



A Self-Adaptive Algorithm for Split Null Point Problems and Fixed Point Problems for Demicontractive Multivalued Mappings

Suthep Suantai¹ · Pachara Jailoka¹

Received: 5 November 2019 / Accepted: 23 September 2020 / Published online: 7 October 2020
© Springer Nature B.V. 2020

Abstract In this work, we study the split null point problem and the fixed point problem in Hilbert spaces. We introduce a self-adaptive algorithm based on the viscosity approximation method without prior knowledge of the operator norm for finding a common solution of the considered problem for maximal monotone mappings and demicontractive multivalued mappings. A strong convergence result of our proposed algorithm is established under some suitable conditions. Some convergence results for the split feasibility problem and the split minimization problem are consequences of our main result. Finally, we also give numerical examples for supporting our main result.

Keywords Fixed points · Split null point problems · Demicontractive mappings · Maximal monotone mappings · Resolvent operators

Mathematics Subject Classification 47H10 · 47J25 · 54H25

1 Introduction

Throughout this work, we assume that \mathcal{H} , \mathcal{H}_1 and \mathcal{H}_2 are real Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and the induced norms $\| \cdot \|$, and let I be the identity operator on a Hilbert space. Denote by \mathbb{N} and \mathbb{R} the set of positive integers and the set of real numbers, respectively.

The *split inverse problem* (SIP) concerns a model of finding a point

$$x^* \in \mathcal{H}_1 \text{ that solves } IP_1 \text{ such that } Ax^* \in \mathcal{H}_2 \text{ solves } IP_2, \quad (1.1)$$

where IP_1 and IP_2 denote inverse problems formulated in \mathcal{H}_1 and \mathcal{H}_2 , respectively, and $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a bounded linear operator. In 1994, Censor and Elfving [7] introduced the

✉ P. Jailoka
pachara.j4@gmail.com

S. Suantai
suthep.s@cmu.ac.th

¹ Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

first instance of the split inverse problem which is the split feasibility problem (SFP). After that, various split problems were introduced and studied such as the split common fixed point problem (SCFPP) [8], the split common null point problem (SCNPP) [5], the split variational inequality problem (SVIP) [11], the split equilibrium problem (SEP) [23], the split minimization problem (SMP), etc. Recently, the split inverse problem has been widely studied by many authors (see [4, 5, 7–9, 11, 15, 17, 23]) due to its model can be applied in various problems related to significant real-world applications. For example, in signal and image processing can be formulated as the equation system:

$$y = Ax + \varepsilon, \tag{1.2}$$

where $y \in \mathbb{R}^M$ is the vector of noisy observations, $x \in \mathbb{R}^N$ is a recovered vector, ε is the noise, and $A : \mathbb{R}^N \rightarrow \mathbb{R}^M$ is a bounded linear observation operator. It is known that (1.2) can be modeled as the constrained least-squares problem:

$$\min_{x \in \mathbb{R}^N} \frac{1}{2} \|y - Ax\|_2^2 \text{ subject to } \|x\|_1 \leq \sigma, \tag{1.3}$$

for any nonnegative real number σ . Then, we can apply SIP model (1.1) to Problem (1.3) as follows: Find a point $x \in \mathbb{R}^N$ such that $\|x\|_1 \leq \sigma$ and $Ax = y$. In addition, Kotzer et al. [24] used the formulation of the split feasibility problem to model the design of a nonlinear synthetic discriminant filter for optical pattern recognition. In [13], the convex feasibility problem in image recovery was discussed. Censor et al. [9, 10] studied the inverse problem in intensity-modulated radiation therapy. López et al. [25] introduced a self-adaptive algorithm for the split feasibility problem and applied to signal processing.

In this paper, we focus our attention on the *split null point problem* (SNPP) which was introduced by Byrne et al. in 2012 (see [5]). This split problem is the problem of finding a point of the null point set of a multivalued mapping in a Hilbert space such that its image under a given bounded linear operator belongs to the null point set of another multivalued mapping in the image space.

Given two multivalued mappings $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$, a bounded linear operator $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, then the SNPP is formulated as finding a point $x^* \in \mathcal{H}_1$ such that

$$x^* \in B_1^{-1}0 \text{ and } Ax^* \in B_2^{-1}0, \tag{1.4}$$

where $B_1^{-1}0 := \{x \in \mathcal{H}_1 : 0 \in B_1x\}$ and $B_2^{-1}0$ are null point sets of B_1 and B_2 , respectively. To study the SNPP (1.4), we often consider in the case of maximal monotone mappings B_1 and B_2 (see [1, 2, 5, 31, 32]). The subdifferential of a lower semicontinuous and convex function is an important example of maximal monotone mappings and its resolvents are often used to construct algorithms for solving the minimization problem of the function. For example in [14], Combettes and Pesquet introduced proximal splitting methods constructed by the resolvent operators of the subdifferential of functions and also discussed in signal processing.

Byrne et al. [5] proposed two iterative algorithms for solving the SNPP (1.4) for two maximal monotone mappings B_1 and B_2 as follows:

$$x_{n+1} = J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n), \quad n \in \mathbb{N}, \tag{1.5}$$

and

$$\begin{cases} u \in \mathcal{H}_1, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n), \quad n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are resolvents of B_1 and B_2 , respectively, A^* is the adjoint operator of A . They also proved a weak convergence result of Algorithm (1.5) and a strong convergence result of Algorithm (1.6) when $\gamma \in (0, 2/\|A\|^2)$, $\alpha_n \in (0, 1)$, $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. We observe that the parameter λ in above mentioned algorithms depend on the norm of the operator A ; however, the calculation of $\|A\|$ is not an easy work in general practice.

In 2015, the problem of finding a common solution of the null point and fixed point problem was first considered by Takahashi et al. [34]. In 2017, Eslamain et al. [16] studied the problem of finding a common solution of the split null point problem and the fixed point problem for maximal monotone mappings and demicontractive mappings, respectively (also see [20, 21]). It is well known that the class of demicontractive mappings [18, 27] includes several common types of classes of mappings occurring in nonlinear analysis and optimization problems.

In our work, inspired and motivated by these researches, we are interested to study the split null point problem and the fixed point problem for multivalued mappings in Hilbert spaces. Our main objective is to construct a strongly convergent algorithm without prior knowledge of the operator norm for solving the considered problem on the class of maximal monotone mappings and the class of demicontractive multivalued mappings. The paper is organized as follows. Basic definitions and some useful lemmas for proving our main result are given in the preliminaries section. Section 3 is our main result. In this section, we introduce a viscosity-type algorithm [28] by selecting the stepsizes in the similar adaptive way to López et al. [25] for finding a common solution of the split null point problem and the fixed point problem for maximal monotone mappings and demicontractive multivalued mappings, and prove a strong convergence result of the proposed algorithm under some suitable conditions. In Sect. 4, we apply our main problem to the split feasibility problem and the split minimization problem. Finally, in Sect. 5, we provide some numerical results to demonstrate the convergence behavior of our algorithm and to support our main theorem.

2 Preliminaries

We denote the weak and strong convergence of a sequence $\{x_n\} \subset \mathcal{H}$ to $x \in \mathcal{H}$ by $x_n \rightharpoonup x$ and $x_n \rightarrow x$, respectively.

Let E be a nonempty closed convex subset of \mathcal{H} . The (metric) projection from \mathcal{H} onto E , denoted by P_E is defined for each $x \in \mathcal{H}$, $P_E x$ is the unique element in E such that

$$\|x - P_E x\| = d(x, E) := \inf\{\|x - z\| : z \in E\}.$$

It is well known that $u = P_E x$ if and only if $\langle x - u, z - u \rangle \leq 0$ for all $z \in E$. A mapping $T : E \rightarrow E$ is called *directed* if $F(T) := \{x \in E : Tx = x\} \neq \emptyset$ and

$$\|x - Tx\|^2 \leq \langle x - Tx, x - p \rangle \quad \text{for all } x \in E, p \in F(T).$$

Let D be a nonempty subset of E . A mapping $f : E \rightarrow E$ is called a β -contraction with respect to D , where $\beta \in [0, 1)$ if $\|f(x) - f(z)\| \leq \beta \|x - z\|$ for all $x \in E, z \in D$; f is called a β -contraction if f is a β -contraction with respect to E .

