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Almost Global Existence for the 3D Prandtl Boundary Layer Equations

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Abstract In this paper, we prove the almost global existence of classical solutions to the 3D Prandtl system with the initial data which lie within ε of a stable shear flow. Using anisotropic Littlewood-Paley energy estimates in tangentially analytic norms and introducing new linearly-good unknowns, we prove that the 3D Prandtl system has a unique solution with the lifespan of which is greater than $\exp(\varepsilon^{-1}/\log(\varepsilon^{-1}))$. This result extends the work obtained by Ignatova and Vicol (Arch. Ration. Mech. Anal. 2:809–848, 2016) on the 2D Prandtl equations to the three-dimensional setting.

Keywords Prandtl equations · Almost global existence · Littlewood-Paley theory

1 Introduction

We consider the following Prandtl boundary layer equations in $\mathbb{R}_+ \times \mathbb{R}^3_+$:

$$\begin{cases} \partial_t u^p + \big(u^p\partial_x + v^p\partial_y + w^p\partial_z\big)u^p + \partial_x p^E = \partial_z^2 u^p, \\ \partial_t v^p + \big(u^p\partial_x + v^p\partial_y + w^p\partial_z\big)v^p + \partial_y p^E = \partial_z^2 v^p, \\ \partial_x u^p + \partial_y v^p + \partial_z w^p = 0, \\ \big(u^p, v^p, w^p\big)\big|_{z=0} = (0,0,0), \\ \lim\limits_{z\to+\infty}\big(u^p, v^p\big) = \big(U^E(t,x,y), V^E(t,x,y)\big), \\ \big(u^p, v^p\big)\big|_{t=0} = \big(u_0(x,y,z), v_0(x,y,z)\big), \end{cases} \tag{1.1}$$

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here and in what follows, $(t, x, y, z) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$, (u^p, v^p) and w^p denote the tangential and normal velocity of the boundary layer flow, the initial data $(u_0, v_0) := (u_0(x, y, z), v_0(x, y, z))$ and the far-field $(U^E(t, x, y), V^E(t, x, y))$ are given. Furthermore, $(U^E(t, x, y), V^E(t, x, y))$ and the given scalar pressure $p^E(t, x, y)$ are the tangential velocity field and pressure on the boundary $\{z = 0\}$ of the Euler flow, satisfying

$$\begin{cases} \partial_t U^E + U^E \partial_x U^E + V^E \partial_y U^E + \partial_x p^E = 0, \\ \partial_t V^E + U^E \partial_x V^E + V^E \partial_y V^E + \partial_y p^E = 0, \quad t > 0, (x, y) \in \mathbb{R}^2. \end{cases} \tag{1.2}$$

This system (1.1) introduced by Prandtl [33] is a model for the first approximation of the velocity field near the boundary in the zero viscosity limit of the initial boundary value problem of the incompressible Navier-Stokes equations, with the non-slip boundary condition. It is then natural to ask whether solutions to the Navier-Stokes system with zero Dirichlet boundary condition do converge to a solution to the Euler system when the viscosity goes to zero. We refer to [13, 17, 35, 36] and references therein for this justification. One of the key steps to justify the zero viscosity limit is to deal with the well-posedness of the Prandtl system. Up to now, whether the Prandtl equation with general data is well-posed in Sobolev spaces or not is still open except for some special cases, for example, the initial data that are monotonic with respect to the normal variable [1, 32, 34, 43] and in the analytic framework [16, 18, 30, 35, 36, 44], see the references therein for the recent progress. Still, there are some results that do not require the monotonicity and analyticity conditions, see [4, 10, 21–23].

Most of the results of the boundary layer equations are developed for the 2D Prandtl system. Under Oleinik's monotonicity assumption $u^p_y > 0$, Oleinik [34] proved the local existence and uniqueness of classical solutions. With the additional favorable condition on the pressure, Xin and Zhang [43] got the global existence of weak solutions. For the real-analytic initial data, Sammartino and Caflisch [35, 36] established the local well-posedness. Later, the analyticity in y variable was removed by Lombardo et al. [30]. Motivated by the fact that the energy method can be well applied to the Navier-Stokes equations, Alexandre et al. [32], Masmoudi and Wong [1] use the energy method independently, where the key observation is some kind of cancellation property in the convection term due to the monotonicity condition. The first result which does not require monotonicity and analyticity was established by Gérard-Varet and Masmoudi [10], and they proved the local well-posedness in the Gevrey space $\frac{7}{4}$ of the two-dimensional Prandtl equations with non-degenerate critical points. Gérard-Varet and Dormy [9] showed that the linearized Prandtl equations are ill-posed without the structural condition and supposed that the optimal Gevrey index is $\sigma = 2$. Motivated by this conjecture, Li and Yang [21] got the well-posedness in all Gevrey space G^σ with $\sigma \in [\frac{3}{2}, 2]$. Chen et al. [4] proved the well-posedness for the linearized Prandtl equation in Gevrey function space G^σ for any index $1 \leq \sigma < 2$. We refer to [3, 7, 12, 19, 38, 40] and references therein for more new results. Finally, let us mention that for the convex initial data, the local well-posedness holds under simple Gevrey regularity in [12]. And Chen et al. [3] proved the well-posedness of the Prandtl equation for the monotonic data in Sobolev spaces with exponential weight and low regularity by using the paralinearization method.

There are some works about the formation of singularity in the Prandtl system, see [5, 6, 8, 20] in particular. The precise description of the formation of singularity is still an open problem. However, in the case of a trivial Euler flow U^E and a trivial Euler pressure p^E, E and Engquist [8] established a finite time singularity for the Prandtl equations. For a more general class of non-trivial inviscid flow, Kukavica, Vicol and Wang [20] established the finite time blowup of the boundary layer thickness. A similar one dimensional reduced problem was considered by their approach using contradiction, and the method did not give details about the singularity. Very recently, Collot et al. [5] provide a precise description of the mechanism that leads to the singularity solutions for the same reduced problem. Dalibard and Masmoudi [6] justify rigorously the "Goldstein singularity" for the stationary Prandtl model.

Compared with the 2D case, the three-dimensional system share similar difficulties, both are troublesome to deal with instability, separation of the boundary layer and so on. However, much less is known about the three-dimensional Prandtl equations, due to the extra difficulties coming from the secondary flow appearing in the three-dimensional boundary layer system and the complicated structure of the boundary layer arising from the multi-dimensional velocity field. As for the well-posedness theories, only partial results have been got in some specific settings, such as in the analytic functional [18, 35, 36] and under some constraint on its flow structure [26]. Without monotonic condition, Liu et al. [25] gave an instability criterion. However, Li and Yang [22] proved the system is locally well-posed in the Gevrey function space with Gevrey index in $(1, 2]$ without asking analyticity or structural constraint which complements the ill-posedness result in [25]. To have an over-around study on the Prandtl boundary layer equation, the interested readers can refer to [14, 33] and the references therein.

In the aforementioned works only local-posedness are achieved. The global weak solutions are established by Xin and Zhang [43], and the global strong solutions are still unclear so far. To get a finer understanding of the Prandtl equations, one must understand its behavior on a longer lifespan. To the best of our knowledge, the long-time existence of the Prandtl equations has few results. Zhang and Zhang [44] proved that when $d = 2$ or $d = 3$, the Prandtl system has a unique solution with the lifespan

$$T_\varepsilon \geq \varepsilon^{-\frac{4}{3}}.$$

Ignatova and Vicol [16] got an almost global existence solution for the 2D Prandtl system of which the lifespan can be extended at least up to time

$$T_\varepsilon \geq \exp\big(\varepsilon^{-1} / \log(\varepsilon^{-1})\big).$$

Therefore, it is interesting to study whether the 3D Prandtl system exists almost globally with the small analytic initial data or not. In this paper, using the Littlewood-Paley theory, we will prove an almost global existence for the 3D Prandtl boundary layer system of the small analytic data with the far-field states that are uniform constant states (κ_1, κ_2), consequently, there is no pressure term $(\partial_x p^E, \partial_y p^E)$ in (1.1), which extend the result of [16] to 3D. In the whole paper, we will consider the following system

$$\begin{cases} \partial_t u^p + (u^p \partial_x + v^p \partial_y + w^p \partial_z) u^p = \partial_z^2 u^p, \\ \partial_t v^p + (u^p \partial_x + v^p \partial_y + w^p \partial_z) v^p = \partial_z^2 v^p, \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^3_+ \\ \partial_x u^p + \partial_y v^p + \partial_z w^p = 0, \\ (u^p, v^p, w^p)|_{z=0} = (0, 0, 0), \\ \lim\limits_{z \to +\infty} (u^p, v^p) = (\kappa_1, \kappa_2), \\ (u^p, v^p)|_{t=0} = (u_0(x, y, z), v_0(x, y, z)). \end{cases} \tag{1.3}$$

Nowadays, many researchers consider some relevant problems motivated by the research of the Prandtl boundary layer system, for example, the justification and the well-posedness/ill-posedness problem for the magnetohydrodynamics boundary layer system, cf. [11, 15, 24, 27–29, 37, 39, 41, 42] and so on.

The rest of this paper is organized as follows. In Sect. 2, we first introduce new linearly-good unknowns to cancel out the bad terms. Then we state our main result at the last subsection and explain the main difficulties and the methods to overcome them. For simplicity, in Sect. 3, we only present some *a priori* estimates to prove the existence part of the main theorem. In Sect. 4, we prove the uniqueness of the solution. At the final part of the paper, for the sake of self-containedness, we will list some important functional tool box used in Sect. 5.1. Next we derive the 3D Prandtl boundary layer system (2.1) in Sect. 5.2. Finally we present some important lemmas in Sect. 5.3 and Sect. 5.4.

2 Statements of the Result

Here we introduce the linear good unknowns and the functional spaces which be used throughout this paper, then state our main result.

