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Global Existence and Long-Time Behavior of Solutions to the Vlasov–Poisson–Fokker–Planck System

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Abstract In this paper, we study the global existence of solutions to the Vlasov–Poisson–Fokker–Planck system in the whole space by using the refined energy method. In the proof, the *a priori* estimates on the macroscopic and microscopic components of solutions are obtained by use of the macroscopic balance laws. As a by-product, the algebraic decay rate of solutions converge to the global Maxwellian, which established by employing the Fourier analysis.

Mathematics Subject Classification 35Q84 · 82D05

Keywords Energy method · Global existence · Decay rate · Macro–micro decomposition · Fourier analysis

1 Introduction

We are concerned with the following Vlasov–Poisson–Fokker–Planck system

$$\begin{cases} \partial_t F + v\cdot\nabla_x F + \nabla_x\phi\cdot\nabla_v F = L_{FP}F, \quad t\geq 0, \quad x,v\in\mathbb{R}^3,\\ \Delta_x\phi = \int_{\mathbb{R}^3} F dv - 1,\\ \lim\limits_{|x|\to\infty}\phi(t,x)=0, \quad \forall\, t\geq 0, \end{cases} \tag{1.1}$$

with initial data

$$F(0,x,v)=F_0(x,v), \tag{1.2}$$

where $F(t,x,v)$ is the distribution function of particles at time $t\geq 0$, position $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ with velocity $v=(v_1,v_2,v_3)\in\mathbb{R}^3$. The potential function $\phi=\phi(t,x)$ is

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coupled with the distribution function $F(t,x,v)$ through the Poisson equation. The Fokker–Planck operator L_{FP} is defined by

$$L_{FP}F = \nabla_v \cdot (\nabla_v F + vF).$$

We next consider the global solutions to Eq. (1.1) near a global Maxwellian $\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}$. The perturbation $f(t,x,v)$ to μ is defined by $F = \mu + \mu^{\frac{1}{2}} f$. Then Eq. (1.1) for the perturbation $f(t,x,v)$ can be rewritten as

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f - \frac{1}{2} v \cdot \nabla_x \phi f - \nabla_x \phi \cdot v \mu^{\frac{1}{2}} = L_{FP} f, \\ \Delta_x \phi = \int_{\mathbb{R}^3} \mu^{\frac{1}{2}} f dv, \qquad \lim_{|x| \to \infty} \phi(t,x) = 0, \end{cases} \tag{1.3}$$

with initial data

$$f(0,x,v) = f_0(x,v) = \mu^{-\frac{1}{2}} (F_0 - \mu). \tag{1.4}$$

The Fokker–Planck operator is given by

$$L_{FP} f = \mu^{-\frac{1}{2}} \nabla_v \cdot [\mu \nabla_v (\mu^{-\frac{1}{2}} f)] = \Delta_v f + \frac{1}{4}(6 - |v|^2) f.$$

For any fixed (t,x), we define the v-orthogonal projection

$$P : L^2(\mathbb{R}^3) \to \text{span}\{\mu^{\frac{1}{2}}, v_j \mu^{\frac{1}{2}}\}, \quad (j = 1,2,3)$$

by

$$Pf(t,x,v) = \{a(t,x) + b(t,x) \cdot v\} \mu^{\frac{1}{2}}, \tag{1.5}$$

where $a, b = (b_1, b_2, b_3)$ are called the coefficient of the macroscopic component of Pf, and evidently, $a(t,x) = \langle \mu^{\frac{1}{2}}, f \rangle$, and $b(t,x) = \langle v \mu^{\frac{1}{2}}, f \rangle$.

For fixed (t,x), $f(t,x,v)$ can be uniquely decomposed as

$$f(t,x,v) = Pf(t,x,v) + \{I - P\} f(t,x,v), \tag{1.6}$$

where I denotes the identity operator, Pf and $\{I - P\}f$ are called the macroscopic and the microscopic component of f, respectively.

For any function $f(t,x,v)$, we denote

$$P_0 f(t,x,v) = a(t,x) \mu^{\frac{1}{2}} \quad \text{and} \quad P_1 f(t,x,v) = b(t,x) \cdot v \mu^{\frac{1}{2}},$$

then P can be written as $P = P_0 \oplus P_1$.

Notations. Throughout this paper, we assume that $N \geq 4$, and C denotes a positive constant which may change from line to line and only depends on η in some place. In addition, $A \sim B$ means that there exists a positive constant $c > 0$ such that $cB \leq A \leq \frac{1}{c} B$. We use $\langle \cdot, \cdot \rangle$ to denote the standard L^2 inner product in $\mathbb{R}^3_v$, and $(\cdot, \cdot)$ to denote the L^2 inner product in $\mathbb{R}^3_x \times \mathbb{R}^3_v$ or $\mathbb{R}^3_x$. The corresponding norms are denoted by $|\cdot|_2$ and $\|\cdot\|$, respectively. Let the multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, and $\beta = (\beta_1, \beta_2, \beta_3)$, we denote $\partial^\alpha_\beta = \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2} \partial^{\alpha_3}_{x_3} \partial^{\beta_1}_{v_1} \partial^{\beta_2}_{v_2} \partial^{\beta_3}_{v_3}$. The length of α and β denote by $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $|\beta| = \beta_1 + \beta_2 + \beta_3$, respectively. $\beta \leq \alpha$ means that $\beta_j \leq \alpha_j$ for $j = 1,2,3$, while $\beta < \alpha$ means that $\beta \leq \alpha$ and $|\beta| < |\alpha|$. We use C_α^β to denote the usual binomial coefficient. Finally, we use H^N to denote the Sobolev space $H^N(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ or $H^N(\mathbb{R}_x^3)$.

For the velocity weight function $\nu = \nu(v)$ is denoted by $\nu(v) = 1 + |v|^2$, we define

$$|f|_\nu^2 = \int_{\mathbb{R}^3} \Big(\nu(v)|f|^2 + |\nabla_v f|^2\Big)dx, \qquad \|f\|_\nu^2 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Big(\nu(v)|f|^2 + |\nabla_v f|^2\Big)dxdv.$$

The instant energy functional

$$\mathcal{E}_N(f(t)) \sim \sum_{|\alpha|+|\beta|\le N} \|\partial_\beta^\alpha f(t)\|^2 + \|\nabla_x \phi(t)\|_{H^N}^2, \tag{1.7}$$

and the dissipation rate

$$\mathcal{D}_N(f(t)) = \sum_{0<|\alpha|\le N} \|\partial^\alpha Pf(t)\|^2 + \|b\|^2 + \sum_{|\alpha|+|\beta|\le N} \|\partial_\beta^\alpha \{I-P\}f(t)\|_\nu^2 + \|\nabla_x \phi(t)\|_{H^N}^2. \tag{1.8}$$

In the following, we define the space $Z_1 = L^2(\mathbb{R}_v^3; L^1(\mathbb{R}_x^3))$ with the norm

$$\|f\|_{Z_1} = \Big(\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3} |f(x,v)|dx\Big)^2 dv\Big)^{\frac{1}{2}}.$$

For any integrable function $f : \mathbb{R}^3 \to \mathbb{R}$, we define the Fourier transform as follows

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^3} e^{-2\pi i x\cdot\xi} f(x)dx,$$

where $x \cdot \xi = \sum_{j=1}^3 x_j \xi_j$, for $\xi \in \mathbb{R}^3$, and $i = \sqrt{-1} \in \mathbb{C}$ is the imaginary unit. And the dot product $a \cdot \bar{b} = (a \mid b)$ for any $a, b \in \mathbb{C}^3$.

Our main result is stated as follows.

Theorem 1.1 *Let* $F_0(x,v) = \mu + \mu^{\frac{1}{2}} f_0(x,v) \ge 0$, *suppose that* $\mathcal{E}_N(f_0)$ *is small enough. Then the system* (1.3) *has a unique global smooth solution* $f(t,x,v)$, *which satisfies*

$$f(t,x,v) \in C([0,\infty); H^N(\mathbb{R}^3 \times \mathbb{R}^3)), \quad F(t,x,v) = \mu + \mu^{\frac{1}{2}} f(t,x,v) \ge 0, \tag{1.9}$$

and the Lyapunov-type inequality

$$\frac{d}{dt}\mathcal{E}_N(f(t)) + \lambda \mathcal{D}_N(f(t)) \le 0 \quad \textit{for any } t \ge 0. \tag{1.10}$$

Moreover, if we further assume that $\|f_0\|_{Z_1}$ *is bounded, then the algebraic decay rate*

$$\mathcal{E}_N(f(t)) \le C(\mathcal{E}_N(f_0) + \|f_0\|_{Z_1}^2)(1+t)^{-\frac{3}{2}}$$

holds for any $t \ge 0$.

Remark 1.1 In [17] and [18], the authors obtained the following Lyapunov-type inequality

$$\frac{d}{dt}\mathcal{E}_N(t) + \mathcal{D}_N(t) \le 0.$$

Since $\mathcal{E}_N(t) \le C\mathcal{D}_N(t)$, it follows that

$$\mathcal{E}_N(t) \le \mathcal{E}_N(0)e^{-Ct},$$

where $\mathcal{E}_N(t) \sim \sum_{|\alpha|\le N}(\|\partial^\alpha f\|^2 + \|\partial^\alpha \nabla\phi\|^2)$ in [17], and $\mathcal{E}_N(t) \sim \sum_{|\alpha|\le N}\|\partial^\alpha f\|^2 + \|\nabla\phi\|^2$ in [18]. For the case of $|\beta| \ne 0$, the method to deal with problems is more complicated. Moreover, our decay result is obtained by using the Fourier analysis, and their is just a direct result of Lyapunov-type inequality.

Remark 1.2 By the elliptic theory, one has

$$\|\nabla_x\phi\|_{H^N}^2 \ge C\|a\|^2,$$

it follows from (1.7) and (1.8) that there exist constants $C_1 > 0$ and $C_2 > 0$, such that

$$\mathcal{D}_N(f(t)) \ge C_1 \sum_{|\alpha|+|\beta|\le N} \|\partial_\beta^\alpha f(t)\|^2 + \|\nabla_x\phi(t)\|_{H^N}^2 \ge C_2\mathcal{E}_N(f(t)).$$

By virtue of (1.10), we can get the same result as [17] and [18].

At the end of this section, we briefly review the existence theory for the Vlasov–Poisson–Fokker–Planck (VPFP for short) system. For this system, there have been many literatures on the global existence of weak solutions, classic solutions, regular solutions, smooth solutions and time-periodic solutions and so on. For example, Carrillo [5] and Victory [22], they constructed the global weak solutions to the VPFP system. For the classical solutions, Hwang and Jang [17] established the global existence and the exponential time decay to the VPFP system by taking advantage of the standard energy method [14]. And in the relativistic sense, Luo and Yu [18] also constructed global solutions of the VPFP system and obtained exponential time decay by using a new energy method developed by Yang and Yu [25–28] through the combination of the Kawashima compensating function and the standard energy method. In [19], Ono established the global existence of regular solutions to the VPFP system. In [2], Bouchut proved the existence and uniqueness of global smooth solutions in $L^1(\mathbb{R}^3)$ and obtained the smoothing effect in [3]. In [11], Duan and Liu studied the existence and uniqueness of the time-periodic solutions to the VPFP system by using Serrin's method. Besides the results mentioned above, the asymptotic behavior and the long-time behavior of solutions to the VPFP system, we can refer to [1, 4, 7] and [21]. For other topics related to the VPFP system, the interested readers can also refer to [6, 9, 10, 13, 16, 20, 23] and references therein.

In this paper, motivated by [12], and by using the refined energy method, which is based on the macro–micro decomposition near global Maxwellians, we can also get the global classical solutions of the VPFP system, and present the algebraic time decay of solutions which is different from the exponential decay results in Luo et al. [18] and Hwang et al. [17]. Compared with [18], we didn't use the Kawashima compensating function, and with [17], the instant energy functional and the dissipation only included the pure spatial derivatives, but we contained the spatial and the velocity derivatives. In such case, we will deal with the complex space-velocity-mixed derivatives estimate. Moreover, it should be pointed out that the time rate of convergence to equilibrium is an important topic in the mathematical theory of the physical background. As Villani [24] said that there exist general structures in which the interaction between a conservative part and a degenerate dissipative part lead to convergence to equilibrium, where this property was called *hypocoercivity*. In Theorem 1.1, we give a concrete example of the hypocoercivity property for the VPFP system in the framework of perturbations.

Finally, before concluding this section, we simply sketch the main ideas used in obtaining our results. Using the macro–micro decomposition and the dissipative properties of L_{FP}, one can get the weighted energy estimates which are the estimates of the microscopic component, and the estimate of the macroscopic component can be obtained by defining a temporal energy functional. It is worth pointing out that we also need to the dissipation of $\nabla_x\phi$, which is different from the reference [12]. Therefore, with the help of the uniform-in-time estimate, the global existence of solutions can be proved by employing the standard continuity argument. On the other hand, we construct a linearized Cauchy problem with a non-homogeneous term to establishing the time decay of solutions by using the Fourier analysis.