We now recall some notations and definitions on multivalued mappings. Let $U : E \rightarrow 2^E$ be a multivalued mapping. An element $p \in E$ is called a *fixed point* of U if $p \in Up$. The set of all fixed points of U is also denoted by $F(U)$. We say that U satisfies the *endpoint*

condition if $Up = \{p\}$ for all $p \in F(U)$. We denote by $CB(E)$ the family of all nonempty closed bounded subsets of E . The Pompeiu-Hausdorff metric on $CB(E)$ is defined by

$$H(C, D) := \max \left\{ \sup_{x \in C} d(x, D), \sup_{z \in D} d(z, C) \right\}$$

for all $C, D \in CB(E)$.

Definition 2.1 Let E be a nonempty closed convex subset of \mathcal{H} . A multivalued mapping $U : E \rightarrow CB(E)$ is said to be

(i) *nonexpansive* if

$$H(Ux, Uz) \leq \|x - z\| \quad \text{for all } x, z \in E,$$

(ii) *quasi-nonexpansive* if $F(U) \neq \emptyset$ and

$$H(Ux, Up) \leq \|x - p\| \quad \text{for all } x \in E, p \in F(U),$$

(iii) *demicontractive* [12, 19] if $F(U) \neq \emptyset$ and there exists $\kappa \in [0, 1)$ such that

$$H(Ux, Up)^2 \leq \|x - p\|^2 + \kappa d(x, Ux)^2 \quad \text{for all } x \in E, p \in F(U).$$

It is observed in Definition 2.1 that the class of demicontractive mappings includes classes of quasi-nonexpansive mappings and nonexpansive mappings with nonempty fixed point sets. We give an example of demicontractive multivalued mappings which is not quasi-nonexpansive as follows.

Example 2.2 [22] Let $\mathcal{H} = \mathbb{R}$. For each $i \in \mathbb{N}$, define $U_i : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ by

$$U_i x = \begin{cases} [-\frac{(2i+1)x}{2}, -(i+1)x], & \text{if } x \leq 0, \\ [-(i+1)x, -\frac{(2i+1)x}{2}], & \text{if } x > 0. \end{cases}$$

Then, U_i is demicontractive with a constant $\kappa_i = \frac{4i^2+8i}{4i^2+12i+9} \in (0, 1)$.

Lemma 2.3 Let E be a nonempty closed convex subset of \mathcal{H} . Let $U : E \rightarrow CB(E)$ be a κ -demicontractive multivalued mapping. Then we have

- (i) $F(U)$ is closed;
- (ii) If U satisfies the endpoint condition, then $F(U)$ is convex.

Proof (i) Let $\{p_n\} \subset F(U)$ be a sequence such that $p_n \rightarrow p$ as $n \rightarrow \infty$. Since U is κ -demicontractive, we have

$$\begin{aligned} d(p, Up) &\leq \|p - p_n\| + d(p_n, Up) \\ &\leq \|p - p_n\| + H(Up_n, Up) \\ &\leq 2\|p - p_n\| + \sqrt{\kappa}d(p, Up). \end{aligned}$$

By taking $n \rightarrow \infty$ into above inequality, we have $d(p, Up) \leq \sqrt{\kappa}d(p, Up)$. This implies that $d(p, Up) = 0$, that is, $p \in F(U)$. Hence $F(U)$ is closed.

(ii) Suppose that $Up = \{p\}$ for all $p \in F(U)$. Let $x, y \in F(U)$ and $\alpha \in (0, 1)$. Take $z := \alpha x + (1 - \alpha)y$, and let $v \in Uz$. By the demicontractivity of U , we have

$$\begin{aligned} d(z, Uz)^2 &\leq \|z - v\|^2 \\ &= \|\alpha(x - v) + (1 - \alpha)(y - v)\|^2 \\ &= \alpha\|x - v\|^2 + (1 - \alpha)\|y - v\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= \alpha d(v, Ux)^2 + (1 - \alpha)d(v, Uy)^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &\leq \alpha H(Uz, Ux)^2 + (1 - \alpha)H(Uz, Uy)^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &\leq \alpha\|x - z\|^2 + (1 - \alpha)\|y - z\|^2 - \alpha(1 - \alpha)\|x - y\|^2 + \kappa d(z, Uz)^2 \\ &= \alpha(1 - \alpha)^2\|x - y\|^2 + \alpha^2(1 - \alpha)\|x - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 + \kappa d(z, Uz)^2 \\ &= \alpha(1 - \alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2 + \kappa d(z, Uz)^2 = \kappa d(z, Uz)^2, \end{aligned}$$

which implies $d(z, Uz)^2 = 0$, i.e., $z \in F(U)$. Therefore, $F(U)$ is convex. □

Definition 2.4 Let E be a nonempty closed convex subset of \mathcal{H} and let $U : E \rightarrow CB(E)$ be a multivalued mapping. We say that $I - U$ is demiclosed at zero if for any sequence $\{x_n\}$ in E which converges weakly to $p \in E$ and the sequence $\{\|x_n - z_n\|\}$ converges strongly to 0, where $z_n \in Ux_n$, then $p \in F(U)$.

Let us recall the maximal monotone mapping.

Definition 2.5 A multivalued mapping $B : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is called maximal monotone if B is monotone, i.e.,

$$\langle x - z, u - w \rangle \geq 0 \quad \text{for all } x, z \in \text{dom}(B), u \in Bx, w \in Bz,$$

where $\text{dom}(B) := \{x \in \mathcal{H} : Bx \neq \emptyset\}$, and the graph $G(B)$ of B ,

$$G(B) := \{(x, u) \in \mathcal{H} \times \mathcal{H} : u \in Bx\},$$

is not properly contained in the graph of any other monotone mapping.

Let $B : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be a maximal monotone mapping and $\lambda > 0$. The resolvent of B with parameter λ is defined by

$$J_\lambda^B := (I + \lambda B)^{-1}.$$

It is well known [3] that $J_\lambda^B : \mathcal{H} \rightarrow \text{dom}(B)$ is single-valued, firmly nonexpansive, i.e., for any $x, z \in \mathcal{H}$,

$$\|J_\lambda^B x - J_\lambda^B z\|^2 \leq \langle J_\lambda^B x - J_\lambda^B z, x - z \rangle.$$

Moreover, $F(J_\lambda^B) = B^{-1}0$ and $I - J_\lambda^B$ is demiclosed at zero.

Let $g : \mathcal{H} \rightarrow (-\infty, \infty]$ be proper. A subdifferential ∂g of g at $x \in \mathcal{H}$ is defined by

$$\partial g(x) := \{u \in \mathcal{H} : g(x) + \langle u, z - x \rangle \leq g(z), \forall z \in \mathcal{H}\}.$$

The set of minimizers of g is defined by

$$\text{Argmin } g := \{x \in \mathcal{H} : g(x) \leq g(z), \forall z \in \mathcal{H}\}.$$

Note that if g is a proper, lower semicontinuous and convex function, then ∂g is a maximal monotone mapping (see [29]). In this case, the resolvent $J_\lambda^{\partial g}$ of ∂g is called the *proximity operator* [3] of λg , and is denoted by $\text{Prox}_{\lambda g}$.

Example 2.6 Let E be a nonempty closed convex subset of \mathcal{H} . Define an indicator function $i_E : \mathcal{H} \rightarrow (-\infty, \infty]$ of E by

$$i_E(x) := \begin{cases} 0, & \text{if } x \in E, \\ \infty, & \text{if } x \notin E. \end{cases}$$

One can see that

$$\partial i_E(x) = N_E(x) := \begin{cases} \{u \in \mathcal{H} : \langle u, z - x \rangle \leq 0, \forall z \in E\}, & \text{if } x \in E, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Since i_E is a proper, lower semicontinuous and convex function, we have $\partial i_E = N_E$ is maximal monotone and $J_\lambda^{\partial i_E} = J_\lambda^{N_E} = P_E$ for all $\lambda > 0$. Note that we call N_E the *normal cone* to E .

