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Riemann Problem for a 2 × 2 Hyperbolic System with Linear Damping

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Abstract In this paper, we study the Riemann problem for a 2×2 nonstrictly hyperbolic system with linear damping. We introduce the special time-dependent viscosity to obtain approximate solutions. Therefore, we solve the Riemann problem (1.1)–(1.2) by limiting viscosity approach.

Mathematics Subject Classification (2010) Primary 35L40 · Secondary 35F50

Keywords Nonstrictly hyperbolic system · Linear damping · Riemann problem · Viscosity limit · Delta shock wave solution

1 Introduction

In this paper, for a constant $\alpha > 0$, we study the Riemann problem to the following hyperbolic system of conservation laws with linear damping

$$\begin{cases} v_t + (uv)_x = 0, \\ u_t + (\frac{u^2}{2})_x = -\alpha u, \end{cases} \tag{1.1}$$

with initial data given by

$$(v(x,0), u(x,0)) = \begin{cases} (v_-, u_-), & \text{if } x < 0, \\ (v_+, u_+), & \text{if } x > 0, \end{cases} \tag{1.2}$$

for arbitrary constant states $(v_\pm, u_\pm)$. It is well known that the system (1.1) is not strictly hyperbolic with eigenvalue $\lambda = u$ and right eigenvector $\mathbf{r} = (1, 0)$. Moreover, $\nabla\lambda \cdot \mathbf{r} = 0$ and

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therefore the system is linearly degenerate. The homogeneous case of the system (1.1) is a particular case of the following triangular system of conservation laws

$$\begin{cases} v_t + (vg(u))_x = 0, \\ u_t + (f(u))_x = 0 \end{cases} \tag{1.3}$$

when $f(u) = \frac{1}{2}u^2$ and $g(u) = f'(u)$. The triangular system of conservation laws (1.3) arises in a wide variety of models in physics and engineering, see for example [12] and the references therein. For this reason, the system (1.1) has been studied by many authors and several rigorous results have been obtained for this. When $\alpha = 0$ the second equation of the system (1.1) is called the Burgers equation and it is used to model various phenomenon such as shock waves in gas dynamics and hydrodynamics turbulence [1, 2, 10, 17]. Finally, the homogeneous case of the system (1.1) is used to model the evolution of density inhomogeneities in matter in the universe [18, B. Late nonlinear stage, 3. Sticky dust].

In 1977, Korchinski [15] in his PhD thesis considered the Riemann problem for system (1.3) with $f(u) = \frac{1}{2}u^2$ and $g(u) = \frac{1}{2}f'(u)$. He motivated by some numerical results has constructed the unique Riemann solution using generalized delta functions to obtain singular shocks in the sense of distributions. This is the delta shock wave concept, which is a generalization of a classic shock wave. In 1990, LeFloch [16] established existence of weak solutions to the Cauchy problem of system (1.3) with $g(u) = f'(u)$ and $f''(u) > 0$. In 1993, Joseph [13] considered the Riemann problem for the homogeneous case of the system (1.1). His work include delta shock wave solutions. He uses a parabolic regularization system to obtained an explicit formulae of the Riemann solutions. So, he constructed the weak limit of the approximation solution and this is defined as a delta shock wave type solution. In 1994, Tan, Zhang and Zheng established in [20] the existence, uniqueness and stability of delta shock waves for a viscous perturbation of the system studied by Korchinski. In 2000, Ercole [9] obtained a delta shock solution as a limit of smooth solutions by the vanishing viscosity method for the Riemann problem to the system (1.3) with $g'(u) > 0$, $f''(u) > 0$ and $f'(u) < g(u)$. More explicitly, Ercole considered the following viscous system

$$\begin{cases} v_t + (vg(u))_x = \varepsilon t v_{xx}, \\ u_t + (f(u))_x = \varepsilon t u_{xx}. \end{cases}$$

This approach was developed by Tupciev in [21] and Dafermos in [5]. Many works related to the triangular system (1.3) can be found in the literature [6, 9, 11, 13, 16, 20, 25] and references cited therein. In 2019, Keita and Bourgault [14, Sect. 4] solved the Riemann problem to the problem (1.1)–(1.2).

Many authors have obtained explicit formulae of the delta shock wave solution by reducing the original system to a system of ordinary differential equations known as the generalized Rankine-Hugoniot conditions. In particular, in [14] the explicit solution to the problem (1.1)–(1.2) is obtained from the generalized Rankine-Hugoniot conditions (see [14, Theorem 4.2]). In this paper, we present another method to solve the Riemann problem to the problem (1.1)–(1.2) which does not require of the generalized Rankine-Hugoniot conditions to obtain the explicit delta shock wave solution. However, the explicit solution here obtained satisfies the generalized Rankine-Hugoniot conditions. As the original physical application of the homogeneous system (1.1) was deduced by the viscosity Burgers equation, this motivate us to introduce the following parabolic regularization

$$\begin{cases} v_t^\varepsilon + (u^\varepsilon v^\varepsilon)_x = \varepsilon e^{-\alpha t} v_{xx}^\varepsilon, \\ u_t^\varepsilon + (\frac{(u^\varepsilon)^2}{2})_x = \varepsilon e^{-\alpha t} u_{xx}^\varepsilon - \alpha u^\varepsilon, \end{cases} \tag{1.4}$$

where $\alpha > 0$ is a constant. In general, regularization methods are importants because one can construct an approximate solution near the Riemann solution, opening the way to further works in areas such as numerical analysis, stability of solutions and many others. Observe that when $\alpha = 0$, the second equation of system (1.4) is the viscosity Burgers equation. The parabolic regularization (1.4) is well motivated by the Burgers equation with time-dependent viscosity

$$u_t + \left(\frac{1}{2}u^2\right)_x = F(t)u_{xx}.$$

where $F(t) > 0$ for $t > 0$. The Burgers equation with time-dependent viscosity was studied as a mathematical model of the propagation of the finite-amplitude sound waves in variable-area ducts, where u is an acoustic variable, with the linear effects of changes in the duct area taken out, and the time-dependent viscosity $F(t)$ is the duct area [3, 8, 24]. The reader can find results concerning the existence, uniqueness and explicit solutions to the Burgers equation with time-dependent viscosity with suitable conditions for $F(t)$ in [3, 4, 8, 19, 23, 24, 26, 27] and references cited therein. The Burgers equation with time-dependent viscosity with linear damping was studied in [22] and their results include explicit solutions for different $F(t)$. Particularly, they have studied the case when $F(t) = e^{-\alpha t}$ with $\alpha > 0$.

The homogeneous system corresponding to the system (1.1) is invariant under uniform stretching of coordinates: $(x,t) \to (\beta x, \beta t)$ with β a constant, hence it admits self-similar solutions defined on the space-time plane and constant along straight-line rays emanating from the origen, which is important to study viscous profiles. Now, compared with the homogeneous system, the structure of solutions for the Riemann problem of (1.1) is more complicated since there is no self-similar solution in the form of $(v(x,t), u(x,t)) = (v(x/t), u(x/t))$ due to the inhomogeneity. However, we are interested in obtain viscous solutions because, in future works, this could help explore viscous profiles in some sense to the system (1.1). Thus, we motivated by the methods and ideas due to Joseph [13], in this work we employ asymptotic expansion [7] of the complementary error function to construct approximate solutions and by limiting viscosity approach to solve the Riemann problem for the system (1.1). Therefore, we show existence of weak solutions for the Riemann problem for the system (1.1). These solutions include delta shock wave solutions.

The outline of the remaining of the paper is as follows. In Sect. 2, we show an auxiliary result to be used in the construct of classical Riemann solutions and in the delta shock wave solution. In Sect. 3, we present the solution of classical Riemann problem. Finally, we study the delta shock waves in the Sect. 4.

