_0\} \cup C \cup \mathcal{A}_1$, where $C$ consists of points with connecting orbits from $E_0$ to $\mathcal{A}_1$ and $\mathcal{A}_1$ attracts all points with initial infection force. Under an additional condition, we employ the approach of Lyapunov functional to find that $\mathcal{A}_1$ just consists of an endemic equilibrium.

**Keywords** Age structure · SEIR model · Uniform persistence · Lyapunov functional · Global stability

**Mathematics Subject Classification (2010)** 35B35 · 35Q92 · 92D30

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# 1 Introduction

It is well-known that the SIR epidemic model is one of the most studied systems in mathematical epidemiology, where the population is split into three classes, susceptible (denoted by $S$), infected (denoted by $I$), and recovered (denoted by $R$). Since the first such model was presented by Kernack and Mckendrick [18], significant achievement has been made in analyzing its variations for different situations. These models include ordinary differential equation models [18, 21], reaction diffusion models [14], delayed models [15, 22], and age-structured models [6, 17, 23]. The main goal in most of the works is to determine the global behavior of solutions in terms of the basic reproduction number $R_0$. Usually, a threshold dynamics is established. Roughly speaking, when $R_0 < 1$ the infection disappears from the population while when $R_0 > 1$ the infection persists (see, for instance, [12, 19–21]).

In the last decade, a large amount of attention has been paid to age-structured models. One of the first such models is presented and studied by Magal et al. [23]. The model they investigated is as follows.

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \mu S(t) - S(t)\int_0^\infty \beta(a) i(t,a)da, \\ \frac{\partial i(t,a)}{\partial t} + \frac{\partial i(t,a)}{\partial a} = -\delta(a) i(t,a), \\ i(t,0) = S(t)\int_0^\infty \beta(a) i(t,a)da, \\ S(0) = S_0 \in \mathbb{R}^+, \quad i(0,\cdot) = i_0 \in L^1_+(\mathbb{R}^+), \end{cases} \tag{1}$$

where $L^1_+(\mathbb{R}^+)$ is the space of integrable and positive functionals. $i(t,a)$ is the density of infected individuals at time $t$ with infection age $a$ (time passed since being infected); $\beta(a)$ and $\delta(a)$ are the transmission coefficient and the exit (or (and) mortality or (and) recovery) rate of infected individuals, respectively; $\Lambda$ is the entering flux into the susceptible class; and $\mu$ represents the mortality rate of the population. For the first time, the approach of Lyapunov functional has been applied to prove the global stability of the endemic equilibrium. This has been further generalized to investigate age-structured SIR models with nonlinear incidences (see, for example, [1, 5]).

Though it is true that the transmission rate $\beta(a)$ can take into account the exposed stage, it has been shown that the age-structured SIR model (1) in general and $\beta(a)$ in particular loses its biological precision in the case of infectious diseases with large latent periods such as tuberculosis and HIV [24], where the latent period can vary from months to years or even decades before the disease becomes contagious. Also, for the purpose of treatment, recognizing the disease in its earliest stage is crucial in obtaining a proper treatment strategy. For example, for brucellosis, if the treatment is retarded, then this infection can lead to infertility for human. Therefore, McCluskey [24] introduced a latent class and investigated the following model,

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \mu S(t) - S(t)\int_0^\infty \beta(a) i(t,a)da, \\ \frac{\partial e(t,a)}{\partial t} + \frac{\partial e(t,a)}{\partial a} = -(\gamma(a) + \mu(a)) e(t,a), \\ \frac{\partial i(t,a)}{\partial t} + \frac{\partial i(t,a)}{\partial a} = -\nu(a) i(t,a), \\ e(t,0) = S(t)\int_0^\infty \beta(a) i(t,a)da, \\ i(t,0) = \int_0^\infty \gamma(a) e(t,a)da, \end{cases} \tag{2}$$

where $e(t,a)$ is the density of exposed individuals at time $t$ with exposure time $a$, individuals who have been in the exposed class can progress to the $I$ class with the rate $\gamma(a)$, $\mu(a)$ is the removal rate from the exposed class, $\nu(a)$ is the rate of individuals removed from infectious class. The asymptotic smoothness of the semi-flow generated by solutions of system (2) and uniform persistence are proven by reformulating the system as a system of Volterra integral equations. The global stability of the endemic equilibrium depending on the basic reproduction number is obtained by constructing suitable Lyapunov functionals. Later on, these results have been generalized to system (2) with immigration of the infected [25] and further to a general separable incidence function $\beta(a)f(S(t))h(i(t,a))$ [31]. Further, other nonlinear incidence functions appear in other works such as $S(t)f(i(t))$ in an SIVR model [30], $\varphi\left(S(t), \int_0^\infty \beta(a)i(t,a)\right)$ in an SVEIR model [29], and other generalizations as the incidence functions used in the papers [1, 7].

Motivated by the above works, in this paper, we further the study of age-structured models by considering the following SEIR model that incorporates ages of latency and infection and also a nonlinear incidence,

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \mu S(t) - G(S(t), J(t)), \\ \frac{\partial e(t,a)}{\partial t} + \frac{\partial e(t,a)}{\partial a} = -(\mu + \zeta_1(a) + \zeta_2(a))e(t,a), \\ \frac{\partial i(t,a)}{\partial t} + \frac{\partial i(t,a)}{\partial t} = -(\mu + \delta_1(a) + \delta_2(a))i(t,a), \\ \frac{dR(t)}{dt} = \int_0^\infty \delta_1(a)i(t,a)da - (\mu + \hat{\mu})R(t), \\ \quad \text{(which is decoupled from the others and hence its evolution} \\ \quad \text{will not be considered in the sequel)} \\ e(t,0) = G(S(t), J(t)), \\ i(t,0) = L(t), \\ S(0) = S_0 \in \mathbb{R}^+, \quad e(0,.) = e_0 \in L^1_+(\mathbb{R}^+), \quad i(0,.) = i_0 \in L^1_+(\mathbb{R}^+), \end{cases} \tag{3}$$

where

$$J(t) = \int_0^\infty \beta(a)i(t,a)da \quad \text{and} \quad L(t) = \int_0^\infty \zeta_1(a)e(t,a)da.$$

Besides the same biological meanings of the same parameters in (2), $\zeta_1(a)$ is the age dependent progression rate from exposed to infected and $\delta_1(a)$ is the recovery rate of infected. Unlike most work in the literature, we have also included extra death rates $\zeta_2(a)$ and $\delta_2(a)$ induced by the disease, and $\hat{\mu}$ in the exposed, infected, and recovered, respectively. The incidence is described by the nonlinear function $G$, where, in epidemiology, $J(t)$ is referred to as the infection force at time $t$. Mathematically, the latent compartment generates a big difficulty in constructing a proper Lyapunov functional. In this paper, we develop new forms that help us to determine the global behavior of the equilibria.

As we will not study the evolution of $R(t)$, for simplicity of notation, we denote $\zeta_1(a) + \zeta_2(a)$ as $\zeta(a)$ and $\delta_1(a) + \delta_2(a)$ as $\delta(a)$. We make the following assumptions on the parameters that are biologically relevant.

(H1(a)) $\Lambda, \mu > 0$.
(H1(b)) $\beta, \zeta_1, \zeta_2, \delta \in L^\infty_+(\mathbb{R}^+)$ with the respective essential upper bounds $\bar{\beta}, \bar{\zeta_1}, \bar{\zeta_2}, \bar{\delta}$. Moreover, $\beta$ and $\zeta$ are uniformly continuous on $\mathbb{R}_+$.
(H1(c)) The function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\begin{aligned} \kappa(a) &= \int_0^a \zeta_1(\tau)\pi(\tau)\beta(a-\tau)\nu(a-\tau)d\tau \\ &\left( = \int_0^a \beta(\tau)\nu(\tau)\zeta_1(a-\tau)\pi(a-\tau)d\tau \right) \end{aligned} \tag{4}$$

is not zero a.e., where

$$\pi(a) = e^{-\int_0^a [\mu+\zeta(s)]ds} \quad \text{and} \quad \nu(a) = e^{-\int_0^a [\mu+\delta(s)]ds} \qquad \text{for } a \geq 0. \tag{5}$$

From **(H1(c))**, we know that $\zeta(a) \leq \bar{\zeta_1} + \bar{\zeta_2} := \bar{\zeta}$. The biological meaning of **(H1(c))** is very clear. Note that $\pi(a)$ and $\nu(a)$ are the survival probabilities of the exposed and infected individuals to latency and infection age $a$, respectively. Given $a > 0$, let us consider the process of a susceptible individual who can get infected and still have infectivity $a$ time units after. During the process, the susceptible can get infected (with latency age 0), who can stay in $E$ class for any $\tau$ units of time (with $\tau \in [0, a]$), then enters $I$ class with rate $\zeta_1(\tau)\pi(\tau)$ (with infection age 0). It follows that the individual survives to infection age $a - \tau$ and hence the possibility of transmitting infection is $\beta(a - \tau)\nu(a - \tau)$. Therefore, $\kappa(a)$ is the ability of transmitting infection of a susceptible individual after $a$ units of time since being infected. If $\kappa$ is zero a.e., then the infection cannot continue. In fact, this can be confirmed later on. If $\kappa$ is zero a.e., then the basic reproduction number $R_0 = 0$ and the disease-free equilibrium is globally asymptotically stable. This is not the situation we are considering.

We mention that nonlinear incidences have been considered by many authors (to name a few, see [1, 3, 7, 10, 29, 30]). Here we assume that the incidence $G(S, J)$ satisfies the following properties.

(H2(a)) $G(S, J)$ is differentiable and $\frac{\partial G(S,J)}{\partial S} > 0$ for $S > 0$ and $J > 0$, $\frac{\partial G(S,J)}{\partial J} > 0$ for $S > 0$ and $J \geq 0$. Moreover, $G(0, J) = G(S, 0) = 0$ for all $S \geq 0$ and $J \geq 0$.

(H2(b)) The function $\dfrac{\partial G}{\partial J}(S, J)$ is continuous at $(S^0, 0)$, where $S^0 = \frac{\Lambda}{\mu}$. Moreover, for given $S \geq 0$, $G(S, \cdot)$ is concave down.

(H2(c)) The function $G$ is locally Lipschitz continuous in $S$ and $J$, *i.e.*, for every $C > 0$ there exists some $K := K_C > 0$ such that

$$|G(S_2, J_2) - G(S_1, J_1)| \leq K(|S_2 - S_1| + |J_2 - J_1|),$$

whenever $0 \leq S_2, S_1, J_2, J_1 \leq C$.