The rest of this paper is organized as follows. In Sect. 2, we employ the macro–micro decomposition to obtaining the *a prior* energy estimate of the macroscopic component by defining a temporal energy functional. In Sect. 3, we list the dissipative properties of the linear Fokker–Planck operator L_{FP} and get the weighted energy estimates, which play an important role in establishing the global existence. Finally, we devote ourselves to obtaining the global existence and the algebraic rate of convergence of solutions in Sect. 4 and Sect. 5, respectively.

2 Macro–Micro Decomposition

In this section, we next shall obtain the dissipation rate of the right macroscopic term. Notice that the following equivalent relation

$$\sum_{0<|\alpha|\le N} \|\partial^\alpha Pf(t)\|^2 \sim \sum_{|\alpha|\le N-1} \|\partial^\alpha \nabla_x(a,b)\|^2,$$

from Eq. (1.1), taking the velocity integration over $\mathbb{R}^3$, and using the collision invariant property, we get the following local macroscopic balance laws

$$\begin{cases} \partial_t \int_{\mathbb{R}^3} F dv + \nabla_x \cdot \int_{\mathbb{R}^3} vF dv = 0, \\ \partial_t \int_{\mathbb{R}^3} vF dv + \nabla_x \cdot \int_{\mathbb{R}^3} v\otimes vF dv - \nabla_x\phi \int_{\mathbb{R}^3} F dv + \int_{\mathbb{R}^3} vF dv = 0. \end{cases} \tag{2.1}$$

By using the perturbed expression of F and the decomposition (1.5), we obtain from the macroscopic balance laws (2.1) and $(1.3)_2$ that

$$\begin{cases} \partial_t a + \nabla_x \cdot b = 0, \\ \partial_t b + \nabla_x a + \nabla_x \cdot \langle (v\otimes v - 1)M^{\frac{1}{2}}, \{I-P\}f\rangle - \nabla_x\phi(1+a) + b = 0, \\ \Delta_x \phi = a. \end{cases} \tag{2.2}$$

Now, we can rewrite $(1.3)_1$ as

$$\begin{aligned} &\partial_t Pf + v\cdot\nabla_x Pf + \nabla_x\phi\cdot\nabla_v Pf - \frac{1}{2}v\cdot\nabla_x\phi Pf - (\nabla_x\phi - b)\cdot vM^{\frac{1}{2}} \\ &\quad = -\partial_t\{I-P\}f + \ell + r, \end{aligned} \tag{2.3}$$

where

$$\ell = -v \cdot \nabla_x \{I-P\}f + L_{FP}\{I-P\}f,$$

and

$$r = \frac{1}{2} v \cdot \nabla_x \phi \{I-P\}f - \nabla_x \phi \cdot \nabla_v \{I-P\}f.$$

Define the high-order moment function $A = (A_{jm})_{3\times 3}$ by

$$A_{jm}(f) = \langle (v_j v_m - 1)\mu^{\frac{1}{2}}, f\rangle.$$

Applying A_{jm} to (2.3), it follows from $(2.2)_1$ that

$$\begin{cases} \partial_t A_{jm}(\{I-P\}f) + \partial_j b_m + \partial_m b_j - (\partial_j \phi b_m + \partial_m \phi b_j) = A_{jm}(\ell + r), \\ \partial_t A_{jj}(\{I-P\}f) + 2\partial_j b_j - 2\partial_j \phi b_j = A_{jj}(\ell + r), \end{cases} \tag{2.4}$$

where the derivation of the system (2.4) similar to [12]. Hence, the details are omitted for simplicity.

In what follows, we introduce the temporal energy functional $\mathcal{E}_0(f)$ by

$$\mathcal{E}_0(f)(t) = \sum_{|\alpha|\le N-1} \sum_{1\le j,m\le 3} \int_{\mathbb{R}^3} \partial^\alpha(\partial_j b_m + \partial_m b_j)\partial^\alpha A_{jm}(\{I-P\}f)dx$$
$$- \sum_{|\alpha|\le N-1} \int_{\mathbb{R}^3} \partial^\alpha a \partial^\alpha \nabla \cdot b dx,$$

to obtaining the dissipation of $\|\nabla_x(a,b)\|^2_{H^{N-1}}$.

Lemma 2.1 *For smooth solutions of the system* (1.3), *we have*

$$\frac{d}{dt}\mathcal{E}_0(f)(t) + \lambda \|\nabla_x(a,b)\|^2_{H^{N-1}} \le C\|\nabla_x \phi\|^2_{H^N} + C\|b\|^2_{H^N} + C\|\{I-P\}f\|^2_{H^N}$$
$$+ C\|\nabla_x\phi\|^2_{H^N}(\|\nabla_x(a,b)\|^2_{H^{N-1}} + \|\{I-P\}f\|^2_{H^N}). \tag{2.5}$$

Proof Using integration by parts and $(2.4)_1$, we have

$$\begin{aligned} &2(\|\nabla_x \partial^\alpha b\|^2 + \|\nabla_x \cdot \partial^\alpha b\|^2) \\ &\quad = \sum_{1\le j,m\le 3} \|\partial^\alpha(\partial_j b_m + \partial_m b_j)\|^2 \\ &\quad = \sum_{1\le j,m\le 3} \int_{\mathbb{R}^3} \partial^\alpha(\partial_j b_m + \partial_m b_j)\partial^\alpha[(\partial_j\phi b_m + \partial_m \phi b_j) \\ &\qquad - \partial_t A_{jm}(\{I-P\}f) + A_{jm}(\ell + r)]\, dx \\ &\quad = -\frac{d}{dt}\sum_{1\le j,m\le 3} \int_{\mathbb{R}^3} \partial^\alpha(\partial_j b_m + \partial_m b_j)\partial^\alpha A_{jm}(\{I-P\}f)\, dx \end{aligned}$$

$$+\sum_{1\le j,m\le 3}\int_{\mathbb{R}^3}\partial^\alpha(\partial_j\partial_t b_m+\partial_m\partial_t b_j)\partial^\alpha A_{jm}(\{I-P\}f)\,dx$$

$$+\sum_{1\le j,m\le 3}\int_{\mathbb{R}^3}\partial^\alpha(\partial_j b_m+\partial_m b_j)\partial^\alpha[(\partial_j\phi b_m+\partial_m\phi b_j)+A_{jm}(\ell+r)]\,dx.$$

We denote the second and third terms of the last equal sign by I_1 and I_2, respectively. For I_1 and I_2, by the Sobolev imbedding, i.e. $L^3(\mathbb{R}^3)\hookrightarrow H^1(\mathbb{R}^3)$ and $L^6(\mathbb{R}^3)\hookrightarrow \dot{H}^1(\mathbb{R}^3)$, it holds that

$$\|\partial^\alpha(\partial_j\phi a)\|^2\le C\|\partial^{\alpha-\alpha'}\partial_j\phi\|_{L^3}^2\|\partial^{\alpha'}a\|_{L^6}^2\le C\sum_{|\alpha|\le N}\|\partial^\alpha\nabla_x\phi\|^2\sum_{|\alpha|\le N-1}\|\partial^\alpha\nabla_x a\|^2.$$

Thus, one has

$$\begin{aligned}
I_1&=-2\sum_{1\le j,m\le 3}\int_{\mathbb{R}^3}\partial^\alpha\partial_t b_j\partial^\alpha\partial_m A_{jm}(\{I-P\}f)\,dx\\
&=2\sum_{1\le j,m\le 3}\int_{\mathbb{R}^3}\partial^\alpha[\partial_j a+\sum_{1\le j,m\le 3}\partial_m A_{jm}(\{I-P\}f)\\
&\quad-\partial_j\phi+b_j-\partial_j\phi a]\partial^\alpha\partial_m A_{jm}(\{I-P\}f)\,dx\\
&\le\eta(\|\partial^\alpha\nabla_x a\|^2+\|\partial^{\alpha+1}b\|^2)+\|\partial^{\alpha+1}\nabla_x\phi\|^2+C\|\nabla_x\phi\|_{H^N}^2\|\nabla_x a\|_{H^{N-1}}^2\\
&\quad+C\|\{I-P\}f\|_{H^N}^2,
\end{aligned}$$

and

$$\begin{aligned}
I_2&\le\frac{1}{2}\sum_{1\le j,m\le 3}\|\partial^\alpha(\partial_j b_m+\partial_m b_j)\|^2+C\sum_{1\le j,m\le 3}\|\partial^\alpha(\partial_j\phi b_m+\partial_m\phi b_j)\|^2\\
&\quad+C\sum_{1\le j,m\le 3}(\|\partial^\alpha A_{jm}(\ell)\|^2+\|\partial^\alpha A_{jm}(r)\|^2)\\
&\le\frac{1}{2}\sum_{1\le j,m\le 3}\|\partial^\alpha(\partial_j b_m+\partial_m b_j)\|^2+C\|\{I-P\}f\|_{H^N}^2\\
&\quad+C\|\nabla_x\phi\|_{H^N}^2(\|\nabla_x\{I-P\}f\|_{H^{N-1}}^2+\|\nabla_x b\|_{H^{N-1}}^2),
\end{aligned}$$

where we used $(2.2)_2$, integration by parts, and the following estimates

$$\begin{aligned}
\|\partial^\alpha A_{jm}(\ell)\|&=\|A_{jm}(\partial^\alpha\ell)\|\\
&=\left\|\left\langle(v_jv_m-1)\mu^{\frac{1}{2}},-v\cdot\nabla_x\partial^\alpha\{I-P\}f+L_{FP}\partial^\alpha\{I-P\}f\right\rangle\right\|\\
&\le\left\|\left\langle-v(v_jv_m-1)\mu^{\frac{1}{2}},\nabla_x\partial^\alpha\{I-P\}f\right\rangle\right\|\\
&\quad+\left\|\left\langle L_{FP}\left((v_jv_m-1)\mu^{\frac{1}{2}}\right),\partial^\alpha\{I-P\}f\right\rangle\right\|\\
&\le C\|\nabla_x\partial^\alpha\{I-P\}f\|+C\|\partial^\alpha\{I-P\}f\|,
\end{aligned}$$

and

$$
\begin{aligned}
\|\partial^\alpha A_{jm}(r)\| &= \|A_{jm}(\partial^\alpha r)\| \\
&= \Big\|\Big\langle (v_j v_m - 1)\mu^{\frac{1}{2}}, \partial^\alpha\Big(\frac{1}{2}v\cdot\nabla_x\phi\{I-P\}f - \nabla_x\phi\cdot\nabla_v\{I-P\}f\Big)\Big\rangle\Big\| \\
&= \sum_{\alpha'\le\alpha} C_\alpha^{\alpha'}\Big\|\Big\langle (v_j v_m - 1)\mu^{\frac{1}{2}}, \frac{1}{2}v\cdot\partial^{\alpha-\alpha'}\nabla_x\phi\partial^{\alpha'}\{I-P\}f \\
&\quad - \nabla_x\partial^{\alpha-\alpha'}\phi\cdot\nabla_v\partial^{\alpha'}\{I-P\}f\Big\rangle\Big\| \\
&\le \sum_{\alpha'\le\alpha} C_\alpha^{\alpha'}\Big(\Big\|\Big\langle \frac{1}{2}v(v_j v_m - 1)\mu^{\frac{1}{2}}, \partial^{\alpha-\alpha'}\nabla_x\phi\partial^{\alpha'}\{I-P\}f\Big\rangle\Big\| \\
&\quad + \Big\|\Big\langle \nabla_v\Big((v_j v_m - 1)\mu^{\frac{1}{2}}\Big), \nabla_x\partial^{\alpha-\alpha'}\phi\partial^{\alpha'}\{I-P\}f\Big\rangle\Big\|\Big) \\
&\le C\|\nabla_x\phi\|_{H^N}\|\nabla_x\{I-P\}f\|_{H^{N-1}}.
\end{aligned}
$$

By collecting the above estimates and taking summation over $|\alpha|\le N-1$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\le N-1}\sum_{1\le j,m\le 3}\int_{\mathbb{R}^3}\partial^\alpha(\partial_j b_m + \partial_m b_j)\partial^\alpha A_{jm}(\{I-P\}f)\,dx \\
&\quad + 2\sum_{|\alpha|\le N-1}(\|\nabla_x\partial^\alpha b\|^2 + \|\nabla_x\cdot\partial^\alpha b\|^2) \\
&\le \eta\sum_{1\le|\alpha|\le N}(\|\partial^\alpha a\|^2 + \|\partial^\alpha b\|^2) + \sum_{1\le|\alpha|\le N}\|\partial^\alpha\nabla_x\phi\|^2 + C\|\{I-P\}f\|_{H^N}^2 \\
&\quad + C\|\nabla_x\phi\|_{H^N}^2(\|\nabla_x\{I-P\}f\|_{H^{N-1}}^2 + \|\nabla_x(a,b)\|_{H^{N-1}}^2).
\end{aligned}
\tag{2.6}
$$