We next give some significant facts and tools for proving our main result.

Lemma 2.7 *Let $x, z \in \mathcal{H}, \vartheta \in \mathbb{R}$. Then the following inequalities hold on \mathcal{H} :*

- (i) $\|x + z\|^2 \leq \|x\|^2 + 2\langle z, x + z \rangle$;
- (ii) $\|\vartheta x + (1 - \vartheta)z\|^2 = \vartheta\|x\|^2 + (1 - \vartheta)\|z\|^2 - \vartheta(1 - \vartheta)\|x - z\|^2$.

Lemma 2.8 ([35]) *Suppose that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \mu_n)a_n + \mu_n\sigma_n + \tau_n, \quad n \in \mathbb{N},$$

where $\{\mu_n\}, \{\sigma_n\}$ and $\{\tau_n\}$ satisfy the following conditions:

- (i) $\{\mu_n\} \subset [0, 1], \sum_{n=1}^\infty \mu_n = \infty$;
- (ii) $\limsup_n \sigma_n \leq 0$ or $\sum_{n=1}^\infty |\mu_n\sigma_n| < \infty$;
- (iii) $\tau_n \geq 0$ for all $n \in \mathbb{N}, \sum_{n=1}^\infty \tau_n < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.9 ([26]) *Let $\{r_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ which satisfies $r_{n_i} < r_{n_i+1}$ for all $i \in \mathbb{N}$. Define a sequence of positive integers $\{\rho(n)\}$ by*

$$\rho(n) := \max\{m \leq n : r_m < r_{m+1}\}$$

for all $n \geq n_0$ (for some n_0 large enough). Then $\{\rho(n)\}$ is a nondecreasing sequence such that $\rho(n) \rightarrow \infty$ as $n \rightarrow \infty$, and it holds that

$$r_{\rho(n)} \leq r_{\rho(n)+1}, \quad r_n \leq r_{\rho(n)+1}.$$

3 Main Result

To prove our main result, we need the following useful lemma.

Lemma 3.1 *Let $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator, and let $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ be a directed mapping with $A^{-1}(F(T)) \neq \emptyset$. If $x \in \mathcal{H}_1$ with $Ax \neq T(Ax)$ and $p \in A^{-1}(F(T))$, then*

$$\|x - \gamma A^*(I - T)Ax - p\|^2 \leq \|x - p\|^2 - (2 - \xi)\xi \frac{\|(I - T)Ax\|^4}{\|A^*(I - T)Ax\|^2}, \tag{3.1}$$

where

$$\gamma := \frac{\xi \|(I - T)Ax\|^2}{\|A^*(I - T)Ax\|^2}$$

and $\xi \in (0, 2)$.

Proof Let $x \in \mathcal{H}_1$ with $Ax \neq T(Ax)$ and $p \in A^{-1}(F(T))$. If $A^*(I - T)Ax = 0$, by the property of a directed mapping T , we have

$$\|(I - T)Ax\|^2 \leq \langle (I - T)Ax, Ax - Ap \rangle = \langle A^*(I - T)Ax, x - p \rangle = 0,$$

which is a contradiction. Thus, $A^*(I - T)Ax \neq 0$ and hence γ is well defined. Since T is directed,

$$\begin{aligned} \|x - \gamma A^*(I - T)Ax - p\|^2 &= \|x - p\|^2 - 2\gamma \langle A^*(I - T)Ax, x - p \rangle \\ &\quad + \gamma^2 \|A^*(I - T)Ax\|^2 \\ &= \|x - p\|^2 - 2\gamma \langle (I - T)Ax, Ax - Ap \rangle \\ &\quad + \gamma^2 \|A^*(I - T)Ax\|^2 \\ &\leq \|x - p\|^2 - 2\gamma \|(I - T)Ax\|^2 + \gamma^2 \|A^*(I - T)Ax\|^2 \\ &= \|x - p\|^2 - (2 - \xi)\xi \frac{\|(I - T)Ax\|^4}{\|A^*(I - T)Ax\|^2}. \quad \square \end{aligned}$$

We now state and prove our main theorem.

Theorem 3.2 *Let \mathcal{H}_1 and \mathcal{H}_2 be two real Hilbert spaces and C be a nonempty closed convex subset of \mathcal{H}_1 . Let $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator. Let $U : C \rightarrow CB(C)$ be a κ -demicontractive multivalued mapping. Let $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$ be maximal monotone mappings such that $\text{dom}(B_1)$ is included in C , and let $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ be resolvents of B_1 and B_2 , respectively for $\lambda > 0$. Assume that $\Omega := F(U) \cap \Gamma \neq \emptyset$, where $\Gamma = \{x \in B_1^{-1}0 : Ax \in B_2^{-1}0\}$. Let $f : C \rightarrow C$ be a β -contraction with respect to Ω . Suppose that $\{x_n\}$ is a sequence generated iteratively by $x_1 \in C$ and*

$$\begin{cases} y_n = J_{\lambda_n}^{B_1}(x_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Ax_n), \\ u_n = (1 - \vartheta_n)y_n + \vartheta_n z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)u_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.2}$$

where $z_n \in U y_n$, the stepsize γ_n is selected in such a way:

$$\gamma_n := \begin{cases} \frac{\xi_n \|(I - J_{\lambda_n}^{B_2})Ax_n\|^2}{\|A^*(I - J_{\lambda_n}^{B_2})Ax_n\|^2}, & \text{if } Ax_n \notin B_2^{-1}0, \\ 1, & \text{otherwise,} \end{cases} \tag{3.3}$$

and the real sequences $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\vartheta_n\}$ and $\{\xi_n\}$ satisfy the following conditions:

- (C1) $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (C2) $\lambda_n \in (0, \infty)$ such that $\liminf_{n \rightarrow \infty} \lambda_n > 0$;
- (C3) $0 < a \leq \vartheta_n \leq b < 1 - \kappa$;
- (C4) $0 < c \leq \xi_n \leq d < 2$.

If U satisfies the endpoint condition and $I - U$ is demiclosed at zero, then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega$, where $x^* = P_{\Omega}f(x^*)$.

Proof By Lemma 2.3, we have $F(U)$ is closed and convex, and hence Ω is also closed and convex. One can show that $P_{\Omega}f$ is a contraction on Ω and it follows from Banach fixed point theorem that $x^* = P_{\Omega}f(x^*)$ for some $x^* \in \Omega$. By characterization of the metric projection P_{Ω} , we have

$$\langle f(x^*) - x^*, p - x^* \rangle \leq 0 \quad \text{for all } p \in \Omega. \tag{3.4}$$

Since $x^* \in \Omega$, we have $Ux^* = \{x^*\}$, $J_{\lambda_n}^{B_1}x^* = x^*$ and $J_{\lambda_n}^{B_2}(Ax^*) = Ax^*$. We first show that $\{x_n\}$ is bounded. By Lemma 2.7 (ii) and the demicontractivity of U with the constant κ , we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \|(1 - \vartheta_n)(y_n - x^*) + \vartheta_n(z_n - x^*)\|^2 \\ &= (1 - \vartheta_n)\|y_n - x^*\|^2 + \vartheta_n\|z_n - x^*\|^2 - \vartheta_n(1 - \vartheta_n)\|y_n - z_n\|^2 \\ &= (1 - \vartheta_n)\|y_n - x^*\|^2 + \vartheta_n d(z_n, Ux^*)^2 - \vartheta_n(1 - \vartheta_n)\|y_n - z_n\|^2 \\ &\leq (1 - \vartheta_n)\|y_n - x^*\|^2 + \vartheta_n H(Uy_n, Ux^*)^2 - \vartheta_n(1 - \vartheta_n)\|y_n - z_n\|^2 \\ &\leq (1 - \vartheta_n)\|y_n - x^*\|^2 + \vartheta_n (\|y_n - x^*\|^2 + \kappa d(y_n, Uy_n)^2) \\ &\quad - \vartheta_n(1 - \vartheta_n)\|y_n - z_n\|^2 \\ &\leq (1 - \vartheta_n)\|y_n - x^*\|^2 + \vartheta_n\|y_n - x^*\|^2 + \vartheta_n \kappa \|y_n - z_n\|^2 \\ &\quad - \vartheta_n(1 - \vartheta_n)\|y_n - z_n\|^2 \\ &= \|y_n - x^*\|^2 - \vartheta_n(1 - \kappa - \vartheta_n)\|y_n - z_n\|^2. \end{aligned} \tag{3.5}$$