Clearly, $G(S, J)$ includes some in the literature as special cases, for example, $f(S(t)) \int_0^\infty h(i(t, a))da$ in [31]. But it also contains cases that the above mentioned papers are inapplicable to, which include the Beddington-DeAngelis [2, 4] incidence

$$G(S, J) = \frac{SJ}{1 + c_1 S + c_2 J}$$

and the Crowley-Martin [26, 28] incidence

$$G(S, J) = \frac{SJ}{1 + c_1 S + c_2 J + c_1 c_2 SJ}.$$

We mention that model (3) is a special case of that studied by Wang *et. al.* [29] with $p = 0$. However, in this and similar works, there are some critical issues. On the one hand, no one has ever proved that $\Omega_0$ (or similar notations) is positively invariant with respect to the solution semiflow. This fact is used in the proof of the uniform weak $\rho$-persistence of the semiflow. During the arguments, translations have been made. After translations, without the invariance of $\Omega_0$, the transferred initial condition may not be in $\Omega_0$. In fact, many claimed the positive invariance after establishing the uniform weak $\rho$-persistence. This results in the first loop in the discussion. In this work, for the first time, we have recognized this issue and solved it successfully. On the other hand, if we do not know that $\Omega_0$ is closed, then we could not get the existence of a global compact attractor in $\Omega_0$ by using [11, Theorem 3.4.6]. Actually, this approach requires that the semiflow be uniform $\rho$-persistence. But, in these works, this is deduced from the existence of a global compact attractor in $\Omega_0$. This gives another loop. Here we shall provide a different approach based on the theory developed by Smith and Thieme [27] to solve this dilemma. We reiterate that the main contribution of this work is to use a concrete example to give a rigorous discussion of epidemic models with age structures. This new approach can be adopted to fill the gaps in existing literature and to deal with some new models.

The main result of this paper is a threshold dynamics determined by the basic reproduction number. We first give some preliminary results on existence of solutions and their bounds in Sect. 2. Section 3 is the main part of this paper, which deals with the global dynamics. We first show the existence of a global attractor, followed is the global stability of the disease-free equilibrium when the basic reproduction number $R_0 < 1$. Then we establish the existence of an endemic equilibrium when $R_0 > 1$. In order to study the global stability of the endemic equilibrium by the approach of Lyapunov functional, we first establish the uniform persistence. Here for the first time, some new arguments are developed. The paper concludes with a brief discussion.

## 2 Preliminary Results

The phase space of (3) is $\Gamma = \mathbb{R}^+ \times L^1_+(\mathbb{R}^+) \times L^1_+(\mathbb{R}^+)$, the positive cone of the Banach space $\mathbb{R} \times L^1(\mathbb{R}^+) \times L^1(\mathbb{R}^+)$ equipped with the norm

$$\|(x, \varphi, \psi)\| = |x| + \int_0^\infty |\varphi(a)|da + \int_0^\infty |\psi(a)|da$$

for $(x, \varphi, \psi) \in \mathbb{R} \times L^1(\mathbb{R}^+) \times L^1(\mathbb{R}^+)$. For $\gamma_0 = (S_0, e_0, i_0) \in \Gamma$, problem (3) has a unique and continuous solution $(S(t), e(t, a), i(t, a))$ on $\mathbb{R}^+$, which satisfies $(S(t), e(t, \cdot), i(t, \cdot)) \in \Gamma$ for all $t \in \mathbb{R}^+$. This can be established through standard arguments (see, for example, [8, 9]). Therefore, we can obtain a continuous semi-flow $\Phi : \mathbb{R}_+ \times \Gamma \to \Gamma$ defined by solutions of (3),

$$\Phi(t, \gamma_0) = (S(t), e(t, .), i(t, .)) \qquad \text{for } (t, \gamma_0) \in \mathbb{R}^+ \times \Gamma,$$

where $(S(t), e(t, \cdot), i(t, \cdot))$ is the solution of (3) with the initial condition $\gamma_0$.

Note that, for $(t, \gamma_0) \in \mathbb{R}^+ \times \Gamma$,

$$\|\Phi(t, \gamma_0)\| = S(t) + \int_0^\infty e(t, a)da + \int_0^\infty i(t, a)da.$$

Then

$$\begin{aligned} &\frac{d}{dt}\|\Phi(t, \gamma_0)\| \\ &= [\Lambda - \mu S(t) - G(S(t), J(t))] + \int_0^\infty \frac{\partial e(t, a)}{\partial t}da + \int_0^\infty \frac{\partial i(t, a)}{\partial t}da \\ &= [\Lambda - \mu S(t) - G(S(t), J(t))] - \int_0^\infty \left[\frac{\partial e(t, a)}{\partial a} + (\mu + \zeta(a))e(t, a)\right]da \end{aligned}$$

$$
\begin{aligned}
&\quad - \int_0^\infty \left[ \frac{\partial i(t,a)}{\partial a} + (\mu + \delta(a)) i(t,a) \right] da \\
&= [\Lambda - \mu S(t) - G(S(t), J(t))] \\
&\quad - \left[ e(t,\infty) - e(t,0) - \int_0^\infty (\mu + \zeta(a)) e(t,a) da - L(t) \right] \\
&\quad - \left[ i(t,\infty) - i(t,0) - \int_0^\infty (\mu + \delta(a)) i(t,a) da \right].
\end{aligned}
$$

This, combined with the boundary conditions in (3), gives

$$
\frac{d\|\Phi(t,\gamma_0)\|}{dt} \le \Lambda - \mu \|\Phi(t,\gamma_0)\|
$$

and hence

$$
\|\Phi(t,\gamma_0)\| \le e^{-\mu t} \|\gamma_0\| + \frac{\Lambda}{\mu} \left(1 - e^{-\mu t}\right). \tag{6}
$$

Then (6) immediately implies the following result on the boundedness of solutions of (3).

**Proposition 1**

(i) $\|\Phi(t,\gamma_0)\| \le \max\{\frac{\Lambda}{\mu}, \|\gamma_0\|\}$ *for* $(t,\gamma_0) \in \mathbb{R}^+ \times \Gamma$.
(ii) $\limsup_{t\to\infty} \|\Phi(t,\gamma_0)\| \le \frac{\Lambda}{\mu}$ *for* $(t,\gamma_0) \in \mathbb{R}^+ \times \Gamma$. *Thus* $\Phi$ *is point-dissipative, that is, there is a bounded set that attracts all points in* $\Gamma$.
(iii) *The set*

$$
\Omega = \left\{ \gamma_0 \in \Gamma : \|\gamma_0\| \le \frac{\Lambda}{\mu} \right\}
$$

*is a positively invariant and attracting set of* $\Phi$.
(iv) *Let* $B \subset \Gamma$ *be bounded. Then* $\{\Phi(t,\gamma) : (t,\gamma) \in \mathbb{R}^+ \times B\}$ *is bounded. In particular,* $\Phi$ *is eventually bounded on* $B$.

Since we are mainly concerned with limiting behavior of solutions of (3), sometimes we assume that the initial values are in $\Omega$ and let $K$ be the local Lipschitz constant of $G$ associated with $C = \max\{\frac{\Lambda}{\mu}, \frac{\Lambda}{\mu}\bar{\beta}\}$ when there is no confusion.

**Proposition 2** $\liminf_{t\to\infty} S(t) \ge \xi \triangleq \frac{\Lambda}{\mu+K}$ *for any solution with initial values in* $\Omega$.

*Proof* For any $\gamma_0 \in \Omega$, we have $\|i(t,\cdot)\|_1 \le \frac{\Lambda}{\mu}$ and hence $J(t) \le \frac{\Lambda}{\mu}\bar{\beta}$ for $t \ge 0$. It follows from this and Lipschitz continuity of $G$ that

$$
\frac{dS(t)}{dt} \ge \Lambda - \mu S(t) - K S(t),
$$

which implies that $\liminf_{t\to\infty} S(t) \ge \xi$. □

We give the expressions of $e(t,a)$ and $i(t,a)$ to conclude this section.

Let $\nu$ and $\pi$ be defined by (5). Integrating the partial differential equations for $e$ and $i$ in (3) respectively along the characteristic lines $t - a = c$, where $c$ is a constant, gives us

$$e(t,a) = \begin{cases} G(S(t-a), J(t-a))\pi(a), & t > a \geq 0 \\ e_0(a-t)\frac{\pi(a)}{\pi(a-t)}, & a > t \geq 0 \end{cases} \tag{7}$$

and

$$i(t,a) = \begin{cases} L(t-a)\nu(a), & t > a \geq 0, \\ i_0(a-t)\frac{\nu(a)}{\nu(a-t)}, & a > t \geq 0. \end{cases} \tag{8}$$

## 3 Global Dynamics

### 3.1 Existence of a Global Attractor

The roughest result on the global dynamics of (3) is the existence of a global attractor. We employ Theorem 2.33 of Smith and Thieme [27] to establish it.

We first verify the asymptotical smoothness of $\Phi$. A semi-flow is called asymptotically smooth if each forward invariant bounded closed set is attracted by a nonempty compact set. The asymptotic smoothness of $\Phi$ is obtained by using the following result, which is a special case of Theorem 2.46 [27].

**Theorem 1** *The semi-flow $\Phi : \mathbb{R}^+ \times \Gamma \to \Gamma$ is asymptotically smooth if there are maps $\Theta$, $\Psi : \mathbb{R}^+ \times \Gamma \to \Gamma$ such that $\Phi(t,\gamma) = \Theta(t,\gamma) + \Psi(t,\gamma)$ and the following hold for any bounded closed set $C$ that is forward invariant under $\Phi$*:

- $\lim_{t\to\infty} \operatorname{diam}\Theta(t,C) = 0$,
- *there exists $t_C$ such that $\Psi(t,C)$ has a compact closure for each $t \geq t_C$.*

The following Frechet-Kolomogorov Theorem characterizes the compactness of subsets in $L^1(\mathbb{R}^+)$.

**Theorem 2** [27] *Let $\mathcal{F}$ be a subset of $L^1(\mathbb{R}^+)$. Then $\mathcal{F}$ has compact closure if and only if the following conditions hold*:

1. $\sup_{f\in\mathcal{F}} \int_0^\infty |f(a)|da < \infty$,
2. $\lim_{r\to\infty} \int_r^\infty |f(a)|da = 0$ *uniformly in* $f \in \mathcal{F}$,
3. $\lim_{h\to 0} \int_0^\infty |f(a+h) - f(a)|da = 0$ *uniformly in* $f \in \mathcal{F}$,
4. $\lim_{h\to 0} \int_0^h |f(a)|da = 0$ *uniformly in* $f \in \mathcal{F}$.

In order to apply Theorems 1 and 2, we need the following result.