Next, we shall give the dissipation of $\sum_{1\le|\alpha|\le N}\|\partial^\alpha a\|^2$.
Using $(2.2)_2$ again, we have

$$
\begin{aligned}
&\|\partial^\alpha\nabla_x a\|^2 \\
&\quad = \sum_{1\le j\le 3}\int_{\mathbb{R}^3}\partial^\alpha\partial_j a\partial^\alpha\partial_j a\,dx \\
&\quad = \sum_{1\le j\le 3}\int_{\mathbb{R}^3}\partial^\alpha\partial_j a\partial^\alpha\Big[-\partial_t b_j + \partial_j\phi - b_j - \sum_{1\le j,m\le 3}\partial_m A_{jm}(\{I-P\}f) + \partial_j\phi a\Big]\,dx \\
&\quad = -\frac{d}{dt}\sum_{1\le j\le 3}\int_{\mathbb{R}^3}\partial^\alpha\partial_j a\partial^\alpha b_j\,dx + \sum_{1\le j\le 3}\int_{\mathbb{R}^3}\partial^\alpha\partial_j\partial_t a\partial^\alpha b_j\,dx \\
&\qquad + \sum_{1\le j\le 3}\int_{\mathbb{R}^3}\partial^\alpha\partial_j a\partial^\alpha\Big[\partial_j\phi - b_j - \sum_{1\le j,m\le 3}\partial_m A_{jm}(\{I-P\}f) + \partial_j\phi a\Big]\,dx := \sum_{j=3}^{5} I_j.
\end{aligned}
$$

For I_3 and I_4, we have $I_3 = \frac{d}{dt}\int_{\mathbb{R}^3} \partial^\alpha a \partial^\alpha \nabla_x \cdot b\, dx$, and

$$I_4 = -\int_{\mathbb{R}^3} \partial^\alpha \partial_t a \partial^\alpha \nabla_x \cdot b\, dx = \|\partial^\alpha \nabla_x \cdot b\|^2.$$

For I_5, similar as I_1, one has

$$I_5 \le \frac{1}{2}\|\nabla_x \partial^\alpha a\|^2 + \|\partial^\alpha \nabla_x \phi\|^2 + \|\partial^\alpha b\|^2 + C\|\partial^\alpha \nabla_x \{I-P\} f\|^2 \\ + C\|\nabla_x \phi\|^2_{H^N}\|\nabla_x a\|^2_{H^{N-1}}.$$

Combining the above estimates and taking summation over $|\alpha| \le N-1$, we obtain

$$\begin{aligned} &-\frac{d}{dt}\sum_{|\alpha|\le N-1}\int_{\mathbb{R}^3} \partial^\alpha a \partial^\alpha \nabla_x \cdot b\, dx + \frac{1}{2}\sum_{1\le|\alpha|\le N}\|\partial^\alpha a\|^2 \\ &\le \|\nabla_x \phi\|^2_{H^{N-1}} + \|b\|^2_{H^N} + C\|\nabla_x\{I-P\}f\|^2_{H^{N-1}} + C\|\nabla_x \phi\|^2_{H^N}\|\nabla_x a\|^2_{H^{N-1}}. \end{aligned} \tag{2.7}$$

Therefore, the desired estimate (2.5) follows from (2.6) and (2.7). □

In what follows, we introduce an equivalent energy functional

$$\mathcal{E}(f)(t) \sim \sum_{|\alpha|+|\beta|\le N}\|\partial^\alpha_\beta f(t)\|^2 + \|\nabla_x \phi\|^2_{H^N} + \mathcal{E}_0(f)(t) - \sum_{|\alpha|\le N}\int_{\mathbb{R}^3} \partial^\alpha b \partial^\alpha \nabla_x \phi dx.$$

We now give the dissipation of $\|\nabla_x \phi\|^2_{H^N}$, let $|\alpha| \le N$, by applying ∂^α to $(2.2)_2$ and taking the inner product of the resulting equation with $\partial^\alpha \nabla_x \phi$ over $\mathbb{R}^3$ and using $(2.2)_3$, we obtain

$$\begin{aligned} \|\partial^\alpha \nabla_x \phi\|^2 &= \int_{\mathbb{R}^3} \partial_t \partial^\alpha b \partial^\alpha \nabla_x \phi dx + \int_{\mathbb{R}^3} \partial^\alpha \nabla_x a \partial^\alpha \nabla_x \phi dx + \int_{\mathbb{R}^3} \nabla_x \cdot A(\{I-P\}\partial^\alpha f)\partial^\alpha \nabla_x \phi dx \\ &\quad + \int_{\mathbb{R}^3} \partial^\alpha b \partial^\alpha \nabla_x \phi dx - \int_{\mathbb{R}^3} \partial^\alpha (a \nabla_x \phi)\partial^\alpha \nabla_x \phi dx := \sum_{i=6}^{10} I_i. \end{aligned}$$

For I_6, one has

$$I_6 = \frac{d}{dt}\int_{\mathbb{R}^3} \partial^\alpha b \partial^\alpha \nabla_x \phi dx - \int_{\mathbb{R}^3} \partial^\alpha b \partial^\alpha \nabla_x \partial_t \phi dx$$

Denote the right second term by I_6' of I_6, using $(2.2)_1$, we have

$$\begin{aligned} I_6' &= \int_{\mathbb{R}^3} \partial^\alpha \nabla_x \cdot b \partial^\alpha \Delta_x^{-1} \partial_t a dx = -\int_{\mathbb{R}^3} \partial^\alpha \nabla_x \cdot b \partial^\alpha \Delta_x^{-1} \nabla_x \cdot b dx \\ &= \int_{\mathbb{R}^3} |\nabla_x \Delta_x^{-1} \partial^\alpha \nabla_x \cdot b|^2 dx \le C\|\partial^\alpha b\|^2, \end{aligned}$$

where we used the operator $\nabla_x \Delta_x^{-1} \nabla_x \cdot$ is bounded from L^2 to L^2.

Thus,

$$I_6 \le \frac{d}{dt}\int_{\mathbb{R}^3} \partial^\alpha b \partial^\alpha \nabla_x \phi dx + C\|\partial^\alpha b\|^2.$$

For I_7, when $\alpha = 0$,

$$\int_{\mathbb{R}^3} \nabla_x a \nabla_x \phi dx \le \frac{1}{2}\|\nabla_x \phi\|^2 + \frac{1}{2}\|\nabla_x a\|^2,$$

when $1 \le |\alpha| \le N$,

$$-\int_{\mathbb{R}^3} \partial^\alpha \nabla_x a \partial^\alpha \nabla_x \phi dx = \|\partial^\alpha a\|^2.$$

For I_8, using integration by parts, we get

$$\begin{aligned}
I_8 &= \sum_{1\le j,m\le 3}\int_{\mathbb{R}^3} \partial_m A_{jm}(\{I-P\}\partial^\alpha f)\partial^\alpha \partial_j \phi dx \\
&= -\sum_{1\le j,m\le 3}\int_{\mathbb{R}^3} A_{jm}(\{I-P\}\partial^\alpha f)\partial^\alpha \partial_j \partial_m \Delta_x^{-1} a dx \\
&\le \eta\|\partial^\alpha \partial_j \partial_m \Delta_x^{-1} a\|^2 + C_\eta \|A(\{I-P\}\partial^\alpha f)\|^2 \le \eta\|\partial^\alpha a\|^2 + C_\eta\|\{I-P\}\partial^\alpha f\|^2.
\end{aligned}$$

For I_9 and I_{10}, by direct computation, one has

$$I_9 \le \eta\|\partial^\alpha \nabla_x \phi\|^2 + C_\eta\|\partial^\alpha b\|^2 \quad \text{and} \quad I_{10} \le C\|a\|_{H^N}\|\nabla_x \phi\|_{H^N}^2.$$

Collecting the above estimates, and using Young's inequality, and then taking summation over $|\alpha| \le N$, we obtain

$$\begin{aligned}
&-\frac{d}{dt}\sum_{|\alpha|\le N}\int_{\mathbb{R}^3} \partial^\alpha b \partial^\alpha \nabla_x \phi dx + \lambda\Big(\sum_{1\le|\alpha|\le N}\|\partial^\alpha a\|^2 + \|\nabla_x \phi\|_{H^N}^2\Big) \\
&\le C\|b\|_{H^N}^2 + C\|\{I-P\}f\|_{H^N}^2 + C\|a\|_{H^N}^2\|\nabla_x \phi\|_{H^N}^2.
\end{aligned} \tag{2.8}$$

Remark 2.1 Here, we need the dissipation of $\nabla_x \phi$, because the estimate (2.5) includes this term, which can't be absorbed by other terms so that we can't establish the Lyapunov-type inequality to proving the global existence of solutions to the VPFP system.

3 Energy Estimates

In this section, we shall establish the energy estimates in order to obtain the global existence of solutions. For the linear Fokker–Planck operator L_{FP}, its the dissipative properties are listed as follows, which its proofs, we can refer to [8] and [12].

Lemma 3.1 *L_{FP} is a self-adjoint operator, and the following conclusions hold*

(1) *There exists a constant $\lambda > 0$ such that the coercivity inequality*

$$-\langle L_{FP} f, f\rangle \geq \lambda |\{I-P\}f|_\nu^2 + |b|^2$$

holds.

(2) *There exists a constant $\lambda_0 > 0$ such that*

$$-\langle L_{FP} f, f\rangle \geq \lambda_0 |\{I-P_0\}f|_\nu^2.$$

Remark 3.1 Using the self-adjoint of L_{FP} and (ii), then (i) holds.

Lemma 3.2 *For smooth solutions of the system* (1.3), *we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|f\|^2 + \|\nabla_x\phi\|^2\Big) + \lambda\|\{I-P\}f\|_\nu^2 + \|b\|^2\\ &\quad\leq C\|\nabla_x\phi\|_{H^N}\Big(\|a\|^2 + \|\nabla_x(a,b)\|^2 + \|\{I-P\}f\|_\nu^2\Big).\end{aligned}\tag{3.1}$$

Proof For Eq. $(1.3)_1$, taking the inner product of it with f over $\mathbb{R}^3\times\mathbb{R}^3$ with respect to x and v, one has

$$\frac{1}{2}\frac{d}{dt}\|f\|^2 - (\nabla_x\phi\cdot v\mu^{\frac{1}{2}}, f) = \frac{1}{2}(v\cdot\nabla_x\phi f, f) + (L_{FP}f, f).\tag{3.2}$$

We now estimate terms in (3.2). For the second term on the left side of (3.2), we get

$$\begin{aligned}-(\nabla_x\phi\cdot v\mu^{\frac{1}{2}}, f) &= -\int_{\mathbb{R}^3}\nabla_x\phi b dx = \int_{\mathbb{R}^3}\phi\nabla_x\cdot b dx = -\int_{\mathbb{R}^3}\phi\partial_t a dx\\ &= -\int_{\mathbb{R}^3}\phi\Delta_x\partial_t\phi dx = \frac{1}{2}\frac{d}{dt}\|\nabla_x\phi\|^2,\end{aligned}$$

where $(2.2)_1$ and $(2.2)_3$ are used.

For the first and second terms on the right side of (3.2), using the decomposition of f in (1.6), we obtain

$$\begin{aligned}\frac{1}{2}(v\cdot\nabla_x\phi f, f) &= \frac{1}{2}(v\cdot\nabla_x\phi, |Pf|^2) + (v\cdot\nabla_x\phi Pf, \{I-P\}f)\\ &\quad+\frac{1}{2}(v\cdot\nabla_x\phi, |\{I-P\}f|^2) := \sum_{i=1}^{3} I_i.\end{aligned}\tag{3.3}$$

Next, we deal with the term I_i $(i=1,2,3)$. For I_2 and I_3, we have

$$\begin{aligned}I_2 &\leq \int_{\mathbb{R}^3}|\nabla_x\phi||Pf|_2|\{I-P\}f|_\nu dx \leq \|\nabla_x\phi\|_{L^3}\||Pf|_2\|_{L^6}\|\{I-P\}f\|_\nu\\ &\leq C\|\nabla_x\phi\|_{H^1}\|\nabla_x(a,b)\|\|\{I-P\}f\|_\nu \leq C\|\nabla_x\phi\|_{H^1}(\|\nabla_x(a,b)\|^2 + \|\{I-P\}f\|_\nu^2),\end{aligned}$$

since

$$\||Pf|_2\|_{L^6} \le |\|Pf\|_{L^6}|_2 \le C|\|\nabla_x Pf\||_2 \le C\|\nabla_x(a,b)\|,$$

where we used the Minkowski inequality and the Sobolev imbedding theorem.