2 Preliminaries

In this section we give an auxiliary result to be used later. Let $(v^\varepsilon(x,t), u^\varepsilon(x,t))$ be an approximate solution of problem (1.1)–(1.2) which is defined by the parabolic approximation (1.4) with initial data given by

$$(v^\varepsilon(x,0), u^\varepsilon(x,0)) = \begin{cases} (v_-, u_-), & \text{if } x < 0 \\ (v_+, u_+), & \text{if } x > 0 \end{cases} \tag{2.1}$$

for arbitrary constant states $(v_\pm, u_\pm)$.

Proposition 2.1 *Let $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ be the solution of the problem (1.4)–(2.1). Then,*

$$u^\varepsilon(x,t) = \frac{u_+ a_+^\varepsilon(x,t) + u_- a_-^\varepsilon(x,t)}{a_+^\varepsilon(x,t) + a_-^\varepsilon(x,t)} e^{-\alpha t}$$

and

$$v^\varepsilon(x,t) = \partial_x W^\varepsilon(x,t), \tag{2.2}$$

where

$$W^\varepsilon(x,t) = \frac{v_+\left(x - \frac{(1-e^{-\alpha t})}{\alpha} u_+\right) a_+^\varepsilon(x,t) + v_-\left(x - \frac{(1-e^{-\alpha t})}{\alpha} u_-\right) a_-^\varepsilon(x,t)}{a_+^\varepsilon(x,t) + a_-^\varepsilon(x,t)} + \frac{(v_+ - v_-)\left(\frac{\varepsilon(1-e^{-\alpha t})}{\alpha\pi}\right)^{1/2} \exp\left(-\frac{\alpha x^2}{4\varepsilon(1-e^{-\alpha t})}\right)}{a_+^\varepsilon(x,t) + a_-^\varepsilon(x,t)},$$

$$a_-^\varepsilon(x,t) = \frac{1}{(4\frac{\varepsilon}{\alpha}\pi(1-e^{-\alpha t}))^{1/2}} \int_{-\infty}^0 \exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})} - \frac{u_- y}{2\varepsilon}\right) dy$$

and

$$a_+^\varepsilon(x,t) = \frac{1}{(4\frac{\varepsilon}{\alpha}\pi(1-e^{-\alpha t}))^{1/2}} \int_0^\infty \exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})} - \frac{u_+ y}{2\varepsilon}\right) dy.$$

Proof Observe that if $(\widehat{v}, \widehat{u})$ solves

$$\begin{cases} \widehat{v}_t + e^{-\alpha t}\widehat{u}_x \widehat{v}_x = \varepsilon e^{-\alpha t}\widehat{v}_{xx}, \\ \widehat{u}_t + \frac{1}{2}e^{-\alpha t}(\widehat{u}_x)^2 = \varepsilon e^{-\alpha t}\widehat{u}_{xx}, \end{cases} \tag{2.3}$$

with initial condition

$$(\widehat{v}(x,0), \widehat{u}(x,0)) = \begin{cases} (v_- x, u_- x), & \text{if } x < 0 \\ (v_+ x, u_+ x), & \text{if } x > 0 \end{cases}$$

then $(v^\varepsilon, u^\varepsilon)$ defined by $(v^\varepsilon, u^\varepsilon) = (\widehat{v}_x, \widehat{u}_x e^{-\alpha t})$ solves the problem (1.4)–(2.1).

Consider the Hopf-Cole transformation [2, 10] and a modified version of it,

$$\begin{cases} \widehat{u} = -2\varepsilon \ln(S^\varepsilon), \\ \widehat{v} = C^\varepsilon e^{\frac{\widehat{u}}{2\varepsilon}}. \end{cases} \tag{2.4}$$

Then, from system (2.3) and the Hopf-Cole transformation, we have

$$\begin{cases} C_t^\varepsilon = \varepsilon e^{-\alpha t} C_{xx}^\varepsilon, \\ S_t^\varepsilon = \varepsilon e^{-\alpha t} S_{xx}^\varepsilon, \end{cases} \tag{2.5}$$

with initial condition given by

$$(C^\varepsilon(x,0), S^\varepsilon(x,0)) = \begin{cases} (v_- x e^{-\frac{u_- x}{2\varepsilon}}, e^{-\frac{u_- x}{2\varepsilon}}), & \text{if } x < 0, \\ (v_+ x e^{-\frac{u_+ x}{2\varepsilon}}, e^{-\frac{u_+ x}{2\varepsilon}}), & \text{if } x > 0. \end{cases} \tag{2.6}$$

We rewrite the solution to the problem (2.5)-(2.6) in terms of the heat kernel as

$$C^{\varepsilon}(x,t)=\frac{1}{(4\frac{\varepsilon}{\alpha}\pi(1-e^{-\alpha t}))^{1/2}}\Bigg[v_{+}\int_{0}^{\infty}y\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_{+}y}{2\varepsilon}\right)dy$$
$$+v_{-}\int_{-\infty}^{0}y\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_{-}y}{2\varepsilon}\right)dy\Bigg] \tag{2.7}$$

$$S^{\varepsilon}(x,t)=\frac{1}{(4\frac{\varepsilon}{\alpha}\pi(1-e^{-\alpha t}))^{1/2}}\Bigg[\int_{0}^{\infty}\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_{+}y}{2\varepsilon}\right)dy$$
$$+\int_{-\infty}^{0}\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_{-}y}{2\varepsilon}\right)dy\Bigg]. \tag{2.8}$$

Observe that

$$\int_{0}^{\infty}\partial_y\left(\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\right)\right)\exp\left(-\frac{u_{+}y}{2\varepsilon}\right)dy$$
$$=\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_{+}y}{2\varepsilon}\right)\Bigg|_{0}^{\infty}$$
$$+\frac{u_{+}}{2\varepsilon}\int_{0}^{\infty}\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_{+}y}{2\varepsilon}\right)dy$$
$$=-\exp\left(-\frac{x^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\right)$$
$$+\frac{u_{+}}{2\varepsilon}\int_{0}^{\infty}\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_{+}y}{2\varepsilon}\right)dy. \tag{2.9}$$

In a similar way we get

$$\int_{-\infty}^{0}\partial_y\left(\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\right)\right)\exp\left(-\frac{u_{+}y}{2\varepsilon}\right)dy$$
$$=\exp\left(-\frac{x^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\right)$$
$$+\frac{u_{-}}{2\varepsilon}\int_{0}^{\infty}\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_{+}y}{2\varepsilon}\right)dy. \tag{2.10}$$

On the other hand, we also have that

$$\int_{0}^{\infty}\partial_y\left(\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\right)\right)\exp\left(-\frac{u_{+}y}{2\varepsilon}\right)dy$$
$$=\int_{0}^{\infty}\frac{(x-y)}{2\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_{+}y}{2\varepsilon}\right)dy$$
$$=\frac{x}{2\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\int_{0}^{\infty}\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_{+}y}{2\varepsilon}\right)dy$$

$$-\frac{1}{2\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\int_0^\infty y\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_+y}{2\varepsilon}\right)dy,$$

and from (2.9) we have that

$$\begin{aligned}
&\int_0^\infty y\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_+y}{2\varepsilon}\right)dy\\
&\quad=\frac{2\varepsilon}{\alpha}(1-e^{-\alpha t})\exp\left(-\frac{x^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\right)\\
&\quad+\left(x-\frac{(1-e^{-\alpha t})}{\alpha}u_+\right)\int_0^\infty \exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_+y}{2\varepsilon}\right)dy.
\end{aligned}\tag{2.11}$$