**Proposition 3** *Let $B \subset \Gamma$ be bounded. Then, for any $\varepsilon > 0$, there exists $t_0 > 0$ such that, for all $t \geq 0$, $h \in (0, t_0)$, and $\gamma_0 \in B$,*

$$|J(t+h) - J(t)| \leq \varepsilon \quad \textit{and} \quad |L(t+h) - L(t)| \leq \varepsilon.$$

*Proof* Let $C > \frac{\Lambda}{\mu}$ be a bound for $B$. For $t \geq 0$, $h > 0$, and $\gamma_0 \in B$, we have

$$\begin{aligned}
&|J(t+h) - J(t)| \\
&= \left| \int_0^\infty \beta(a) i(t+h,a)da - \int_0^\infty \beta(a) i(t,a)da \right| \\
&= \left| \int_0^h \beta(a) i(t+h,a)da + \int_h^\infty \beta(a) i(t+h,a)da - \int_0^\infty \beta(a) i(t,a)da \right| \\
&= \left| \int_0^h \beta(a) L(t+h-a)\nu(a)da + \int_0^\infty \beta(a+h) i(t+h,a+h)da \right. \\
&\quad \left. - \int_0^\infty \beta(a) i(t,a)da \right|.
\end{aligned}$$

By Proposition 1, for $t \geq 0$, we have $\|e(t,\cdot)\|_1 \leq C$ and $\|i(t,\cdot)\|_1 \leq C$, and hence $L(t) \leq \bar{\zeta} C$. Then we get

$$|J(t+h) - J(t)| \leq \bar{\beta}\bar{\zeta}Ch + \left| \int_0^\infty \beta(a+h) i(t+h,a+h)da - \int_0^\infty \beta(a) i(t,a)da \right|.$$

It follows easily from (8) that $i(t+h,a+h) = i(t,a)e^{-\int_a^{a+h}(\mu+\delta(s))ds}$. Thus

$$\begin{aligned}
&|J(t+h) - J(t)| \\
&\leq \bar{\beta}\bar{\zeta}Ch + \left| \int_0^\infty \beta(a+h) i(t,a) e^{-\int_a^{a+h}(\mu+\delta(s))ds} da - \int_0^\infty \beta(a) i(t,a)da \right| \\
&= \bar{\beta}\bar{\zeta}Ch + \left| \int_0^\infty (\beta(a+h) e^{-\int_a^{a+h}(\mu+\delta(s))ds} - \beta(a)) i(t,a)da \right| \\
&\leq \bar{\beta}\bar{\zeta}Ch + \int_0^\infty \beta(a+h) \left| e^{-\int_a^{a+h}(\mu+\delta(s))ds} - 1 \right| i(t,a)da \\
&\quad + \int_0^\infty |\beta(a+h) - \beta(a)| i(t,a)da.
\end{aligned}$$

Noting $0 \geq -\int_a^{a+h}(\mu+\delta(s))ds \geq -\mu h$, we have $1 \geq e^{-\int_a^{a+h}(\mu+\delta(s))ds} \geq e^{-\mu h} \geq 1 - \mu h$, where the last inequality comes from the fact that $e^x$ lies above its tangent. Then $\beta(a+h) \left| e^{-\int_a^{a+h}(\mu+\delta(s))ds} - 1 \right| \leq \bar{\beta}\mu h$. Therefore,

$$J(t+h) - J(t) \leq \bar{\beta}\bar{\zeta}Ch + \bar{\beta}\mu Ch + \int_0^\infty |\beta(a+h) - \beta(a)| i(t,a)da.$$

Now it is clear from the uniform continuity of $\beta$ that there exists $t_1 > 0$ such that

$$|J(t+h) - J(t)| \leq \varepsilon \qquad \text{for all } t \geq 0,\ h \in (0, t_1), \text{ and } \gamma_0 \in B.$$

Similarly, we can obtain a $t_2 > 0$ such that

$$|L(t+h) - L(t)| \leq \varepsilon \qquad \text{for all } t \geq 0,\ h \in (0, t_2), \text{ and } \gamma_0 \in B.$$

Letting $t_0 = \min\{t_1, t_2\}$ finishes the proof. □

Now, we are ready to prove the asymptotic smoothness of $\Phi$.

**Theorem 3** *$\Phi$ is asymptotically smooth.*

*Proof* To apply Theorem 1, we decompose $\Phi$ as follows. Define $\Theta, \Psi : \mathbb{R}^+ \times \Gamma \to \Gamma$ respectively by

$$\Theta(t, \gamma_0) = (0, \tilde{e}(t, \cdot), \tilde{i}(t, \cdot)), \qquad \Psi(t, \gamma_0) = (S(t), \hat{e}(t, \cdot), \hat{i}(t, \cdot))$$

for $(t, \gamma_0) \in \mathbb{R}^+ \times \Gamma$, where

$$\tilde{e}(t,a) = \begin{Bmatrix} 0, & t \geq a \geq 0 \\ e(t,a), & a \geq t \geq 0 \end{Bmatrix} = \begin{Bmatrix} 0, & t \geq a \geq 0, \\ e_0(a-t)\frac{\pi(a)}{\pi(a-t)}, & a \geq t \geq 0, \end{Bmatrix}$$

$$\tilde{i}(t,a) = \begin{Bmatrix} 0, & t \geq a \geq 0 \\ i(t,a), & a \geq t \geq 0 \end{Bmatrix} = \begin{Bmatrix} 0, & t \geq a \geq 0, \\ i_0(a-t)\frac{\nu(a)}{\nu(a-t)}, & a \geq t \geq 0, \end{Bmatrix}$$

and

$$\hat{e}(t,a) = e(t,a) - \tilde{e}(t,a), \qquad \hat{i}(t,a) = i(t,a) - \tilde{i}(t,a).$$

Then $\Phi = \Theta + \Psi$. Note that for $t \geq 0$,

$$\begin{aligned} \hat{e}(t,a) &= \begin{Bmatrix} e(t,a), & t \geq a \geq 0 \\ 0, & a \geq t \geq 0 \end{Bmatrix} \\ &= \begin{Bmatrix} G(S(t-a), J(t-a))\pi(a), & t \geq a \geq 0 \\ 0, & a \geq t \geq 0 \end{Bmatrix} \end{aligned}$$

and

$$\hat{i}(t,a) = \begin{Bmatrix} i(t,a), & t \geq a \geq 0 \\ 0, & a \geq t \geq 0 \end{Bmatrix} = \begin{Bmatrix} L(t-a)\nu(a), & t \geq a \geq 0 \\ 0, & a \geq t \geq 0 \end{Bmatrix}$$

Let $B \subset \Gamma$ be bounded with a bound $C$. Without loss of generality, we assume $C > \frac{\Lambda}{\mu}$. First, we have

$$\begin{aligned} &||\Theta(t, \gamma_0)|| \\ &= |0| + \int_0^\infty \tilde{e}(t,a)da + \int_0^\infty \tilde{i}(t,a)da, \\ &= \int_t^\infty e_0(a-t)\frac{\pi(a-t)}{\pi(a)}da + \int_t^\infty i_0(a-t)\frac{\nu(a-t)}{\nu(a)}da, \\ &= \int_0^\infty e_0(s)\frac{\pi(s+t)}{\pi(s)}ds + \int_0^\infty i_0(s)\frac{\nu(s+t)}{\nu(s)}ds, \\ &= \int_0^\infty e_0(s)e^{-\int_s^{s+t}(\mu+\zeta(\sigma))d\sigma}ds + \int_0^\infty i_0(s)e^{-\int_s^{s+t}(\mu+\delta(\sigma))d\sigma}ds, \\ &\leq e^{-\mu t}\int_0^\infty e_0(s)ds + e^{-\mu t}\int_0^\infty i_0(s)ds \\ &\leq e^{-\mu t}\|\gamma_0\|. \end{aligned}$$

Then we immediately see that the first condition in Theorem 1 holds.

Next, we verify the second condition of Theorem 1. By Proposition 1, $S(t) \in [0, C]$ for $t \geq 0$ and hence $\{S(t) : \gamma_0 \in B\}$ is precompact for every $t \geq 0$. In the following we only show $\{\hat{e}(t, \cdot) : \gamma_0 \in B\}$ is precompact in $L^1(\mathbb{R}^+)$ for every $t \geq 0$ as the proof of $\{\hat{i}(t, \cdot) : \gamma_0 \in B\}$ being precompact is similar.

Observe that $\hat{e}(t, a) \leq KCe^{-\mu a}$ by the Lipschtiz continuity of $G$ and $S(t) \leq C$. Then conditions 1, 2, and 4 of Theorem 2 immediately follows from this. Now, we verify the third condition of this theorem. For $h \in (0, t)$, we have

$$
\begin{aligned}
&\int_0^\infty \left|\hat{e}(t, a+h) - \hat{e}(t, a)\right| da \\
&= \int_0^{t-h} |G(S(t-a-h), J(t-a-h))\pi(a+h) \\
&\quad -G(S(t-a), J(t-a))\pi(a)|da, \\
&\quad + \int_{t-h}^{t} |0 - G(S(t-a), J(t-a))\pi(a)|da, \\
&\leq K\bar{\beta}Ch + \int_0^{t-h} G(S(t-a-h), J(t-a-h))\, |\pi(a+h) - \pi(a)|\, da, \\
&\quad + \int_0^{t-h} |G(S(t-a-h), J(t-a-h)) - G(S(t-a), J(t-a))|\, \pi(a)da \\
&\leq K\bar{\beta}Ch + K\bar{\beta}C \int_0^{t-h} |\pi(a+h) - \pi(a)|\, da \\
&\quad +K \int_0^{t-h} \big(|S(t-a-h) - S(t-a)| + |J(t-a-h) - J(t-a)|\big)\pi(a)da.
\end{aligned}
$$

Recall that $\pi(a)$ is a decreasing function. It follows that

$$
\begin{aligned}
\int_0^{t-h} |\pi(a+h) - \pi(a)|da &= \int_0^{t-h} \pi(a) - \pi(a+h)da, \\
&= \int_0^{t-h} \pi(a)da - \int_0^{t-h} \pi(a+h)da, \\
&= \int_0^{t-h} \pi(a)da - \int_h^{t} \pi(s)ds, \\
&= \int_0^{h} \pi(a)da - \int_{t-h}^{t} \pi(a)da, \\
&\leq \int_0^{h} \pi(a)da, \\
&\leq h.
\end{aligned}
$$

Therefore,

$$
\int_0^\infty \left|\hat{e}(t, a+h) - \hat{e}(t, a)\right| da \leq 2K\bar{\beta}Ch + K \int_0^{t-h} \big(|S(t-a-h) - S(t-a)|
$$

$$+|J(t-a-h)-J(t-a)|\big)da.$$

Now Proposition 1 and the first equation of (3) imply that $|\frac{dS(t)}{dt}| \leq \Lambda + \mu C + G(C, \bar{\beta}C)$ for all $t \geq 0$ and $\gamma_0 \in B$. This, combined with Proposition 3, immediately show that $\lim_{h\to 0}\int_0^\infty \hat{e}(t,a+h) - \hat{e}(t,a)|da = 0$ uniformly on $B$. Thus we have shown that $\Psi(t,B)$ is precompact for all $t \geq 0$. By applying Theorem 1, we know that $\Phi$ is asymptotically smooth. □

By now, we have shown that $\Phi$ is point-dissipative, eventually bounded on bounded sets, and asymptotically smooth (see Proposition 1 and Theorem 3), namely, we have verified all the assumptions of Theorem 2.33 [27]. So we have arrived at the main result of this subsection.