2.1 The Linear Good Unknowns

Motivated by [16], to deal with (1.1) better, we write $u^p(t,x,y,z)$ and $v^p(t,x,y,z)$ as perturbations $u(t,x,y,z)$ and $v(t,x,y,z)$ of the lifts $\kappa_1\varphi(t,z)$ and $\kappa_2\varphi(t,z)$ respectively via

$$u^p(t,x,y,z)=\kappa_1\varphi(t,z)+u(t,x,y,z), \qquad v^p(t,x,y,z)=\kappa_2\varphi(t,z)+v(t,x,y,z),$$

where

$$\varphi(t,z)=\frac{1}{\sqrt{\pi}}\int_0^{z/\sqrt{\langle t\rangle}}\exp\Big(-\frac{\tilde{z}^2}{4}\Big)d\tilde{z},$$

and denote by $\langle t\rangle=1+t$. Then we introduce new linearly-good unknowns

$$g_1(t,x,y,z)=\partial_z u(t,x,y,z)+\frac{z}{2\langle t\rangle}u(t,x,y,z),$$

$$g_2(t,x,y,z)=\partial_z v(t,x,y,z)+\frac{z}{2\langle t\rangle}v(t,x,y,z).$$

Through some simple calculations, we have that the equations for the good unknowns $g=(g_1,g_2)^\top$ in $\{t>0,(x,y)\in\mathbb{R}^2,z\in\mathbb{R}^+\}$ are

$$\begin{cases} \partial_t g-\partial_z^2 g+\frac{1}{\langle t\rangle}g+\kappa_1\partial_z\varphi\nabla_h^\perp v+\kappa_1\varphi\partial_x g+g_1\nabla_h^\perp v-\frac{z}{2\langle t\rangle}u\nabla_h^\perp v+u\partial_x g \\ \quad -\kappa_2\partial_z\varphi\nabla_h^\perp u+\kappa_2\varphi\partial_y g-g_2\nabla_h^\perp u+\frac{z}{2\langle t\rangle}v\nabla_h^\perp u+v\partial_y g+w\partial_z g-\frac{1}{2\langle t\rangle}w\mathbf{u}=0, \\ u=U(g_1), \qquad v=U(g_2), \quad \text{and} \quad w=W(g_1,g_2), \\ \partial_z g|_{z=0}=\lim\limits_{z\to+\infty}g=0, \\ g|_{t=0}=(g_{10},g_{20}), \end{cases} \tag{2.1}$$

where

$$\mathbf{u} = (u, v)^\top, \qquad \nabla_h = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}, \qquad \nabla_h^\perp = \begin{pmatrix} -\partial_y \\ \partial_x \end{pmatrix},$$
$$U(g_i) \doteq \exp\left(\frac{-z^2}{4\langle t\rangle}\right) \int_0^z g_i(t, x, y, \widetilde{z}) \exp\left(\frac{\widetilde{z}^2}{4\langle t\rangle}\right) d\widetilde{z}, \quad i = 1, 2, \tag{2.2}$$

$$W(g_1, g_2) \doteq -\int_0^z \big[U(\partial_x g_1) + U(\partial_y g_2)\big] d\widetilde{z}. \tag{2.3}$$

For the sake of self-containedness, we will provide several specific calculations in Sect. 5.2. Note that we may recover $u = U(g_1)$, $v = U(g_2)$ and $w = W(g_1, g_2)$ via linear operators U and W that are nonlocal in z, cf. (2.2) and (2.3). So g_1 and g_2 are only prognostic variables in this problem, and the system (2.1) is equivalent to (1.3). Thus, solving the 3D Prandtl boundary layer system (1.3) is equivalent to solving (2.1).

2.2 The Functional Spaces

Next, let us introduce some functional spaces we are going to use. In view of the definition of the operator U in (2.2) and the choice of φ, it is natural to use the function $\psi(t, z)$ determined by

$$\psi(t, z) \doteq \frac{\alpha z^2}{4\langle t\rangle}, \tag{2.4}$$

for this $\alpha \in [\frac{1}{4}, \frac{1}{2}]$ to be chosen later, see Theorem 2.1. To overcome a technical obstacle, the Poincaré inequality doesn't hold in the unbounded domain. It is natural to define the Gaussian weight as following

$$e^\psi = \exp\left(\frac{\alpha z^2}{4\langle t\rangle}\right).$$

As in [2], for any local bounded function Φ on $\mathbb{R}^+ \times \mathbb{R}^2$, we define

$$u_\Phi(t, x, y, z) = \mathscr{F}^{-1}_{\xi\to(x,y)}\big(e^{\Phi(t,\xi)}\widehat{u}(t, \xi, z)\big), \tag{2.5}$$

$$\widetilde{u_\Phi}(t, x, y, z) = \mathscr{F}^{-1}_{\xi\to(x,y)}\big(\big|e^{\Phi(t,\xi)}\widehat{u}(t, \xi, z)\big|\big), \tag{2.6}$$

where and in all that follows, $\mathscr{F}u$ and $\widehat{u}$ always denote the partial Fourier transform of the distribution u with respect to (x, y) variables, $\widehat{u}(\xi, z) = \mathscr{F}_{(x,y)\to\xi}(u)(\xi, z)$, and $\xi = (\xi_1, \xi_2)$. Define the phase function Φ as following

$$\Phi(t, \xi) \doteq \big(\tau_0 - \lambda\theta(t)\big)|\xi|, \tag{2.7}$$

where λ is a large enough positive constant which will be chosen precise later, see (3.25), and $\theta(t)$ is a key quantity to describe the analytic band of (g_1, g_2):

$$\begin{cases} \frac{d}{dt}\theta(t) = \|(g_{1\Phi}, g_{2\Phi})\|_{B_{1,\alpha}} + (|\kappa_1| + |\kappa_2|)\langle t\rangle^{\frac{1}{4}} \|e^\psi \partial_z\varphi\|_{L^2_z}, \\ \theta|_{t=0} = 0, \end{cases} \tag{2.8}$$

where $B_{1,\alpha}$ is defined in Definition 2.1.

In order to define the functional spaces of the solution, motivated by [16, 32], it is convenient to define the following spaces. Firstly, let us recall from [44] that

$$\Delta_k^h u = \mathscr{F}^{-1}\big(\varphi(2^{-k}|\xi|)\widehat{u}\big), \qquad S_k^h u = \mathscr{F}^{-1}\big(\chi(2^{-k}|\xi|)\widehat{u}\big), \tag{2.9}$$

where $\chi(\tau)$, $\varphi(\tau)$ are smooth functions such that

$$\operatorname{Supp}\varphi \subset \left\{\tau \in \mathbb{R} \Big/ \frac{3}{4} \le |\tau| \le \frac{8}{3}\right\} \quad \text{and} \quad \forall \tau > 0,\ \sum_{j\in\mathbb{Z}} \varphi\big(2^{-j}\tau\big) = 1,$$

$$\operatorname{Supp}\chi \subset \left\{\tau \in \mathbb{R} \Big/ |\tau| \le \frac{4}{3}\right\} \quad \text{and} \quad \chi(\tau) + \sum_{j\ge 0} \varphi\big(2^{-j}\tau\big) = 1.$$

Definition 2.1 Let s be in $\mathbb{R}$. For g in $\mathscr{S}_h'(\mathbb{R}_+^3)$, which means that g is in $\mathscr{S}'(\mathbb{R}_+^3)$ and satisfies $\lim\limits_{k\to-\infty} \|S_k^h g\|_{L^\infty} = 0$, we set

$$\|g_\Phi\|_{X_{s,\alpha}} = \big\|\big(2^{ks}\|e^{\psi}\Delta_k^h g_\Phi\|_{L_+^2}\big)_k\big\|_{\ell^1(\mathbb{Z})}.$$

For compactness of notation, for a function g_Φ such that $g_\Phi, \zeta g_\Phi, \partial_z g_\Phi \in X_{s,\alpha}$, we use the time weighted norm

$$\|g_\Phi\|_{B_{s,\alpha}} = \langle t\rangle^{\frac14}\|g_\Phi\|_{X_{s,\alpha}} + \langle t\rangle^{\frac14}\|\zeta g_\Phi\|_{X_{s,\alpha}} + \langle t\rangle^{\frac34}\|\partial_z g_\Phi\|_{X_{s,\alpha}}, \tag{2.10}$$

where $\zeta = \zeta(t,z) = \frac{z}{\langle t\rangle^{1/2}}$ the heat self-similar variable.

In order to obtain a better description of the regularizing effect of the transport diffusion equation, we need to use Chemin-Lerner type spaces $\widetilde{L}_T^p(X_{s,\alpha}(\mathbb{R}_+^3))$.

Definition 2.2

$$\|g\|_{\widetilde{L}_T^p(X_{s,\alpha})} \doteq \sum_{k\in\mathbb{Z}} 2^{ks}\left(\int_0^T \|e^{\psi}\Delta_k^h g_\Phi(t)\|_{L_+^2}^p\,dt\right)^{\frac1p},$$

with the usual change if $p = \infty$.

Moreover, in order to overcome the difficulty that one can not use Gronwall's type argument in the framework of Chemin-Lerner space $\widetilde{L}_T^2(X_{s,\alpha})$, so we need to use the weighted Chemin-Lerner norm, which was introduced by Marius and Zhang in [31].