Using (2.10), it is easy to see that

$$\begin{aligned}
&\int_{-\infty}^0 y\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_+y}{2\varepsilon}\right)dy\\
&\quad=-\frac{2\varepsilon}{\alpha}(1-e^{-\alpha t})\exp\left(-\frac{x^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\right)\\
&\quad+\left(x-\frac{(1-e^{-\alpha t})}{\alpha}u_-\right)\int_{-\infty}^0 \exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_+y}{2\varepsilon}\right)dy.
\end{aligned}\tag{2.12}$$

Also, notice that

$$\begin{aligned}
&\partial_x\left(\int_0^\infty \exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_+y}{2\varepsilon}\right)dy\right)\\
&\qquad=-\int_0^\infty \partial_y\left(\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\right)\right)\exp\left(-\frac{u_+y}{2\varepsilon}\right)dy
\end{aligned}$$

and

$$\begin{aligned}
&\partial_x\left(\int_{-\infty}^0 \exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_+y}{2\varepsilon}\right)dy\right)\\
&\qquad=-\int_{-\infty}^0 \partial_y\left(\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\right)\right)\exp\left(-\frac{u_+y}{2\varepsilon}\right)dy,
\end{aligned}$$

thus from (2.9) we get

$$\begin{aligned}
&\partial_x\left(\int_0^\infty \exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_+y}{2\varepsilon}\right)dy\right)\\
&=\exp\left(-\frac{x^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\right)-\frac{u_+}{2\varepsilon}\int_0^\infty \exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_+y}{2\varepsilon}\right)dy
\end{aligned}\tag{2.13}$$

and from (2.10),

$$\partial_x\left(\int_{-\infty}^{0}\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_+ y}{2\varepsilon}\right)dy\right)$$
$$=-\exp\left(-\frac{x^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\right)-\frac{u_-}{2\varepsilon}\int_{-\infty}^{0}\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_+ y}{2\varepsilon}\right)dy. \quad (2.14)$$

Define

$$a_-^{\varepsilon}(x,t)=\frac{1}{(4\frac{\varepsilon}{\alpha}\pi(1-e^{-\alpha t}))^{1/2}}\int_{-\infty}^{0}\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_- y}{2\varepsilon}\right)dy,$$
$$a_+^{\varepsilon}(x,t)=\frac{1}{(4\frac{\varepsilon}{\alpha}\pi(1-e^{-\alpha t}))^{1/2}}\int_{0}^{\infty}\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_+ y}{2\varepsilon}\right)dy,$$

thus, from (2.7), (2.11) and (2.12) we get

$$C^{\varepsilon}(x,t)$$
$$=v_+\left[\left(\frac{\varepsilon(1-e^{-\alpha t})}{\alpha\pi}\right)^{1/2}\exp\left(-\frac{x^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\right)+\left(x-\frac{(1-e^{-\alpha t})}{\alpha}u_+\right)a_+^{\varepsilon}(x,t)\right]$$
$$+v_-\left[-\left(\frac{\varepsilon(1-e^{-\alpha t})}{\alpha\pi}\right)^{1/2}\exp\left(-\frac{x^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}\right)+\left(x-\frac{(1-e^{-\alpha t})}{\alpha}u_-\right)a_-^{\varepsilon}(x,t)\right], \quad (2.15)$$

and from (2.8) we have

$$S^{\varepsilon}(x,t)=a_+^{\varepsilon}(x,t)+a_-^{\varepsilon}(x,t). \quad (2.16)$$

From (2.16), (2.13) and (2.14) we have

$$S_x^{\varepsilon}(x,t)=-\frac{u_+}{2\varepsilon}a_+^{\varepsilon}(x,t)-\frac{u_-}{2\varepsilon}a_-^{\varepsilon}(x,t). \quad (2.17)$$

From the Hopf-Cole transformation (2.4), we get that $v^{\varepsilon}(x,t)=\widehat{v}_x(x,t)=(C^{\varepsilon}/S^{\varepsilon})_x$ and $u^{\varepsilon}(x,t)=\widehat{u}_x(x,t)e^{-\alpha t}=-2\varepsilon e^{-\alpha t}S_x^{\varepsilon}/S^{\varepsilon}$, and using the equalities (2.15), (2.16) and (2.17), the proof is complete. □

Now, observe that

$$a_+^{\varepsilon}(x,t)$$
$$=\left(\frac{\alpha}{4\varepsilon\pi(1-e^{-\alpha t})}\right)^{1/2}\int_0^{\infty}\exp\left(-\frac{(x-y)^2}{4\frac{\varepsilon}{\alpha}(1-e^{-\alpha t})}-\frac{u_+ y}{2\varepsilon}\right)dy$$
$$=\left(\frac{\alpha}{4\varepsilon\pi(1-e^{-\alpha t})}\right)^{1/2}\exp\left(\frac{-x^2+(x-x_+(t))^2}{A_{\varepsilon}(t)}\right)\int_0^{\infty}\exp\left(\frac{-(y+x_+(t)-x)^2}{A_{\varepsilon}(t)}\right)dy$$
$$=\frac{1}{\pi^{1/2}}\exp\left(\frac{-x^2+(x-x_+(t))^2}{A_{\varepsilon}(t)}\right)\int_{(A_{\varepsilon}(t))^{-1/2}(x_+(t)-x)}^{\infty}\exp\left(-y^2\right)dy$$

where $A_\varepsilon(t) = \frac{4\varepsilon(1-e^{-\alpha t})}{\alpha}$ and $x_+(t) = \frac{(1-e^{-\alpha t})}{\alpha}u_+$. In a similar way we have

$$a_-^\varepsilon(x,t) = \frac{1}{\pi^{1/2}} \exp\left(\frac{-x^2 + (x - x_-(t))^2}{A_\varepsilon(t)}\right) \int_{(A_\varepsilon(t))^{-1/2}(x-x_-(t))}^{\infty} \exp\left(-y^2\right) dy$$

with $x_-(t) = \frac{(1-e^{-\alpha t})}{\alpha}u_-$.

Let us introduce the following temporary notation

$$I_+^{\varepsilon,t} = \int_{(A_\varepsilon(t))^{-1/2}(x_+(t)-x)}^{\infty} \exp\left(-y^2\right) dy$$

and

$$I_-^{\varepsilon,t} = \int_{(A_\varepsilon(t))^{-1/2}(x-x_-(t))}^{\infty} \exp\left(-y^2\right) dy.$$

As $\varepsilon \to 0+$, using the asymptotic expansion of complementary error function (or error function, see [7]) we have

$$I_+^{\varepsilon,t} = \begin{cases} \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{n!} \left(\frac{(A_\varepsilon(t))^{1/2}}{2(x_+(t)-x)}\right)^{2n+1} \exp\left(-\frac{(x_+(t)-x)^2}{A_\varepsilon(t)}\right), \\ \quad \text{if } x_+(t) - x > (A_\varepsilon(t))^{1/2}, \\ \frac{\pi^{1/2}}{2}, \quad \text{if } x_+(t) = x, \\ \pi^{1/2} - \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{n!} \left(\frac{(A_\varepsilon(t))^{1/2}}{2(x-x_+(t))}\right)^{2n+1} \exp\left(-\frac{(x_+(t)-x)^2}{A_\varepsilon(t)}\right), \\ \quad \text{if } x_+(t) - x < -(A_\varepsilon(t))^{1/2} \end{cases}$$

and

$$I_-^{\varepsilon,t} = \begin{cases} \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{n!} \left(\frac{(A_\varepsilon(t))^{1/2}}{2(x-x_-(t))}\right)^{2n+1} \exp\left(-\frac{(x_-(t)-x)^2}{A_\varepsilon(t)}\right), \\ \quad \text{if } x - x_-(t) > (A_\varepsilon(t))^{1/2}, \\ \frac{\pi^{1/2}}{2}, \quad \text{if } x_-(t) = x, \\ \pi^{1/2} - \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{n!} \left(\frac{(A_\varepsilon(t))^{1/2}}{2(x_-(t)-x)}\right)^{2n+1} \exp\left(-\frac{(x_-(t)-x)^2}{A_\varepsilon(t)}\right), \\ \quad \text{if } x - x_-(t) < -(A_\varepsilon(t))^{1/2} \end{cases}$$