**Theorem 4** *The semi-flow $\Phi$ has a global attractor $\mathcal{A}$ in $\Gamma$, which attracts any bounded set of $\Gamma$.*

Clearly, the global attractor $\mathcal{A} \subset \Omega$. Since it is invariant, it can only contain points with total $\Phi$-trajectories through them. A total $\Phi$-trajectory is a function $\gamma(t) = (S(t), e(t,.), i(t,.))$ such that $\gamma(t+r) = \Phi(t, \gamma(r))$ for all $t \in \mathbb{R}$ and $r \geq 0$. For a total $\Phi$-trajectory, we can get

$$\begin{aligned} &\tfrac{dS(t)}{dt} = \Lambda - \mu S(t) - G(S(t), J(t)),\\ &e(t,a) = e(t-a,0)\pi(a) = G(S(t-a), J(t-a))\pi(a),\\ &i(t,a) = i(t-a,0)\nu(a) = L(t-a)\nu(a),\\ &J(t) = \textstyle\int_0^\infty \beta(a)L(t-a)\nu(a)da,\\ &L(t) = \textstyle\int_0^\infty \zeta_1(a)G(S(t-a), J(t-a))\pi(a)da, \end{aligned} \tag{9}$$

for all $t \in \mathbb{R}$ and $a \in \mathbb{R}_+$.

In the following, we describe the global attractor $\mathcal{A}$.

## 3.2 The Global Stability of the Disease-Free Equilibrium

System (3) always has the disease-free equilibrium $E_0 = (S^0, 0, 0)$. Recall that $S^0 = \frac{\Lambda}{\mu}$.

We first study the local stability of $E_0$. Through linearization, we can get the characteristic equation at an arbitrary equilibrium $E_* = (S_*, e_*, i_*)$,

$$\begin{aligned} &(\lambda+\mu)\left[\frac{\partial G(S_*, J_*)}{\partial J}\int_0^\infty e^{-\lambda a}\beta(a)\nu(a)da\int_0^\infty e^{-\lambda a}\zeta_1(a)\pi(a)da - 1\right]\\ &-\frac{\partial G(S_*, J_*)}{\partial S} = 0, \end{aligned} \tag{10}$$

where $J_* = \int_0^\infty \beta(a)i_*(a)da$. Then $E_*$ is locally (asymptotically) stable if all roots of (10) have negative real parts and otherwise it is unstable. For detail, we refer to [16], for example. Before moving forward, we introduce the basic reproduction number $R_0$ defined by

$$R_0 = \frac{\partial G(S^0, 0)}{\partial J}\int_0^\infty \beta(a)\nu(a)da\int_0^\infty \zeta_1(a)\pi(a)da. \tag{11}$$

Recall that $\pi(a)$ and $\nu(a)$ are respectively the survival probabilities of the exposed and infected individuals to latency and infection age $a$. It follows that $\int_0^\infty \beta(a)\nu(a)da$ is the average transmission rate and $\int_0^\infty \zeta_1(a)\pi(a)da$ is the conversion rate from exposed to infected. As $\frac{\partial G(S^0,0)}{\partial J}$ is the initial spread rate, it follows that $R_0$ is the average number of second infected cases generated by introducing one infected individual into a population of susceptible only. This agrees with the definition of basic reproduction number.

**Proposition 4** *The disease-free equilibrium $E_0$ is locally asymptotically stable if $R_0 < 1$ while it is unstable if $R_0 > 1$.*

*Proof* Since $G(S^0, 0) = 0$, it follows that $\frac{\partial G(S^0,0)}{\partial S} = 0$. Then by (10), the characteristic equation at $E_0$ is

$$(\lambda + \mu)\left[\frac{\partial G(S^0, 0)}{\partial J}\int_0^\infty e^{-\lambda a}\beta(a)\nu(a)da\int_0^\infty e^{-\lambda a}\zeta_1(a)\pi(a)da - 1\right] = 0.$$

Obviously, the stability of $E_0$ is determined by roots of $F_0(\lambda) = 0$, where

$$F_0(\lambda) = \frac{\partial G(S^0, 0)}{\partial J}\int_0^\infty e^{-\lambda a}\beta(a)\nu(a)da\int_0^\infty e^{-\lambda a}\zeta_1(a)\pi(a)da - 1.$$

If $R_0 > 1$, then $F_0(0) = R_0 - 1 > 0$. Since $\lim_{\lambda\to\infty} F_0(\lambda) = -1 < 0$. By the intermediate value theorem, $F_0(\lambda) = 0$ has a positive root. Thus $E_0$ is unstable if $R_0 > 1$. Now, assume that $R_0 < 1$. We claim that all roots of $F_0(\lambda) = 0$ have negative real parts. If this does not hold, then there exists $\lambda_0$ with $\mathrm{Re}(\lambda_0) \geq 0$ such that $F_0(\lambda_0) = 0$. Thus we have

$$\begin{aligned}
1 &= \left|\frac{\partial G(S^0, 0)}{\partial J}\int_0^\infty e^{-\lambda_0 a}\beta(a)\nu(a)da\int_0^\infty e^{-\lambda_0 a}\zeta_1(a)\pi(a)da\right|,\\
&\leq \frac{\partial G(S^0, 0)}{\partial J}\int_0^\infty \left|e^{-\lambda_0 a}\beta(a)\nu(a)\right|da\int_0^\infty \left|e^{-\lambda_0 a}\zeta_1(a)\pi(a)\right|da,\\
&\leq \frac{\partial G(S^0, 0)}{\partial J}\int_0^\infty \beta(a)\nu(a)da\int_0^\infty \zeta_1(a)\pi(a)da,\\
&= R_0,
\end{aligned}$$

a contradiction to $R_0 < 1$. This proves the claim and hence $E_0$ is locally asymptotically stable when $R_0 < 1$. □

Actually, $E_0$ is globally stable when it is locally stable. We shall apply the Fluctuation Lemma [13] to prove it. For a bounded function $f : \mathbb{R}^+ \to \mathbb{R}$, we denote

$$f_\infty = \liminf_{t\to\infty} f(t) \qquad \text{and} \qquad f^\infty = \limsup_{t\to\infty} f(t).$$

The following technical result will be useful in the coming discussion.

**Lemma 1** [16] *Suppose that $k \in L^1_+(\mathbb{R}^+)$ and $f : \mathbb{R}^+ \to \mathbb{R}$ is a bounded function. Then*

$$\limsup_{t\to\infty}\int_0^t k(\theta)f(t-\theta)d\theta \leq f^\infty\|k\|_1.$$

**Theorem 5** *Suppose $R_0 < 1$. Then the disease-free equilibrium $E_0$ is globally asymptotically stable in $\Gamma$.*

*Proof* By Proposition 4, it suffices to show that $E_0$ is attractive in $\Gamma$. Let $\gamma_0 \in \Gamma$. As $\Omega$ is attractive and positively invariant, without loss of generality, we can assume that $\gamma_0 \in \Omega$. By Proposition 1, $\|\Phi(t, \gamma_0)\| \le \frac{\Lambda}{\mu}$ and $J(t)$ and $L(t)$ are also bounded.

We first show $J^\infty = L^\infty = 0$. By (8),

$$\begin{aligned} J(t) &= \int_0^\infty \beta(a)i(t,a)da = \int_0^t \beta(a)i(t,a)da + \int_t^\infty \beta(a)i(t,a)da, \\ &= \int_0^t \beta(a)L(t-a)\nu(a)da + \int_t^\infty \beta(a)i_0(a-t)\frac{\nu(a)}{\nu(a-t)}da, \\ &\le \int_0^t \beta(a)L(t-a)\nu(a)da + e^{-\mu t}\bar{\beta}\int_t^\infty i_0(a-t)da, \\ &\le \int_0^t \beta(a)L(t-a)\nu(a)da + e^{-\mu t}\bar{\beta}\int_0^\infty i_0(a)da. \end{aligned}$$

Applying Lemma 1, we get

$$J^\infty \le L^\infty \int_0^\infty \beta(a)\nu(a)da. \tag{12}$$

Similarly, using the monotonicity and concavity of $G(S, J)$, we can get

$$\begin{aligned} L(t) &= \int_0^t \zeta_1(a)G(S(t-a), J(t-a))\pi(a)da \\ &\quad + \int_t^\infty \zeta_1(a)e_0(a-t)\frac{\pi(a)}{\pi(a-t)}da \\ &\le \int_0^t \zeta_1(a)G(S^0, J(t-a))\pi(a)da + e^{-\mu t}\bar{\zeta}_1\int_0^\infty e_0(a)da \\ &\le \frac{\partial G(S^0,0)}{\partial J}\int_0^t \zeta_1(a)\pi(a)J(t-a)da + e^{-\mu t}\bar{\zeta}_1\int_0^\infty e_0(a)da. \end{aligned}$$

Applying Lemma 1 again gives us

$$L^\infty \le \frac{\partial G(S^0,0)}{\partial J}J^\infty \int_0^\infty \zeta_1(a)\pi(a)da. \tag{13}$$

It follows from (12) and (13) that

$$J^\infty \le R_0 J^\infty.$$

Since $R_0 < 1$ and $J^\infty \ge 0$, we have $J^\infty = 0$ and hence $L^\infty = 0$ from (13).

Secondly, similar arguments as those for estimating $J^\infty$ and $L^\infty$ will produce $\|e(t,\cdot)\|_1 \to 0$ and $\|i(t,\cdot)\|_1 \to 0$ as $t \to \infty$. For example,

$$\|i(t,\cdot)\|_1 = \int_0^t L(t-a)\nu(a)da + \int_t^\infty i_0(a-t)\frac{\nu(a)}{\nu(a-t)}da$$

$$\leq \int_0^t L(t-a)\nu(a)da + e^{-\mu t}\int_0^\infty i_0(a)da.$$

With the help of Lemma 1, we have $\limsup_{t\to\infty}\|i(t,\cdot)\|_1 \leq L^\infty\|\nu\|_1 = 0$ and hence $\lim_{t\to\infty}\|i(t,\cdot)\|_1 = 0$.

Finally, we show that $\lim_{t\to\infty} S(t) = S^0$. By the Fluctuation Lemma, there exists a sequence $\{t_n\}$ such that $t_n \to \infty$, $S(t_n) \to S_\infty$, and $\frac{dS(t_n)}{dt} \to 0$ as $n \to \infty$. Using the Lipschitz continuity of $G$, we have

$$\frac{dS(t_n)}{dt} = \Lambda - \mu S(t_n) - G(S(t_n), J(t_n)) \geq \Lambda - \mu S(t_n) - KJ(t_n). \tag{14}$$

Note that $\lim_{t\to\infty} J(t) = 0$ as $\lim_{t\to\infty}\|i(t,\cdot)\|_1 = 0$. Letting $n\to\infty$ in (14) gives

$$0 \geq \Lambda - \mu S_\infty,$$

or $S_\infty \geq S^0$. As $S^\infty \leq S^0$, it follows that $\lim_{t\to\infty} S(t) = S^0$.