Definition 2.3 Let $f(t) \in L_{loc}^1(\mathbb{R}_+)$ be a nonnegative function. We define

$$\|g\|_{\widetilde{L}_{t,f}^p(X_{s,\alpha})} \doteq \sum_{k\in\mathbb{Z}} 2^{ks}\left(\int_0^t f(t')\|e^{\psi}\Delta_k^h g_\Phi(t')\|_{L_+^2}^p\,dt'\right)^{\frac1p},$$

$$\|g\|_{\widetilde{L}_{t,f}^p(B_{s,\alpha})} \doteq \sum_{k\in\mathbb{Z}} 2^{ks}\Bigg[\int_0^t f(t')\Big(\langle t'\rangle^{\frac p4}\big(\|e^{\psi}\Delta_k^h g_\Phi\|_{L_+^2}^p + \|\zeta e^{\psi}\Delta_k^h g_\Phi\|_{L_+^2}^p\big)$$

$$+\langle t'\rangle^{\frac{3p}{4}}\|e^{\psi}\Delta_k^h\partial_z g_\Phi\|_{L_+^2}^p\Big)dt'\Bigg]^{\frac1p},$$

where $p < \infty$. When $p = \infty$, we define

$$\|g\|_{\widetilde{L}^\infty_{T,f(t)}(X_{s,\alpha})} \doteq \sum_{k\in\mathbb{Z}} 2^{ks} \sup_{t\in[0,T]} f(t) \left\| e^{\psi} \Delta_k^h g_\Phi(t) \right\|_{L^2_+}.$$

In this paper, $(a|b)_{L^2_+} \doteq \int_{\mathbb{R}^3_+} a(x,y,z)\overline{b}(x,y,z)dxdydz$ stands for the L^2 inner product of a, b on $\mathbb{R}^3_+$. We denote $\{d_k\}_{k\in\mathbb{Z}}$ to be generic elements in the sphere of $\ell^1(\mathbb{Z})$, $\|(g_1, g_2)\| = \|g_1\| + \|g_2\|$, $\dot{g}(t) \doteq \frac{d}{dt} g(t)$, $m = (m_1, m_2)$, $\nabla_h^m \doteq \partial_x^{m_1}\partial_y^{m_2}$, $\nabla_h \doteq (\partial_x, \partial_y)^\top$. Throughout this paper, we use $A \lesssim B$ to denote $A \lesssim CB$ for some absolute positive constant C, whose meaning may change from line to line. For convenience, we provide some tool box about the Littlewood-Paley theory, see Sect. 5.1.

2.3 Main Result

Denote $|D_h|$ the Fourier multiplier with the symbol $|\xi|$. Then our main result is stated as follows.

Theorem 2.1 *There exist $C_* > 0$ and $\varepsilon_* > 0$ such that for any $\varepsilon \in (0, \varepsilon_*]$, when $e^{\tau_0|D_h|} g_{i0} \in X_{1,\frac{1}{2}}$, $i = 1, 2$, and*

$$\left\| e^{\tau_0|D_h|} g_{10} \right\|_{X_{1,\frac{1}{2}}} + \left\| e^{\tau_0|D_h|} g_{20} \right\|_{X_{1,\frac{1}{2}}} \le \varepsilon, \tag{2.11}$$

with

$$\tau_0 \ge \frac{C_*}{\ln\frac{1}{\varepsilon}} + C_*\big(|\kappa_1| + |\kappa_2|\big)\exp\left(\frac{1}{\varepsilon\ln\frac{1}{\varepsilon}}\right), \tag{2.12}$$

the system (2.1) has a unique solution (g_1, g_2) on $[0, T_\varepsilon]$, where

$$T_\varepsilon \ge \exp\left(\frac{\varepsilon^{-1}}{\ln(\varepsilon^{-1})}\right).$$

Furthermore, the phase function $\Phi(t,\xi)$ in (2.7) of the solution (g_1, g_2) satisfies

$$\Phi(t,\xi) \ge \frac{3\tau_0}{4}|\xi|, \tag{2.13}$$

for all $t \in [0, T_\varepsilon]$, and the solution (g_1, g_2) obeys the bounds

$$\left\|(g_{1\Phi}, g_{2\Phi})\right\|_{\widetilde{L}^\infty_{T_\varepsilon, f_3(t)}(X_{1,\alpha})} \le C\varepsilon, \tag{2.14}$$

$$\left\|(g_{1\Phi}, g_{2\Phi})\right\|_{\widetilde{L}^2_{T_\varepsilon, f_2(t)}(B_{1,\alpha})} \le \frac{\varepsilon C\sqrt{C_0}}{\sqrt{\delta}}, \tag{2.15}$$

where $\delta = \varepsilon\ln\frac{1}{\varepsilon}$, $\alpha = \frac{1-\delta}{2}$, $f_2(t) = \langle t\rangle^{1-2\delta}$, $f_3(t) = \langle t\rangle^{\frac{5}{4}-\delta}$, and C_0 is a positive constant.

Remark 2.1 Here, we could choose that $\varepsilon_* = \frac{1}{200}$ and $C_* = \max\{16eC\sqrt{C_0}\lambda + 8\lambda C_1, 4eC^4\sqrt{C_0C_2}\}$, see (3.27) and (4.14). If κ_1 and κ_2 are small enough such that

$$|\kappa_1| + |\kappa_2| \le \frac{1}{\ln\frac{1}{\varepsilon}}\exp\left(-\frac{1}{\varepsilon\ln\frac{1}{\varepsilon}}\right), \tag{2.16}$$

the condition on τ_0 (2.12) reduces to

$$\tau_0 \geq \frac{2C_*}{\ln \frac{1}{\varepsilon}}.$$

Specially, we can choose that

$$\kappa_1 = \kappa_2 = 0.$$

Remark 2.2 We note that a similar path is followed by [16], and that energy estimates for the ensuring linear problem lead to the maximal time of existence $\mathcal{O}(\exp(\varepsilon^{-1}/\ln \varepsilon^{-1}))$. However, our method is different from [16], we use anisotropic Littlewood-Paley energy estimates in tangentially analytic norms, inspired by the ones previously used by Zhang and Zhang [44]. Besides, we should impose a smallness assumption on the far-field states but [16] just need it to be a uniform constant. And the assumption about the initial value of radius is different. In [16] it is independent of the far-field state, while in our paper, the initial value of the analytic radius is related with the far-field states. The lifespan of (1.1) is at least $\mathcal{O}(\varepsilon^{-\frac{4}{3}})$ in the result of Zhang and Zhang for the 3D Prandtl system. In [44], the authors raise the question of "whether the lifespan obtained in Theorem 1.1 is sharp". In [16], the authors give a positive answer to this question in the two dimensional space, and prove that the Prandtl system has an almost global existence. In this paper we give a positive answer to this question in the three dimensional space.

In the final part of this subsection, we will explain the main difficulties and the ideas introduced in this paper. The degeneracy in the viscosity dissipation coupled with the loss of derivative in the non local term is the main difficulty in the well-posedness theories. Thus, the main enemies to getting a longer lifespan are the terms $w\partial_z\varphi$ and $w\partial_z^2\varphi$ in the perturbation equation (5.4) and the perturbed quantity equation (5.5) respectively. As $w = -\int_0^z (\partial_x u + \partial_y v)d\tilde{z}$, this term loses one tangential derivative. In the case of 2D, these bad terms can be handled under the Oleinik's monotonicity assumption using some kind of cancellation properties, cf. [1, 16, 32]. A good unknown inspired by [1, 32] plays a fundamental role in our proof. The main idea of our paper is to introduce the new linearly-good unknowns $g_1 = \partial_z u - u\frac{\partial_z^2\varphi}{\partial_z\varphi}$ and $g_2 = \partial_z v - v\frac{\partial_z^2\varphi}{\partial_z\varphi}$ to produce damping terms $\frac{1}{\langle t\rangle}g_1$ and $\frac{1}{\langle t\rangle}g_2$ respectively in the linear equation (2.1). This change of variable is directly motivated by [16, 32] when $\partial_z\varphi > 0$. When the far-field states (κ_1, κ_2) obey the fine condition in (2.12), especially, they satisfy (2.16). We can show there exists a unique almost global existence solution for this new system.

To capture the sharp time decay from the heat equation, and to explore certain cancellations in the nonlinear terms of the Prandtl equations, we choose the boundary lift $\varphi(t, z)$ to be the Gauss error function $\mathrm{erf}(\frac{z}{\sqrt{4\langle t\rangle}})$, see (5.3). The equations of u and v contain terms that are linear and quadratic in φ. We choose the lift φ so that the quadratic terms in φ vanish, and we only need to deal with the linear ones in φ.

A main part of the proof is the *a priori* estimate (3.22) below. Our idea is to solve the partial differential equation (2.1) for the prognostic variable (g_1, g_2) with a nonlinear ODE (2.8) for the analytic band $\theta(t)$ simultaneously. The phase function $\Phi(t, \xi)$ does not decrease to less than $\frac{3\tau_0}{4}|\xi|$ within the lifespan follows from the time integrability of the dissipative terms presented on the left side of (3.22).

3 The Proof of the Existence Part of Theorem 2.1

This section is to derive some *a priori* estimates of the solution for the 3D Prandtl boundary layer system on the time interval $[0, \exp(\frac{\varepsilon^{-1}}{\ln(\varepsilon^{-1})})]$, which are crucial for proving Theorem 2.1.

The general method to prove the existence result for the nonlinear partial equations is first to construct a sequence of approximate solutions, then perform uniform estimates for such approximate sequence, and pass to the limit of the approximate problem, see [16]. For simplicity, here we only present the *a priori* estimates for smooth enough solutions of (2.1) in the analytical framework.

The main result of this part can be stated as follows.