By the firm nonexpansivity of $J_{\lambda_n}^{B_2}$, we have $J_{\lambda_n}^{B_2}$ is a directed mapping. If $Ax_n \notin B_2^{-1}0$, then $J_{\lambda_n}^{B_2}(Ax_n) \neq Ax_n$ and it follows from Lemma 3.1 that

$$\begin{aligned} \|y_n - x^*\|^2 &= \|J_{\lambda_n}^{B_1}(x_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Ax_n) - x^*\|^2 \\ &\leq \|x_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Ax_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - (2 - \xi_n)\xi_n \frac{\|(I - J_{\lambda_n}^{B_2})Ax_n\|^4}{\|A^*(I - J_{\lambda_n}^{B_2})Ax_n\|^2} \end{aligned} \tag{3.6}$$

$$\leq \|x_n - x^*\|^2 - \frac{(2 - \xi_n)\xi_n}{\|A\|^2} \|(I - J_{\lambda_n}^{B_2})Ax_n\|^2. \tag{3.7}$$

Note that in the case of $Ax_n \in B_2^{-1}0$, we have $J_{\lambda_n}^{B_2}(Ax_n) = Ax_n$. This implies that $\|y_n - x^*\|^2 = \|J_{\lambda_n}^{B_1}x_n - J_{\lambda_n}^{B_1}x^*\|^2 \leq \|x_n - x^*\|^2$ and hence (3.7) still holds. By substituting (3.7) into (3.5), we have

$$\begin{aligned} \|u_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \frac{(2 - \xi_n)\xi_n}{\|A\|^2} \|(I - J_{\lambda_n}^{B_2})Ax_n\|^2 \\ &\quad - \vartheta_n(1 - \kappa - \vartheta_n)\|y_n - z_n\|^2. \end{aligned} \tag{3.8}$$

It follows that $\|u_n - x^*\| \leq \|x_n - x^*\|$. Thus, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n(f(x_n) - x^*) + (1 - \alpha_n)(u_n - x^*)\| \\ &\leq \alpha_n\|f(x_n) - x^*\| + (1 - \alpha_n)\|u_n - x^*\| \\ &\leq \alpha_n(\|f(x_n) - f(x^*)\| + \|f(x^*) - x^*\|) + (1 - \alpha_n)\|x_n - x^*\| \\ &\leq \alpha_n(\beta\|x_n - x^*\| + \|f(x^*) - x^*\|) + (1 - \alpha_n)\|x_n - x^*\| \\ &= (1 - \alpha_n(1 - \beta))\|x_n - x^*\| + \alpha_n(1 - \beta)\frac{\|f(x^*) - x^*\|}{1 - \beta} \\ &\leq \max\left\{\|x_n - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \beta}\right\}. \end{aligned}$$

By mathematical induction, we obtain

$$\|x_n - x^*\| \leq \max\left\{\|x_1 - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \beta}\right\}$$

for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is bounded. This implies that $\{f(x_n)\}$ and $\{y_n\}$ are also bounded. Now, from (3.8), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n(f(x_n) - x^*) + (1 - \alpha_n)(u_n - x^*)\|^2 \\ &\leq \alpha_n\|f(x_n) - x^*\|^2 + (1 - \alpha_n)\|u_n - x^*\|^2 \\ &\leq \alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \frac{(2 - \xi_n)\xi_n}{\|A\|^2} \|(I - J_{\lambda_n}^{B_2})Ax_n\|^2 \\ &\quad - \vartheta_n(1 - \kappa - \vartheta_n)\|y_n - z_n\|^2. \end{aligned} \tag{3.9}$$

By (3.9), we obtain the following two inequalities

$$\frac{(2 - \xi_n)\xi_n}{\|A\|^2} \|(I - J_{\lambda_n}^{B_2})Ax_n\|^2 \leq \alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \tag{3.10}$$

and

$$\vartheta_n(1 - \kappa - \vartheta_n)\|y_n - z_n\|^2 \leq \alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.11}$$

We next divide the rest of the proof into two cases.

Case 1: Assume that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}_{n \geq n_0}$ is either nonincreasing or nondecreasing. Since $\{\|x_n - x^*\|\}$ is bounded, then it converges and $\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \rightarrow 0$ as $n \rightarrow \infty$. Inequalities (3.10), (3.11) and our control conditions (C1)–(C4) yield

$$\lim_{n \rightarrow \infty} \|(I - J_{\lambda_n}^{B_2})Ax_n\| = 0 \tag{3.12}$$

and

$$\lim_{n \rightarrow \infty} \|y_n - z_n\| = 0. \tag{3.13}$$

If $Ax_n \in B_2^{-1}0$, then $\gamma_n \|A^*(I - J_{\lambda_n}^{B_2})Ax_n\| = 0$. Thus, we assume that $Ax_n \notin B_2^{-1}0$. By substituting (3.6) into (3.5), we have

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (2 - \xi_n)\gamma_n \|(I - J_{\lambda_n}^{B_2})Ax_n\|^2,$$

which implies that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \alpha_n \|f(x_n) - x^*\|^2 + (1 - \alpha_n) \|u_n - x^*\|^2 \\ &\leq \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - (2 - \xi_n)\gamma_n \|(I - J_{\lambda_n}^{B_2})Ax_n\|^2 \end{aligned}$$

or

$$(2 - \xi_n)\gamma_n \|(I - J_{\lambda_n}^{B_2})Ax_n\|^2 \leq \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

It follows from (C1), (C2) and (C4) that $\gamma_n \|(I - J_{\lambda_n}^{B_2})Ax_n\|^2 \rightarrow 0$ as $n \rightarrow \infty$. This implies that

$$\lim_{n \rightarrow \infty} \gamma_n \|A^*(I - J_{\lambda_n}^{B_2})Ax_n\| = 0. \tag{3.14}$$

By the firm nonexpansivity of $J_{\lambda_n}^{B_1}$ and Lemma 3.1, we have

$$\begin{aligned} \|y_n - x^*\|^2 &= \|J_{\lambda_n}^{B_1}(x_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Ax_n) - J_{\lambda_n}^{B_1}x^*\|^2 \\ &\leq \langle J_{\lambda_n}^{B_1}(x_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Ax_n) - J_{\lambda_n}^{B_1}x^*, x_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Ax_n - x^* \rangle \\ &= \langle y_n - x^*, x_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Ax_n - x^* \rangle \\ &= \frac{1}{2} \left(\|y_n - x^*\|^2 + \|x_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Ax_n - x^*\|^2 \right. \\ &\quad \left. - \|y_n - x_n - \gamma_n A^*(J_{\lambda_n}^{B_2} - I)Ax_n\|^2 \right) \\ &\leq \frac{1}{2} \left(\|y_n - x^*\|^2 + \|x_n - x^*\|^2 - \|y_n - x_n - \gamma_n A^*(J_{\lambda_n}^{B_2} - I)Ax_n\|^2 \right) \\ &= \frac{1}{2} \left(\|y_n - x^*\|^2 + \|x_n - x^*\|^2 - \|y_n - x_n\|^2 - \gamma_n^2 \|A^*(J_{\lambda_n}^{B_2} - I)Ax_n\|^2 \right. \\ &\quad \left. + 2\gamma_n \langle y_n - x_n, A^*(J_{\lambda_n}^{B_2} - I)Ax_n \rangle \right) \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2} \left(\|y_n - x^*\|^2 + \|x_n - x^*\|^2 - \|y_n - x_n\|^2 - \gamma_n^2 \|A^*(J_{\lambda_n}^{B_2} - I)Ax_n\|^2 \right. \\ &\quad \left. + 2\gamma_n \|y_n - x_n\| \|A^*(J_{\lambda_n}^{B_2} - I)Ax_n\| \right), \end{aligned}$$

which implies that

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|y_n - x_n\|^2 + 2\gamma_n \|y_n - x_n\| \|A^*(I - J_{\lambda_n}^{B_2})Ax_n\|. \tag{3.15}$$