And

$$I_3 \le \frac{1}{2}\|\nabla_x\phi\|_{L^\infty}\|\{I-P\}f\|_\nu^2 \le C\|\nabla_x^2\phi\|_{H^1}\|\{I-P\}f\|_\nu^2.$$

For I_1, one has

$$\begin{aligned}
I_1 &= \frac{1}{2}(v\cdot\nabla_x\phi, [(a+b\cdot v)\mu^{\frac{1}{2}}]^2) = (\nabla_x\phi, ab\cdot vvM)\\
&= \sum_{ij}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\partial_i\phi\mu ab_j v_j v_i dxdv = \sum_j\iint_{\mathbb{R}^3\times\mathbb{R}^3}\partial_j\phi\mu ab_j v_j^2 dxdv\\
&= \int_{\mathbb{R}^3}\nabla_x\phi\cdot badx\int_{\mathbb{R}^3}\frac{|v|^2}{3}Mdv = \int_{\mathbb{R}^3}\nabla_x\phi\cdot badx\\
&\le \|\nabla_x\phi\|_{L^3}\|b\|_{L^6}\|a\| \le C\|\nabla_x\phi\|_{H^1}\|\nabla_x b\|\|a\| \le C\|\nabla_x\phi\|_{H^1}(\|\nabla_x b\|^2+\|a\|^2).
\end{aligned}$$

By Lemma 3.1, we see that

$$-(L_{FP}f, f) \ge \lambda\|\{I-P\}f\|_\nu^2 + \|b\|^2.$$

Combining the above estimates, we obtain the desired estimated (3.1). This completes the proof of Lemma 3.2. □

Lemma 3.3 *For smooth solutions of the system* (1.3), *we have*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{1\le|\alpha|\le N}(\|\partial^\alpha f\|^2+\|\nabla_x\partial^\alpha\phi\|^2)+\lambda\sum_{1\le|\alpha|\le N}\|\partial^\alpha\{I-P\}f\|_\nu^2+\sum_{1\le|\alpha|\le N}\|\partial^\alpha b\|^2\\
&\quad\le C\|\nabla_x\phi\|_{H^N}(\sum_{1\le|\alpha|\le N}\|\partial^\alpha\{I-P\}f\|_\nu^2+\sum_{1\le|\alpha|\le N-1}\|\partial^\alpha\nabla_v\{I-P\}f\|^2\\
&\qquad+\|\nabla_x(a,b)\|_{H^{N-1}}^2).
\end{aligned} \tag{3.4}$$

Proof Applying $\partial^\alpha(1\le|\alpha|\le N)$ to Eq. $(1.3)_1$, we get

$$\begin{aligned}
&\partial_t\partial^\alpha f + v\cdot\nabla_x\partial^\alpha f + \nabla_x\phi\cdot\nabla_v\partial^\alpha f - \partial^\alpha(\frac{1}{2}v\cdot\nabla_x\phi f)\\
&\quad-\nabla_x\partial^\alpha\phi\cdot v\mu^{\frac{1}{2}} + \sum_{\alpha'<\alpha}C_\alpha^{\alpha'}\partial^{\alpha-\alpha'}\nabla_x\phi\cdot\nabla_v\partial^{\alpha'}f = L_{FP}\partial^\alpha f.
\end{aligned} \tag{3.5}$$

Taking the inner product of (3.5) with $\partial^\alpha f$ over $\mathbb{R}^3\times\mathbb{R}^3$, similarly, one has

$$(-\nabla_x\partial^\alpha\phi\cdot v\mu^{\frac{1}{2}}, \partial^\alpha f) = (-\nabla_x\partial^\alpha\phi, \partial^\alpha b) = \frac{1}{2}\frac{d}{dt}\|\nabla_x\partial^\alpha\phi\|^2.$$

For the fourth term of the left side of (3.5), using the decomposition of f, we have

$$\begin{aligned}
&(\partial^\alpha(\frac{1}{2}v\cdot\nabla_x\phi f),\partial^\alpha f)\\
&\quad=(\partial^\alpha(\frac{1}{2}v\cdot\nabla_x\phi Pf),\partial^\alpha f)\\
&\qquad+(\partial^\alpha(\frac{1}{2}v\cdot\nabla_x\phi\{I-P\}f),\partial^\alpha Pf)+(\partial^\alpha(\frac{1}{2}v\cdot\nabla_x\phi\{I-P\}f),\partial^\alpha\{I-P\}f).
\end{aligned}\tag{3.6}$$

The terms of (3.6) can be estimated as follows

$$\sum_{|\alpha'|\le|\alpha|}C_\alpha^{\alpha'}(\frac{1}{2}v\cdot\nabla_x\partial^{\alpha-\alpha'}\phi\partial^{\alpha'}Pf,\partial^\alpha f)\le C\sum_{|\alpha'|\le|\alpha|}\int_{\mathbb{R}^3}|\nabla_x\partial^{\alpha-\alpha'}\phi||\partial^{\alpha'}(a,b)||\partial^\alpha f|_\nu dx:=I_4.$$

When $|\alpha'|\le\frac{N}{2}$, one has

$$\begin{aligned}
I_4&\le\sup_{x\in\mathbb{R}^3}|\partial^{\alpha'}(a,b)|\|\nabla_x\partial^{\alpha-\alpha'}\phi\|\|\partial^\alpha f\|_\nu\\
&\le C\|\nabla_x\phi\|_{H^N}\sum_{1\le|\alpha|\le N}(\|\partial^\alpha\{I-P\}f\|_\nu^2+\|\partial^\alpha(a,b)\|^2),
\end{aligned}$$

where the Sobolev imbedding is used. When $|\alpha'|\ge\frac{N}{2}$, we similarly obtain

$$\begin{aligned}
I_4&\le\sup_{x\in\mathbb{R}^3}|\nabla_x\partial^{\alpha-\alpha'}\phi|\|\partial^{\alpha'}(a,b)\|\|\partial^\alpha f\|_\nu\\
&\le C\|\nabla_x\phi\|_{H^N}\sum_{1\le|\alpha|\le N}(\|\partial^\alpha\{I-P\}f\|_\nu^2+\|\partial^\alpha(a,b)\|^2).
\end{aligned}$$

For the second term on the right side of (3.6), using the same procedure as above, we get

$$\begin{aligned}
&(\partial^\alpha(\frac{1}{2}v\cdot\nabla_x\phi\{I-P\}f),\partial^\alpha Pf)\\
&\quad\le C\sum_{|\alpha'|\le|\alpha|}\int_{\mathbb{R}^3}|\nabla_x\partial^{\alpha-\alpha'}\phi||\partial^{\alpha'}\{I-P\}f|_\nu|\partial^\alpha(a,b)|dx\\
&\quad\le C\|\nabla_x\phi\|_{H^N}\sum_{1\le|\alpha|\le N}(\|\partial^\alpha\{I-P\}f\|_\nu^2+\|\partial^\alpha(a,b)\|^2),
\end{aligned}$$

and

$$\begin{aligned}
&(\partial^\alpha(\frac{1}{2}v\cdot\nabla_x\phi\{I-P\}f),\partial^\alpha\{I-P\}f)\\
&\quad\le C\sum_{|\alpha'|\le|\alpha|}\int_{\mathbb{R}^3}|\nabla_x\partial^{\alpha-\alpha'}\phi||\partial^{\alpha'}\{I-P\}f|_\nu|\partial^\alpha\{I-P\}f|_\nu dx\\
&\quad\le C\|\nabla_x\phi\|_{H^N}\sum_{1\le|\alpha|\le N}\|\partial^\alpha\{I-P\}f\|_\nu^2.
\end{aligned}$$

Thus, we obtain from the above estimates that

$$(\partial^\alpha(\frac{1}{2}v\cdot\nabla_x\phi f),\partial^\alpha f)\le C\|\nabla_x\phi\|_{H^N}\sum_{1\le|\alpha|\le N}(\|\partial^\alpha\{I-P\}f\|_\nu^2+\|\partial^\alpha(a,b)\|^2).$$

For the last term on the left side of (3.5), one has

$$\begin{aligned}&\sum_{\alpha'<\alpha}C_\alpha^{\alpha'}(\partial^{\alpha-\alpha'}\nabla_x\phi\cdot\nabla_v\partial^{\alpha'}f,\partial^\alpha f)\\&\quad=\sum_{\alpha'<\alpha}C_\alpha^{\alpha'}(\partial^{\alpha-\alpha'}\nabla_x\phi\cdot\nabla_v\partial^{\alpha'}Pf,\partial^\alpha f)\\&\qquad+\sum_{\alpha'<\alpha}C_\alpha^{\alpha'}(\partial^{\alpha-\alpha'}\nabla_x\phi\cdot\nabla_v\partial^{\alpha'}\{I-P\}f,\partial^\alpha f):=I_5+I_6.\end{aligned}$$

For I_5 and I_6, similarly, we have

$$\begin{aligned}I_5&\le\int_{\mathbb{R}^3}|\partial^{\alpha-\alpha'}\nabla_x\phi||\partial^{\alpha'}(a,b)||\partial^\alpha f|_2dx\\&\le C\|\nabla_x\phi\|_{H^N}(\sum_{1\le|\alpha|\le N}\|\partial^\alpha\{I-P\}f\|_\nu^2+\|\nabla_x(a,b)\|_{H^{N-1}}^2),\end{aligned}$$

and

$$\begin{aligned}I_6&\le\int_{\mathbb{R}^3}|\partial^{\alpha-\alpha'}\nabla_x\phi||\nabla_v\partial^{\alpha'}\{I-P\}f|_2|\partial^\alpha f|_2dx\\&\le C\|\nabla_x\phi\|_{H^N}(\sum_{1\le|\alpha'|\le N-1}\|\partial^{\alpha'}\nabla_v\{I-P\}f\|^2+\sum_{1\le|\alpha|\le N}\|\partial^\alpha\{I-P\}f\|_\nu^2\\&\quad+\|\nabla_x(a,b)\|_{H^{N-1}}^2).\end{aligned}$$

By Lemma 3.1, one has

$$-(L_{FP}\partial^\alpha f,\partial^\alpha f)\ge\lambda\|\{I-P\}\partial^\alpha f\|_\nu^2+\|\partial^\alpha b\|.$$

Collecting the above estimates, and then taking summation over $1\le|\alpha|\le N$, which yields the desired estimate (3.4). We have thus completed the proof of Lemma 3.3. □

Remark 3.2 From Lemmas 3.2 and 3.3, we can obtain the dissipation of $\|b\|_{H^N}^2$.

Lemma 3.4 *Let* $1\le k\le N$, *for smooth solutions of the system* (1.3), *we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|+|\beta|\le N,|\beta|=k}\|\partial_\beta^\alpha\{I-P\}f\|^2+\lambda\sum_{|\alpha|+|\beta|\le N,|\beta|=k}\|\partial_\beta^\alpha\{I-P\}f\|_\nu^2\\&\quad\le C\|\nabla_x\phi\|_{H^N}(\sum_{|\alpha|+|\beta|\le N}\|\partial_\beta^\alpha\{I-P\}f\|^2+\|\nabla_x(a,b)\|_{H^{N-1}}^2)\\&\qquad+C\sum_{|\alpha|\le N-k+1}\|\partial^\alpha\{I-P\}f\|_\nu^2+C\sum_{|\alpha|\le N-k}\|\partial^\alpha\nabla_x(a,b)\|^2\\&\qquad+C\chi_{\{2\le k\le N\}}\sum_{|\alpha|+|\beta|\le N,1\le|\beta|\le k-1}\|\partial_\beta^\alpha\{I-P\}f\|_\nu^2,\end{aligned}\tag{3.7}$$

where χ_A *denotes the characteristic function of a set* A.

Proof Applying the microscopic projection $\{I-P\}$ to Eq. (1.3), one has

$$\partial_t\{I-P\}f+\{I-P\}v\cdot\nabla_x f+\{I-P\}(\nabla_x\phi\cdot\nabla_v f-\frac{1}{2}v\cdot\nabla_x\phi f)=L_{FP}\{I-P\}f,$$

where $\{I-P\}P=0$ and $\{I-P\}v\mu^{\frac{1}{2}}=0$ are used.

We can further obtain from the above equation that

$$\begin{aligned}
&\partial_t\{I-P\}f+v\cdot\nabla_x\{I-P\}f+\nabla_x\phi\cdot\nabla_v\{I-P\}f\\
&\quad=L_{FP}\{I-P\}f+\frac{1}{2}v\cdot\nabla_x\phi\{I-P\}f\\
&\qquad+P(v\cdot\nabla_x\{I-P\}f-\frac{1}{2}v\cdot\nabla_x\phi\{I-P\}f+\nabla_x\phi\cdot\nabla_v\{I-P\}f)\\
&\qquad-\{I-P\}(v\cdot\nabla_x Pf-\frac{1}{2}v\cdot\nabla_x\phi Pf+\nabla_x\phi\cdot\nabla_v Pf),
\end{aligned}\tag{3.8}$$

where we used $\{I-P\}L_{FP}=L_{FP}\{I-P\}$.