Proposition 3.1 (*A priori* estimates) *Let* $\delta = \varepsilon \ln \frac{1}{\varepsilon}$ *and* $\alpha = \frac{1-\delta}{2}$. *Suppose* g_i *with* $g_{i\Phi} \in L^\infty([0, T_1]; X_{1,\alpha})$ *and* $g_{i\Phi} \in \widetilde{L}^2_{T_1}(B_{1,\alpha})$, $i = 1, 2$ *is the solution to the Prandtl boundary layer system* (2.1), *where* $T_1 = \exp(\frac{\varepsilon^{-1}}{\ln(\varepsilon^{-1})})$. *Then there exist* $C_* > 0$ *and* $\varepsilon_* > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_*]$, *when the initial data* $e^{\tau_0|D_h|} g_{i0} \in X_{1,\frac{1}{2}}$, $i = 1, 2$, *satisfying* (2.11), *coupling with* τ_0 *satisfying* (2.12), *the estimates*

$$\left\|(g_{1\Phi}, g_{2\Phi})\right\|_{\widetilde{L}^\infty_{T_1, f_3(t)}(X_{1,\alpha})} \le C\varepsilon, \tag{3.1}$$

$$\left\|(g_{1\Phi}, g_{2\Phi})\right\|_{\widetilde{L}^2_{T_1, f_2(t)}(B_{1,\alpha})} \le \frac{\varepsilon C\sqrt{C_0}}{\sqrt{\delta}}, \tag{3.2}$$

and

$$\Phi(t, \xi) \ge \frac{3\tau_0}{4}|\xi|, \tag{3.3}$$

hold for any $\epsilon \in (0, \epsilon_*]$ *and any* $t \in [0, T_1]$, *where* $f_2(t) = \langle t\rangle^{1-2\delta}$, $f_3(t) = \langle t\rangle^{\frac{5}{4}-\delta}$ *and* C_0 *is a positive constant.*

3.1 *A Priori* Estimates

In what follows, we shall always assume that $t < T^*$ with T^* being determined as following:

$$T^* \doteq \sup\left\{t \in (0, T_1];\ \theta(s) < \frac{\tau_0}{\lambda},\ \forall s \in [0, t]\right\}. \tag{3.4}$$

In view of (2.1), (2.5) and (2.7), it is easy to observe that $g = (g_1, g_2)^\top$ and $\mathbf{F} = (F_1, F_2)^\top$ satisfy

$$\begin{aligned}
&\partial_t g_\Phi + \lambda\dot{\theta}|D_h|g_\Phi - \partial_z^2 g_\Phi + \frac{1}{\langle t\rangle} g_\Phi + \kappa_1\partial_z\varphi\nabla_h^\perp v_\Phi + \kappa_1\varphi\partial_x g_\Phi \\
&\quad - \kappa_2\partial_z\varphi\nabla_h^\perp u_\Phi + \kappa_2\varphi\partial_y g_\Phi = \mathbf{F},
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
F_i = &-\big(g_1\nabla_h^\perp v\big)_\Phi + \frac{z}{2\langle t\rangle}\big(u\nabla_h^\perp v\big)_\Phi - (u\partial_x g)_\Phi + \big(g_2\nabla_h^\perp u\big)_\Phi - \frac{z}{2\langle t\rangle}\big(v\nabla_h^\perp u\big)_\Phi - (v\partial_y g)_\Phi \\
&- (w\partial_z g)_\Phi + \frac{1}{2\langle t\rangle}(w\mathbf{u})_\Phi, \quad i = 1, 2.
\end{aligned} \tag{3.6}$$

Applying the dyadic operator Δ_k^h to (3.5), and then taking the L^2_+ inner product of the simplified equation (3.5) with $e^{2\psi}\Delta_k^h g_\Phi$, and using integration by parts for the linear terms, the boundary condition of g_i in (2.1) and the cancellation

$$\kappa_i \int_{\mathbb{R}^3_+} e^{\psi}\varphi\Delta_k^h\nabla_h g_{i\Phi} e^{\psi}\Delta_k^h g_{i\Phi}\,dxdydz = 0, \quad i = 1, 2,$$

we obtain

$$\begin{aligned}
&\sum_{i=1}^{2}\Big(\frac{1}{2}\frac{d}{dt}\|e^{\psi}\Delta_k^h g_{i\Phi}\|^2_{L^2_+} + c\lambda\frac{d\theta(t)}{dt}2^k\|e^{\psi}\Delta_k^h g_{i\Phi}\|^2_{L^2_+} + \|e^{\psi}\Delta_k^h\partial_z g_{i\Phi}\|^2_{L^2_+}\\
&\quad + \frac{\alpha(1-2\alpha)}{4\langle t\rangle}\|\zeta e^{\psi}\Delta_k^h g_{i\Phi}\|^2_{L^2_+} + \frac{2-\alpha}{2\langle t\rangle}\|e^{\psi}\Delta_k^h g_{i\Phi}\|^2_{L^2_+}\Big)\\
&\le \big|\big(e^{\psi}\Delta_k^h\mathbf{F}_\Phi \mid e^{\psi}\Delta_k^h g_\Phi\big)_{L^2_+}\big| + \big|\big(-\kappa_1\partial_z\varphi\nabla_h^{\perp}v_\Phi \mid e^{\psi}\Delta_k^h g_\Phi\big)_{L^2_+}\big|\\
&\quad + \big|\big(\kappa_2\partial_z\varphi\nabla_h^{\perp}u_\Phi \mid e^{\psi}\Delta_k^h g_\Phi\big)_{L^2_+}\big|.
\end{aligned} \tag{3.7}$$

To bound the linear terms, we need the following lemma.

Lemma 3.1 *Under the conditions in Proposition 3.1, for any fixed $\alpha \in [\frac{1}{4}, \frac{1}{2}]$, for arbitrary $\beta \in [0, 1]$, we have*

$$\begin{aligned}
&\sum_{i=1}^{2}\Big(\|e^{\psi}\Delta_k^h\partial_z g_{i\Phi}\|^2_{L^2_+} + \frac{\alpha(1-2\alpha)}{4\langle t\rangle}\|\zeta e^{\psi}\Delta_k^h g_{i\Phi}\|^2_{L^2_+} + \frac{2-\alpha}{2\langle t\rangle}\|e^{\psi}\Delta_k^h g_{i\Phi}\|^2_{L^2_+}\Big)\\
&\ge \sum_{i=1}^{2}\Big(\frac{\beta}{2}\|e^{\psi}\Delta_k^h\partial_z g_{i\Phi}\|^2_{L^2_+} + \frac{\alpha(1-2\alpha)}{4\langle t\rangle}\|\zeta e^{\psi}\Delta_k^h g_{i\Phi}\|^2_{L^2_+}\\
&\quad + \frac{2+(1-\beta)\alpha}{2\langle t\rangle}\|e^{\psi}\Delta_k^h g_{i\Phi}\|^2_{L^2_+}\Big),
\end{aligned} \tag{3.8}$$

and

$$|\kappa_i|\big|\big(e^{\psi}\partial_z\varphi\Delta_k^h U(\nabla_h g_j)_\Phi \mid e^{\psi}\Delta_k^h g_{q\Phi}\big)_{L^2_+}\big| \le 2^k\dot{\theta}\|e^{\psi}\Delta_k^h g_{j\Phi}\|_{L^2_+}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L^2_+}, \tag{3.9}$$

where $t \in [0, T^*]$, for any $k \in \mathbb{Z}$, $i, j, q = 1, 2$.

It is trivial to prove (3.8) by using the Treves inequality, so we omit the detail, see Lemma 3.3 in [16]. The proof of (3.9), see Lemma 5.2. Hence, using (3.7) and Lemma 3.1, we obtain

$$\begin{aligned}
&\sum_{i=1}^{2}\Big(\frac{1}{2}\frac{d}{dt}\|e^{\psi}\Delta_k^h g_{i\Phi}\|^2_{L^2_+} + c\lambda\frac{d\theta(t)}{dt}2^k\|e^{\psi}\Delta_k^h g_{i\Phi}\|^2_{L^2_+} + \frac{\beta}{2}\|e^{\psi}\Delta_k^h\partial_z g_{i\Phi}\|^2_{L^2_+}\\
&\quad + \frac{\alpha(1-2\alpha)}{4\langle t\rangle}\|\zeta e^{\psi}\Delta_k^h g_{i\Phi}\|^2_{L^2_+} + \frac{2+(1-\beta)\alpha}{2\langle t\rangle}\|e^{\psi}\Delta_k^h g_{i\Phi}\|^2_{L^2_+}\Big)\\
&\le \sum_{i=1}^{2}\big|\big(e^{\psi}\Delta_k^h F_{i\Phi} \mid e^{\psi}\Delta_k^h g_{i\Phi}\big)_{L^2_+}\big| + 2^k\dot{\theta}(t)\big(\|e^{\psi}\Delta_k^h g_{1\Phi}\|_{L^2_+} + \|e^{\psi}\Delta_k^h g_{2\Phi}\|_{L^2_+}\big)^2.
\end{aligned} \tag{3.10}$$

To bound the nonlinear terms, we need the following key estimates. The proof is presented in Sect. 5.3 for simplicity, as it follows the same line as Lemma 3.1 in [16].

Lemma 3.2 (Bounds for the diagnostic variables) *Let* $\psi(t,z)$ *be given by* (2.4), *with* $\alpha \in [\frac{1}{4}, \frac{1}{2}]$. *Define* U *and* W *by* (2.2) *and* (2.3) *respectively. Under the conditions in Proposition* 3.1, *for* $k \in \mathbb{Z}$, $i = 1, 2$, *we have*

$$\big\|\Delta_k^h U(g_{i\Phi})\big\|_{L_z^\infty L_{x,y}^2} \le C\langle t\rangle^{\frac{1}{4}}\big\|e^{\psi}\Delta_k^h g_{i\Phi}\big\|_{L_{x,y,z}^2}, \tag{3.11}$$

$$\begin{aligned}&\big\|\Delta_k^h W(g_{1\Phi}, g_{2\Phi})\big\|_{L_z^\infty L_{x,y}^2}\\ &\quad \le C\langle t\rangle^{\frac{3}{4}}2^k\big\|e^{\psi}\Delta_k^h g_{1\Phi}\big\|_{L_{x,y,z}^2} + C\langle t\rangle^{\frac{3}{4}}2^k\big\|e^{\psi}\Delta_k^h g_{2\Phi}\big\|_{L_{x,y,z}^2},\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\big\|e^{\psi}\Delta_k^h U(g_{i\Phi})\big\|_{L_{x,y,z}^2}\\ &\quad \le C\big\|e^{\psi}\Delta_k^h g_{i\Phi}\big\|_{L_{x,y,z}^2}^{\frac{1}{2}}\Big(\langle t\rangle^{\frac{3}{4}}\big\|e^{\psi}\Delta_k^h\partial_z g_{i\Phi}\big\|_{L_{x,y,z}^2}^{\frac{1}{2}} + \langle t\rangle^{\frac{1}{2}}\big\|\zeta e^{\psi}\Delta_k^h g_{i\Phi}\big\|_{L_{x,y,z}^2}^{\frac{1}{2}}\Big),\end{aligned} \tag{3.13}$$

$$\big\|e^{\psi}\Delta_k^h U(g_{i\Phi})\big\|_{L_{x,y}^2 L_z^\infty} \le C\langle t\rangle^{\frac{1}{4}}\big\|e^{\psi}\Delta_k^h g_{i\Phi}\big\|_{L_{x,y,z}^2}. \tag{3.14}$$

Basing on these key estimates, we have the following important lemma to bound the nonlinear terms for every $k \in \mathbb{Z}$. By the definition of $\Phi(t,\xi)$, for any $t < T^*$, there holds the following convex inequality

$$\Phi(t,\xi) \le \Phi(t,\xi-\eta) + \Phi(t,\eta), \quad \forall \xi, \eta \in \mathbb{R}^2. \tag{3.15}$$

The key lemma is stated as following.