So, we have that

$$a_+^\varepsilon(x,t) = \begin{cases} \frac{Q_+}{\pi^{1/2}} \exp\left(\frac{-x^2}{A_\varepsilon(t)}\right), & \text{if } x_+(t) - x > (A_\varepsilon(t))^{1/2}, \\ \frac{1}{2} \exp\left(\frac{-x^2}{A_\varepsilon(t)}\right), & \text{if } x_+(t) = x, \\ \exp\left(\frac{-x^2+(x_+(t)-x)^2}{A_\varepsilon(t)}\right) + \frac{Q_+}{\pi^{1/2}} \exp\left(\frac{-x^2}{A_\varepsilon(t)}\right), & \text{if } x_+(t) - x < -(A_\varepsilon(t))^{1/2} \end{cases} \tag{2.18}$$

and

$$a_-^\varepsilon(x,t) = \begin{cases} -\frac{Q_-}{\pi^{1/2}} \exp\left(\frac{-x^2}{A_\varepsilon(t)}\right), & \text{if } x - x_-(t) > (A_\varepsilon(t))^{1/2}, \\ \frac{1}{2} \exp\left(\frac{-x^2}{A_\varepsilon(t)}\right), & \text{if } x_-(t) = x, \\ \exp\left(\frac{-x^2+(x_-(t)-x)^2}{A_\varepsilon(t)}\right) - \frac{Q_-}{\pi^{1/2}} \exp\left(\frac{-x^2}{A_\varepsilon(t)}\right), & \text{if } x - x_-(t) < -(A_\varepsilon(t))^{1/2} \end{cases} \tag{2.19}$$

where

$$Q_{\pm} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{n!} \left(\frac{(A_\varepsilon(t))^{1/2}}{2(x_\pm(t) - x)} \right)^{2n+1}.$$

3 Classical Riemann Problem

In this section, we study the Riemann problem to the system (1.1) with initial data (1.2) when $u_- \le u_+$. Therefore, we are interested in analyzing the behavior of the solutions $(v^\varepsilon, u^\varepsilon)$ of the problem (1.4)–(2.1) as $\varepsilon \to 0+$.

Theorem 3.1 *Suppose $u_- \le u_+$. Let $(v^\varepsilon(x,t), u^\varepsilon(x,t))$ be the solution of the problem* (1.4)–(2.1). *Then the limit*

$$\lim_{\varepsilon \to 0+} (v^\varepsilon(x,t), u^\varepsilon(x,t)) = (v(x,t), u(x,t))$$

exists in the sense of distributions and $(v(x,t), u(x,t))$ solves (1.1)–(1.2). *Moreover, if $u_- < u_+$ then*

$$(v(x,t), u(x,t)) = \begin{cases} (v_-, u_- e^{-\alpha t}), & \text{if } x < x_-(t), \\ (0, \frac{\alpha x e^{-\alpha t}}{1 - e^{-\alpha t}}), & \text{if } x_-(t) < x < x_+(t), \\ (v_+, u_+ e^{-\alpha t}), & \text{if } x > x_+(t), \end{cases}$$

and when $u_- = u_+$ then

$$(v(x,t), u(x,t)) = \begin{cases} (v_-, u_- e^{-\alpha t}), & \text{if } x < x_-(t), \\ (v_+, u_- e^{-\alpha t}), & \text{if } x > x_-(t). \end{cases}$$

Proof Using approximations of (2.18) and (2.19), we have that

- if $x - x_-(t) < -(A_\varepsilon(t))^{1/2}$, then

$$W^\varepsilon(x,t) \approx \frac{-\frac{v_+(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}} \exp\left(\frac{-x^2}{A_\varepsilon(t)}\right) + v_-(x - x_-(t)) b_-^\varepsilon + (v_+ - v_-) \frac{(A_\varepsilon)^{1/2}}{2\pi^{1/2}} \exp(\frac{-x^2}{A_\varepsilon})}{\frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}(x_+(t) - x)} \exp(\frac{-x^2}{A_\varepsilon(t)}) + b_-^\varepsilon},$$

  where $b_-^\varepsilon(x,t) = \exp\left(\frac{-x^2 + (x_-(t) - x)^2}{A_\varepsilon(t)}\right) - \frac{1}{\pi^{1/2}} \frac{(A_\varepsilon(t))^{1/2}}{2(x_-(t) - x)} \exp\left(\frac{-x^2}{A_\varepsilon(t)}\right)$, and simplifying we have

$$W^\varepsilon(x,t) \approx \frac{v_-(x - x_-(t)) \exp\left(\frac{(x_-(t) - x)^2}{A_\varepsilon(t)}\right)}{\exp\left(\frac{(x_-(t) - x)^2}{A_\varepsilon(t)}\right) + \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}} \left(\frac{1}{x - x_-(t)} - \frac{1}{x - x_+(t)}\right)} \tag{3.1}$$

- if $x_-(t) + (A_\varepsilon(t))^{1/2} < x < x_+(t) - (A_\varepsilon(t))^{1/2}$, then

$$W^\varepsilon(x,t) \approx \frac{v_+(x - x_+(t)) C_+^\varepsilon + v_-(x - x_-(t)) C_-^\varepsilon + (v_+ - v_-) \frac{(A_\varepsilon)^{1/2}}{2\pi^{1/2}} \exp(\frac{-x^2}{A_\varepsilon})}{C_+^\varepsilon + C_-^\varepsilon},$$

where $C_{\pm}^{\varepsilon}(x,t) = \pm\frac{1}{\pi^{1/2}}\left(\frac{(A_\varepsilon(t))^{1/2}}{2(x_\pm(t)-x)} - \frac{(A_\varepsilon(t))^{3/2}}{4(x_\pm(t)-x)^3}\right)\exp\left(\frac{-x^2}{A_\varepsilon(t)}\right)$, and so

$$W^\varepsilon(x,t) \approx \frac{A_\varepsilon(t)\left(\frac{v_+}{(x_+(t)-x)^2} - \frac{v_-}{(x_-(t)-x)^2}\right)}{2\left(\frac{1}{x_+(t)-x} - \frac{1}{x_-(t)-x}\right) + A_\varepsilon(t)\left(\frac{1}{(x_-(t)-x)^3} - \frac{1}{(x_+(t)-x)^3}\right)} \tag{3.2}$$

- if $x_+(t) - x < -(A_\varepsilon(t))^{1/2}$, then

$$W^\varepsilon(x,t) \approx \frac{v_+(x - x_+(t))b_+^\varepsilon + \frac{v_-(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\exp\left(\frac{-x^2}{A_\varepsilon(t)}\right) + (v_+ - v_-)\frac{(A_\varepsilon)^{1/2}}{2\pi^{1/2}}\exp(\frac{-x^2}{A_\varepsilon})}{b_+^\varepsilon + \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}(x-x_-(t))}\exp(\frac{-x^2}{A_\varepsilon(t)})},$$

where $b_+^\varepsilon(x,t) = \exp\left(\frac{-x^2+(x_+(t)-x)^2}{A_\varepsilon(t)}\right) - \frac{1}{\pi^{1/2}}\frac{(A_\varepsilon(t))^{1/2}}{2(x-x_+(t))}\exp\left(\frac{-x^2}{A_\varepsilon(t)}\right)$, and

$$W^\varepsilon(x,t) \approx \frac{v_+(x - x_+(t))\exp\left(\frac{(x_+(t)-x)^2}{A_\varepsilon(t)}\right)}{\exp\left(\frac{(x_+(t)-x)^2}{A_\varepsilon(t)}\right) + \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\left(\frac{1}{x-x_-(t)} - \frac{1}{x-x_+(t)}\right)}. \tag{3.3}$$