For $1\le k\le N$, we apply $\partial^\alpha_\beta(|\alpha|+|\beta|\le N,|\beta|=k)$ to Eq. (3.8), and take the inner product of the resulting equation with $\partial^\alpha_\beta\{I-P\}f$ over $\mathbb{R}^3\times\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta\{I-P\}f\|^2\\
&\quad=-\int_{\mathbb{R}^3}\langle\partial^\alpha_\beta(v\cdot\nabla_x\{I-P\}f+\nabla_x\phi\cdot\nabla_v\{I-P\}f),\partial^\alpha_\beta\{I-P\}f\rangle dx\\
&\qquad+\int_{\mathbb{R}^3}\langle\partial^\alpha_\beta(L_{FP}\{I-P\}f),\partial^\alpha_\beta\{I-P\}f\rangle dx+\int_{\mathbb{R}^3}\frac{1}{2}\langle\partial^\alpha_\beta(v\cdot\nabla_x\phi\{I-P\}f),\\
&\qquad\partial^\alpha_\beta\{I-P\}f\rangle dx+\int_{\mathbb{R}^3}\langle\partial^\alpha_\beta(P(v\cdot\nabla_x\{I-P\}f-\frac{1}{2}v\cdot\nabla_x\phi\{I-P\}f\\
&\qquad+\nabla_x\phi\cdot\nabla_v\{I-P\}f)),\partial^\alpha_\beta\{I-P\}f\rangle dx-\int_{\mathbb{R}^3}\langle\partial^\alpha_\beta(\{I-P\}(v\cdot\nabla_x Pf\\
&\qquad-\frac{1}{2}v\cdot\nabla_x\phi Pf+\nabla_x\phi\cdot\nabla_v Pf)),\partial^\alpha_\beta\{I-P\}f\rangle dx:=\sum_{j=7}^{11}I_j.
\end{aligned}\tag{3.9}$$

Next, we estimate terms in (3.9) one by one. For I_7, using integration by parts, we get

$$\begin{aligned}
I_7=&-\sum_{1\le|\beta'|\le|\beta|}C_\beta^{\beta'}\int_{\mathbb{R}^3}\langle\partial_{\beta'}v\cdot\nabla_x\partial^\alpha_{\beta-\beta'}\{I-P\}f,\partial^\alpha_\beta\{I-P\}f\rangle dx\\
&-\sum_{1\le|\alpha'|\le|\alpha|}C_\alpha^{\alpha'}\int_{\mathbb{R}^3}\langle\nabla_x\partial^{\alpha'}\phi\cdot\nabla_v\partial_\beta^{\alpha-\alpha'}\{I-P\}f,\partial^\alpha_\beta\{I-P\}f\rangle dx\\
:=&\,I_7'+I_7''.
\end{aligned}$$

We deal with the term I_7' and I_7'' as follows

$$I_7' \le \eta\|\partial_\beta^\alpha\{I-P\}f\|^2 + C_\eta\Big(\sum_{|\alpha|\le N-k}\|\partial^\alpha\nabla_x\{I-P\}f\|^2 + \chi_{\{2\le k\le N\}}\sum_{|\alpha|+|\beta|\le N, 1\le|\beta|\le k-1}\|\partial_\beta^\alpha\{I-P\}f\|^2\Big),$$

and

$$\begin{aligned}
I_7'' &\le C\sum_{1\le|\alpha'|\le|\alpha|}\int_{\mathbb{R}^3}|\nabla_x\partial^{\alpha'}\phi||\nabla_v\partial_\beta^{\alpha-\alpha'}\{I-P\}f|_2|\partial_\beta^\alpha\{I-P\}f|_2dx\\
&\le C\sum_{|\alpha|\le N-1}\|\partial^\alpha\nabla_x\phi\|\sum_{|\beta|=1}\sup_{x\in\mathbb{R}^3}\|\nabla_v\partial^\beta\{I-P\}f\|\|\partial_\beta^\alpha\{I-P\}f\|\\
&\quad + C\sum_{1\le|\alpha'|\le N-2}\|\partial^{\alpha'}\nabla_x\phi\|_\infty\sum_{1\le|\alpha'|\le|\alpha|}\|\nabla_v\partial_\beta^{\alpha-\alpha'}\{I-P\}f\|\|\partial_\beta^\alpha\{I-P\}f\|\\
&\le C\sum_{|\alpha|\le N-1}\|\partial^\alpha\nabla_x\phi\|\sum_{|\alpha|\le1,|\beta|=1}\|\nabla_x\nabla_v\partial^\beta\{I-P\}f\|\|\partial_\beta^\alpha\{I-P\}f\|\\
&\quad + C\sum_{2\le|\alpha|\le N}\|\partial^\alpha\nabla_x\phi\|\sum_{|\alpha|+|\beta|\le N}\|\partial_\beta^\alpha\{I-P\}f\|\|\partial_\beta^\alpha\{I-P\}f\|\\
&\le C\|\nabla_x\phi\|_{H^N}\sum_{|\alpha|+|\beta|\le N}\|\partial_\beta^\alpha\{I-P\}f\|^2.
\end{aligned}$$

For I_8, using the commutator operator, i.e. $[A,B]=AB-BA$, one has

$$\begin{aligned}
&\int_{\mathbb{R}^3}\langle\partial_\beta^\alpha L_{FP}\{I-P\}f,\partial_\beta^\alpha\{I-P\}f\rangle dx\\
&\quad=\int_{\mathbb{R}^3}\langle\partial^\alpha[\partial_\beta,-|v|^2]\{I-P\}f,\partial_\beta^\alpha\{I-P\}f\rangle dx\\
&\qquad+\int_{\mathbb{R}^3}\langle L_{FP}\partial_\beta^\alpha\{I-P\}f,\partial_\beta^\alpha\{I-P\}f\rangle dx := I_8'+I_8'',
\end{aligned}$$

where we used $[\partial_\beta, L_{FP}]=[\partial_\beta,-|v|^2]$.

For I_8' and I_8'', we obtain

$$\begin{aligned}
I_8' &\le \eta\|\partial_\beta^\alpha\{I-P\}f\|^2 + C_\eta\|[\partial_\beta,-|v|^2]\partial^\alpha\{I-P\}f\|^2\\
&\le \eta\|\partial_\beta^\alpha\{I-P\}f\|^2 + C_\eta\Big(\sum_{|\alpha|\le N-k}\|\partial^\alpha\{I-P\}f\|_v^2\\
&\qquad+\chi_{\{2\le k\le N\}}\sum_{|\alpha|+|\beta|\le N,1\le|\beta|\le k-1}\|\partial_\beta^\alpha\{I-P\}f\|_v^2\Big),
\end{aligned}$$

and

$$-I_8'' \geq \lambda_0 \|\{I-P_0\}\partial_\beta^\alpha\{I-P\}f\|_\nu^2 \geq \frac{\lambda_0}{2}\|\partial_\beta^\alpha\{I-P\}f\|_\nu^2 - \lambda_0\|P_0\partial_\beta^\alpha\{I-P\}f\|_\nu^2$$

$$\geq \frac{\lambda_0}{2}\|\partial_\beta^\alpha\{I-P\}f\|_\nu^2 - C\|\partial^\alpha\{I-P\}f\|_\nu^2,$$

where we have used Lemma 3.1.

For I_9, similar to Lemma 3.3, we have

$$\frac{1}{2}\int_{\mathbb{R}^3}\langle\partial_\beta^\alpha(v\cdot\nabla_x\phi\{I-P\}f), \partial_\beta^\alpha\{I-P\}f\rangle dx$$

$$= \frac{1}{2}\sum_{|\alpha'|\leq|\alpha|, |\beta'|\leq|\beta|} C_\alpha^{\alpha'}C_\beta^{\beta'}\int_{\mathbb{R}^3}\langle\partial_{\beta'}v\cdot\nabla_x\partial^{\alpha'}\phi\partial_{\beta-\beta'}^{\alpha-\alpha'}\{I-P\}f, \partial_\beta^\alpha\{I-P\}f\rangle dx$$

$$\leq C\|\nabla_x\phi\|_{H^N}\sum_{|\alpha|+|\beta|\leq N}\|\partial_\beta^\alpha\{I-P\}f\|^2.$$

For I_{10} and I_{11}, one has

$$I_{11} \leq \eta\|\partial_\beta^\alpha\{I-P\}f\|^2 + C_\eta\|\nabla_x(a,b)\|_{H^{N-k}}^2 + C\|\nabla_x\phi\|_{H^N}\|\nabla_x(a,b)\|_{H^{N-1}}^2,$$

and

$$I_{10} \leq \eta\|\partial_\beta^\alpha\{I-P\}f\|^2 + C_\eta\sum_{|\alpha'|\leq N-k}\|\nabla_x\partial^{\alpha'}\{I-P\}f\|^2$$

$$+ C\|\nabla_x\phi\|_{H^N}\sum_{|\alpha'|+|\beta'|\leq N}\|\partial_{\beta'}^{\alpha'}\{I-P\}f\|^2.$$

Combining the above estimates, and taking summation over $|\alpha|+|\beta|\leq N$, $|\beta|=k$, and then choosing $\eta>0$ small enough, we obtain the desired estimate (3.7). We have thus proved Lemma 3.4. □

4 Global Existence

In this section, in order to obtain the global existence and uniqueness of solutions to the system (1.3)–(1.4), we first study the local existence and uniqueness of it. The iterative sequence $\{f^n(t,x,v)\}_{n=0}^\infty$ of solutions to the following system

$$\begin{cases} \partial_t f^{n+1} + v\cdot\nabla_x f^{n+1} + \nabla_x\phi^n\cdot\nabla_v f^{n+1} = L_{FP}f^{n+1} + \frac{1}{2}v\cdot\nabla_x\phi^n f^{n+1} + \nabla_x\phi^{n+1}\cdot v\mu^{\frac{1}{2}}, \\ \Delta_x\phi^n = \int_{\mathbb{R}^3}\mu^{\frac{1}{2}}f^n dv, \quad \lim_{|x|\to\infty}\phi^n(t,x)=0, \\ f^{n+1}(0,x,v) = f_0(x,v). \end{cases} \tag{4.1}$$

Here $n\geq 0$, and $f^0=0$ is the starting value of iteration. The solution space $X(0,T;M)$ defined by

$$X(0,T;M) = \{f\in C([0,\infty); H^N(\mathbb{R}^3\times\mathbb{R}^3)) : \sup_{0\leq t\leq T}\mathcal{E}_N(f(t))\leq M, \mu+\sqrt{\mu}f\geq 0\}.$$

The main result of this section is as follows.

Theorem 4.1 *Let $N \geq 4$, there exist $\epsilon_0 > 0$, $T^* > 0$ and $M_0 > 0$ such that if $f_0 \in H^N(\mathbb{R}^3 \times \mathbb{R}^3)$ with $F_0 = \mu + \sqrt{\mu} f_0 \geq 0$ and $\mathcal{E}_N(f_0) \leq \epsilon_0$, then for each $n \geq 1$, f^n is well-defined with $f^n \in X(0, T^*; M_0)$. Moreover, one has the following conclusions*

(1) *$\{f^n\}_{n\geq 0}$ is a Cauchy sequence in the Banach space $C([0, T^*]; H^{N-1}(\mathbb{R}^3 \times \mathbb{R}^3))$.*
(2) *The corresponding limit function denoted by f belongs to $X(0, T^*; M_0)$, and f is a solution to the Cauchy problem (1.3)–(1.4).*
(3) *f is unique in $X(0, T^*; M_0)$ for the Cauchy problem (1.3)–(1.4).*

Proof Using induction, we assume that $f^n \in X(0, T^*; M_0)$ holds true for $n \geq 0$. In order to take the forthcoming calculations, one can suppose that f^n is smooth enough. Otherwise, one can study the following regularized iterative system

$$\begin{cases} \partial_t f^{n+1,\epsilon} + v \cdot \nabla_x f^{n+1,\epsilon} + \nabla_x \phi^{n,\epsilon} \cdot \nabla_v f^{n+1,\epsilon} \\ \quad = L_{FP} f^{n+1,\epsilon} + \frac{1}{2} v \cdot \nabla_x \phi^{n,\epsilon} f^{n+1,\epsilon} + \nabla_x \phi^{n+1,\epsilon} \cdot v \mu^{\frac{1}{2}}, \\ \Delta_x \phi^{n,\epsilon} = \int_{\mathbb{R}^3} \mu^{\frac{1}{2}} f^{n,\epsilon} dv, \qquad \lim_{|x|\to\infty} \phi^{n,\epsilon}(t,x) = 0, \\ f^{n+1,\epsilon}(0,x,v) = f_0^\epsilon(x,v), \end{cases}$$

for any $\epsilon > 0$, where f_0^ϵ is a smooth approximation of f_0. One can carry out the following same procedures for $f^{n,\epsilon}$ and pass to the limit by letting $\epsilon \to 0$.