Lemma 3.3 *Under the conditions in Proposition* 3.1, *for* $i, j, q = 1, 2$, $t \in [0, T^*]$, *we have*,

$$\int_0^t \langle t'\rangle^{\frac{5}{2}-2\delta}\big|\big(e^{\psi}\Delta_k^h\big(g_i U(\nabla_h g_j)\big)_\Phi \mid e^{\psi}\Delta_k^h g_{q\Phi}\big)_{L_+^2}\big|dt' \lesssim d_k^2 2^{-2k}A_{ijq}, \tag{3.16}$$

$$\int_0^t \langle t'\rangle^{\frac{5}{2}-2\delta}\Big|\frac{1}{\langle t'\rangle}\big(e^{\psi}z\Delta_k^h\big(U(g_i)U(\nabla_h g_j)\big)_\Phi \mid e^{\psi}\Delta_k^h g_{q\Phi}\big)_{L_+^2}\Big|dt' \lesssim d_k^2 2^{-2k}A_{ijq}, \tag{3.17}$$

$$\int_0^t \langle t'\rangle^{\frac{5}{2}-2\delta}\big|\big(e^{\psi}\Delta_k^h\big(U(g_i)\nabla_h g_j\big)_\Phi \mid e^{\psi}\Delta_k^h g_{q\Phi}\big)_{L_+^2}\big|dt' \lesssim d_k^2 2^{-2k}A_{ijq}, \tag{3.18}$$

$$\int_0^t \langle t'\rangle^{\frac{5}{2}-2\delta}\big|\big(e^{\psi}\Delta_k^h\big(W(g_1,g_2)\partial_z g_j\big)_\Phi \mid e^{\psi}\Delta_k^h g_{q\Phi}\big)_{L_+^2}\big|dt' \lesssim d_k^2 2^{-2k}A_{12q}, \tag{3.19}$$

$$\int_0^t \langle t'\rangle^{\frac{5}{2}-2\delta}\big|\big(e^{\psi}\Delta_k^h\big(W(g_1,g_2)U(g_j)\big)_\Phi \mid e^{\psi}\Delta_k^h g_{q\Phi}\big)_{L_+^2}\big|dt' \lesssim d_k^2 2^{-2k}A_{12q}. \tag{3.20}$$

where $A_{ijq} = \|(g_{i\Phi}, g_{j\Phi})\|_{\widetilde{L}^2_{t,f_1(t)}(X_{\frac{3}{2},\alpha})}\|g_{q\Phi}\|_{\widetilde{L}^2_{t,f_1(t)}(X_{\frac{3}{2},\alpha})}$, *and* $f_1(t) = \langle t\rangle^{\frac{5}{2}-2\delta}\dot{\theta}(t)$.

Proof Applying Bony's decomposition (5.1) for $g_i U(\nabla_h g_j)$ for the (x,y) variables, the property of $\Phi(t,\xi)$ in (3.15), the support properties of the Fourier transform, Hölder's inequality, (3.11) in Lemma 3.2 and the anisotropic Bernstein type inequality in Lemma 5.1, we get

$$
\begin{aligned}
&\int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta} \big|\big(e^{\psi}\Delta_k^h\big(g_i U(\nabla_h g_j)\big)_\Phi \mid e^{\psi}\Delta_k^h g_{q\Phi}\big)_{L^2_+}\big| dt' \\
&\quad\le \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta} \big(\big\|e^{\psi}\Delta_k^h\big(T^h_{g_i}U(\nabla_h g_j)\big)_\Phi\big\|_{L^2_+} + \big\|e^{\psi}\Delta_k^h\big(T'_{U(\nabla_h g_j)}g_i\big)_\Phi\big\|_{L^2_+}\big)\big\|e^{\psi}\Delta_k^h g_{q\Phi}\big\|_{L^2_+} dt' \\
&\quad\le \sum_{|k-k'|\le 4} \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta} \big\|e^{\psi} S^h_{k'-1}\widetilde{g_{i\Phi}}\big\|_{L^2_z L^\infty_{x,y}} \big\|\Delta^h_{k'}U(\nabla_h g_{j\Phi})\big\|_{L^\infty_z L^2_{x,y}} \big\|e^{\psi}\Delta_k^h g_{q\Phi}\big\|_{L^2_+} dt' \\
&\qquad + \sum_{k'\ge k-3} \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta} \big\|S^h_{k'+2}U(\widetilde{\nabla_h g_{j\Phi}})\big\|_{L^\infty_z L^\infty_{x,y}} \big\|e^{\psi}\Delta^h_{k'} g_{i\Phi}\big\|_{L^2_z L^2_{x,y}} \big\|e^{\psi}\Delta_k^h g_{q\Phi}\big\|_{L^2_+} dt' \\
&\quad\le C \sum_{\substack{|k-k'|\le 4\\ l\le k'-2}} 2^{l+k'} \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta+\frac{1}{4}} \big\|e^{\psi}\Delta^h_l g_{i\Phi}\big\|_{L^2_+} \big\|e^{\psi}\Delta^h_{k'} g_{j\Phi}\big\|_{L^2_+} \big\|e^{\psi}\Delta_k^h g_{q\Phi}\big\|_{L^2_+} dt' \\
&\qquad + C \sum_{\substack{k'\ge k-3\\ l\le k'+1}} 2^{2l} \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta+\frac{1}{4}} \big\|e^{\psi}\Delta^h_{k'} g_{i\Phi}\big\|_{L^2_+} \big\|e^{\psi}\Delta^h_l g_{j\Phi}\big\|_{L^2_+} \big\|e^{\psi}\Delta_k^h g_{q\Phi}\big\|_{L^2_+} dt' \\
&\quad\le C \sum_{|k-k'|\le 4} 2^{k'} \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta+\frac{1}{4}} \|g_{i\Phi}\|_{X_{1,\alpha}} \big\|e^{\psi}\Delta^h_{k'} g_{j\Phi}\big\|_{L^2_+} \big\|e^{\psi}\Delta_k^h g_{q\Phi}\big\|_{L^2_+} dt' \\
&\qquad + C \sum_{k'\ge k-3} 2^{k'} \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta+\frac{1}{4}} \|g_{j\Phi}\|_{X_{1,\alpha}} \big\|e^{\psi}\Delta^h_{k'} g_{i\Phi}\big\|_{L^2_+} \big\|e^{\psi}\Delta_k^h g_{q\Phi}\big\|_{L^2_+} dt' \\
&\quad\le C \sum_{|k-k'|\le 4} 2^{k'} \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta} \dot\theta(t') \big\|e^{\psi}\Delta^h_{k'} g_{j\Phi}\big\|_{L^2_+} \big\|e^{\psi}\Delta_k^h g_{q\Phi}\big\|_{L^2_+} dt' \\
&\qquad + C \sum_{k'\ge k-3} 2^{k'} \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta} \dot\theta(t') \big\|e^{\psi}\Delta^h_{k'} g_{i\Phi}\big\|_{L^2_+} \big\|e^{\psi}\Delta_k^h g_{q\Phi}\big\|_{L^2_+} dt' \\
&\quad\lesssim d_k^2 2^{-2k} \big\|(g_{i\Phi}, g_{j\Phi})\big\|_{\widetilde{L}^2_{T^*, f_1(t)}(X_{\frac{3}{2},\alpha})} \|g_{q\Phi}\|_{\widetilde{L}^2_{T^*, f_1(t)}(X_{\frac{3}{2},\alpha})}.
\end{aligned}
$$

Thus, we obtain (3.16). Next, we follow the same line to prove (3.17)–(3.20), for simplicity, we provide the detail proof in Sect. 5.4. □

At this step, we will prove the following key lemma to bound the nonlinear terms.

Lemma 3.4 *Under the conditions in Proposition 3.1, we get*

$$
\int_0^t \langle t'\rangle^{\frac{5}{2}-2\delta} \big|\big(e^{\psi}\Delta_k^h \boldsymbol{F}_\Phi \mid e^{\psi}\Delta_k^h g_\Phi\big)_{L^2_+}\big| dt' \lesssim d_k^2 2^{-2k} \big\|(g_{1\Phi}, g_{2\Phi})\big\|^2_{\widetilde{L}^2_{t, f_1(t)}(X_{\frac{3}{2},\alpha})}, \quad (3.21)
$$

where $\boldsymbol{F}$ is defined by (3.6), $t \in [0, T^*]$.