So, if $u_- < u_+$ then from (3.1), (3.2) and (3.3) we have

$$\lim_{\varepsilon\to 0+} W^\varepsilon(x,t) = W(x,t) = \begin{cases} v_-(x - x_-(t)), & \text{if } x < x_-(t),\\ 0, & \text{if } x_-(t) < x < x_+(t),\\ v_+(x - x_+(t)), & \text{if } x > x_+(t),\end{cases}$$

Since $W^\varepsilon(x,t)$ is bounded on compact subsets of $\mathbb{R}\times\mathbb{R}_+ = \{(x,t) : x \in \mathbb{R}, t > 0\}$ and $W^\varepsilon(x,t) \to W(x,t)$ pointwise as $\varepsilon \to 0+$, then $W^\varepsilon(x,t) \to W(x,t)$ in the sense of distributions and so $W_x^\varepsilon(x,t)$ converges in the distributional sense to $W_x(x,t)$. From (2.2) we have that $\lim_{\varepsilon\to 0+} v^\varepsilon(x,t) = v(x,t)$ exists in the sense of distribution and

$$v(x,t) = W_x(x,t) = \begin{cases} v_-, & \text{if } x < x_-(t),\\ 0, & \text{if } x_-(t) < x < x_+(t),\\ v_+, & \text{if } x > x_+(t).\end{cases}$$

On the other hand, if $x_-(t) + (A_\varepsilon(t))^{1/2} < x < x_+(t) - (A_\varepsilon(t))^{1/2}$, then

$$u^\varepsilon(x,t) = \frac{\frac{u_+}{x_+(t)-x} + \frac{u_-}{x-x_-(t)} + \sum_{n=1}^\infty \frac{(-1)^n(2n)!(A_\varepsilon(t))^n}{n!4^n}\left(\frac{u_+}{(x_+(t)-x)^{2n+1}} + \frac{u_-}{(x-x_-(t))^{2n+1}}\right)}{\frac{1}{x_+(t)-x} + \frac{1}{x-x_-(t)} + + \sum_{n=1}^\infty \frac{(-1)^n(2n)!(A_\varepsilon(t))^n}{n!4^n}\left(\frac{1}{(x_+(t)-x)^{2n+1}} + \frac{1}{(x-x_-(t))^{2n+1}}\right)} e^{-\alpha t}. \tag{3.4}$$

If $x - x_-(t) < -(A_\varepsilon(t))^{1/2}$, then

$$u^\varepsilon(x,t) \approx \frac{\frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\left(\frac{u_+}{x_+(t)-x} - \frac{u_-}{x_-(t)-x}\right) + u_-\exp\left(\frac{(x_-(t)-x)^2}{A_\varepsilon(t)}\right)}{\exp\left(\frac{(x_-(t)-x)^2}{A_\varepsilon(t)}\right) + \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\left(\frac{1}{x-x_-(t)} - \frac{1}{x-x_+(t)}\right)} e^{-\alpha t} \tag{3.5}$$

and if $x_+(t) - x < -(A_\varepsilon(t))^{1/2}$, then

$$u^\varepsilon(x,t) \approx \frac{u_+\exp\left(\frac{(x_+(t)-x)^2}{A_\varepsilon(t)}\right) + \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\left(\frac{u_+}{x_+(t)-x} - \frac{u_-}{x_-(t)-x}\right)}{\exp\left(\frac{(x_+(t)-x)^2}{A_\varepsilon(t)}\right) + \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\left(\frac{1}{x-x_-(t)} - \frac{1}{x-x_+(t)}\right)} e^{-\alpha t}. \tag{3.6}$$

Thus, for $u_- < u_+$, from (3.4), (3.5) and (3.6) we have

$$\lim_{\varepsilon\to 0+} u^\varepsilon(x,t) = u(x,t) = \begin{cases} u_- e^{-\alpha t}, & \text{if } x < x_-(t), \\ \frac{\alpha x e^{-\alpha t}}{1-e^{-\alpha t}}, & \text{if } x_-(t) < x < x_+(t), \\ u_+ e^{-\alpha t}, & \text{if } x > x_+(t), \end{cases}$$

Since $u^\varepsilon(x,t)$ is bounded on compact subsets of $\mathbb{R}\times\mathbb{R}_+ = \{(x,t) : x\in\mathbb{R}, t>0\}$ and $u^\varepsilon(x,t)\to u(x,t)$ pointwise as $\varepsilon\to 0+$, then $u^\varepsilon(x,t)\to u(x,t)$ in the sense of distribution. Proceeding as before, when $u_- = u_+$, it is easy to see that

$$\lim_{\varepsilon\to 0+} (v^\varepsilon(x,t), u^\varepsilon(x,t)) = (v(x,t), u(x,t)) = \begin{cases} (v_-, u_- e^{-\alpha t}), & \text{if } x < x_-(t), \\ (v_+, u_- e^{-\alpha t}), & \text{if } x > x_-(t). \end{cases}$$

Finally, it is easy to see that $(v(x,t), u(x,t))$ solves (1.1) and we thus omit the details. □

Remark 3.2 Observe that $\lim_{\alpha\to 0+} \frac{\alpha x e^{-\alpha t}}{1-e^{-\alpha t}} = \frac{x}{t}$, $\lim_{\alpha\to 0+} x_\pm(t) = u_\pm t$ and

$$\lim_{\alpha\to 0+} (v(x,t), u(x,t)) = \begin{cases} (v_-, u_-), & \text{if } x < u_- t, \\ (0, x/t), & \text{if } u_- t < x < u_+ t, \\ (v_+, u_+), & \text{if } x > u_+ t, \end{cases}$$

i.e., when $\alpha\to 0+$, $(v(x,t), u(x,t))$ converges to the solution given by Joseph in [13] for homogeneous case of the system (1.1).

4 Delta Shock Wave Solutions

In this section, we study the Riemann problem to the system (1.1) with initial data (1.2) when $u_- > u_+$. In this case the solution is not bounded and the solution containing a weighted delta measure supported on a smooth curve [14]. Therefore, we introduce the following definition:

Definition 4.1 A two-dimensional weighted delta function $w(s)\delta_L$ supported on a smooth curve $L = \{(x(s), t(s)) : a < s < b\}$, for $w\in L^1((a,b))$, is defined as

$$\langle w(\cdot)\delta_L, \varphi(\cdot,\cdot)\rangle = \int_a^b w(s)\varphi(x(s), t(s))ds,$$

for any test function $\varphi\in C_0^\infty(\mathbb{R}\times[0,\infty))$.

We also need to define a delta shock wave solution to the Riemann problem (1.1) – (1.2).