Since U is κ -demicontractive, it follows from (3.15) that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \alpha_n \|f(x_n) - x^*\|^2 + (1 - \alpha_n) \|u_n - x^*\|^2 \\ &\leq \alpha_n \|f(x_n) - x^*\|^2 + (1 - \alpha_n) \left((1 - \vartheta_n) \|y_n - x^*\|^2 + \vartheta_n \|z_n - x^*\|^2 \right) \\ &\leq \alpha_n \|f(x_n) - x^*\|^2 + (1 - \vartheta_n) \|y_n - x^*\|^2 + \vartheta_n d(z_n, Ux^*)^2 \\ &\leq \alpha_n \|f(x_n) - x^*\|^2 + (1 - \vartheta_n) \|y_n - x^*\|^2 + \vartheta_n H(Uy_n, Ux^*)^2 \\ &\leq \alpha_n \|f(x_n) - x^*\|^2 + (1 - \vartheta_n) \|y_n - x^*\|^2 + \vartheta_n \|y_n - x^*\|^2 \\ &\quad + \vartheta_n \kappa d(y_n, Uy_n)^2 \\ &\leq \alpha_n \|f(x_n) - x^*\|^2 + \|y_n - x^*\|^2 + \vartheta_n \kappa \|y_n - z_n\|^2 \\ &\leq \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|y_n - x_n\|^2 \\ &\quad + 2\gamma_n \|y_n - x_n\| \|A^*(I - J_{\lambda_n}^{B_2})Ax_n\| + \vartheta_n \kappa \|y_n - z_n\|^2. \end{aligned} \tag{3.16}$$

By (3.12), (3.13), (3.14) and (3.16), we deduce that

$$\begin{aligned} \|y_n - x_n\|^2 &\leq \alpha_n \|f(x_n) - x^*\|^2 + 2\gamma_n \|y_n - x_n\| \|A^*(I - J_{\lambda_n}^{B_2})Ax_n\| + \vartheta_n \kappa \|y_n - z_n\|^2 \\ &\quad + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, which implies that $\|y_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. We next show that

$$\limsup_{n \rightarrow \infty} \langle f(x^*) - x^*, x_n - x^* \rangle \leq 0.$$

To show this, let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{j \rightarrow \infty} \langle f(x^*) - x^*, x_{n_j} - x^* \rangle = \limsup_{n \rightarrow \infty} \langle f(x^*) - x^*, x_n - x^* \rangle.$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ and $p \in \mathcal{H}_1$ such that $x_{n_{j_k}} \rightharpoonup p$. Without loss of generality, we can assume that $x_{n_j} \rightharpoonup p$. Since A is a bounded linear operator, we have $\langle z, Ax_{n_j} - Ap \rangle = \langle A^*z, x_{n_j} - p \rangle \rightarrow 0$ as $j \rightarrow \infty$, for all $z \in \mathcal{H}_2$, this implies that $Ax_{n_j} \rightharpoonup Ap$. From (3.12) and by the demiclosedness of $I - J_{\lambda_n}^{B_2}$ at zero, we get $Ap \in F(J_{\lambda_n}^{B_2}) = B_2^{-1}0$. Since $x_{n_j} \rightharpoonup p$ and $\|y_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$, we have $y_{n_j} \rightharpoonup p$. From (3.13) and by the demiclosedness of $I - U$ at zero, we obtain $p \in F(U)$. Now let us show that $p \in B_1^{-1}0$. From $y_n = J_{\lambda_n}^{B_1}(x_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Ax_n)$, then we can easily prove that

$$\frac{1}{\lambda_n} (x_n - y_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Ax_n) \in B_1 y_n.$$

By the monotonicity of B_1 , we have

$$\langle y_n - v, \frac{1}{\lambda_n}(x_n - y_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Ax_n) - w \rangle \geq 0$$

for all $(v, w) \in G(B_1)$. Thus, we also have

$$\langle y_{n_j} - v, \frac{1}{\lambda_{n_j}}(x_{n_j} - y_{n_j} - \gamma_n A^*(I - J_{\lambda_{n_j}}^{B_2})Ax_{n_j}) - w \rangle \geq 0 \tag{3.17}$$

for all $(v, w) \in G(B_1)$. Since $y_{n_j} \rightarrow p$, $\|x_{n_j} - y_{n_j}\| \rightarrow 0$ and $\|(I - J_{\lambda_{n_j}}^{B_2})Ax_{n_j}\| \rightarrow 0$ as $j \rightarrow \infty$, then by taking the limit as $j \rightarrow \infty$ in (3.17) yields

$$\langle p - v, -w \rangle \geq 0$$

for all $(v, w) \in G(B_1)$. By the maximal monotonicity of B_1 , we get $0 \in B_1 p$, i.e., $p \in B_1^{-1}0$. Therefore, $p \in \Omega$. Since x^* satisfies inequality (3.4), we have

$$\limsup_{n \rightarrow \infty} \langle f(x^*) - x^*, x_n - x^* \rangle = \lim_{j \rightarrow \infty} \langle f(x^*) - x^*, x_{n_j} - x^* \rangle = \langle f(x^*) - x^*, p - x^* \rangle \leq 0.$$

By Lemma 2.7 (i), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)(u_n - x^*) + \alpha_n(f(x_n) - x^*)\|^2 \\ &\leq (1 - \alpha_n)^2 \|u_n - x^*\|^2 + 2\alpha_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle \\ &= (1 - \alpha_n)^2 \|u_n - x^*\|^2 + 2\alpha_n \langle f(x_n) - f(x^*), x_{n+1} - x^* \rangle \\ &\quad + 2\alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \beta \|x_n - x^*\| \|x_{n+1} - x^*\| \\ &\quad + 2\alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + \alpha_n \beta (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\ &\quad + 2\alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle. \end{aligned}$$

Thus,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \frac{(1 - \alpha_n)^2 + \alpha_n \beta}{1 - \alpha_n \beta} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n \beta} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\ &= \left(1 - \frac{(1 - \beta)\alpha_n}{1 - \alpha_n \beta}\right) \|x_n - x^*\|^2 + \frac{(\alpha_n - (1 - \beta))\alpha_n}{1 - \alpha_n \beta} \|x_n - x^*\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n \beta} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\ &\leq \left(1 - \frac{(1 - \beta)\alpha_n}{1 - \alpha_n \beta}\right) \|x_n - x^*\|^2 \\ &\quad + \frac{(1 - \beta)\alpha_n}{1 - \alpha_n \beta} \left\{ \left(\frac{\alpha_n}{1 - \beta} - 1\right) M + \frac{2}{1 - \beta} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \right\} \end{aligned}$$

$$= (1 - \mu_n)\|x_n - x^*\|^2 + \mu_n\sigma_n, \tag{3.18}$$

where $M = \sup\{\|x_n - x^*\|^2 : n \in \mathbb{N}\}$, $\mu_n = \frac{(1-\beta)\alpha_n}{1-\alpha_n\beta}$, and $\sigma_n = \left(\frac{\alpha_n}{1-\beta} - 1\right)M + \frac{2}{1-\beta}\langle f(x^*) - x^*, x_{n+1} - x^* \rangle$. Clearly, $\{\mu_n\} \subset [0, 1]$, $\sum_{n=1}^\infty \mu_n = \infty$ and $\limsup_n \sigma_n \leq 0$. Hence, by applying Lemma 2.8 to (3.18), we can conclude that $x_n \rightarrow x^*$ as $n \rightarrow \infty$.