In summary, we have shown that $\lim_{t\to\infty}\Phi(t,\gamma_0) = E_0$. This completes the proof. □

### 3.3 Existence and Local Stability of Endemic Equilibria

Theorem 5 implies that $\mathcal{A} = \{E_0\}$ when $R_0 < 1$. Thus the infection dynamics is quite simple in this case, namely, the disease will become extinct.

In the following, we consider the case where $R_0 > 1$. We begin with the existence of equilibria other than the disease-free equilibrium.

Let $(S^*, e^*, i^*) \in \Gamma$ (and in fact in $\Omega$) be an equilibrium of (3) or equivalently of $\Phi$. Then

$$0 = \Lambda - \mu S^* - G(S^*, J^*), \tag{15a}$$

$$\frac{de^*(a)}{da} = -(\mu + \zeta(a))e^*(a), \tag{15b}$$

$$\frac{di^*(a)}{da} = -(\mu + \delta(a))i^*(a), \tag{15c}$$

$$e^*(0) = G(S^*, J^*), \tag{15d}$$

$$i^*(0) = \int_0^\infty \zeta_1(a)e^*(a)da, \tag{15e}$$

$$J^* = \int_0^\infty \beta(a)i^*(a)da.$$

It follows from (15b) and (15c) that

$$e^*(a) = e^*(0)\pi(a) \qquad \text{and} \qquad i^*(a) = i^*(0)\nu(a), \tag{16}$$

respectively. This and (15e) give us

$$i^*(0) = e^*(0)\int_0^\infty \zeta_1(a)\pi(a)da, \tag{17}$$

which implies that an equilibrium rather than $E_0$ must be endemic. Moreover, (15d) combined with (15a) tells us that

$$S^* = \frac{\Lambda - e^*(0)}{\mu},$$

which also implies that $e^*(0) \leq \Lambda$. Substituting what we have got into (15d), we obtain $e^*(0) \in [0, \Lambda)$ is a zero of $P(x)$, where

$$P(x) = x - G\left(\frac{\Lambda - x}{\mu}, x \int_0^\infty \zeta_1(a)\pi(a)da \int_0^\infty \beta(a)\nu(a)da\right).$$

Clearly, $P(0) = 0$, which means that (3) always has the disease-free equilibrium $E_0$. Noting $P'(0) = 1 - R_0 < 0$, we know that $P(x)$ is negative for $x > 0$ sufficiently small. This, together with $P(\Lambda) = \Lambda > 0$, implies the existence of a positive zero of $P$ in $(0, \Lambda)$. To summarize, we have shown the following result.

**Proposition 5** *Suppose $R_0 > 1$. Then besides the disease-free equilibrium $E_0$, (3) also has at least one endemic equilibrium.*

Note that at that moment the assumptions on $G$ may not be enough to guarantee the uniqueness of endemic equilibria.

**Proposition 6** *Suppose that $R_0 > 1$ and $E^* = (S^*, e^*, i^*)$ is an endemic equilibrium of (3). Then $E^*$ is locally asymptotically stable.*

*Proof* Recall that the characteristic equation at $E^*$ is given by (10), that is,

$$(\lambda + \mu)\left[\frac{\partial G(S^*, J^*)}{\partial J} \int_0^\infty e^{-\lambda a}\beta(a)\nu(a)da \int_0^\infty e^{-\lambda a}\zeta_1(a)\pi(a)da - 1\right] = \frac{\partial G(S^*, J^*)}{\partial S}, \tag{18}$$

where $J^* = \int_0^\infty \beta(a)i^*(a)da$. We claim that all roots of (18) have negative real parts. By way of contradiction, suppose that (18) has a root $\lambda_0$ with $\mathrm{Re}(\lambda_0) \geq 0$. Then we can rewrite (18) to get

$$\begin{aligned} 1 &= \left| \frac{(\lambda_0+\mu)\frac{\partial G(S^*,J^*)}{\partial J} \int_0^\infty e^{-\lambda_0}\zeta_1(a)\pi(a)da \int_0^\infty e^{-\lambda_0 a}\beta(a)\nu(a)da}{\lambda_0+\mu+\frac{\partial G(S^*,J^*)}{\partial S}} \right| \\ &< \frac{\partial G(S^*,J^*)}{\partial J} \int_0^\infty \zeta_1(a)\pi(a)da \int_0^\infty \beta(a)\pi(a)da. \end{aligned} \tag{19}$$

However, with the assistance of the concavity of $G$ and $e^*(0) = G(S^*, J^*)$, we get

$$e^*(0) \geq \frac{\partial G(S^*, J^*)}{\partial J} J^*.$$

It follows from (16) and (17) that

$$J^* = \int_0^\infty \beta(a)i^*(a)\nu(a)da = \int_0^\infty \beta(a)\nu(a)da \int_0^\infty \zeta_1(a)\pi(a)da e^*(0).$$

Thus

$$e^*(0) \geq \frac{\partial G(S^*, J^*)}{\partial J} \int_0^\infty \beta(a)\nu(a)da \int_0^\infty \zeta_1(a)\pi(a)da e^*(0),$$

or

$$1 \geq \frac{\partial G(S^*, J^*)}{\partial J} \int_0^\infty \beta(a)\nu(a)da \int_0^\infty \zeta_1(a)\pi(a)da.$$

This contradicts with (19) and hence the claim is proved. By the claim, we see that $E^*$ is locally asymptotically stable. □

We mention that, later on, for the global stability of the endemic equilibrium, we do not need to prove its local stability first. However, in Theorem 7, we need an additional condition and here we only require $R_0 > 1$.

In order to study the global stability of the endemic equilibria, we need the uniform persistence.

### 3.4 Uniform Persistence

The uniform persistence is established by applying Theorem 5.2 in [27]. For this purpose, we define $\rho : \Gamma \to \mathbb{R}^+$ by

$$\rho(\gamma_0) = \int_0^\infty \beta(a) i_0(a) da \qquad \text{where } \gamma_0 = (S_0, e_0, i_0) \in \Gamma.$$

Denote

$$\Gamma_0 = \{\gamma \in \Gamma : \rho(\Phi(t, \gamma)) = 0 \text{ for all } t \geq 0\}.$$

Clearly $\Gamma_0 \neq \emptyset$ as $E_0 \in \Gamma_0$. We say that $\Phi$ is uniformly weakly $\rho$-persistent, if there exists $\eta > 0$ such that

$$\limsup_{t\to\infty} \rho(\Phi(t, \gamma_0)) > \eta \qquad \text{whenever } \rho(\gamma_0) > 0,$$

and is uniformly (strongly) $\rho$-persistent if we can replace lim sup by lim inf above.

Note that $\rho(\Phi(t, \gamma_0)) = J(t)$ for all $t \geq 0$. Using (7) and (8), we obtain

$$\begin{aligned}
&J(t)\\
&= \int_0^t \beta(a) i(t, a) da + \int_t^\infty \beta(a) i(t, a) da,\\
&= \int_0^t \beta(a) i(t-a, 0) \nu(a) da + \int_t^\infty \beta(a) i_0(a-t) \frac{\nu(a)}{\nu(a-t)} da,\\
&= \int_0^t \beta(a) \nu(a) L(t-a) da + \int_t^\infty \beta(a) i_0(a) \frac{\nu(a)}{\nu(a-t)} da,\\
&= \int_0^t \beta(a) \nu(a) \int_0^\infty \zeta_1(\sigma) e(t-a, \sigma) d\sigma da + \int_t^\infty \beta(a) i_0(a-t) \frac{\nu(a)}{\nu(a-t)} da,\\
&= \int_0^t \beta(a) \nu(a) \left[ \int_0^{t-a} \zeta_1(\sigma) e(t-a, \sigma) d\sigma + \int_{t-a}^\infty \zeta_1(\sigma) e(t-a, \sigma) d\sigma \right] da\\
&\quad + \int_t^\infty \beta(a) i_0(a-t) \frac{\nu(a)}{\nu(a-t)} da,\\
&= \int_0^t \beta(a) \nu(a) \int_0^{t-a} \zeta_1(\sigma) \pi(\sigma) G(S(t-a-\sigma), J(t-a-\sigma)) d\sigma da + \tilde{R}(t),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{R}(t) = &\int_0^t \beta(a) \nu(a) \int_{t-a}^\infty \zeta_1(\sigma) e_0(\sigma - t + a) \frac{\pi(\sigma)}{\pi(\sigma - t + a)} d\sigma da\\
&+ \int_t^\infty \beta(a) i_0(a-t) \frac{\nu(a)}{\nu(a-t)} da.
\end{aligned}$$

For the first part in the above expression for $J(t)$, we first make the change $\tau = a + \sigma$ to get

$$\int_0^t \beta(a)\nu(a) \int_0^{t-a} \zeta_1(\sigma)\pi(\sigma)G(S(t-a-\sigma), J(t-a-\sigma))d\sigma da$$
$$= \int_0^t \beta(a)\nu(a) \int_a^t \zeta_1(\tau-a)G(S(t-\tau), J(t-\tau))\pi(\tau-a)d\tau da.$$

Then we change the order of integration to obtain

$$\int_0^t \beta(a)\nu(a) \int_0^{t-a} \zeta_1(\sigma)\pi(\sigma)G(S(t-a-\sigma), J(t-a-\sigma))d\sigma da$$
$$= \int_0^t G(S(t-\tau), J(t-\tau)) \left[ \int_0^\tau \beta(a)\zeta_1(\tau-a)\nu(a)\pi(\tau-a)da \right] d\tau.$$

Therefore, we have arrived at

$$J(t) = \int_0^t G(S(t-\tau), J(t-\tau))\kappa(\tau)d\tau + \tilde{R}(t), \tag{20}$$

where $\kappa$ is defined in (4).

The following result is needed in the proof of the uniform weak $\rho$-persistence of $\Phi$.