Definition 4.2 A distribution pair (v,u) is a delta shock wave solution of the problem (1.1) and (1.2) in the sense of distributions if there exist a smooth curve L and a function $w\in C^1(L)$ such that v and u are represented in the following form

$$v = \widetilde{v}(x,t) + w\delta_L \quad \text{and} \quad u = \widetilde{u}(x,t),$$

$\widetilde{v}, \widetilde{u} \in L^\infty(\mathbb{R} \times (0, \infty); \mathbb{R})$ and

$$\begin{cases} \langle v, \varphi_t \rangle + \langle uv, \varphi_x \rangle = 0, \\ \langle u, \varphi_t \rangle + \langle \frac{1}{2}u^2, \varphi_x \rangle = \langle \alpha u, \varphi \rangle, \end{cases}$$

for all the test functions $\varphi \in C_0^\infty(\mathbb{R} \times (0, \infty))$, where $u|_L = u_\delta(t)$ and

$$\langle v, \varphi \rangle = \int_0^\infty \int_{\mathbb{R}} \widetilde{v}\varphi dx dt + \langle w\delta_L, \varphi \rangle$$

and

$$\langle vG(u), \varphi \rangle = \int_0^\infty \int_{\mathbb{R}} \widetilde{v}G(\widetilde{v})\varphi dx dt + \langle wG(u_\delta)\delta_L, \varphi \rangle.$$

With the previous definitions, we are going to find a solution with discontinuity $x = x(t)$ for (1.1) of the form

$$(v(x,t), u(x,t)) = \begin{cases} (v_-(x,t), u_-(x,t)), & \text{if } x < x(t), \\ (w(t)\delta_L, u_\delta(t)), & \text{if } x = x(t), \\ (v_+(x,t), u_+(x,t)), & \text{if } x > x(t), \end{cases}$$

where $v_\pm(x,t)$, $u_\pm(x,t)$ are piecewise smooth solutions of system (1.1), $\delta(\cdot)$ is the Dirac measure supported on the curve $x(t) \in C^1$, and $x(t)$, $w(t)$ and $u_\delta(t)$ are to be determined.

Theorem 4.3 *Suppose $u_- > u_+$. Let $(v^\varepsilon(x,t), u^\varepsilon(x,t))$ be the solution of the problem (1.4)–(2.1). Then the limit*

$$\lim_{\varepsilon \to 0+} (v^\varepsilon(x,t), u^\varepsilon(x,t)) = (v(x,t), u(x,t))$$

exists in the sense of distributions and $(v(x,t), u(x,t))$ solves (1.1)–(1.2). Moreover,

$$(v(x,t), u(x,t)) = \begin{cases} (v_-, u_- e^{-\alpha t}), & \textit{if } x < x(t), \\ (w(t)\delta_{x=x(t)}, \frac{u_-+u_+}{2} e^{-\alpha t}), & \textit{if } x = x(t), \\ (v_+, u_+ e^{-\alpha t}), & \textit{if } x > x(t), \end{cases}$$

where $w(t) = \frac{(v_-+v_+)(u_--u_+)(1-e^{-\alpha t})}{2\alpha}$ and $x(t) = \frac{u_-+u_+}{2}\frac{1-e^{-\alpha t}}{\alpha}$.

Proof As $u_- > u_+$ then $x_-(t) > x_+(t)$. Therefore, from (2.18) and (2.19) we have that

- if $x - x_-(t) > (A_\varepsilon(t))^{1/2}$ then

$$W^\varepsilon(x,t) \approx \frac{v_+(x - x_+(t)) \exp\left(\frac{(x_+(t)-x)^2}{A_\varepsilon(t)}\right)}{\exp\left(\frac{(x_+(t)-x)^2}{A_\varepsilon(t)}\right) - \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\left(\frac{1}{x-x_+(t)} - \frac{1}{x-x_-(t)}\right)}$$

and

$$u^\varepsilon(x,t) \approx \frac{u_+ \exp\left(\frac{(x_+(t)-x)^2}{A_\varepsilon(t)}\right) - \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\left(\frac{u_+}{x-x_+(t)} - \frac{u_-}{x-x_-(t)}\right)}{\exp\left(\frac{(x_+(t)-x)^2}{A_\varepsilon(t)}\right) - \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\left(\frac{1}{x-x_+(t)} - \frac{1}{x-x_-(t)}\right)} e^{-\alpha t}$$

- if $x_+(t) - x < -(A_\varepsilon(t))^{1/2}$ and $x = x_-(t)$ then

$$W^\varepsilon(x,t) \approx \frac{v_+(x - x_+(t)) \exp\left(\frac{-x^2+(x_+(t)-x)^2}{A_\varepsilon(t)}\right) - v_- \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}} \exp\left(-\frac{x^2}{A_\varepsilon(t)}\right)}{\exp\left(\frac{-x^2+(x_+(t)-x)^2}{A_\varepsilon(t)}\right) - \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}(x-x_+(t))} \exp\left(-\frac{x^2}{A_\varepsilon(t)}\right) + \frac{1}{2}\exp\left(-\frac{x^2}{A_\varepsilon(t)}\right)}$$

and

$$u^\varepsilon(x,t) \approx \frac{u_+ \exp\left(\frac{-x^2+(x_+(t)-x)^2}{A_\varepsilon(t)}\right) - \frac{u_+(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}(x-x_+(t))} \exp\left(-\frac{x^2}{A_\varepsilon(t)}\right) + \frac{u_-}{2}\exp\left(-\frac{x^2}{A_\varepsilon(t)}\right)}{\exp\left(\frac{-x^2+(x_+(t)-x)^2}{A_\varepsilon(t)}\right) - \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}(x-x_+(t))} \exp\left(-\frac{x^2}{A_\varepsilon(t)}\right) + \frac{1}{2}\exp\left(-\frac{x^2}{A_\varepsilon(t)}\right)} e^{-\alpha t}$$

- if $x_+(t) + (A_\varepsilon(t))^{1/2} \le x \le x_-(t) - (A_\varepsilon(t))^{1/2}$ then

$$W^\varepsilon(x,t) \approx \frac{v_+(x - x_+(t)) \exp\left(\frac{(x_+(t)-x)^2}{A_\varepsilon(t)}\right) + v_-(x - x_-(t)) \exp\left(\frac{(x_-(t)-x)^2}{A_\varepsilon(t)}\right)}{\exp\left(\frac{(x_+(t)-x)^2}{A_\varepsilon(t)}\right) + \exp\left(\frac{(x_-(t)-x)^2}{A_\varepsilon(t)}\right) + \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\left(\frac{1}{x_+(t)-x} - \frac{1}{x_-(t)-x}\right)}$$

and

$$u^\varepsilon(x,t) \approx \frac{u_+ \exp\left(\frac{(x_+(t)-x)^2}{A_\varepsilon(t)}\right) + u_- \exp\left(\frac{(x_-(t)-x)^2}{A_\varepsilon(t)}\right) + \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\left(\frac{u_+}{x_+(t)-x} - \frac{u_-}{x_-(t)-x}\right)}{\exp\left(\frac{(x_+(t)-x)^2}{A_\varepsilon(t)}\right) + \exp\left(\frac{(x_-(t)-x)^2}{A_\varepsilon(t)}\right) + \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\left(\frac{1}{x_+(t)-x} - \frac{1}{x_-(t)-x}\right)} e^{-\alpha t}$$

- if $x_+(t) - x > (A_\varepsilon(t))^{1/2}$ then

$$W^\varepsilon(x,t) \approx \frac{v_-(x - x_-(t)) \exp\left(\frac{(x_-(t)-x)^2}{A_\varepsilon(t)}\right)}{\exp\left(\frac{(x_-(t)-x)^2}{A_\varepsilon(t)}\right) + \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\left(\frac{1}{x_+(t)-x} - \frac{1}{x_-(t)-x}\right)}$$

and

$$u^\varepsilon(x,t) \approx \frac{u_- \exp\left(\frac{(x_-(t)-x)^2}{A_\varepsilon(t)}\right) + \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\left(\frac{u_+}{x_+(t)-x} - \frac{u_-}{x_-(t)-x}\right)}{\exp\left(\frac{(x_-(t)-x)^2}{A_\varepsilon(t)}\right) + \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}}\left(\frac{1}{x_+(t)-x} - \frac{1}{x_-(t)-x}\right)} e^{-\alpha t}$$