Case 2: Suppose that $\{\|x_n - x^*\|\}$ is not a monotone sequence. Then there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\|x_{n_i} - x^*\| < \|x_{n_{i+1}} - x^*\|$ for all $i \in \mathbb{N}$. We now define a positive integer sequence $\{\rho(n)\}$ by

$$\rho(n) := \max\{m \leq n : \|x_m - x^*\| < \|x_{m+1} - x^*\|\}$$

for all $n \geq n_0$ (for some n_0 large enough). By Lemma 2.9, we have $\{\rho(n)\}$ is a nondecreasing sequence such that $\rho(n) \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\|x_{\rho(n)} - x^*\|^2 - \|x_{\rho(n)+1} - x^*\|^2 \leq 0$$

for all $n \geq n_0$. From (3.10), we obtain

$$\lim_{n \rightarrow \infty} \|(I - J_{\lambda, \rho(n)}^{B_2})Ax_{\rho(n)}\| = 0. \tag{3.19}$$

From (3.11), we have

$$\lim_{n \rightarrow \infty} \|y_{\rho(n)} - z_{\rho(n)}\| = 0. \tag{3.20}$$

By (3.19), (3.20) and by the same proof as in Case 1, we obtain

$$\limsup_{n \rightarrow \infty} \langle f(x^*) - x^*, x_{\rho(n)} - x^* \rangle \leq 0.$$

By the same computation as in Case 1, we deduce that

$$\|x_{\rho(n)+1} - x^*\|^2 \leq (1 - \mu_{\rho(n)})\|x_{\rho(n)} - x^*\|^2 + \mu_{\rho(n)}\sigma_{\rho(n)}, \tag{3.21}$$

where $\mu_{\rho(n)} = \frac{(1-\beta)\alpha_{\rho(n)}}{1-\alpha_{\rho(n)}\beta}$, $\sigma_{\rho(n)} = \left(\frac{\alpha_{\rho(n)}}{1-\beta} - 1\right)M + \frac{2}{1-\beta}\langle f(x^*) - x^*, x_{\rho(n)+1} - x^* \rangle$ and $M = \sup\{\|x_{\rho(n)} - x^*\|^2 : n \in \mathbb{N}\}$. Clearly, $\limsup_n \sigma_{\rho(n)} \leq 0$. Since $\|x_{\rho(n)} - x^*\|^2 \leq \|x_{\rho(n)+1} - x^*\|^2$, it follows from (3.21) that $\|x_{\rho(n)} - x^*\|^2 \leq \sigma_{\rho(n)}$. This implies that $\|x_{\rho(n)} - x^*\| \rightarrow 0$ as $n \rightarrow \infty$. It follows from Lemma 2.9 and (3.21) that

$$0 \leq \|x_n - x^*\| \leq \|x_{\rho(n)+1} - x^*\| \rightarrow 0$$

as $n \rightarrow \infty$. Hence $\{x_n\}$ converges strongly to x^* . This completes the proof. □

Remark 3.3 We have some observations on Theorem 3.2.

- (i) The stepsize γ_n defined by (3.3) does not depend on $\|A\|$ (in fact, it depends on x_n).
- (ii) Taking $f(x) = u$ for some $u \in C$, Algorithm (3.2) becomes the Halpern-type algorithm. In particular, if $u = 0$ (in the case that $0 \in C$), then $\{x_n\}$ converges to x^* , where x^* is the minimum norm solution in Ω .

4 Applications

4.1 The Split Feasibility Problem

Let C and Q are nonempty closed convex subsets of \mathcal{H}_1 and \mathcal{H}_2 , respectively, and let $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator. Recall that the *split feasibility problem* (SFP) is to find a point

$$x^* \in C \text{ such that } Ax^* \in Q. \tag{4.1}$$

Applying Theorem 3.2, we obtain a strongly convergent algorithm which is independent of the operator norms for finding a common solution of the SFP (4.1) and the fixed point problem for demicontractive multivalued mappings as follows.

Theorem 4.1 *Let C and Q be nonempty closed convex subsets of \mathcal{H}_1 and \mathcal{H}_2 , respectively, and let $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator. Let $U : C \rightarrow CB(C)$ be a κ -demicontractive multivalued mapping. Assume that $\Omega := F(U) \cap A^{-1}(Q) \neq \emptyset$. Let $f : C \rightarrow C$ be a β -contraction with respect to Ω . Suppose that $\{x_n\}$ is a sequence generated iteratively by $x_1 \in C$ and*

$$\begin{cases} y_n = P_C(x_n - \gamma_n A^*(I - P_Q)Ax_n), \\ u_n = (1 - \vartheta_n)y_n + \vartheta_n z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)u_n, \quad n \in \mathbb{N}, \end{cases} \tag{4.2}$$

where $z_n \in Uy_n$, the stepsize γ_n is selected in the same way as in [25]:

$$\gamma_n := \begin{cases} \frac{\xi_n \|(I - P_Q)Ax_n\|^2}{\|A^*(I - P_Q)Ax_n\|^2}, & \text{if } Ax_n \notin Q, \\ 1, & \text{otherwise,} \end{cases} \tag{4.3}$$

and the sequences $\{\alpha_n\}$, $\{\vartheta_n\}$ and $\{\xi_n\}$ satisfy (C1), (C3) and (C4) in Theorem 3.2. If U satisfies the endpoint condition and $I - U$ is demiclosed at zero, then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega$, where $x^* = P_\Omega f(x^*)$.

Proof Setting $B_1 := N_C = \partial i_C$ and $B_2 := N_Q = \partial i_Q$, we have B_1 and B_2 are maximal monotone. We also have $J_\lambda^{B_1} = P_C$ and $J_\lambda^{B_2} = P_Q$ for $\lambda > 0$, and $B_1^{-1}0 = C$ and $B_2^{-1}0 = Q$. Thus, the result is obtained directly by Theorem 3.2. □

4.2 The Split Minimization Problem

Let $g_1 : \mathcal{H}_1 \rightarrow (-\infty, \infty]$ and $g_2 : \mathcal{H}_2 \rightarrow (-\infty, \infty]$ be two proper, lower semicontinuous and convex functions, and let $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator. The *split minimization problem* (SMP) is to find a point $x^* \in \mathcal{H}_1$ such that

$$x^* \in \text{Argmin } g_1 \text{ and } Ax^* \in \text{Argmin } g_2. \tag{4.4}$$

Applying Theorem 3.2, we get a strongly convergent algorithm which is independent of the operator norms for finding a common solution of the SMP (4.4) and the fixed point problem for demicontractive multivalued mappings as follows.

Theorem 4.2 *Let $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator. Let $U : \mathcal{H}_1 \rightarrow CB(\mathcal{H}_1)$ be a κ -demicontractive multivalued mapping. Let $g_1 : \mathcal{H}_1 \rightarrow (-\infty, \infty]$ and $g_2 : \mathcal{H}_2 \rightarrow (-\infty, \infty]$ be two proper, lower semicontinuous and convex functions. Assume that $\Omega := F(U) \cap \Gamma \neq \emptyset$, where $\Gamma = \{x \in \text{Argmin } g_1 : Ax \in \text{Argmin } g_2\}$. Let $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ be a β -contraction with respect to Ω . Suppose that $\{x_n\}$ is a sequence generated iteratively by $x_1 \in \mathcal{H}_1$ and*

$$\begin{cases} y_n = \text{Prox}_{\lambda_n g_1}(x_n - \gamma_n A^*(I - \text{Prox}_{\lambda_n g_2})Ax_n), \\ u_n = (1 - \vartheta_n)y_n + \vartheta_n z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)u_n, \end{cases} \quad n \in \mathbb{N}, \tag{4.5}$$

where $z_n \in U y_n$, the stepsize γ_n is selected in such a way:

$$\gamma_n := \begin{cases} \frac{\xi_n \| (I - \text{Prox}_{\lambda_n g_2})Ax_n \|^2}{\| A^*(I - \text{Prox}_{\lambda_n g_2})Ax_n \|^2}, & \text{if } Ax_n \notin \text{Argmin } g_2, \\ 1, & \text{otherwise,} \end{cases} \tag{4.6}$$

and the sequences $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\vartheta_n\}$ and $\{\xi_n\}$ satisfy (C1)–(C4) in Theorem 3.2. If U satisfies the endpoint condition and $I - U$ is demiclosed at zero, then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega$, where $x^* = P_\Omega f(x^*)$.