**Lemma 2** *Let $\gamma_0 \in \Gamma$ such that $\rho(\gamma_0) > 0$. Then there exists a $T \geq 0$ such that $\rho(\Phi(t, \gamma_0)) > 0$ for all $t \geq T$.*

*Proof* Since $J(0) > 0$, there exists a $\delta > 0$ such that $J(t) > 0$ for $t \in [0, \delta]$. Moreover, there exists $\varsigma > 0$ such that $J(t) \leq \varsigma$ for $t \geq 0$. Note that $S(t) > 0$ for $t > 0$ and $\liminf_{t\to\infty} S(t) > 0$ (which can be proved in a similar manner as that for Proposition 2). Thus $\underline{\xi} \triangleq \inf_{t\in[\frac{\delta}{2},\infty)} S(t) > 0$. Define $\hat{\gamma} : \mathbb{R} \to \Gamma$ by $\hat{\gamma}(t) = \Phi(t + \frac{\delta}{2}, \gamma_0) = \Phi(t, (\Phi(\frac{\delta}{2}, \gamma_0))$ for $t \geq 0$. Then by (20), we get

$$\begin{aligned} \hat{J}(t) &= \int_0^t G(\hat{S}(t-\tau), \hat{J}(t-\tau))\kappa(\tau)d\tau + \hat{\tilde{R}}(t), \\ &\geq \int_0^t G(\underline{\xi}, \hat{J}(t-\tau))\kappa(\tau)d\tau + \hat{\tilde{R}}(t), \\ &\geq \int_0^t \frac{\partial G(\underline{\xi}, \varsigma)}{\partial J} \hat{J}(t-\tau)\kappa(\tau)d\tau + \hat{\tilde{R}}(t). \end{aligned}$$

Here we have used the monotonicity of $G$ in $S$ and of the concavity in $J$. Note that $\hat{\tilde{R}}(t)$ is continuous at 0 and $\hat{\tilde{R}}(0) = J(\frac{\delta}{2}) > 0$. Under assumption **(H1(c))**, by Corollary B.6 [27], there exists $\hat{T} > 0$ (which only depends on $\kappa$) such that $\hat{J}(t) = J(t + \frac{\delta}{2}) > 0$ for $t \geq \hat{T}$. The desired result holds with $T = \hat{T} + \frac{\delta}{2}$. □

**Proposition 7** *Suppose that $R_0 > 1$. Then $\Phi$ is uniformly weakly $\rho$-persistent.*

*Proof* Since $R_0 > 1$ and $\frac{\partial G(S,J)}{\partial J}$ is continuous at $(S^0, 0)$ (see assumption **(H2(b))**), we can choose $\varepsilon > 0$ small enough such that

$$\Lambda > G(S^0 + \varepsilon, \varepsilon) - \mu\varepsilon$$

and

$$\frac{\partial G(\frac{\Lambda - G(S^0+\varepsilon,\varepsilon)}{\mu} - \varepsilon, \varepsilon)}{\partial J} \mathcal{L}[\beta\nu](\varepsilon)\mathcal{L}[\zeta_1\pi](\varepsilon) > 1. \tag{21}$$

Here $\mathcal{L}[\cdot]$ represents the Laplace transform.

With contradictive argument, suppose that $\Phi$ is not uniformly weakly $\rho$-persistent. Then there exists $\gamma_0$ with $\rho(\gamma_0) > 0$ such that

$$\limsup_{t\to\infty} \rho(\Phi(t, \gamma_0)) = \liminf_{t\to\infty} J(t) < \varepsilon.$$

This and Proposition 1 imply that there exists $t_0 \geq 0$ such that

$$J(t) \leq \varepsilon \quad \text{and} \quad S(t) \leq S^0 + \varepsilon \qquad \text{for } t \geq t_0.$$

Without loss of generality, we assume that $t_0 = 0$ as we can replace $\gamma_0$ with $\Phi(\max\{t_0, T\}, \gamma_0)$, where $T$ is the number guaranteed in Lemma 2. Note that Lemma 2 also implies that $\rho(\Phi(\max\{t_0, T\}, \gamma_0)) > 0$. Then for $t \geq 0$, it follows from the monotonicity of $G$ that

$$S'(t) \geq \Lambda - \mu S(t) - G(S^0 + \varepsilon, \varepsilon),$$

which implies that $\liminf_{t\to\infty} S(t) \geq \frac{\Lambda - G(S^0+\varepsilon,\varepsilon)}{\mu}$. Without loss of generality again, we can assume that $S(t) \geq \frac{\Lambda - G(S^0+\varepsilon,\varepsilon)}{\mu} - \varepsilon$ for $t \geq 0$. Then, for $t \geq 0$, it follows from the monotonicity and concavity of $G$ and (20) that

$$\begin{aligned} J(t) &\geq \int_0^t G\left(\frac{\Lambda - G(S^0+\varepsilon,\varepsilon)}{\mu} - \varepsilon, J(t-\tau)\right)\kappa(\tau)d\tau \\ &\geq \frac{\partial G(\frac{\Lambda - G(S^0+\varepsilon,\varepsilon)}{\mu} - \varepsilon, \varepsilon)}{\partial J} \int_0^t J(t-\tau)\kappa(\tau)d\tau. \end{aligned}$$

Taking the Laplace transforms of both sides of the above inequality gives

$$\mathcal{L}[J](s) \geq \frac{\partial G(\frac{\Lambda - G(S^0+\varepsilon,\varepsilon)}{\mu} - \varepsilon, \varepsilon)}{\partial J} \mathcal{L}[J](s)\mathcal{L}[\kappa](s)$$

for any $s \geq 0$. Since $J(0) > 0$ as mentioned before, we have $\mathcal{L}[J](s) > 0$ for all $s \geq 0$. Therefore, taking $s = \varepsilon$ in the previous inequality involving Laplace transforms gives

$$\frac{\partial G(\frac{\Lambda - G(S^0+\varepsilon,\varepsilon)}{\mu} - \varepsilon, \varepsilon)}{\partial J} \mathcal{L}[\kappa](\varepsilon) \leq 1.$$

Now, we first change the order of integration and then make the change of variable $\tau - a = s$ to get

$$\mathcal{L}[\kappa](\varepsilon) = \int_0^\infty e^{-\varepsilon\tau} \int_0^\tau \beta(a)\nu(a)\zeta_1(\tau-a)\pi(\tau-a)dad\tau = \mathcal{L}[\beta\nu](\varepsilon)\mathcal{L}[\zeta_1\pi](\varepsilon).$$

Therefore, we can get

$$\frac{\partial G(\frac{\Lambda - G(S^0+\varepsilon,\varepsilon)}{\mu} - \varepsilon, \varepsilon)}{\partial J} \mathcal{L}[\beta\nu](\varepsilon)\mathcal{L}[\zeta_1\pi](\varepsilon) \leq 1,$$

a contradiction to (21). This completes the proof. □

In the proof of Proposition 7, we have used the fact that $\mathcal{L}[J](s) > 0$ for $s \geq 0$, which requires $J(t) > 0$ for some $t \geq 0$. Unfortunately, this is not guaranteed without Lemma 2 since we only have $J(0) > 0$ but we have made translations in the discussion. In the literature so far, to the best of knowledge, no researchers have realized this and it is just taken for granted.

In order to apply Theorem 5.2 [27], we still need to check the hypothesis **(H1)** in it.

**Lemma 3** *Let $\gamma(t) = (S(t), e(t, \cdot), i(t, \cdot))$ be a total $\Phi$-trajectory. If $J(t) = 0$ for all $t \leq 0$ then $J(t) = 0$ for all $t \geq 0$.*

*Proof* It follows from (9) and the Lipschitz continuity of $G$ that

$$\begin{aligned} J(t) &= \int_0^\infty \beta(a) i(t-a, 0)\nu(a)da, \\ &= \int_0^\infty \beta(a)\nu(a) \int_0^\infty \zeta_1(\sigma) G(S(t-a-\sigma), J(t-a-\sigma))\pi(\sigma)d\sigma da \\ &\leq K \int_0^\infty \beta(a)\nu(a) \int_0^\infty \zeta_1(\sigma)\pi(\sigma) J(t-a-\sigma)d\sigma da. \end{aligned}$$

Making a change of variable $t - a \to a$ gives us

$$J(t) \leq K \int_{-\infty}^t \beta(t-a)\nu(t-a) \int_0^\infty \zeta_1(\sigma)\pi(\sigma) J(a-\sigma)d\sigma da.$$

With another change of variable $a - \sigma \to \sigma$, we get

$$J(t) \leq K \int_{-\infty}^t \beta(t-a)\nu(t-a) \int_{-\infty}^a \zeta_1(a-\sigma)\pi(a-\sigma) J(\sigma)d\sigma da.$$

Since $J(t) = 0$ for all $t \leq 0$, $\pi(a) \leq 1$, $\nu(a) \leq 1$, we get

$$J(t) \leq K\bar{\beta}\bar{\zeta} \int_0^t \int_0^a J(\sigma)d\sigma da.$$

Let $\mathcal{J}(t) = \int_0^t J(s)ds$. Then

$$\frac{d\mathcal{J}(t)}{dt} = J(t) \leq K\bar{\beta}\bar{\zeta} \int_0^t \mathcal{J}(a)da.$$

Since $\mathcal{J}(0) = 0$, Gronwall's inequality leads to $\mathcal{J}(t) = 0$ for $t \geq 0$ and hence $J(t) = 0$ for $t \geq 0$. □

**Lemma 4** *Let $\gamma(t) = (S(t), e(t, \cdot), i(t, \cdot))$ be a total $\Phi$-trajectory in $\mathcal{A}$. Then $S(t)$ is strictly positive and (either $J(t) = 0$ for all $t \in \mathbb{R}$ or $J(t)$ is positive on $\mathbb{R}$).*

*Proof* Since $\gamma(t) \in \mathcal{A}$, by Propositions 1 and 2, we know that $S(t) \geq \xi > 0$ and $J(t) \leq \bar{\beta}S^0$ for $t \in \mathbb{R}$. For the second part, from Lemma 3 by performing appropriate direction, we see the following alternatives: either $J(t) = 0$ for all $t \geq r$ if $J(t) = 0$ for all $t \leq r$ or there exists a sequence $\{t_n\}$ satisfying $t_n \to -\infty$ as $n \to \infty$ and $J(t_n) > 0$. We assume the second.

For every $n$, define $J_n, S_n : \mathbb{R} \to \mathbb{R}$ by $J_n(t) = J(t + t_n)$ and $S_n(t) = S(t + t_n)$ for $t \in \mathbb{R}$. Then similar arguments as those in the proof of Lemma 2 will yield a $T > 0$ (independent of $n$) such that $J_n(t) > 0$ for $t \geq T$ and every $n$. This is a contradiction and hence the proof is completed. □

By now we have verified all the assumption of Theorem 5.7 [27]. It is not difficult to show that $\lim_{t\to\infty} \Phi(t, \gamma_0) = E_0$ for $\gamma_0 \in \Gamma_0$. Therefore, we have obtained the following result.