- if $x - x_-(t) < -(A_\varepsilon(t))^{1/2}$ and $x = x_+(t)$ then

$$W^\varepsilon(x,t) \approx \frac{v_+ \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}} \exp\left(-\frac{x^2}{A_\varepsilon(t)}\right) + v_-(x - x_-(t)) \exp\left(\frac{-x^2+(x_-(t)-x)^2}{A_\varepsilon(t)}\right)}{\frac{1}{2}\exp\left(-\frac{x^2}{A_\varepsilon(t)}\right) + \exp\left(\frac{-x^2+(x_-(t)-x)^2}{A_\varepsilon(t)}\right) - \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}(x_-(t)-x)} \exp\left(-\frac{x^2}{A_\varepsilon(t)}\right)}$$

and

$$u^\varepsilon(x,t) \approx \frac{\frac{u_+}{2}\exp\left(-\frac{x^2}{A_\varepsilon(t)}\right) + u_- \exp\left(\frac{-x^2+(x_-(t)-x)^2}{A_\varepsilon(t)}\right) - \frac{u_-(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}(x_-(t)-x)} \exp\left(-\frac{x^2}{A_\varepsilon(t)}\right)}{\frac{1}{2}\exp\left(-\frac{x^2}{A_\varepsilon(t)}\right) + \exp\left(\frac{-x^2+(x_-(t)-x)^2}{A_\varepsilon(t)}\right) - \frac{(A_\varepsilon(t))^{1/2}}{2\pi^{1/2}(x_-(t)-x)} \exp\left(-\frac{x^2}{A_\varepsilon(t)}\right)} e^{-\alpha t}.$$

Therefore, we have that

$$\lim_{\varepsilon \to 0+} W^\varepsilon(x,t) = \begin{cases} v_-(x - x_-(t)), & \text{if } (x - x_+(t))^2 - (x - x_-(t))^2 < 0, \\ v_+(x - x_+(t)), & \text{if } (x - x_+(t))^2 - (x - x_-(t))^2 > 0. \end{cases}$$

But $(x-x_+(t))^2-(x-x_-(t))^2=2(x_-(t)-x_+(t))(x-\frac{x_-(t)+x_+(t)}{2})$ and as $u_->u_+$ then we have

$$\lim_{\varepsilon\to 0+} W^\varepsilon(x,t)=\begin{cases} v_-(x-x_-(t)), & \text{if } x<\frac{x_-(t)+x_+(t)}{2},\\ v_+(x-x_+(t)), & \text{if } x>\frac{x_-(t)+x_+(t)}{2}.\end{cases}$$

Since $W^\varepsilon(x,t)$ is bounded on compact subsets of $\mathbb{R}\times\mathbb{R}_+=\{(x,t):x\in\mathbb{R},t>0\}$ and $W^\varepsilon(x,t)\to W(x,t)$ pointwise as $\varepsilon\to 0+$, then $W^\varepsilon(x,t)\to W(x,t)$ in the sense of distribution and so $W^\varepsilon_x(x,t)$ converges in the distributional sense to $W_x(x,t)$. For simplicity of notation, we write $x(t)$ instead of $\frac{x_-(t)+x_+(t)}{2}$. From (2.2) we have that $\lim_{\varepsilon\to 0+} v^\varepsilon(x,t)=v(x,t)$ exists in the sense of distribution and

$$v(x,t)=W_x(x,t)=\begin{cases} v_-, & \text{if } x<x(t),\\ (x_-(t)-x_+(t))\frac{v_-+v_+}{2}\delta_{x=x(t)}, & \text{if } x=x(t),\\ v_+, & \text{if } x>x(t).\end{cases}$$

Also, we have that

$$u(x,t)=\begin{cases} u_-e^{-\alpha t}, & \text{if } x<x(t),\\ \frac{u_-+u_+}{2}e^{-\alpha t}, & \text{if } x=x(t),\\ u_+e^{-\alpha t}, & \text{if } x>x(t).\end{cases}$$

Now, we have to show that $(v(x,t),u(x,t))$ solves (1.1). Thus, for any test function $\varphi\in C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$ we have

$$\begin{aligned}
\langle v,\varphi_t\rangle+\langle uv,\varphi_x\rangle &= \int_0^\infty\int_{\mathbb{R}}(v\varphi_t+uv\varphi_x)dxdt\\
&\quad+\int_0^\infty \frac{v_-+v_+}{2}(x_-(t)-x_+(t))\left(\varphi_t+\frac{u_-+u_+}{2}e^{-\alpha t}\varphi_x\right)dt\\
&=\int_0^\infty\int_{-\infty}^{x(t)}(v_-\varphi_t+u_-v_-e^{-\alpha t}\varphi_x)dxdt\\
&\quad+\int_0^\infty\int_{x(t)}^{\infty}(v_+\varphi_t+u_+v_+e^{-\alpha t}\varphi_x)dxdt\\
&\quad+\int_0^\infty \frac{v_-+v_+}{2}(x_-(t)-x_+(t))\left(\varphi_t+\frac{u_-+u_+}{2}e^{-\alpha t}\varphi_x\right)dt\\
&=-\oint -\left(u_-v_-e^{-\alpha t}\varphi\right)dt+(v_-\varphi)dx\\
&\quad+\oint -\left(u_+v_+e^{-\alpha t}\varphi\right)dt+(v_+\varphi)dx\\
&\quad+\int_0^\infty \frac{v_-+v_+}{2}(x_-(t)-x_+(t))\left(\varphi_t+\frac{u_-+u_+}{2}e^{-\alpha t}\varphi_x\right)dt\\
&=\int_0^\infty\left((u_-v_--u_+v_+)e^{-\alpha t}-(v_--v_+)\frac{dx(t)}{dt}\right)\varphi dt\\
&\quad-\int_0^\infty\frac{d}{dt}\left(\frac{v_-+v_+}{2}(x_-(t)-x_+(t))\right)\varphi dt=0,
\end{aligned}$$

and

$$\begin{aligned}
\langle u,\varphi_t\rangle+\langle\frac{1}{2}u^2,\varphi_x\rangle &= \int_0^\infty\int_{\mathbb{R}}(u\varphi_t+\frac{1}{2}u^2\varphi_x)dxdt\\
&= \int_0^\infty\int_{-\infty}^{x(t)}(u_-e^{-\alpha t}\varphi_t+\frac{1}{2}u_-^2e^{-2\alpha t}\varphi_x)dxdt\\
&\quad+\int_0^\infty\int_{x(t)}^{\infty}(u_+e^{-\alpha t}\varphi_t+\frac{1}{2}u_+^2e^{-2\alpha t}\varphi_x)dxdt\\
&= -\oint-\left(\frac{1}{2}u_-^2e^{-2\alpha t}\varphi\right)dt+(u_-e^{-\alpha t}\varphi)dx\\
&\quad-\int_0^\infty\int_{-\infty}^{x(t)}-\alpha u_-e^{-\alpha t}\varphi dxdt\\
&\quad+\oint-\left(\frac{1}{2}u_+^2e^{-2\alpha t}\varphi\right)dt+(u_+e^{-\alpha t}\varphi)dx\\
&\quad-\int_0^\infty\int_{x(t)}^{\infty}-\alpha u_+e^{-\alpha t}\varphi dxdt\\
&= \int_0^\infty\left(\frac{1}{2}(u_-^2-u_+^2)e^{-\alpha t}-(u_--u_+)\frac{dx(t)}{dt}\right)\varphi e^{-\alpha t}dt\\
&\quad+\langle\alpha u,\varphi\rangle=\langle\alpha u,\varphi\rangle.
\end{aligned}$$

The proof is complete. □

Remark 4.4 1. In [14, Sect. 4], Keita and Bourgault have been solved the Riemann problem for (1.1) with initial data (1.2). They show that the delta shock wave solution is unique under the following entropy condition [14, Definition 3.3]

$$u_+e^{-\alpha t}<\sigma(t)<u_-e^{-\alpha t}.$$

Observe that us delta shock wave solution satisfies the following over-compressive condition

$$u_+e^{-\alpha t}<\frac{u_-+u_+}{2}e^{-\alpha t}<u_-e^{-\alpha t},\quad \text{for all } t>0,$$

i.e., so the delta shock wave here obtained is unique.