Proof Taking $C = \mathcal{H}_1$, $B_1 := \partial g_1$ and $B_2 := \partial g_2$, we have B_1 and B_2 are maximal monotone. One can show that $\text{Argmin } g_1 = (\partial g_1)^{-1}0 = B_1^{-1}0$ and $\text{Argmin } g_2 = (\partial g_2)^{-1}0 = B_2^{-1}0$. Therefore, the result is obtained immediately by Theorem 3.2. \square

5 Numerical Examples

We first give a numerical example of Theorem 3.2 to demonstrate the convergence behavior of Algorithm (3.2).

Example 5.1 Let $\mathcal{H}_1 = C = \mathbb{R}$ and $\mathcal{H}_2 = \mathbb{R}^3$ with the usual norms. Define a multivalued mapping $U : \mathbb{R} \rightarrow CB(\mathbb{R})$ by

$$Ux := \begin{cases} [-\frac{11x}{2}, -6x], & \text{if } x \leq 0, \\ [-6x, -\frac{11x}{2}], & \text{if } x > 0. \end{cases}$$

By Example 2.2, U is demicontractive with a constant $\kappa = \frac{140}{169}$. Let $B_1 : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ be defined by

$$B_1(x) := \begin{cases} \{u \in \mathbb{R} : z^2 + xz - 2x^2 \geq (z - x)u, \forall z \in [-9, 3]\}, & \text{if } x \in [-9, 3], \\ \emptyset, & \text{otherwise.} \end{cases}$$

By [33, Theorem 4.2], B_1 is maximal monotone. Define a maximal monotone mapping $B_2 : \mathbb{R}^3 \rightarrow 2^{\mathbb{R}^3}$ by $B_2 := \partial g$, where $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a function defined by

$$g(x, y, z) = \frac{|2x - 5y + 3z|^2}{2}.$$

Table 1 Numerical experiment of Algorithm (5.1)

| $f(x) = x/2$ | | |
|--------------|-----------|-----------|
| n | x_n | E_n |
| 1 | 10 | – |
| 2 | 1.2416232 | 8.7583768 |
| 3 | 0.1269221 | 1.1147011 |
| 4 | 0.0112343 | 0.1156878 |
| 5 | 0.0008861 | 0.0103483 |
| 6 | 0.0000635 | 0.0008226 |
| 7 | 0.0000042 | 0.0000593 |
| 8 | 0.0000003 | 0.0000039 |
| 9 | 1.501E-08 | 0.0000002 |
| 10 | 8.247E-10 | 1.418E-08 |

Thus, the explicit forms of the resolvents of B_1 and B_2 can be written by $J_1^{B_1}x = x/4$ and $J_1^{B_2} = \text{Prox}_g = P^{-1}$, where

$$P = \begin{pmatrix} 5 & -10 & 6 \\ -10 & 26 & -15 \\ 6 & -15 & 10 \end{pmatrix},$$

see [14, 30]. Define a bounded linear operator $A : \mathbb{R} \rightarrow \mathbb{R}^3$ by $Ax := (15x, 6x, -27x)$. Let $\Omega := F(U) \cap \Gamma$, where $\Gamma = \{x \in B_1^{-1}0 : Ax \in B_2^{-1}0\}$. Put the following control sequences: $\alpha_n = \frac{1}{n+3}$, $\vartheta_n = \frac{n}{6n+1}$, $\xi_n = \sqrt{\frac{3n+1}{n+2}}$, $\lambda_n = 1$ and take $z_n = -\frac{23}{4}y_n$. Thus, Algorithm (3.2) in our main theorem becomes

$$x_{n+1} = \frac{1}{n+3}f(x_n) + \frac{1}{16} \left(\frac{n+2}{n+3} \right) \left(\frac{-3n+4}{6n+1} \right) (x_n - \gamma_n A^\top (I - P^{-1}) Ax_n), \quad (5.1)$$

where

$$\gamma_n = \begin{cases} \sqrt{\frac{3n+1}{n+2} \frac{\|(I-P^{-1})Ax_n\|^2}{\|A^\top(I-P^{-1})Ax_n\|^2}}, & \text{if } Ax_n \neq P^{-1}(Ax_n), \\ 1, & \text{otherwise,} \end{cases}$$

and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a contraction with respect to Ω .

Let us start with the initial point $x_1 = 10$ and the stopping criterion for our testing process is set as: $E_n := |x_n - x_{n-1}| < 10^{-7}$. In Table 1, a numerical experiment of Algorithm (5.1) is shown by taking $f(x) = x/2$. In Table 2, we show the numbers of iterations of Algorithm (5.1) by considering different contractions in the form of $f(x) = \beta x$ ($0 < \beta < 1$) and constant functions.

Remark 5.2 By testing the convergence behavior of Algorithm (5.1) in Example 5.1, we observe that

- (i) Algorithm (5.1) converges to a solution, i.e., $x_n \rightarrow 0 \in \Omega$.
- (ii) A contraction f in our algorithm influences the convergence behavior. Namely, selecting non-constant functions make our algorithm more efficient than using constant functions in terms of the number of iterations and the approximate solution. So, our algorithm is more general and desirable than the Halpern-type method.

Table 2 The numbers of iterations of Algorithm (5.1) by choosing different contractions

| Choices of f | n (No. of iterations) | x_n | E_n |
|------------------|-------------------------|-------|----------------------|
| $f(x) = \beta x$ | $\beta = 0.1$ | 7 | 1.197E-09 5.407E-08 |
| | $\beta = 0.3$ | 9 | 4.349E-10 1.049E-08 |
| | $\beta = 0.5$ | 10 | 8.247E-10 1.418E-08 |
| | $\beta = 0.7$ | 11 | 8.357E-10 1.150E-08 |
| | $\beta = 0.9$ | 11 | 8.164E-09 9.002E-08 |
| $f(x) = u$ | $u = 0.1$ | 1010 | 0.0001010 9.987E-08 |
| | $u = 0.5$ | 2259 | 0.0002260 9.999E-08 |
| | $u = 0.9$ | 3032 | 0.0003031 9.995E-08 |
| | $u = 1$ | 3196 | 0.0003196 9.995E-08 |
| | $u = -1.5$ | 3914 | -0.0003914 9.998E-08 |

Next, we give an example in the infinite-dimensional space L^2 for supporting Theorem 3.2.

Example 5.3 Let $\mathcal{H}_1 = C = \mathcal{H}_2 = L^2([0, 1])$. Let $x \in L^2([0, 1])$. Define a bounded linear operator $A : L^2([0, 1]) \rightarrow L^2([0, 1])$ by

$$(Ax)(t) := \int_0^1 (t^2 + 1)x(t)dt.$$

Define a mapping $U : L^2([0, 1]) \rightarrow L^2([0, 1])$ by

$$(Ux)(t) := \sin(x(t)).$$

Then, U is demicontractive. Let

$$E_1 = \left\{ u \in L^2([0, 1]) : \int_0^1 w(t)u(t)dt \leq 0 \right\},$$

where $0 \neq w \in L^2([0, 1])$, and let

$$E_2 = \{u \in L^2([0, 1]) : u \geq 0\}.$$

Define two maximal monotone mappings $B_1, B_2 : L^2([0, 1]) \rightarrow 2^{L^2([0, 1])}$ by $B_1 := N_{E_1}$ and $B_2 := N_{E_2}$ (see Example 2.6). We can write the explicit forms of the resolvents of B_1 and B_2 as follows:

$$J_\lambda^{B_1} x = P_{E_1} x = \begin{cases} x - \frac{\int_0^1 w(t)x(t)dt}{\int_0^1 w^2(t)dt} w, & \text{if } x \notin E_1, \\ x, & \text{if } x \in E_1 \end{cases}$$

and $J_\lambda^{B_2} x = P_{E_2} x = x_+$, where $x_+(t) = \max\{x(t), 0\}$ (see [6]). Now, Algorithm (3.2) has the following form:

$$\begin{cases} y_n = P_{E_1}(x_n - \gamma_n A^*(I - P_{E_2})Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)((1 - \vartheta_n)y_n + \vartheta_n U y_n), \end{cases} \tag{5.2}$$

where

$$\gamma_n = \begin{cases} \frac{\xi_n \|(I - P_{E_2})Ax_n\|^2}{\|A^*(I - P_{E_2})Ax_n\|^2}, & \text{if } Ax_n \notin E_2, \\ 1, & \text{otherwise.} \end{cases}$$

By choosing a contraction $f : L^2([0, 1]) \rightarrow L^2([0, 1])$ and the control sequences $\{\alpha_n\}$, $\{\vartheta_n\}$ and $\{\xi_n\}$ satisfying the conditions (C1), (C3) and (C4) in Theorem 3.2, it can guarantee that the sequence $\{x_n\}$ generated by (5.2) converges strongly to $0 \in \Omega := F(U) \cap E_1 \cap A^{-1}(E_2)$.