**Theorem 6** *The global attractor $\mathcal{A}$ is the disjoint union*

$$\mathcal{A} = \{E_0\} \cup C \cup \mathcal{A}_1,$$

*where $C$ and $\mathcal{A}_1$ are invariant sets and $\mathcal{A}_1$ is compact. Moreover, the following hold.*

(i) *$\mathcal{A}_1$ attracts all solutions with initial conditions belonging to $\Gamma \setminus \Gamma_0$ and is uniformly $\rho$-positive, i.e., there exists some $\eta > 0$ such that*

$$\int_0^\infty \beta(a) i_0(a) da \geq \eta \qquad \text{for all } \gamma_0 = (S_0, e_0, i_0) \in \mathcal{A}_1.$$

*In particular, $\mathcal{A}_1$ is stable.*

(ii) *If $\gamma_0 \in \Gamma \setminus \mathcal{A}_1$ and $\gamma$ is a total $\Phi$-trajectory through $\gamma_0$ with precompact range, then* $\lim_{t\to-\infty} \gamma(t) = E_0$.

*If $\gamma_0 \in \Gamma \setminus \{E_0\}$ and $\gamma$ is a total $\Phi$-trajectory through $\gamma_0$ with precompact range, then $\gamma(t) \to \mathcal{A}_1$ as $t \to \infty$.*

*In particular, the set $C$ consists of those $\gamma_0 \in \mathcal{A}$ through which there exists a total $\Phi$-trajectory $\gamma$ with $\gamma(-t) \to E_0$ and $\gamma(t) \to \mathcal{A}_1$ as $t \to \infty$.*

**Corollary 1** *Suppose $R_0 > 1$. Let $\gamma(t) = (S(t)e(t, \cdot), i(t, \cdot))$ be a total $\Phi$-trajectory in $\mathcal{A}_1$. Then there exists $\epsilon_0 > 0$ such that $S(t), e(t, 0), i(t, 0) \geq \varepsilon_0$ for all $t \in \mathbb{R}$.*

*Proof* Since $\mathcal{A}_1$ is invariant, by Proposition 2, we have $S(t) \geq \xi$ for $t \in \mathbb{R}$. Moreover, by Theorem (6), there exists $\eta > 0$ such that $\rho(\gamma(t)) = J(t) > \eta$ for all $t \in \mathbb{R}$. Then we have

$$e(t, 0) = G(S(t), J(t)) \geq G(\xi, \eta) \qquad \text{for } t \in \mathbb{R},$$

by the monotonicity of $G(S, J)$ with respect to both $S$ and $J$. Thus

$$\begin{aligned} i(t, 0) &= \int_0^\infty \zeta_1(a) e(t, a) da \\ &= \int_0^\infty \zeta_1(a) e(t - a, 0) \pi(a) da \\ &\geq G(\xi, \eta) \int_0^\infty \zeta_1(a) \pi(a) da, \end{aligned}$$

for $t \in \mathbb{R}$. Letting $\varepsilon_0 = \min\{\xi, G(\xi, \eta), G(\xi, \eta) \int_0^\infty \zeta(a)\pi(a)da\}$ completes the proof. □

### 3.5 Global Stability of Endemic Equilibria

Let $E^* = (S^*, e^*, i^*)$ be an endemic equilibrium of (3). In this subsection, we prove the global stability of $E^*$ under an additional assumption. During the discussion, we need some further information on solutions and a technical lemma.

**Lemma 5** *For each solution $\gamma(t) = (S(t), e(t,\cdot), i(t,\cdot))$ of* (3), *we have*

$$\int_0^\infty \zeta_1(a)e^*(a)\left[\frac{e(t,a)}{e^*(a)} - \frac{i(t,0)}{i^*(0)}\right]da = 0, \tag{22}$$

$$\int_0^\infty \beta(a)i^*(a)\left[\frac{i(t,a)}{i^*(a)} - \frac{J(t)}{J^*}\right]da = 0. \tag{23}$$

*Proof* Using the boundary condition given in (3), we observe that

$$\begin{aligned}0 &= i(t,0) - \frac{i^*(0)i(t,0)}{i^*(0)}\\ &= \int_0^\infty \zeta_1(a)e(t,a)da - \int_0^\infty \zeta_1(a)e^*(a)da\frac{i(t,0)}{i^*(0)}\\ &= \int_0^\infty \zeta_1(a)e^*(a)\left[\frac{e(t,a)}{e^*(a)} - \frac{i(t,0)}{i^*(0)}\right]da.\end{aligned}$$

This proves (22). Similarly we have

$$\begin{aligned}0 &= J(t) - \frac{J^*J(t)}{J^*}\\ &= \int_0^\infty \beta(a)i(t,a)da - \int_0^\infty \beta(a)i^*(a)da\frac{J(t)}{J^*}\\ &= \int_0^\infty \beta(a)i^*(a)\left[\frac{i(t,a)}{i^*(a)} - \frac{J(t)}{J^*}\right]da,\end{aligned}$$

which gives (23). □

**Lemma 6** *Let $\chi$ be a non-negative, bounded Lebesgue measurable function. Suppose that $z_1$ and $z_2$ are non-zero solutions of*

$$\frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} = -\chi(a)z(t,a),$$

*for $t \in \mathbb{R}$ and $a > 0$ with $z_i(t,0) = Z_i(t) > 0$ for all $t \in \mathbb{R}$ and $i = 1, 2$. Define*

$$F(t) = \int_0^\infty \epsilon(a)H\Big(\frac{z_1(t,a)}{z_2(t,a)}\Big)da,$$

*where $H$ is continuous and $\epsilon(a) = \int_a^\infty \omega(a)da$ with $\epsilon, \omega \in L^1$. Then*

$$\frac{dF(t)}{dt} = \int_0^\infty \omega(a)\left[H\Big(\frac{z_1(t,0)}{z_2(t,0)}\Big) - H\Big(\frac{z_1(t,a)}{z_2(t,a)}\Big)\right]da.$$

*Proof* Let $\vartheta(a) = e^{-\int_0^a \chi(s)ds}$ for $a \geq 0$. Then

$$z_i(t,a) = Z_i(t-a)\vartheta(a) \qquad \text{for all } t \in \mathbb{R},\ a \geq 0, \text{ and } i = 1, 2. \tag{24}$$

Thus $z_i(t,a)$ is positive for all $t \in \mathbb{R}$, $a \geq 0$, and $i = 1, 2$. It follows that

$$\frac{dF(t)}{dt} = \frac{d}{dt}\int_0^\infty \epsilon(a) H\Big(\frac{Z_1(t-a)}{Z_2(t-a)}\Big) da.$$

Making the substitution $\sigma = t - a$ gives us

$$\begin{aligned}\frac{dF(t)}{dt} &= \frac{d}{dt}\Big(\int_{-\infty}^t \epsilon(t-\sigma) H\Big(\frac{Z_1(\sigma)}{Z_2(\sigma)}\Big) d\sigma\Big),\\ &= \epsilon(0) H\Big(\frac{Z_1(t)}{Z_2(t)}\Big) + \int_{-\infty}^t \epsilon'(t-\sigma) H\Big(\frac{Z_1(\sigma)}{Z_2(\sigma)}\Big) d\sigma,\\ &= \epsilon(0) H\Big(\frac{Z_1(t)}{Z_2(t)}\Big) + \int_0^\infty \epsilon'(a) H\Big(\frac{Z_1(t-a)}{Z_2(t-a)}\Big) d\sigma.\end{aligned}$$

For the last equality, we have used the substitution $t - \sigma = a$. Now, with the help of (24), $\varepsilon(0) = \int_0^\infty \omega(a)da$, and $\varepsilon'(a) = -\omega(a)$, we immediately get the required result. □

Now, we are ready to state and prove the main result of this subsection.

**Theorem 7** *Assume that* $R_0 > 1$. *Let* $E^* = (S^*, e^*, i^*)$ *be an endemic equilibrium of* (3) *satisfying*

(H3) *For* $S > 0$,

$$\begin{cases} \frac{x}{J^*} < \frac{G(S,x)}{G(S,J^*)} < 1 & \textit{for } 0 < x < J^*,\\ 1 < \frac{G(S,x)}{G(S,J^*)} < \frac{x}{J^*} & \textit{for } x > J^*.\end{cases}$$

*Then* $E^*$ *is globally asymptotically stable in* $\Gamma \setminus \Gamma_0$.

*Remark 1* There are many functionals that can verify the condition $(H_3)$ as example Beddington–DeAngelis incidence functional and Crowely–Martin incidence functional and the Holling $(I - III)$ incidence functional. In [19–21], it is mentioned that the concave incidence functionals with respect to the first variable verifies the condition $(H_3)$. Obviously, the previously mentioned incidence functionals verifies this condition.

*Proof* By Theorem 6, it suffices to show that $\mathcal{A}_1 = \{E^*\}$.

For $\gamma_0 = (S_0, e_0, i_0) \in \mathcal{A}_1$, we define a Lyapunov functional $V(\gamma_0) = V_S(\gamma_0) + V_E(\gamma_0) + V_I(\gamma_0)$ with

$$\begin{aligned} V_S(\gamma_0) &= AB(S_0 - \int_{S^*}^{S_0} \frac{G(S^*, J^*)}{G(\xi, J^*)} d\xi),\\ V_E(\gamma_0) &= B\int_0^\infty \zeta_E(a) g\Big(\frac{e_0(a)}{e^*(a)}\Big) da,\\ V_I(\gamma_0) &= \int_0^\infty \beta_I(a) g\Big(\frac{i_0(a)}{i^*(a)}\Big) da,\end{aligned}$$

where

$$A = \int_0^\infty \zeta_1(a)\pi(a)da,$$

$$B = \int_0^\infty \beta(a)\nu(a)da,$$

$$\zeta_E(a) = \int_a^\infty \zeta_1(s)e^*(s)ds,$$

$$\beta_I(a) = \int_a^\infty \beta(s)i^*(s)ds,$$

$$g(x) = x - 1 - \ln x, \quad x > 0.$$

Note that $V$ is well-defined by Corollary 1. Denote $\gamma(t) = (S(t), e(t,\cdot), i(t,\cdot))$ to be the total $\Phi$-trajectory in $\mathcal{A}_1$ with $\gamma(0) = \gamma_0$. In the following we shall find the derivatives of $V_S$, $V_E$, and $V_I$ one by one before combining them to get the derivative of $V$.