Applying $\partial^\alpha (|\alpha| \leq N)$ to $(4.1)_1$, multiplying the result equation by $\partial^\alpha f^{n+1}$, and then taking integration over $\mathbb{R}^3 \times \mathbb{R}^3$ with respect to x, v, one has

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\partial^\alpha f^{n+1}\|^2 + \|\partial^\alpha \nabla_x \phi^{n+1}\|^2) + \lambda_0 \|\{I - P_0\}\partial^\alpha f^{n+1}\|_\nu^2 \\ &= \sum_{\alpha'<\alpha} C_\alpha^{\alpha'} \int_{\mathbb{R}^3} \langle \partial^{\alpha-\alpha'} \nabla_x \phi^n \cdot \nabla_v \partial^{\alpha'} f^{n+1}, \partial^\alpha f^{n+1} \rangle dx + \frac{1}{2} \int_{\mathbb{R}^3} \langle \partial^\alpha (v \cdot \nabla_x \phi^n f^{n+1}), \partial^\alpha f^{n+1} \rangle dx \\ &\leq C \|\nabla_x \phi^n\|_{H_x^N} \sum_{|\alpha|+|\beta|\leq N} \|\partial_\beta^\alpha f^{n+1}\|_\nu^2 \end{aligned}$$

By a sample calculation, and taking summation over $|\alpha| \leq N$, one has

$$\begin{aligned} &\frac{d}{dt}(\sum_{|\alpha|\leq N} \|\partial^\alpha f^{n+1}\|^2 + \|\nabla_x \phi^{n+1}\|_{H_x^N}^2) + \lambda_0 \sum_{|\alpha|\leq N} \|\partial^\alpha f^{n+1}\|_\nu^2 \\ &\quad \leq C \|\nabla_x \phi^n\|_{H_x^N} \sum_{|\alpha|+|\beta|\leq N} \|\partial_\beta^\alpha f^{n+1}\|_\nu^2 + C \|f^{n+1}\|_{L_v^2(H_x^N)}^2. \end{aligned} \tag{4.2}$$

Applying $\{I - P\}$ and $\partial_\beta^\alpha (|\alpha| + |\beta| \leq N)$ to $(4.1)_1$, successively, and then multiplying the result by $\partial_\beta^\alpha \{I - P\} f^{n+1}$, similarly as before, we obtain that

$$\begin{aligned} &\frac{d}{dt} \sum_{|\alpha|+|\beta|\leq N} \|\partial_\beta^\alpha \{I - P\} f^{n+1}\|^2 + \lambda_0 \sum_{|\alpha|+|\beta|\leq N} \|\partial_\beta^\alpha \{I - P\} f^{n+1}\|_\nu^2 \\ &\quad \leq C \|\nabla_x \phi^n\|_{H_x^N} \sum_{|\alpha|+|\beta|\leq N} \|\partial_\beta^\alpha f^{n+1}\|_\nu^2 + C \sum_{|\alpha|+|\beta|\leq N} \|\partial_\beta^\alpha \{I - P\} f^{n+1}\|_\nu^2. \end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3), one has

$$\begin{aligned}
&\frac{d}{dt}\Big(\sum_{|\alpha|\le N}\|\partial^\alpha f^{n+1}\|^2+\|\nabla_x\phi^{n+1}\|_{H_x^N}^2+\sum_{|\alpha|+|\beta|\le N}\|\partial_\beta^\alpha\{I-P\}f^{n+1}\|^2\Big)\\
&\quad+\lambda_0\Big(\sum_{|\alpha|\le N}\|\partial^\alpha f^{n+1}\|_\nu^2+\sum_{|\alpha|+|\beta|\le N}\|\partial_\beta^\alpha\{I-P\}f^{n+1}\|_\nu^2\Big)\\
&\le C\|\nabla_x\phi^n\|_{H_x^N}\sum_{|\alpha|+|\beta|\le N}\|\partial_\beta^\alpha f^{n+1}\|_\nu^2+C\sum_{|\alpha|+|\beta|\le N}\|\partial_\beta^\alpha\{I-P\}f^{n+1}\|_\nu^2+C\|f^{n+1}\|_{L_v^2(H_x^N)}^2.
\end{aligned} \tag{4.4}$$

Defining $M_n(T)=\sup_{0\le t\le T}\mathcal{E}_N(f^n(t))$, by (4.4), then for any $0\le t\le T\le T^*$, we have

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}_N(f^{n+1}(t))+\lambda_0\sum_{|\alpha|+|\beta|\le N}\|\partial_\beta^\alpha f^{n+1}\|_\nu^2\\
&\quad\le C_1\sqrt{M_n(T)}\sum_{|\alpha|+|\beta|\le N}\|\partial_\beta^\alpha f^{n+1}\|_\nu^2+C_2M_{n+1}(T),
\end{aligned} \tag{4.5}$$

where we used the fact that

$$\mathcal{E}_N(f^{n+1}(t))\sim\sum_{|\alpha|\le N}\|\partial^\alpha f^{n+1}\|^2+\|\nabla_x\phi^{n+1}\|_{H_x^N}^2+\sum_{|\alpha|+|\beta|\le N}\|\partial_\beta^\alpha\{I-P\}f^{n+1}\|^2,$$

and

$$\sum_{|\alpha|+|\beta|\le N}\|\partial_\beta^\alpha f^{n+1}\|_\nu^2\sim\sum_{|\alpha|\le N}\|\partial^\alpha f^{n+1}\|_\nu^2+\sum_{|\alpha|+|\beta|\le N}\|\partial_\beta^\alpha\{I-P\}f^{n+1}\|_\nu^2.$$

Letting $C_1\sqrt{M_n(T)}\le C_1M_0\le\frac{\lambda_0}{2}$, and taking time integration to (4.5), one has

$$M_{n+1}(T)+\frac{\lambda_0}{2}\int_0^T\sum_{|\alpha|+|\beta|\le N}\|\partial_\beta^\alpha f^{n+1}\|_\nu^2ds\le\mathcal{E}_N(f_0)+C_2TM_{n+1}(T). \tag{4.6}$$

Choosing $T^*=\frac{1}{2C_2}$ and $\epsilon_0=\frac{1}{2}M_0$, then

$$M_{n+1}(T^*)\le 2\epsilon_0^2+2C_2T^*M_{n+1}(T^*)\le\frac{1}{2}M_0^2+\frac{1}{2}M_0^2=M_0^2,$$

therefore, we obtain that $\sup_{0\le t\le T^*}\mathcal{E}_N(f^{n+1}(t))\le M_0$.

Next, similarly to (4.5), for any $0\le s\le t\le T^*$, one has

$$\begin{aligned}
|\mathcal{E}_N(f^{n+1}(t))-\mathcal{E}_N(f^{n+1}(s))|&=\Big|\int_s^t\frac{d}{d\theta}\mathcal{E}_N(f^{n+1}(\theta))d\theta\Big|\\
&\le CM_0\sum_{|\alpha|+|\beta|\le N}\int_s^t\|\partial_\beta^\alpha f^{n+1}\|_\nu^2d\theta+CM_0^2|t-s|,
\end{aligned} \tag{4.7}$$

by (4.6), $\|\partial^\alpha_\beta f^{n+1}\|^2_\nu$ is integrable over $[0,T^*]$, thus $f^n \in X(0,T^*;M_0)$ holds true for $n+1$. Therefore, by the inductive hypothesis, the conclusion holds true.

Now, we study the following system

$$\begin{cases} \partial_t(f^{n+1}-f^n) + v\cdot\nabla_x(f^{n+1}-f^n) + \nabla_x\phi^n\cdot\nabla_v(f^{n+1}-f^n) + \nabla_x(\phi^n-\phi^{n-1})\cdot\nabla_v f^n \\ \quad = L_{FP}(f^{n+1}-f^n) + \frac{1}{2}v\cdot\nabla_x\phi^n(f^{n+1}-f^n) \\ \qquad + \frac{1}{2}v\cdot\nabla_x(\phi^n-\phi^{n-1})f^n + \nabla_x(\phi^{n+1}-\phi^n)\cdot v\mu^{\frac{1}{2}}, \\ \Delta_x(\phi^n-\phi^{n-1}) = \int_{\mathbb{R}^3}\mu^{\frac{1}{2}}(f^n-f^{n-1})dv, \quad \lim_{|x|\to\infty}(\phi^n-\phi^{n-1}) = 0, \\ (f^{n+1}-f^n)(0,x,v) = 0. \end{cases}$$

Similarly to (4.5), one has

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}_{N-1}(f^{n+1}(t)-f^n(t)) + \lambda_0 \sum_{|\alpha|+|\beta|\le N-1} \|\partial^\alpha_\beta(f^{n+1}-f^n)\|^2_\nu \\ &\quad \le C\|\nabla_x\phi^n\|_{H^{N-1}_x} \sum_{|\alpha|+|\beta|\le N-1} \|\partial^\alpha_\beta(f^{n+1}-f^n)\|^2_\nu \\ &\qquad + C\|\nabla_x(\phi^n-\phi^{n-1})\|_{H^{N-1}_x} \sum_{|\alpha|+|\beta|\le N} \|\partial^\alpha_\beta f^n\|_\nu \sum_{|\alpha|+|\beta|\le N-1} \|\partial^\alpha_\beta(f^{n+1}-f^n)\|_\nu \\ &\qquad + C \sum_{|\alpha|+|\beta|\le N-1} \|\partial^\alpha_\beta(f^{n+1}-f^n)\|^2 \le C\mathcal{E}_{N-1}(f^n(t)-f^{n-1}(t)), \end{aligned}$$

where we used the Sobolev embedding $H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$. Since $\mathcal{E}_N(f_0)$, T^*, M_0 are sufficiently small, it follows from (4.6) that

$$\sup_n \int_0^{T^*} \sum_{|\alpha|+|\beta|\le N} \|\partial^\alpha_\beta f^{n+1}\|^2_\nu ds$$

is also sufficiently small. Thus, there exists a constant $\mu<1$, such that

$$\sup_{0\le t\le T^*} \mathcal{E}_{N-1}(f^{n+1}(t)-f^n(t)) \le \mu \sup_{0\le t\le T^*} \mathcal{E}_{N-1}(f^n(t)-f^{n-1}(t)), \tag{4.8}$$

which implies that $\{f^n\}_{n\ge 0}$ is a Cauchy sequence in the Banach space $C([0,T^*];H^{N-1}(\mathbb{R}^3\times\mathbb{R}^3))$. Therefore, there exists a limit function $f\in C([0,T^*];H^{N-1}(\mathbb{R}^3\times\mathbb{R}^3))$, such that f is a solution to the Cauchy problem (1.3)–(1.4) by letting $n\to\infty$. From the fact that the pointwise convergence of f^n to f by the Sobolev embedding theorem, the lower semi-continuity of the norms, and $f^n\in X(0,T^*;M_0)$, it follows that

$$F(t,x,v) = \mu + \mu^{\frac{1}{2}} f(t,x,v) \ge 0, \qquad \sup_{0\le t\le T^*} \mathcal{E}_N(f(t)) \le M_0.$$

Similarly to the proof of (4.7), one has $f\in C([0,T^*];H^N(\mathbb{R}^3\times\mathbb{R}^3))$, thus, one can conclude that $f\in X(0,T^*;M_0)$.

Finally, let $g\in X(0,T^*;M_0)$ be another solution to the Cauchy problem (1.3)–(1.4). Taking the similar process of the proof of (4.8), one has

$$\sup_{0\le t\le T^*} \mathcal{E}_N(f(t)-g(t)) \le \mu \sup_{0\le t\le T^*} \mathcal{E}_N(f(t)-g(t))$$

for $\mu<1$. Then, one can deduce that $f\equiv g$. This completes the proof of Theorem 4.1. □

In this moment, in order to obtain the uniform-in-time estimate, we assume that the Cauchy problem of the system (1.3)–(1.4) has a smooth solution $f(t,x,v)$ over $0 \le t \le T$ for $0 < T < \infty$, which satisfies

$$\sup_{0\le t\le T} \mathcal{E}_N(f(t)) \le \epsilon_0, \tag{4.9}$$

where ϵ_0 is a sufficiently small constant. Now, we can apply Lemmas 3.2–3.4 to $f(t,x,v)$ and give proof of the first part in Theorem 1.1.

Proof of global existence and uniqueness in Theorem 1.1 First, from (3.1) and (3.4), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le N}(\|\partial^\alpha f\|^2 + \|\nabla_x\partial^\alpha\phi\|^2) + \lambda\sum_{|\alpha|\le N}\|\partial^\alpha\{I-P\}f\|_\nu^2 + \lambda\sum_{|\alpha|\le N}\|\partial^\alpha b\|^2\\
&\le C\|\nabla_x\phi\|_{H^N}\Big(\sum_{|\alpha|\le N}\|\partial^\alpha\{I-P\}f\|_\nu^2 + \sum_{1\le|\alpha|\le N-1}\|\partial^\alpha\nabla_v\{I-P\}f\|^2\\
&\quad + \|\nabla_x(a,b)\|_{H^{N-1}}^2 + \|b\|^2\Big).
\end{aligned} \tag{4.10}$$

By adding $M_0\times$(2.8) to (2.5), and then adding $M_1\times$(4.10) to the resulting equation, we get

$$\begin{aligned}
&\frac{d}{dt}\Big[\frac{M_1}{2}(\|\partial^\alpha f\|^2 + \|\nabla_x\partial^\alpha\phi\|^2) + \mathcal{E}_0(f)(t) - M_0\sum_{|\alpha|\le N}\int_{\mathbb{R}^3}\partial^\alpha b\partial^\alpha\nabla_x\phi dx\Big]\\
&\quad + \lambda\sum_{|\alpha|\le N}\|\partial^\alpha b\|^2 + \lambda\sum_{|\alpha|\le N}\|\partial^\alpha\{I-P\}f\|_\nu^2\\
&\quad + \lambda\|\nabla_x(a,b)\|_{H^{N-1}}^2 + \lambda\|\nabla_x\phi\|_{H^N}^2\\
&\le C(\epsilon_0+\sqrt{\epsilon_0})\mathcal{D}_N(f)(t) + C\sqrt{\epsilon_0}\sum_{1\le|\alpha|\le N-1}\|\partial^\alpha\nabla_v\{I-P\}f\|^2,
\end{aligned} \tag{4.11}$$

where M_0 and M_1 large enough.