Proof For u, v and w can be computed directly from the linear operators U and W respectively. Thus, applying Lemma 3.3 to the nonlinear terms, using Cauchy-Schwarz's inequality, we immediately obtain Lemma 3.4. □

3.2 Proof of Proposition 3.1

Proof of Proposition 3.1 Without loss of generality, we assume that $\varepsilon \le \varepsilon_* = \frac{1}{200}$. Let

$$\delta = \varepsilon \ln \frac{1}{\varepsilon}, \qquad \alpha = \frac{1-\delta}{2}, \qquad \beta = \frac{\delta}{1-\delta}.$$

At this step, upon multiplying by the term $\langle t \rangle^{\frac{5}{2}-2\delta}$ in (3.10), we arrive at, for all $0 < t < T^*$,

$$\begin{aligned}
&\sum_{i=1}^{2}\Big[\frac{d}{dt}\Big(\frac{1}{2}\langle t\rangle^{\frac{5}{2}-2\delta}\big\|e^{\psi}\Delta_k^h g_{i\Phi}\big\|_{L^2_+}^2\Big) + c\lambda\langle t\rangle^{\frac{5}{2}-2\delta}\frac{d\theta(t)}{dt}2^k\big\|e^{\psi}\Delta_k^h g_{i\Phi}\big\|_{L^2_+}^2\\
&\quad + \frac{\delta\langle t\rangle^{\frac{5}{2}-2\delta}}{C_0}\Big(\frac{1}{\langle t\rangle}\big\|e^{\psi}\Delta_k^h g_{i\Phi}\big\|_{L^2_+}^2 + \big\|e^{\psi}\Delta_k^h\partial_z g_{i\Phi}\big\|_{L^2_+}^2 + \frac{1}{\langle t\rangle}\big\|\zeta e^{\psi}\Delta_k^h g_{i\Phi}\big\|_{L^2_+}^2\Big)\Big]\\
&\le \langle t\rangle^{\frac{5}{2}-2\delta}\sum_{i=1}^{2}\big|\big(e^{\psi}\Delta_k^h F_{i\Phi}\ \big|\ e^{\psi}\Delta_k^h g_{i\Phi}\big)_{L^2_+}\big| + \langle t\rangle^{\frac{5}{2}-2\delta}2^k\dot\theta(t)\big\|e^{\psi}\Delta_k^h(g_{1\Phi},g_{2\Phi})\big\|_{L^2_+}^2, \qquad (3.22)
\end{aligned}$$

for a universal constant C_0 which is independent of α and δ. Integrating the above bound on $[0, t]$, using Lemma 3.4, we get,

$$\begin{aligned}
&\sum_{i=1}^{2}\Big\{\sup_{t\in[0,T^*]}\langle t\rangle^{\frac{5}{2}-2\delta}\big\|e^{\psi}\Delta_k^h g_{i\Phi}\big\|_{L^2_+}^2 + \int_0^{T^*}\Big[c\lambda 2^k\langle t'\rangle^{\frac{5}{2}-2\delta}\frac{d\theta(t')}{dt'}\big\|e^{\psi}\Delta_k^h g_{i\Phi}\big\|_{L^2_+}^2\\
&\quad + \frac{\delta\langle t'\rangle^{\frac{5}{2}-2\delta}}{C_0}\Big(\frac{1}{\langle t'\rangle}\big\|e^{\psi}\Delta_k^h g_{i\Phi}\big\|_{L^2_+}^2 + \big\|e^{\psi}\Delta_k^h\partial_z g_{i\Phi}\big\|_{L^2_+}^2 + \frac{1}{\langle t'\rangle}\big\|\zeta e^{\psi}\Delta_k^h g_{i\Phi}\big\|_{L^2_+}^2\Big)\Big]dt'\Big\}\\
&\le C d_k^2 2^{-2k}\big\|(g_{1\Phi},g_{2\Phi})\big\|_{\widetilde{L}^2_{T^*,f_1(t)}(X_{\frac{3}{2},\alpha})}^2 + \Big(\big\|e^{\frac{\alpha z^2}{4}}e^{\tau_0|D_h|}g_{10}\big\|_{L^2_+}^2 + \big\|e^{\frac{\alpha z^2}{4}}e^{\tau_0|D_h|}g_{20}\big\|_{L^2_+}^2\Big).
\end{aligned} \qquad (3.23)$$

Taking square root of the above inequality and multiplying the resulting inequality by 2^k and summing over $K \in \mathbb{Z}$, we find,

$$\begin{aligned}
&\big\|(g_{1\Phi},g_{2\Phi})\big\|_{\widetilde{L}^\infty_{T^*,f_3(t)}(X_{1,\alpha})} + \sqrt{c\lambda}\big\|(g_{1\Phi},g_{2\Phi})\big\|_{\widetilde{L}^2_{T^*,f_1(t)}(X_{\frac{3}{2},\alpha})} + \frac{\sqrt{\delta}}{\sqrt{C_0}}\big\|(g_{1\Phi},g_{2\Phi})\big\|_{\widetilde{L}^2_{T^*,f_2(t)}(B_{1,\alpha})}\\
&\le \sqrt{C}\big\|(g_{1\Phi},g_{2\Phi})\big\|_{\widetilde{L}^2_{T^*,f_1(t)}(X_{\frac{3}{2},\alpha})} + C\big\|e^{\tau_0|D_h|}g_{10}\big\|_{X_{1,\frac{1}{2}}} + C\big\|e^{\tau_0|D_h|}g_{20}\big\|_{X_{1,\frac{1}{2}}}, \qquad (3.24)
\end{aligned}$$

where $f_3(t) = \langle t\rangle^{5/4-\delta}$. Taking λ to be a large enough positive constant so that

$$c\lambda \ge C, \qquad (3.25)$$

thus (3.24) reduces to

$$
\begin{aligned}
&\left\|(g_{1\Phi},g_{2\Phi})\right\|_{\widetilde{L}^{\infty}_{T^*,f_3(t)}(X_{1,\alpha})}+\frac{\sqrt{\delta}}{\sqrt{C_0}}\left\|(g_{1\Phi},g_{2\Phi})\right\|_{\widetilde{L}^{2}_{T^*,f_2(t)}(B_{1,\alpha})}\\
&\quad\le C\left\|e^{\tau_0|D_h|}g_{10}\right\|_{X_{1,\frac{1}{2}}}+C\left\|e^{\tau_0|D_h|}g_{20}\right\|_{X_{1,\frac{1}{2}}}\le C\varepsilon. \qquad (3.26)
\end{aligned}
$$

Hence in view of (2.8) and (2.10), using Hölder's inequality, by some simple calculations, for all $t\in[0,T^*]$, we arrive

$$
\begin{aligned}
&\int_0^t\big(|\kappa_1|+|\kappa_2|\big)\langle t'\rangle^{\frac{1}{4}}\left\|e^{\psi}\partial_z\varphi\right\|_{L^2_z}dt'+\int_0^t\langle t'\rangle^{\frac{1}{4}}\left\|(g_{1\Phi},g_{2\Phi})\right\|_{X_{1,\alpha}}dt'\\
&\qquad+\int_0^t\langle t'\rangle^{\frac{3}{4}}\left\|(\partial_z g_{1\Phi},\partial_z g_{2\Phi})\right\|_{X_{1,\alpha}}dt'+\int_0^t\langle t'\rangle^{\frac{1}{4}}\left\|(\zeta g_{1\Phi},\zeta g_{2\Phi})\right\|_{X_{1,\alpha}}dt'\\
&\le C_1T_1\big(|\kappa_1|+|\kappa_2|\big)+\left\|(g_{1\Phi},g_{2\Phi})\right\|_{\widetilde{L}^{2}_{T^*,f_2(t)}(B_{1,\alpha})}\left(\int_0^t\langle t'\rangle^{-1+2\delta}dt'\right)^{\frac{1}{2}}\\
&\le\frac{C\sqrt{C_0}\varepsilon\langle T_1\rangle^{\delta}}{\delta}+C_1T_1\big(|\kappa_1|+|\kappa_2|\big)\\
&\le\frac{\tau_0}{4\lambda},
\end{aligned}
$$

where τ_0 satisfies (2.12) with

$$
C_*\ge 16eC\sqrt{C_0}\lambda+8\lambda C_1\quad\text{and}\quad 0<\varepsilon<\frac{1}{200}. \qquad (3.27)
$$

Then we immediately get

$$
\Phi(t,\xi)\ge\frac{3\tau_0}{4}|\xi|,\quad\text{on }\big[0,T^*\big].
$$

Using the classical bootstrap argument, we can prove that $T_1=T^*$ and (3.1)–(3.3) hold for any $t\in[0,T_1]$. Thus, we conclude the proof of Proposition 3.1. □

From Proposition 3.1, one can easily get the existence of the solution for the system (2.1) on $[0,T_1]$. The procedure of the proof of the existence part of Theorem 2.1 is classical, we refer the reader to [16] for more details. The core of the argument is the energy estimates stated in Proposition 3.1. In fact, such estimates are derived on a sequence of approximation solutions of the 3D Prandtl boundary layer system, so we omit the details for simplicity.