Firstly,

$$\begin{aligned} \frac{d}{dt}V_S(\gamma(t)) =& AB\mu\left(1 - \frac{G(S^*, J^*)}{G(S(t), J^*)}\right)(S^* - S(t)) \\ &+ \left(1 - \frac{G(S^*, J^*)}{G(S(t), J^*)}\right) ABG(S^*, J^*) \\ &- \left(1 - \frac{G(S^*, J^*)}{G(S(t), J^*)}\right) ABG(S(t), J(t)). \end{aligned} \tag{25}$$

Next, with the help of Lemma 6, we can get

$$\begin{aligned} \frac{dV_E}{dt}(\gamma(t)) &= B\int_0^\infty \zeta_1(a)e^*(a)\left[g\left(\frac{e(t,0)}{e^*(0)}\right) - g\left(\frac{e(t,a)}{e^*(a)}\right)\right]da, \\ &= B\int_0^\infty \zeta_1(a)e^*(a)\left[\frac{e(t,0)}{e^*(0)} - \frac{e(t,a)}{e^*(a)} + \ln\frac{e(t,a)}{e^*(a)} - \ln\frac{e(t,0)}{e^*(0)}\right]da. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \frac{dV_I}{dt}(\gamma(t)) &= \int_0^\infty \beta(a)i^*(a)\left[g\left(\frac{i(t,0)}{i^*(0)}\right) - g\left(\frac{i(t,a)}{i^*(a)}\right)da\right], \\ &= \int_0^\infty \beta(a)i^*(a)\left[\frac{i(t,0)}{i^*(0)} - \frac{i(t,a)}{i^*(a)} + \ln\frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)}{i^*(0)}\right]da. \end{aligned}$$

Now, we can use (22) and (23) to cancel terms $\frac{e(t,a)}{e^*(a)}$ and $\frac{i(t,a)}{i^*(a)}$ to get

$$\frac{dV_E}{dt}(\gamma(t)) = B\int_0^\infty \zeta_1(a)e^*(a)\left[\frac{e(t,0)}{e^*(0)} + \ln\frac{e(t,a)}{e^*(a)} - \ln\frac{e(t,0)}{e^*(0)} - \frac{i(t,0)}{i^*(0)}\right]da, \tag{26}$$

and

$$\frac{dV_I}{dt}(\gamma(t)) = \int_0^\infty \beta(a)i^*(a)\left[\frac{i(t,0)}{i^*(0)} + \ln\frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)}{i^*(0)} - \frac{J(t)}{J^*}\right]da. \tag{27}$$

Note that

$$\int_0^\infty \beta(a)i^*(a)da = \int_0^\infty \beta(a)i^*(0)v(a)da = Bi^*(0) = B\int_0^\infty \zeta_1(a)e^*(a)da.$$

This, combined with (25), (26), and (27), gives us

$$\begin{aligned}\frac{dV}{dt}(\gamma(t)) = AB\mu\left(1-\frac{G(S^*,J^*)}{G(S(t),J^*)}\right)(S^*-S(t))\\ +\left(1-\frac{G(S^*,J^*)}{G(S(t),J^*)}\right)ABG(S^*,J^*)\\ -\left(1-\frac{G(S^*,J^*)}{G(S(t),J^*)}\right)ABG(S(t),J(t))\\ +B\int_0^\infty \zeta_1(a)e^*(a)\left[\ln\frac{e(t,a)}{e^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right]da\\ +B\int_0^\infty \zeta_1(a)e^*(a)\left[\frac{G(S(t),J(t))}{G(S^*,J^*)}-\ln\frac{G(S(t),J(t))}{G(S^*,J^*)}\right]da\\ +\int_0^\infty \beta(a)i^*(a)\left[\ln\frac{i(t,a)}{i^*(a)}-\frac{J(t)}{J^*}\right]da.\end{aligned}$$

Using $\ln\frac{G(S(t),J(t))}{G(S^*,J^*)} = \ln\frac{G(S(t),J(t))}{G(S(t),J^*)} + \ln\frac{G(S(t),J^*)}{G(S^*,J^*)}$, and adding and subtracting $\int_0^\infty \beta(a)i^*(a)\ln\frac{J(t)}{J^*}da$ into the last integral, we can rewrite $\frac{dV}{dt}(\gamma(t))$ as

$$\begin{aligned}&\frac{dV}{dt}(\gamma(t))\\ &= AB\mu\left(1-\frac{G(S^*,J^*)}{G(S(t),J^*)}\right)(S^*-S(t)) + \left(1-\frac{G(S^*,J^*)}{G(S(t),J^*)}\right)ABG(S^*,J^*)\\ &-\left(1-\frac{G(S^*,J^*)}{G(S(t),J^*)}\right)ABG(S(t),J(t))\\ &+B\int_0^\infty \zeta_1(a)e^*(a)\left[\frac{G(S(t),J(t))}{G(S^*,J^*)}-\ln\frac{G(S(t),J(t))}{G(S(t),J^*)}-\ln\frac{G(S(t),J^*)}{G(S^*,J^*)}\right]da\\ &+B\int_0^\infty \zeta_1(a)e^*(a)\left[\ln\frac{e(t,a)}{e^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right]da\\ &+\int_0^\infty \beta(a)i^*(a)\left[\ln\frac{i(t,a)J^*}{i^*(a)J(t)}-\frac{J(t)}{J^*}+\ln\frac{J(t)}{J^*}+1-1\right]da.\end{aligned}$$

With a simple calculation and using properties of logarithms, we can get

$$\begin{aligned}\frac{dV}{dt}(\gamma(t)) = AB\mu\left(1-\frac{G(S^*,J^*)}{G(S(t),J^*)}\right)(S^*-S(t))\\ +\left(1-\frac{G(S^*,J^*)}{G(S(t),J^*)}-\ln\frac{G(S(t),J^*)}{G(S^*,J^*)}\right)ABG(S^*,J^*)\\ +ABG(S^*,J^*)\left(-1+\frac{G(S(t),J(t))}{G(S(t),J^*)}-\ln\frac{G(S(t),J(t))}{G(S(t),J^*)}\right)\end{aligned}$$

$$+B\int_0^\infty \zeta_1(a)e^*(a)\left[\ln\frac{e(t,a)i^*(0)}{e^*(a)i(t,0)}\right]da$$
$$+\int_0^\infty \beta(a)i^*(a)\left[\ln\frac{i(t,a)J^*}{i^*(a)J(t)} - g\Big(\frac{J(t)}{J^*}\Big)\right]da.$$

Equality (22) implies $\int_0^\infty \zeta_1(a)e^*(a)\left[1-\frac{e(t,a)i^*(0)}{e^*(a)i(t,0)}\right]da = 0$ whereas (23) implies $\int_0^\infty \beta(a)i^*(a)\left[1-\frac{i(t,a)J^*}{i^*(a)J(t)}\right]da = 0$. Adding them to the above expression for $\frac{dV}{dt}(\gamma(t))$ gives

$$\begin{aligned}
\frac{dV}{dt}(\gamma(t)) &= AB\mu\left(1-\frac{G(S^*,J^*)}{G(S(t),J^*)}\right)(S^*-S(t))\\
&\quad+\left(1-\frac{G(S^*,J^*)}{G(S(t),J^*)}-\ln\frac{G(S(t),J^*)}{G(S^*,J^*)}\right)ABG(S^*,J^*)\\
&\quad+ABG(S^*,J^*)\left(g\Big(\frac{G(S(t),J(t))}{G(S(t),J^*)}\Big)-g\Big(\frac{J(t)}{J^*}\Big)\right)\\
&\quad+B\int_0^\infty \zeta_1(a)e^*(a)\left[\ln\frac{e(t,a)i^*(0)}{e^*(a)i(t,0)}+1-\frac{e(t,a)i^*(0)}{e^*(a)i(t,0)}\right]da\\
&\quad+\int_0^\infty \beta(a)i^*(a)\left[\ln\frac{i(t,a)J^*}{i^*(a)J(t)}+1-\frac{i(t,a)J^*}{i^*(a)J(t)}\right]da\\
&= AB\mu\left(1-\frac{G(S^*,J^*)}{G(S(t),J^*)}\right)(S^*-S(t))\\
&\quad+\left(1-\frac{G(S^*,J^*)}{G(S(t),J^*)}-\ln\frac{G(S(t),J^*)}{G(S^*,J^*)}\right)ABG(S^*,J^*)\\
&\quad+ABG(S^*,J^*)\left(g\Big(\frac{G(S(t),J(t))}{G(S(t),J^*)}\Big)-g\Big(\frac{J(t)}{J^*}\Big)\right)\\
&\quad-B\int_0^\infty \zeta_1(a)e^*(a)g\Big(\frac{e(t,a)i^*(0)}{e^*(a)i(t,0)}\Big)da\\
&\quad-\int_0^\infty \beta(a)i^*(a)g\Big(\frac{i(t,a)J^*}{i^*(a)J(t)}\Big)da.
\end{aligned}$$

Note that $g$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$. By **(H3)**, we have

$$\begin{cases} \frac{J(t)}{J^*} < \frac{G(S(t),J(t))}{G(S(t),J^*)} < 1 & \text{if } 0 < J(t) < J^*,\\ 1 < \frac{G(S(t),J(t))}{G(S(t),J^*)} < \frac{J(t)}{J^*} & \text{if } J(t) > J^*. \end{cases}$$

It follows that $g(\frac{G(S(t),J(t))}{G(S(t),J^*)}) \le g(\frac{J(t)}{J^*})$. Furthermore, since $G$ is non-decreasing with respect to $S$ and $g(x) \ge 0$ for $x > 0$, we easily see that $\frac{dV}{dt}(\gamma(t)) \le 0$.

Now, suppose that $\frac{dV}{dt}(\gamma(t)) = 0$. Then $S(t) = S^*$ for all $t \in \mathbb{R}$. This combined with the first equation of (9) gives

$$\Lambda - \mu S^* = G(S^*, J(t)).$$

This and (15a) together imply that $G(S^*, J(t)) = G(S^*, J^*)$ for all $t \in \mathbb{R}$. This, together with the second equation in (9), yields $e(t, a) = G(S^*, J^*)\pi(a) = e^*(a)$ (see (15d) and (16)) for all $t \in \mathbb{R}$ and $a \geq 0$. Similarly, we can get from (9), (16), and (17) that $i(t, a) = i^*(a)$ for all $t \in \mathbb{R}$ and $a \geq 0$. By Theorem 2.53 [27], $\mathcal{A}_1 = \{E^*\}$. This completes the proof. □

## 4 Discussion

It is well-known that the incidence rate is responsible for describing the way, speed, and severity of infection. In [24], McCluskey studied (2) (where the bilinear incidence is used) and established the global behavior. We know for certainty that the transmission varies from one population to another (more generally from one country to another) for the same infection, where culture, environment, treatment availability, and health life style play important roles in the spread of infection. In order to model such big variation, we considered a broad class of general nonlinear incidence $G(S, J)$.

Moreover, tuberculosis and other similar infectious diseases have large latency periods. It is crucial to diagnose the infection in the earliest stage. This point of view has been considered for the first time in the case of age-structured models by McCluskey [24]. Our main interest here was to generalize the results of McCluskey [24] for (2) to the case with a wide class of incidence. More precisely, we got a threshold dynamics determined completely by the basic reproduction number. When $R_0 < 1$, the disease-free equilibrium is globally stable and when $R_0 > 1$, the system is uniformly persistent and the endemic equilibrium is globally asymptotically stable (under some biological meaningful condition). How $R_0$ depends on the choice of the incidence function is clearly shown in the expression (11). We reiterate it again that, in the literature, there are some gaps in establishing the uniform weak $\rho$-persistence and the uniform $\rho$-persistence. Here these gaps have been filled.

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