On the other hand, it follows from (4.9) and the linear combination of (3.7) over $1\le k\le N$ that

$$\begin{aligned}
&\frac{d}{dt}\sum_{1\le k\le N}C_k\sum_{|\alpha|+|\beta|\le N,|\beta|=k}\|\partial_\beta^\alpha\{I-P\}f\|^2 + \lambda\sum_{|\alpha|+|\beta|\le N,|\beta|\ge 1}\|\partial_\beta^\alpha\{I-P\}f\|_\nu^2\\
&\le C\sqrt{\epsilon_0}\mathcal{D}_N(f)(t) + C\sum_{|\alpha|\le N}\|\partial^\alpha\{I-P\}f\|_\nu^2 + C\|\nabla_x(a,b)\|_{H^{N-1}}^2,
\end{aligned} \tag{4.12}$$

for some properly positive constants C_k.

By letting ϵ_0 small enough, the further linear combination of (4.12) and (4.11) yields the following Lyapunov-type inequality

$$\frac{d}{dt}\mathcal{E}_N(f(t)) + \lambda\mathcal{D}_N(f(t)) \le 0, \tag{4.13}$$

for any $0\le t\le T$.

Now, by taking time integration to (4.13), we obtain

$$\sup_{0\le t\le T}\Big(\mathcal{E}_N(f(t))+\lambda\int_0^t \mathcal{D}_N(f(s))ds\Big)\le \mathcal{E}_N(f_0).$$

By the standard continuity argument, the global existence and uniqueness of solutions to the system (1.3)–(1.4) follows from the above uniform-in-time estimate together with the local existence obtained in Theorem 4.1, and for any $t\ge 0$, (1.9)–(1.10) hold. The concrete details can refer to [14], here we omit it for simplicity. This completes the proof of global existence and uniqueness in Theorem 1.1. □

5 Time Decay

In this section, we devote ourselves to establishing the time decay of solutions $f(t,x,v)$ to the VPFP system (1.1)–(1.2). Now, we consider the following Cauchy problem

$$\begin{cases}\partial_t f=\mathbf{B}f+h, \quad t>0,\ x\in\mathbb{R}^3,\\ f(0,x,v)=f_0(x,v),\end{cases}\tag{5.1}$$

where $h=h(t,x,v)$ is given, and the linear operator $\mathbf{B}$ is defined by

$$\mathbf{B}f=L_{FP}f-v\cdot\nabla_x f+b\cdot vM^{\frac12},\quad L_{FP}f=\Delta_v f+\frac14(6-|v|^2)f.$$

If $h=0$, we denote $e^{\mathbf{B}t}$ as the solution operator to the Cauchy problem $(5.1)_1$, then the solution to the Cauchy problem (5.1) can be written as follows

$$f(t)=e^{\mathbf{B}t}f_0+\int_0^t e^{\mathbf{B}(t-s)}h(s)ds.$$

With the above preparation, we have the following decay results.

Theorem 5.1 (1) *Let $\alpha\ge\alpha'\ge 0$, and the initial data f_0 satisfies $\partial^\alpha f_0\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ and $\partial^{\alpha'}f_0\in Z_1$, we have*

$$\|\partial^\alpha e^{\mathbf{B}t}f_0\|\le C(1+t)^{-\frac{2k+3}{4}}(\|\partial^\alpha f_0\|+\|\partial^{\alpha'}f_0\|_{Z_1}),\quad \text{for any } t>0,\tag{5.2}$$

where $k=|\alpha-\alpha'|$ and the constant $C>0$ only depends on m.

(2) *Let $\alpha\ge\alpha'\ge 0$, and the non-homogeneous term h satisfies $\nu^{-\frac12}\partial^\alpha h\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ and $\nu^{-\frac12}\partial^{\alpha'}h\in Z_1$, and if further assume that*

$$\int_{\mathbb{R}^3}\mu^{\frac12}h\,dv=\int_{\mathbb{R}^3}v\mu^{\frac12}h\,dv=0,\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3.\tag{5.3}$$

Then, we have

$$\Big\|\partial^\alpha\int_0^t e^{\mathbf{B}(t-s)}h(s)ds\Big\|^2\le C\int_0^t(1+t-s)^{-\frac{2k+3}{2}}\Big(\|\nu^{-\frac12}\partial^\alpha h(s)\|^2+\|\nu^{-\frac12}\partial^{\alpha'}h(s)\|_{Z_1}^2\Big)ds,\tag{5.4}$$

for any $t>0$, where $k=|\alpha-\alpha'|$ and the constant $C>0$ only depends on k.

Remark 5.1 Notice that, it from the assumption (5.3) in (ii) follows that $Ph = 0$, which will be used later.

In what follows, we can further rewrite (5.1) as

$$\partial_t Pf + v \cdot \nabla_x Pf = \partial_t \{I - P\} f - v \cdot \nabla_x \{I - P\} f + L_{FP} f + h.$$

Similarly as before, we obtain the following macroscopic balance laws:

$$\begin{cases} \partial_t a + \nabla_x \cdot b = 0, \\ \partial_t b_j + \partial_j a + \sum_{1 \le m \le 3} \partial_m A_{jm}(\{I - P\} f) = 0, \\ \partial_j b_m + \partial_m b_j = -\partial_t A_{jm}(\{I - P\} f) + A_{jm}(\ell + h), \end{cases} \tag{5.5}$$

where $\ell = -v \cdot \nabla_x \{I - P\} f + L_{FP} f$.

Before providing proof of Theorem 5.1, by using the above balance laws, we have an important fact that the macroscopic coefficient $b = (b_1, b_2, b_3)$ satisfies an elliptic-type equation, which is initially observed in Guo [15]. Next, we describe it in the following lemma and omit its proof for brevity.

Lemma 5.1 *For* $1 \le m \le 3$, *we have*

$$\begin{aligned} &- \Delta_x b_m + \partial_t \Big[\frac{1}{2} \sum_j \partial_m A_{jj}(\{I - P\} f) - \sum_j \partial_j A_{jm}(\{I - P\} f) \Big] \\ &= \frac{1}{2} \sum_j \partial_m A_{jj}(\ell + h) - \sum_j \partial_j A_{jm}(\ell + h), \quad t \ge 0, \ x \in \mathbb{R}^3. \end{aligned} \tag{5.6}$$

Proof of Theorem 5.1 This proof is similar to Theorem 3.1 of [12], but we prove it for the convenience of the readers. By applying the Fourier transform to (5.6) with respect to x, one has

$$\begin{aligned} &|\xi|^2 \hat{b}_m + \partial_t \Big[\frac{1}{2} \sum_j i\xi_m A_{jj}(\{I - P\} \hat{f}) - \sum_j i\xi_j A_{jm}(\{I - P\} \hat{f}) \Big] \\ &= \frac{1}{2} \sum_j i\xi_m A_{jj}(\hat{\ell} + \hat{h}) - \sum_j i\xi_j A_{jm}(\hat{\ell} + \hat{h}), \end{aligned} \tag{5.7}$$

and then by taking the inner product of the resulting equation with $\bar{\hat{b}}_m$ yields

$$\begin{aligned} &|\xi|^2 |\hat{b}_m|^2 + \partial_t \Big(\frac{1}{2} \sum_j i\xi_m A_{jj}(\{I - P\} \hat{f}) - \sum_j i\xi_j A_{jm}(\{I - P\} \hat{f} \mid \hat{b}_m \Big) \\ &= \Big(\frac{1}{2} \sum_j i\xi_m A_{jj}(\hat{\ell} + \hat{h}) - \sum_j i\xi_j A_{jm}(\hat{\ell} + \hat{h}) \mid \hat{b}_m \Big) \\ &\quad + \Big(\frac{1}{2} \sum_j i\xi_m A_{jj}(\{I - P\} \hat{f}) - \sum_j i\xi_j A_{jm}(\{I - P\} \hat{f} \mid \partial_t \hat{b}_m \Big) \\ &:= I_1 + I_2. \end{aligned} \tag{5.8}$$

The estimates of I_1 and I_2 are as follows.

$$\begin{aligned} I_1 &\le \frac{1}{2}|\xi|^2|\hat{b}_m|^2 + C\sum_{jm}(|A_{jm}(\hat{\ell})|^2 + |A_{jm}(\hat{h})|^2) \\ &\le \frac{1}{2}|\xi|^2|\hat{b}_m|^2 + C(1+|\xi|^2)|\{I-P\}\hat{f}|_2^2 + C|\nu^{-\frac{1}{2}}\hat{h}|_2^2, \end{aligned}$$

where we have used the following estimates

$$\begin{aligned} |A_{jm}(\hat{\ell})| &= \Big|\int_{\mathbb{R}^3}(v_jv_m-1)\mu^{\frac{1}{2}}(-iv\cdot\xi\{I-P\}\hat{f} + L_{FP}\{I-P\}\hat{f})dv\Big| \\ &= \Big|\int_{\mathbb{R}^3}[-iv\cdot\xi(v_jv_m-1)\mu^{\frac{1}{2}} + L_{FP}((v_jv_m-1)\mu^{\frac{1}{2}})]\{I-P\}\hat{f}dv\Big| \\ &\le |-iv\cdot\xi(v_jv_m-1)\mu^{\frac{1}{2}} + L_{FP}((v_jv_m-1)\mu^{\frac{1}{2}})|_2|\{I-P\}\hat{f}|_2 \\ &\le C(1+|\xi|)|\{I-P\}\hat{f}|_2, \end{aligned}$$

and

$$|A_{jm}(\hat{h})| = \Big|\int_{\mathbb{R}^3}(v_jv_m-1)\mu^{\frac{1}{2}}\hat{h}dv\Big| \le C|\nu^{-\frac{1}{2}}\hat{h}|_2.$$

For I_2, using the Fourier transform of $(5.5)_2$, one has

$$\partial_t\hat{b}_j + i\xi_j\hat{a} + \sum_{1\le m\le 3} i\xi_m A_{jm}(\{I-P\}\hat{f}) = 0, \tag{5.9}$$

thus, we have

$$\begin{aligned} I_2 &= \Big(\frac{1}{2}\sum_j i\xi_m A_{jj}(\{I-P\}\hat{f}) - \sum_j i\xi_j A_{jm}(\{I-P\}\hat{f} \mid -i\xi_m\hat{a} - \sum_{1\le k\le 3} i\xi_k A_{mk}(\{I-P\}\hat{f})\Big) \\ &\le \eta|\xi|^2|\hat{a}|^2 + C_\eta(1+|\xi|^2)|\{I-P\}\hat{f}|_2^2. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &\frac{1}{2}|\xi|^2|\hat{b}_m|^2 + \partial_t\Big(\frac{1}{2}\sum_j i\xi_m A_{jj}(\{I-P\}\hat{f}) - \sum_j i\xi_j A_{jm}(\{I-P\}\hat{f} \mid \hat{b}_m\Big) \\ &\quad\le \eta|\xi|^2|\hat{a}|^2 + C(1+|\xi|^2)|\{I-P\}\hat{f}|_2^2 + C|\nu^{-\frac{1}{2}}\hat{h}|_2^2. \end{aligned} \tag{5.10}$$

For the dissipation of $|\xi|^2|\hat{a}|^2$, by taking the inner product of (5.9) with $-i\xi_j\bar{\hat{a}}$, we get

$$(-\partial_t i\xi\cdot\hat{b} \mid \hat{a}) + |\xi|^2|\hat{a}|^2 + \sum_{1\le j,m\le 3}\Big(i\xi_j\xi_m A_{jm}(\{I-P\}\hat{f}) \mid \hat{a}\Big) = 0 \tag{5.11}$$

For the first and third terms on the left side of (5.11), using the Fourier transform of $(5.5)_1$, one has

$$(-\partial_t i\xi\cdot\hat{b}_j \mid \hat{a}) = \partial_t(-i\xi\cdot\hat{b}_j \mid \hat{a}) + (i\xi\cdot\hat{b}_j \mid \partial_t\hat{a}) = \partial_t(-i\xi\cdot\hat{b}_j \mid \hat{a}) - |\xi\cdot\hat{b}|^2,$$

and

$$\Big|\sum_{1\le j,m\le 3}\Big(i\xi_j\xi_m A_{jm}(\{I-P\}\hat{f})\mid \hat{a}\Big)\Big|$$
$$\le \eta|\xi|^2|\hat{a}|^2+C_\eta|\xi|^2\sum_{1\le j,m\le 3}|A_{jm}(\{I-P\}\hat{f})|^2\le \eta|\xi|^2|\hat{a}|^2+C_\eta|\xi|^2|\{I-P\}\hat{f}|^2.$$

Thus, taking the real part of (5.11), we obtain

$$\partial_t Re(-i\xi\cdot\hat{b}_j\mid\hat{a})+\frac{1}{2}|\xi|^2|\hat{a}|^2\le C|\xi|^2|\hat{b}|^2+C|\xi|^2|\{I-P\}\hat{f}|^2. \tag{5.12}$$

Define

$$E(\hat{f})=\frac{3}{2}\sum_{1\le j,m\le 3}\frac{i\xi_m}{1+|\xi|^2}(A_{jj}(\{I-P\}\hat{f}\mid\hat{b}_m)$$
$$-3\sum_{1\le j,m\le 3}\frac{i\xi_j}{1+|\xi|^2}(A_{jm}(\{I-P\}\hat{f}\mid\hat{b}_m)-\frac{i\xi}{1+|\xi|^2}\cdot(\hat{b}\mid\hat{a}),$$

then, it follows from (5.11) and (5.12) that

$$\frac{d}{dt}ReE(\hat{f})+\frac{|\xi|^2}{4(1+|\xi|^2)}(|\hat{a}|^2+|\hat{b}|^2)\le C|\{I-P\}\hat{f}|^2+C|\nu^{-\frac{1}{2}}\hat{h}|_2^2. \tag{5.13}$$

On the other hand, by using $L_{FP}f=L_{FP}\{I-P\}f-b\cdot v\mu^{\frac{1}{2}}$, and taking the Fourier transform $(5.1)_1$, which yield

$$\partial_t\hat{f}+iv\cdot\xi\hat{f}=L_{FP}\{I-P\}\hat{f}+\hat{h},$$

and further taking the product of the above equation with $\bar{\hat{f}}$, we obtain

$$\frac{1}{2}\frac{d}{dt}|\hat{f}|^2+\lambda|\{I-P\}\hat{f}|_\nu^2\le|\langle\hat{h},\bar{\hat{f}}\rangle|,$$

where we used (iii) of Lemma 3.1 and $\{I-P_0\}\{I-P\}=\{I-P\}$.