4 Uniqueness

This section is devoted to the proof of the uniqueness part of Theorem 2.1. Let $(g_1^{(1)},g_2^{(1)})$ and $(g_1^{(2)},g_2^{(2)})$ be two solutions of (2.1) on $[0,T_2]$ with the same initial data (g_{10},g_{20}), satisfying the conditions in Theorem 2.1. Without loss of generality, we assume that $T_2\le T_1$. Specially, considering the following ODE

$$
\begin{cases}
-\frac{d}{dt}\theta^i(t)+\|(g^{(i)}_{1\Phi^i},g^{(i)}_{2\Phi^i})\|_{B_{1,\alpha}}+(|\kappa_1|+|\kappa_2|)\langle t\rangle^{\frac{1}{4}}\|e^{\psi}\partial_z\varphi\|_{L^2_z}=0,\\
\theta^i(0)=0,
\end{cases} \qquad (4.1)
$$

where

$$\Phi^i(t,\xi) \doteq \big(\tau_0 - \lambda\theta^i(t)\big)|\xi|, \quad i = 1, 2, \tag{4.2}$$

we assume that the solutions $(g_1^{(i)}, g_2^{(i)})$, $i = 1, 2$, satisfy

$$\big\|\big(g^{(i)}_{1\Phi^i}, g^{(i)}_{2\Phi^i}\big)\big\|_{\widetilde{L}^\infty_{T_2, f_3(t)}(X_{1,\alpha})} \le C\varepsilon, \tag{4.3}$$

$$\big\|\big(g^{(i)}_{1\Phi^i}, g^{(i)}_{2\Phi^i}\big)\big\|_{\widetilde{L}^2_{T_2, f_2(t)}(B_{1,\alpha})} \le \frac{\varepsilon C\sqrt{C_0}}{\sqrt{\delta}}, \tag{4.4}$$

and

$$\Phi^i(t,\xi) \ge \frac{3\tau_0}{4}|\xi|, \qquad \theta^i(t) \le \frac{\tau_0}{4\lambda}, \tag{4.5}$$

for all $t \in [0, T_2]$.

Denote $\overline{g_1} = g_1^{(1)} - g_1^{(2)}$, $\overline{g_2} = g_2^{(1)} - g_2^{(2)}$. From (2.1), we have

$$\begin{cases} \partial_t\overline{g} - \partial_z^2\overline{g} + \frac{1}{\langle t\rangle}\overline{g} + \kappa_1\partial_z\varphi\nabla_h^\perp\overline{v} + \kappa_1\varphi\partial_x\overline{g} - \kappa_2\partial_z\varphi\nabla_h^\perp\overline{u} + \kappa_2\varphi\partial_y\overline{g} = \overline{F}, \\ \overline{g}|_{z=0} = \overline{g}|_{z=\infty} = (0,0), \qquad \overline{g}|_{t=0} = (0,0), \end{cases} \tag{4.6}$$

where

$$\begin{aligned} \overline{F}_i = & -g_1^{(1)}\nabla_h^\perp\overline{v} - \overline{g_1}\nabla_h^\perp v^{(2)} + \frac{z}{2\langle t\rangle}u^{(1)}\nabla_h^\perp\overline{v} + \frac{z}{2\langle t\rangle}\overline{u}\nabla_h^\perp v^{(2)} - u^{(1)}\partial_x\overline{g} - \overline{u}\partial_x g^{(2)} \\ & + g_2^{(1)}\nabla_h^\perp\overline{u} + \overline{g_2}\nabla_h^\perp u^{(2)} - \frac{z}{2\langle t\rangle}v^{(1)}\nabla_h^\perp\overline{u} - \frac{z}{2\langle t\rangle}\overline{v}\nabla_h^\perp u^{(2)} - v^{(1)}\partial_y\overline{g} - \overline{v}\partial_y g^{(2)} \\ & - \overline{w}\partial_z g^{(1)} - w^{(2)}\partial_z\overline{g} + \frac{1}{2\langle t\rangle}\overline{w}\mathbf{u}^{(1)} + \frac{1}{2\langle t\rangle}w^{(2)}\overline{\mathbf{u}}, \quad i = 1, 2, \end{aligned}$$

$\overline{\mathbf{u}} = (\overline{u}, \overline{v})^\top$, $\mathbf{u}^{(1)} = (u^{(1)}, v^{(1)})^\top$, $g^{(j)} = (g_1^{(j)}, g_2^{(j)})$, $j = 1, 2$, $\overline{g} = (\overline{g_1}, \overline{g_2})$, $\overline{u} = u^{(1)} - u^{(2)} = U(\overline{g_1})$, $\overline{v} = v^{(1)} - v^{(2)} = U(\overline{g_2})$, and $\overline{w} = w^{(1)} - w^{(2)} = W(\overline{g_1}, \overline{g_2})$. Denote the phase function $\widetilde{\Phi}$ as following

$$\widetilde{\Phi}(t,\xi) \doteq \left(\frac{3\tau_0}{4} - \lambda\Theta(t)\right)|\xi|, \quad \Theta(t) \doteq \big(\theta^1 + \theta^2\big)(t). \tag{4.7}$$

From (4.5), we have

$$\Theta(t) \le \frac{\tau_0}{2\lambda}, \quad \forall t \in [0, T_2],$$

so that there holds

$$\widetilde{\Phi}(t,\xi) \le \widetilde{\Phi}(t,\xi-\eta) + \widetilde{\Phi}(t,\eta), \quad \forall \xi,\eta \in \mathbb{R}^2.$$

By the definition of $X_{1,\alpha}$,

$$\|g^{(i)}_{j\widetilde{\Phi}}\|_{X_{1,\alpha}} = \sum_{k\in\mathbb{Z}} 2^k \|e^\psi \Delta_k^h g^{(i)}_{j\widetilde{\Phi}}\|_{L^2_+}, \quad i, j \in \{1, 2\},$$

using (2.5) and (2.9), one has

$$\begin{aligned}2^{2k}\|e^{\psi}\Delta_k^h g^{(i)}_{j\widetilde{\Phi}}\|^2_{L^2_+} &= 2^{2k}\int_0^\infty\int_{\mathbb{R}^2} e^{2\psi}\varphi^2\big(2^{-k}|\xi|\big)e^{2\widetilde{\Phi}(t,\xi)}\big|\widehat{g^{(i)}_j}(t,\xi,z)\big|^2 d\xi dz\\ &\le C\int_0^\infty\int_{\mathbb{R}^2} e^{2\psi}\varphi^2\big(2^{-k}|\xi|\big)e^{2\Phi^i(t,\xi)}|\xi|^2 e^{-\frac{\tau_0}{2}|\xi|}\big|\widehat{g^{(i)}_j}(t,\xi,z)\big|^2 d\xi dz\\ &\le \frac{C_2}{\tau_0^2}\int_0^\infty\int_{\mathbb{R}^2} e^{2\psi}\varphi^2\big(2^{-k}|\xi|\big)e^{2\Phi^i(t,\xi)}\big|\widehat{g^{(i)}_j}(t,\xi,z)\big|^2 d\xi dz\\ &= \frac{C_2}{\tau_0^2}\|e^{\psi}\Delta_k^h g^{(i)}_{j\Phi^i}(t)\|^2_{L^2_+},\end{aligned}$$

and

$$\|g^{(i)}_{j\widetilde{\Phi}}\|_{X_{2,\alpha}} \le \frac{\sqrt{C_2}}{\tau_0}\|g^{(i)}_{j\Phi^i}\|_{X_{1,\alpha}}. \tag{4.8}$$

For $\widetilde{\Phi}(t,\xi)$ defined in (4.7) and the equations of $\overline{g_1}$ and $\overline{g_2}$ in (4.6), we get

$$\begin{aligned}&\partial_t\overline{g}_{\widetilde{\Phi}} + \lambda\frac{d\Theta(t)}{dt}|D_h|\overline{g}_{\widetilde{\Phi}} - \partial_z^2\overline{g}_{\widetilde{\Phi}} + \frac{1}{\langle t\rangle}\overline{g}_{\widetilde{\Phi}} + \kappa_1\partial_z\varphi\nabla_h^{\perp}\overline{v}_{\widetilde{\Phi}} + \kappa_1\varphi\partial_x\overline{g}_{\widetilde{\Phi}}\\ &- \kappa_2\partial_z\varphi\nabla_h^{\perp}\overline{u}_{\widetilde{\Phi}} + \kappa_2\varphi\partial_x\overline{g}_{\widetilde{\Phi}} = \overline{F}_{\widetilde{\Phi}}.\end{aligned}$$

Applying Δ_k^h to the above equation and taking L^2_+ inner product of the resulting equation with $e^{2\psi}\Delta_k^h\overline{g}_{\widetilde{\Phi}}$. and using integration by parts for the linear terms, the boundary condition of $\overline{g}$ in (4.6) and the cancellation

$$\kappa_i\int_{\mathbb{R}^3_+} e^{\psi}\varphi\Delta_k^h\nabla_h\overline{g_i}_{\widetilde{\Phi}}e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}dxdydz = 0, \quad i=1,2,$$

we obtain

$$\begin{aligned}&\sum_{i=1}^2\Big(\frac12\frac{d}{dt}\|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|^2_{L^2_+} + c\lambda\frac{d\Theta(t)}{dt}2^k\|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|^2_{L^2_+} + \|e^{\psi}\Delta_k^h\partial_z\overline{g_i}_{\widetilde{\Phi}}\|^2_{L^2_+}\\ &\quad + \frac{\alpha(1-2\alpha)}{4\langle t\rangle}\|\zeta e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|^2_{L^2_+} + \frac{2-\alpha}{2\langle t\rangle}\|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|^2_{L^2_+}\Big)\\ &\le \big|\big(e^{\psi}\Delta_k^h\overline{F}_{\Phi}\,\big|\,e^{\psi}\Delta_k^h\overline{g}_{\Phi}\big)_{L^2_+}\big| + \big|\big(-\kappa_1\partial_z\varphi\nabla_h^{\perp}\overline{v}_{\widetilde{\Phi}}\,\big|\,e^{\psi}\Delta_k^h\overline{g}_{\widetilde{\Phi}}\big)_{L^2_+}\big|\\ &\quad + \big|\big(\kappa_2\partial_z\varphi\nabla_h^{\perp}\overline{u}_{\widetilde{\Phi}}\,\big|\,e^{\psi}\Delta_k^h\overline{g}_{\widetilde{\Phi}}\big)_{L^2_+}\big|.\end{aligned} \tag{4.9}$$

Bounding the linear terms with the same trick as Lemma 3.1, we omit the detail for simplicity. Hence, using (4.9), we obtain

$$\begin{aligned}&\sum_{i=1}^2\Big(\frac12\frac{d}{dt}\|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|^2_{L^2_+} + c\lambda\frac{d\Theta(t)}{dt}2^k\|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|^2_{L^2_+} + \frac{\beta}{2}\|e^{\psi}\Delta_k^h\partial_z\overline{g_i}_{\widetilde{\Phi}}\|^2_{L^2_+}\\ &\quad + \frac{\alpha(1-2\alpha)}{4\langle t\rangle}\|\zeta e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|^2_{L^2_+} + \frac{2+(1-\beta)\alpha}{2\langle t\rangle}\|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|^2_{L^2_+}\Big)\\ &\le \sum_{i=1}^2\big|\big(e^{\psi}\Delta_k^h\overline{F}_{i\widetilde{\Phi}}\,\big|\,e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\big)_{L^2_+}\big| + 2^k\dot{\Theta}(t)\big(\|e^{\psi}\Delta_k^h\overline{g_1}_{\widetilde{\Phi}}\|_{L^2_+} + \|e^{\psi}\Delta_k^h\overline{g_2}_{\widetilde{\Phi}}\|_{L^2_+}\big)^2.\end{aligned} \tag{4.10}$$

At this step, we will show a key lemma to bound the nonlinear term $w^{(2)}\partial_z\overline{g}$ which is dealt with the different trick from the nonlinear terms of the existence part, the proof is presented in final part of Sect. 5.4. Other nonlinear terms in $\overline{F}$ have the same bound by the same method which Lemma 3.4 used, so we omit the proof.