2. Observe that a pair (v,u) is a delta shock wave solution with discontinuity $x=x(t)$ for the problem (1.1)–(1.2) of the form

$$(v(x,t),u(x,t))=\begin{cases}(v_-(t),u_-(t)), & \text{if } x<x(t),\\ (w(t)\delta_L,u_\delta(t)), & \text{if } x=x(t),\\ (v_+(t),u_+(t)), & \text{if } x>x(t),\end{cases}$$

where $v_\pm(t)$, $u_\pm(t)$ are piecewise smooth solutions of system (1.1), $\delta(\cdot)$ is the Dirac measure supported on the curve $x(t)\in C^1$, and $x(t)$, $w(t)$ and $u_\delta(t)$ are C^1, if and only if the following generalized Rankine-Hugoniot conditions are satisfied

$$\begin{cases} \frac{dx(t)}{dt} = u_\delta(t), \\ \frac{dw(t)}{dt} = -[v]u_\delta(t) + [uv], \\ 0 = -[u]u_\delta(t) + \frac{1}{2}[u^2]. \end{cases}$$

3. Notice that $\lim_{\alpha\to 0+} w(t) = \frac{1}{2}(v_- + v_+)(u_- - u_+)t = \widetilde{w}(t)$, $\lim_{\alpha\to 0+} x(t) = \frac{1}{2}(u_- + u_+)t$ and

$$\lim_{\alpha\to 0+} (v(x,t), u(x,t)) = \begin{cases} (v_-, u_-), & \text{if } x < \frac{1}{2}(u_- + u_+)t, \\ (\widetilde{w}(t)\delta_{x=\frac{1}{2}(u_-+u_+)t}, \frac{1}{2}(u_- + u_+)), & \text{if } x = \frac{1}{2}(u_- + u_+)t, \\ (v_+, u_+), & \text{if } x > \frac{1}{2}(u_- + u_+)t, \end{cases}$$

i.e., when $\alpha \to 0+$, $(v(x,t), u(x,t))$ converges to the solution given by Joseph in [13] for homogeneous case of the system (1.1).

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References

1. Burgers, J.M.: A mathematical model illustrating the theory of turbulence. Adv. Appl. Mech. **1**, 171–179 (1948)
2. Cole, J.D.: On a quasilinear parabolic equation occurring in aerodynamics. Q. Appl. Math. **9**, 225–236 (1951)
3. Crighton, D.G.: Model equations of nonlinear acoustics. Annu. Rev. Fluid Mech. **11**, 11–33 (1979)
4. Crighton, D.G., Scott, J.F.: Asymptotic solution of model equations in nonlinear acoustics. Philos. Trans. R. Soc. A, Math. Phys. Eng. Sci. **292**, 101–134 (1979)
5. Dafermos, C.M.: Solutions of the Riemann problem for a class of hyperbolic systems of conservation laws by the viscosity method. Arch. Ration. Mech. Anal. **52**, 1–9 (1973)
6. Danilov, V.G., Mitrovic, D.: Delta shock wave formation in the case of triangular hyperbolic system of conservation laws. J. Differ. Equ. **245**, 3704–3734 (2008)
7. Dingle, R.B.: Asymptotic Expansions: Their Derivation and Interpretation. Academic Press, London and New York (1973)
8. Doyle, J., Englefield, M.J.: Similarity solutions of a generalized Burgers equation. IMA J. Appl. Math. **44**, 145–153 (1990)
9. Ercole, G.: Delta-shock waves as self-similar viscosity limits. Q. Appl. Math. **LVIII**(1), 177–199 (2000)
10. Hopf, E.: The partial differential equation $u_t + uu_x = \mu u_{xx}$. Commun. Pure Appl. Math. **3**, 201–230 (1950)
11. Huang, F.: Existence and uniqueness of discontinuous solutions for a class nonstrictly hyperbolic systems. In: Chen, G-Q., Li, Y., Zhu, X., Chao, D. (eds.) Advances in Nonlinear Partial Differential Equations and Related Areas, pp. 187–208. World Scientific, Beijing (1998)
12. Isaacson, E.L., Temple, B.: Analysis of a singular hyperbolic system of conservation laws. J. Differ. Equ. **65**, 250–268 (1986)
13. Joseph, K.T.: A Riemann problem whose viscosity solution contain δ-measures. Asymptot. Anal. **7**, 105–120 (1993)
14. Keita, S., Bourgault, Y.: Eulerian droplet model: delta-shock waves and solution of the Riemann problem. J. Math. Anal. Appl. **472**(1), 1001–1027 (2019)

15. Korchinski, D.J.: Solution of a Riemann problem for a 2 × 2 system of conservation laws possessing no classical weak solution. PhD thesis, Adelphi University (1977)
16. LeFloch, P.: An existence and uniqueness result for two nonstrictly hyperbolic systems. In: Keyfitz, B.L., Shearer, M. (eds.) Nonlinear Evolution Equations That Change Type. The IMA Volumes in Mathematics and Its Applications, vol. 27, pp. 126–138. Springer, New York (1990)
17. Sarrico, C.O.R.: New solutions for the one-dimensional nonconservative inviscid Burgers equation. J. Math. Anal. Appl. **317**, 496–509 (2006)
18. Shandarin, S.F., Zeldovich, Y.B.: Large-scale structure of the universe: turbulence, intermittency, structures in a self-gravitating medium. Rev. Mod. Phys. **61**, 185–220 (1989)
19. Scott, J.F.: The long time asymptotics of solution to the generalized Burgers equation. Proc. R. Soc. Lond. A **373**, 443–456 (1981)
20. Tan, D., Zhang, T., Zheng, Y.: Delta shock waves as limits of vanishing viscosity for hyperbolic systems of conversation laws. J. Differ. Equ. **112**, 1–32 (1994)
21. Tupciev, V.A.: On the method of introducing viscosity in the study of problems involving decay of a discontinuity. Sov. Math. Dokl. **14**, 978–982 (1973)
22. Vaganana, B.M., Kumaran, M.S.: Kummer function solutions of damped Burgers equations with time-dependent viscosity by exact linearization. Nonlinear Anal., Real World Appl. **9**, 2222–2233 (2008)
23. Wang, J.H., Zhang, H.: A new viscous regularization of the Riemann problem for Burger's equation. J. Partial Differ. Equ. **13**, 253–263 (2000)
24. Wang, J., Zhang, H.: Existence and decay rates of solutions to the generalized Burgers equation. J. Math. Anal. Appl. **284**, 213–235 (2003)
25. Yang, H., Zhang, Y.: New developments of delta shock waves and its applications in systems of conservation laws. J. Differ. Equ. **252**, 5951–5993 (2012)
26. Zhang, H.: Global existence and asymptotic behaviour of the solution of a generalized Burger's equation with viscosity. Comput. Math. Appl. **41**(5–6), 589–596 (2001)
27. Zhang, H., Wang, X.: Large-time behavior of smooth solutions to a nonuniformly parabolic equation. Comput. Math. Appl. **47**(2–3), 353–363 (2004)