For the right side term, since the condition $Ph=0$, we have

$$|\langle\hat{h},\bar{\hat{f}}\rangle|=|\langle\hat{h},\overline{\{I-P\}\hat{f}}\rangle|\le\eta|\{I-P\}\hat{f}|_\nu^2+C_\eta|\nu^{-\frac{1}{2}}\hat{h}|_2^2,$$

thus, we get

$$\frac{d}{dt}|\hat{f}|^2+\lambda|\{I-P\}\hat{f}|_\nu^2\le C|\nu^{-\frac{1}{2}}\hat{h}|_2^2. \tag{5.14}$$

By taking K large enough, we set $\widetilde{E}(\hat{f})=K|\hat{f}|^2+ReE(\hat{f})$, and combine (5.13) and (5.14), then

$$\frac{d}{dt}\widetilde{E}(\hat{f})+\lambda\Big[|\{I-P\}\hat{f}|_\nu^2+\frac{|\xi|^2}{1+|\xi|^2}(|\hat{a}|^2+|\hat{b}|^2)\Big]\le C|\nu^{-\frac{1}{2}}\hat{h}|_2^2,\quad t>0,\ \xi\in\mathbb{R}^3. \tag{5.15}$$

Notice that $\widetilde{E}(\hat{f}) \sim |\hat{f}|_2^2 \le |\{I-P\}\hat{f}|_\nu^2 + |\hat{a}|^2 + |\hat{b}|^2$, hence, we have

$$\frac{d}{dt}\widetilde{E}(\hat{f}) + \lambda \frac{|\xi|^2}{1+|\xi|^2}\widetilde{E}(\hat{f}) \le C|\nu^{-\frac{1}{2}}\hat{h}|_2^2,$$

which implies that

$$\widetilde{E}(\hat{f}) \le e^{-\lambda \frac{|\xi|^2}{1+|\xi|^2}t}\widetilde{E}(\hat{f}_0) + C\int_0^t e^{-\lambda \frac{|\xi|^2}{1+|\xi|^2}(t-s)}|\nu^{-\frac{1}{2}}\hat{h}|_2^2 ds,$$

where we used the Gronwall inequality. Finally, we get

$$|\hat{f}|_2^2 \le Ce^{-\lambda \frac{|\xi|^2}{1+|\xi|^2}t}|\hat{f}_0|_2^2 + C\int_0^t e^{-\lambda \frac{|\xi|^2}{1+|\xi|^2}(t-s)}|\nu^{-\frac{1}{2}}\hat{h}|_2^2 ds, \quad t>0,\ \xi\in\mathbb{R}^3. \tag{5.16}$$

Next, let $h=0$, from (5.16), then

$$\begin{aligned}\|\partial^\alpha e^{\mathbf{B}t} f_0\|^2 &= \int_{\mathbb{R}^3} |\xi^\alpha|^2|\hat{f}|_2^2 d\xi \le C\int_{\mathbb{R}^3} |\xi^\alpha|^2 e^{-\lambda \frac{|\xi|^2}{1+|\xi|^2}t}|\hat{f}_0|_2^2 d\xi \\ &\le C\int_{|\xi|\le 1} |\xi^{\alpha-\alpha'}|^2 e^{-\lambda \frac{|\xi|^2}{1+|\xi|^2}t}|\xi^{\alpha'}|^2|\hat{f}_0|_2^2 d\xi + C\int_{|\xi|\ge 1} |\xi^\alpha|^2 e^{-\lambda \frac{|\xi|^2}{1+|\xi|^2}t}|\hat{f}_0|_2^2 d\xi \\ &:= CI_3 + CI_4.\end{aligned}$$

For I_3, we have

$$\begin{aligned}|I_3| &\le \int_{|\xi|\le 1} |\xi^{\alpha-\alpha'}|^2 e^{-\lambda t\frac{|\xi|^2}{2}}|\xi^{\alpha'}\hat{f}_0|_2^2 d\xi \\ &\le \sup_{|\xi|\le 1}|\xi^{\alpha'}\hat{f}_0|_2^2 \int_{|\xi|\le 1} |\xi^{\alpha-\alpha'}|^2 e^{-\lambda t\frac{|\xi|^2}{2}} d\xi \le C(1+t)^{-\frac{2m+3}{2}}\|\partial^{\alpha'} f_0\|_{Z_1}^2,\end{aligned}$$

since

$$\int_{|\xi|\le 1} |\xi^{\alpha-\alpha'}|^2 e^{-\lambda t\frac{|\xi|^2}{2}} d\xi \le C(1+t)^{-\frac{2m+3}{2}},$$

and

$$\sup_{|\xi|\le 1}|\xi^{\alpha'}\hat{f}_0|_2^2 \le \sup_{\xi\in\mathbb{R}^3}|\xi^{\alpha'}\hat{f}_0|_2^2 \le |\sup_{\xi\in\mathbb{R}^3}\xi^{\alpha'}\hat{f}_0|_2^2 \le |\|\partial^{\alpha'} f_0\|_{L^1_x}|_{L^2_v}^2 = \|\partial^{\alpha'} f_0\|_{Z_1}^2,$$

where we used the property of the Fourier transform.

By a simple calculation, $I_4 \le e^{-\lambda t/2}\|\partial^\alpha f_0\|^2$.

Thus, it holds that

$$\begin{aligned}\|\partial^\alpha e^{\mathbf{B}t} f_0\|^2 &\le Ce^{-\lambda t/2}\|\partial^\alpha f_0\|^2 + C(1+t)^{-\frac{2m+3}{2}}\|\partial^{\alpha'} f_0\|_{Z_1}^2 \\ &\le C(1+t)^{-\frac{2m+3}{2}}(\|\partial^\alpha f_0\|^2 + \|\partial^{\alpha'} f_0\|_{Z_1}^2).\end{aligned}$$

On the other hand, let the initial data $f_0 = 0$, then

$$f(t) = \int_0^t e^{\mathbf{B}(t-s)} h(s) ds.$$

From (5.16), we obtain

$$\|\partial^\alpha \int_0^t e^{\mathbf{B}(t-s)} h(s) ds\|^2 \leq \int_0^t \int_{\mathbb{R}^3} |\xi^\alpha|^2 e^{-\lambda \frac{|\xi|^2}{1+|\xi|^2}(t-s)} |\nu^{-\frac{1}{2}} \hat{h}|_2^2 d\xi ds. \tag{5.17}$$

By the same argument as above, it follows from (5.17) that the estimate (5.4) holds. The proof of Theorem 5.1 is now complete. □

Proof of time decay in Theorem 1.1 Let $h = h_1 + h_2$, where

$$h_1 = \frac{1}{2} v \cdot \nabla_x \phi \{I - P\} f - \nabla_x \phi \cdot \nabla_v \{I - P\} f,$$

and

$$h_2 = \frac{1}{2} v \cdot \nabla_x \phi P f - \nabla_x \phi \cdot \nabla_v P f + (\nabla_x \phi - b) \cdot v M^{\frac{1}{2}},$$

where from integration by parts, it holds that

$$\int_{\mathbb{R}^3} \mu^{\frac{1}{2}} h_1 dv = \int_{\mathbb{R}^3} v \mu^{\frac{1}{2}} h_1 dv = 0.$$

Next, we set

$$\mathcal{E}_\infty(t) = \sup_{0 \leq s \leq t} (1+s)^{\frac{3}{2}} \mathcal{E}_N(f(s)).$$

By Theorem 5.1 and the Lyapunov-type inequality (1.10), one has

$$\begin{aligned}
\|f\|^2 &\leq (\mathcal{E}_N(f_0) + \|f_0\|_{Z_1}^2)(1+t)^{-\frac{3}{2}} + \int_0^t (1+t-s)^{-\frac{3}{2}} (\|\nu^{-\frac{1}{2}} h_1(s)\|^2 \\
&\quad + \|\nu^{-\frac{1}{2}} h_2(s)\|_{Z_1}^2) ds + \left(\int_0^t (1+t-s)^{-\frac{3}{4}} (\|h_1(s)\| + \|h_2(s)\|_{Z_1}) ds \right)^2 \\
&\leq (\mathcal{E}_N(f_0) + \|f_0\|_{Z_1}^2)(1+t)^{-\frac{3}{2}} + \int_0^t (1+t-s)^{-\frac{3}{2}} \mathcal{E}_N(f(s))^2 ds \\
&\quad + \left(\int_0^t (1+t-s)^{-\frac{3}{4}} \mathcal{E}_N(f(s)) ds \right)^2 \\
&\leq (\mathcal{E}_N(f_0) + \|f_0\|_{Z_1}^2)(1+t)^{-\frac{3}{2}} + \|f_0\|_{H^N}^2 \int_0^t (1+t-s)^{-\frac{3}{2}} \mathcal{E}_N(f(s)) ds
\end{aligned} \tag{5.18}$$

$$+\|f_0\|_{H^N}^{\frac{1}{3}}\Big(\int_0^t (1+t-s)^{-\frac{3}{4}}\mathcal{E}_N(f(s))^{\frac{5}{6}}ds\Big)^2$$

$$\le (\mathcal{E}_N(f_0)+\|f_0\|_{Z_1}^2)(1+t)^{-\frac{3}{2}}+\|f_0\|_{H^N}^2\mathcal{E}_\infty\int_0^t (1+t-s)^{-\frac{3}{2}}(1+s)^{-\frac{3}{2}}ds$$

$$+\|f_0\|_{H^N}^{\frac{1}{3}}\mathcal{E}_\infty^{\frac{5}{3}}\Big(\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{5}{4}}ds\Big)^2$$

$$\le \Big(\mathcal{E}_N(f_0)+\|f_0\|_{Z_1}^2+\|f_0\|_{H^N}^2\mathcal{E}_\infty+\|f_0\|_{H^N}^{\frac{1}{3}}\mathcal{E}_\infty^{\frac{5}{3}}\Big)(1+t)^{-\frac{3}{2}},$$

where we have used the following estimates

$$\|\nu^{-\frac{1}{2}}h_1(s)\|^2+\|\nu^{-\frac{1}{2}}h_2(s)\|_{Z_1}^2 \le \mathcal{E}_N(f(s))^2, \text{ and } \|h_1(s)\|+\|h_2(s)\|_{Z_1}\le \mathcal{E}_N(f(s)),$$

$$\int_0^t (1+t-s)^{-\frac{3}{2}}(1+s)^{-\frac{3}{2}}ds \le (1+t)^{-\frac{3}{2}}, \text{ and } \int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{5}{4}}ds \le (1+t)^{-\frac{3}{4}}.$$

From (1.10), one has

$$\frac{d}{dt}\mathcal{E}_N(f(t))+\lambda\mathcal{E}_N(f(t))\le C\|a\|^2,$$

by the Gronwall inequality, then it follows from (5.18) that

$$\mathcal{E}_\infty \le \mathcal{E}_N(f_0)+\|f_0\|_{Z_1}^2+\|f_0\|_{H^N}^2\mathcal{E}_\infty+\|f_0\|_{H^N}^{\frac{1}{3}}\mathcal{E}_\infty^{\frac{5}{3}}, \quad \text{for any } t\ge 0.$$

Since the smallness of $\|f_0\|_{H^N}$, we get

$$\sup_{t\ge 0}\mathcal{E}_\infty \le \mathcal{E}_N(f_0)+\|f_0\|_{Z_1}^2.$$

Thus, we have completed the proof of time decay in Theorem 1.1. □

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