6 Conclusion

In this paper, we consider a problem of finding a common solution of the split null point problem and the fixed point problem for multivalued mappings in Hilbert spaces. We focus on the class of maximal monotone mappings and the class of demicontractive multivalued mappings which including several common types of classes of mappings occurring in nonlinear analysis and optimization problems. Various algorithms were introduced for solving the problem and most of them depend on the norms of the bounded linear operators; however, the calculation of the operator norm is not an easy work in general practice. We present a viscosity-type algorithm which is independent of the operator norms for solving the problem by selecting the stepsizes in the similar adaptive way to López et al. [25], and obtain some sufficient conditions for strong convergence of our proposed algorithm. Moreover, our main result can be applied to solving the split feasibility problem and the split minimization problem. Numerical examples are also given to support our main result.

Acknowledgements The authors would like to thank the referees for valuable comments and suggestions for improving this work. P. Jailoka was supported by Post-Doctoral Fellowship of Chiang Mai University, Thailand. We also would like to thank Chiang Mai University for the financial support.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

1. Alofi, A., Alsulami, S.M., Takahashi, W.: The split common null point problem and Halpern-type strong convergence theorem in Hilbert spaces. *J. Nonlinear Convex Anal.* **16**, 775–789 (2015)
2. Alsulami, S.M., Takahashi, W.: The split common null point problem for maximal monotone mappings in Hilbert spaces and applications. *J. Nonlinear Convex Anal.* **15**, 793–808 (2014)
3. Bauschke, H.H., Combettes, P.L.: *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. Springer, New York (2011)
4. Byrne, C.: A unified treatment of some iterative algorithms in signal processing and image reconstruction. *Inverse Probl.* **20**, 103–120 (2004)
5. Byrne, C., Censor, Y., Gibali, A., Reich, S.: The split common null point problem. *J. Nonlinear Convex Anal.* **13**, 759–775 (2012)
6. Cegielski, A.: *Iterative Methods for Fixed Point Problems in Hilbert Spaces*. Lecture Notes in Mathematics, vol. 2057. Springer, Heidelberg (2012)
7. Censor, Y., Elfving, T.: A multiprojection algorithm using Bregman projections in a product space. *Numer. Algorithms* **8**, 221–239 (1994)
8. Censor, Y., Segal, A.: The split common fixed point problem for directed operators. *J. Convex Anal.* **16**, 587–600 (2009)
9. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications for inverse problems. *Inverse Probl.* **21**, 2071–2084 (2005)

10. Censor, Y., Borteld, T., Martin, B., Trofimov, A.: A unified approach for inversion problems in intensity-modulated radiation therapy. *Phys. Med. Biol.* **51**, 2353–2365 (2006)
11. Censor, Y., Gibali, A., Reich, S.: Algorithms for the split variational inequality problem. *Numer. Algorithms* **59**, 301–323 (2012)
12. Chidume, C.E., Bello, A.U., Ndambomve, P.: Strong and Δ -convergence theorems for common fixed points of a finite family of multivalued demicontractive mappings in CAT(0) spaces. *Abstr. Appl. Anal.* **2014**, 805168 (2014). <https://doi.org/10.1155/2014/805168>
13. Combettes, P.L.: The convex feasibility problem in image recovery. In: Hawkes, P. (ed.) *Advances in Imaging and Electron Physics*, pp. 155–270. Academic Press, New York (1996)
14. Combettes, P.L., Pesquet, J.C.: Proximal splitting methods in signal processing. In: *Fixed-Point Algor. Inv. Probl. Sci. Eng.*, vol. 49, pp. 185–212 (2011)
15. Eslamian, M., Vahidi, J.: Split common fixed point problem of nonexpansive semigroup. *Mediterr. J. Math.* **13**, 1177–1195 (2016)
16. Eslamian, M., Eskandani, G.Z., Raeisi, M.: Split common null point and common fixed point problems between Banach spaces and Hilbert spaces. *Mediterr. J. Math.* **14**, 119 (2017)
17. Gibali, A.: A new split inverse problem and an application to least intensity feasible solutions. *Pure Appl. Funct. Anal.* **2**, 243–258 (2017)
18. Hicks, T.L., Kubicek, J.D.: On the Mann iteration process in a Hilbert space. *J. Math. Anal. Appl.* **59**, 498–504 (1977)
19. Isiogugu, F.O., Osilike, M.O.: Convergence theorems for new classes of multivalued hemicontractive-type mappings. *Fixed Point Theory Appl.* **2014**, 93 (2014). <https://doi.org/10.1186/1687-1812-2014-93>
20. Jailoka, P., Suantai, S.: Split null point problems and fixed point problems for demicontractive multivalued mappings. *Mediterr. J. Math.* **15**, 204 (2018). <https://doi.org/10.1007/s00009-018-1251-4>
21. Jailoka, P., Suantai, S.: Split common fixed point and null point problems for demicontractive operators in Hilbert spaces. *Optim. Methods Softw.* **34**(2), 248–263 (2019)
22. Jailoka, P., Suantai, S.: The split common fixed point problem for multivalued demicontractive mappings and its applications. *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat.* **113**(2), 689–706 (2019)
23. Kazmi, K.R., Rizvi, S.H.: Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem. *J. Egypt. Math. Soc.* **21**, 44–51 (2013)
24. Kotzer, T., Cohen, N., Shamir, J.: Extended and alternative projections onto convex sets: theory and applications. Technical Report No. EE 900, Dept. of Electrical Engineering, Technion, Haifa, Israel (November 1993)
25. López, G., Martín-Márquez, V., Wang, F., Xu, H.K.: Solving the split feasibility problem without prior knowledge of matrix norms. *Inverse Probl.* **28**, 8 (2012). <https://doi.org/10.1088/0266-5611/28/8/085004>
26. Maingé, P.E.: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. *Set-Valued Anal.* **16**, 899–912 (2008)
27. Mărușter, Ș.: The solution by iteration of nonlinear equations in Hilbert spaces. *Proc. Am. Math. Soc.* **63**, 69–73 (1977)
28. Moudafi, A.: Viscosity approximation methods for fixed-points problems. *J. Math. Anal. Appl.* **241**, 46–55 (2000)
29. Rockafellar, R.T.: On the maximal monotonicity of subdifferential mappings. *Pac. J. Math.* **33**, 209–216 (1970)
30. Singthong, U., Suantai, S.: Equilibrium problems and fixed point problems for nonspreading-type mappings in Hilbert space. *Int. J. Nonlinear Anal. Appl.* **2**, 51–61 (2011)
31. Takahashi, W.: The split common null point problem in Banach spaces. *Arch. Math.* **104**, 357–365 (2015)
32. Takahashi, S., Takahashi, W.: The split common null point problem and the shrinking projection method in Banach spaces. *Optimization* **65**, 281–287 (2016)
33. Takahashi, S., Takahashi, W., Toyoda, M.T.: Strong convergence theorem for maximal monotone operators with nonlinear mappings in Hilbert spaces. *J. Optim. Theory Appl.* **147**, 27–41 (2010)
34. Takahashi, W., Xu, H.K., Yao, J.C.: Iterative methods for generalized split feasibility problems in Hilbert spaces. *Set-Valued Var. Anal.* **23**, 205–221 (2015)
35. Xu, H.K.: Iterative algorithms for nonlinear operators. *J. Lond. Math. Soc.* **66**, 240–256 (2002)