Lemma 4.1 *Denote $\overline{g_i} = g_i^{(1)} - g_i^{(2)}$, and $\widetilde{\Phi}(t,\xi)$ as (4.7), for $i = 1, 2$, we have*

$$\begin{aligned}
&\int_0^{T_2} \langle t'\rangle^{\frac{5}{2}-2\delta} \big| \big(e^{\psi}\Delta_k^h\big(w^{(2)}\partial_z\overline{g_i}\big)_{\widetilde{\Phi}} \mid e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\big)_{L^2_+}\big| dt' \\
&\le d_k^2 2^{-2k}\frac{\delta}{4C^2C_0}\|\overline{g_i}_{\widetilde{\Phi}}\|^2_{\widetilde{L}^2_{T_2,f_2(t)}(B_{1,\alpha})} + \frac{C^4C_0C_2}{\delta\tau_0^2} d_k^2 2^{-2k}\big(\big\|\big(g^{(2)}_{1\Phi^2}, g^{(2)}_{2\Phi^2}\big)\big\|_{\widetilde{L}^\infty_{T_2,f_3(t)}(X_{1,\alpha})}\big)^2 \\
&\quad\times\left(\|\overline{g_i}_{\widetilde{\Phi}}\|_{\widetilde{L}^\infty_{T_2,f_3(t)}(X_{1,\alpha})}\frac{\langle T_2\rangle^{\delta}-1}{\sqrt{\delta}}\right)^2,
\end{aligned} \tag{4.11}$$

where C_2 is a large universal constant.

Upon multiplying by the term $\langle t\rangle^{\frac{5}{2}-2\delta}$ in (4.10), integrating the resulting equations on $[0, t]$, bounding most of the nonlinear terms as Lemma 3.4 and using Lemma 4.1, finally, we arrive at,

$$\begin{aligned}
&\sum_{i=1}^{2}\Bigg\{\sup_{t\in[0,T_2]}\langle t\rangle^{\frac{5}{2}-2\delta}\big\|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\big\|^2_{L^2_+} + c\lambda 2^k\int_0^{T_2}\langle t'\rangle^{\frac{5}{2}-2\delta}\dot{\Theta}(t')\big\|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\big\|^2_{L^2_+}dt' \\
&\quad+\frac{\delta}{C_0}\int_0^{T_2}\langle t'\rangle^{\frac{5}{2}-2\delta}\left(\frac{1}{\langle t'\rangle}\big\|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\big\|^2_{L^2_+} + \big\|e^{\psi}\Delta_k^h\partial_z\overline{g_i}_{\widetilde{\Phi}}\big\|^2_{L^2_+} + \frac{1}{\langle t'\rangle}\big\|\zeta e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\big\|^2_{L^2_+}\right)dt'\Bigg\} \\
&\le Cd_k^2 2^{-2k}\big\|(\overline{g_1}_{\widetilde{\Phi}},\overline{g_2}_{\widetilde{\Phi}})\big\|^2_{\widetilde{L}^2_{T_2,\widetilde{f}(t)}(X_{\frac{3}{2},\alpha})} + d_k^2 2^{-2k}\frac{\delta}{4C^2C_0}\|\overline{g_i}_{\widetilde{\Phi}}\|^2_{\widetilde{L}^2_{T_2,f_2(t)}(B_{1,\alpha})} \\
&\quad+\frac{C^4C_0C_2}{\delta\tau_0^2}d_k^2 2^{-2k}\big(\big\|\big(g^{(2)}_{1\Phi^2}, g^{(2)}_{2\Phi^2}\big)\big\|_{\widetilde{L}^\infty_{T_2,f_3(t)}(X_{1,\alpha})}\big)^2\left(\|\overline{g_i}_{\widetilde{\Phi}}\|_{\widetilde{L}^\infty_{T_2,f_3(t)}(X_{1,\alpha})}\frac{\langle T_2\rangle^{\delta}-1}{\sqrt{\delta}}\right)^2,
\end{aligned}$$

for some universal constants C_0 which are independent of α and δ. Where we denote $\widetilde{f}(t) = \langle t\rangle^{\frac{5}{2}-2\delta}\dot{\Theta}(t)$.

Taking square root of the resulting inequality and multiplying it by 2^k, and by summing over the final inequality for $k \in \mathbb{Z}$, thus, the estimate becomes

$$\begin{aligned}
&\big\|(\overline{g_1}_{\widetilde{\Phi}},\overline{g_2}_{\widetilde{\Phi}})\big\|_{\widetilde{L}^\infty_{T_2,f_3(t)}(X_{1,\alpha})} + \sqrt{c\lambda}\big\|(\overline{g_1}_{\widetilde{\Phi}},\overline{g_2}_{\widetilde{\Phi}})\big\|_{\widetilde{L}^2_{T_2,\widetilde{f}(t)}(X_{\frac{3}{2},\alpha})} + \frac{\sqrt{\delta}}{\sqrt{C_0}}\big\|(\overline{g_1}_{\widetilde{\Phi}},\overline{g_2}_{\widetilde{\Phi}})\big\|_{\widetilde{L}^2_{T_2,f_2(t)}(B_{1,\alpha})} \\
&\le\sqrt{C}\big\|(\overline{g_1}_{\widetilde{\Phi}},\overline{g_2}_{\widetilde{\Phi}})\big\|_{\widetilde{L}^2_{T_2,\widetilde{f}(t)}(X_{\frac{3}{2},\alpha})} + \frac{\sqrt{\delta}}{2\sqrt{C_0}}\big\|(\overline{g_1}_{\widetilde{\Phi}},\overline{g_2}_{\widetilde{\Phi}})\big\|_{\widetilde{L}^2_{T_2,f_2(t)}(B_{1,\alpha})} \\
&\quad+\frac{C^3\sqrt{C_0C_2}(\langle T_2\rangle^{\delta}-1)}{\delta\tau_0}\big\|\big(g^{(2)}_{1\Phi^2}, g^{(2)}_{2\Phi^2}\big)\big\|_{\widetilde{L}^\infty_{T_2,f_3(t)}(X_{1,\alpha})}\big\|(\overline{g_1}_{\widetilde{\Phi}},\overline{g_2}_{\widetilde{\Phi}})\big\|_{\widetilde{L}^\infty_{T_2,f_3(t)}(X_{1,\alpha})}.
\end{aligned} \tag{4.12}$$

From (4.3), $\tau_0 \geq \frac{C_*}{\ln \frac{1}{\varepsilon}}$, the definition of T_1 in Proposition 3.1 and $\delta = \varepsilon \ln \frac{1}{\varepsilon}$, taking λ large enough as (3.25) and $\langle T_2 \rangle^\delta \leq \langle T_1 \rangle^\delta \leq 2e$, we have

$$\begin{aligned}
&\left\| (\overline{g_1}_{\widetilde{\Phi}}, \overline{g_2}_{\widetilde{\Phi}}) \right\|_{\widetilde{L}^\infty_{T_2, f_3(t)}(X_{1,\alpha})} + \frac{\sqrt{\delta}}{2\sqrt{C_0}} \left\| (\overline{g_1}_{\widetilde{\Phi}}, \overline{g_2}_{\widetilde{\Phi}}) \right\|_{\widetilde{L}^2_{T_2, f_2(t)}(B_{1,\alpha})} \\
&\quad \leq \frac{1}{2} \frac{C_*}{\tau_0 \ln \frac{1}{\varepsilon}} \frac{4eC^4\sqrt{C_0 C_2}}{C_*} \left\| (\overline{g_1}_{\widetilde{\Phi}}, \overline{g_2}_{\widetilde{\Phi}}) \right\|_{\widetilde{L}^\infty_{T_2, f_3(t)}(X_{1,\alpha})} \leq \frac{1}{2} \left\| (\overline{g_1}_{\widetilde{\Phi}}, \overline{g_2}) \right\|_{\widetilde{L}^\infty_{T_2, f_3(t)}(X_{1,\alpha})},
\end{aligned} \tag{4.13}$$

when

$$0 < \varepsilon \leq \frac{1}{200} \quad \text{and} \quad C_* \geq 4eC^4\sqrt{C_0 C_2}, \tag{4.14}$$

and

$$\left\| (\overline{g_1}_{\widetilde{\Phi}}, \overline{g_2}_{\widetilde{\Phi}}) \right\|_{\widetilde{L}^\infty_{T_2, f_3(t)}(X_{1,\alpha})} + \frac{\sqrt{\delta}}{\sqrt{C_0}} \left\| (\overline{g_1}_{\widetilde{\Phi}}, \overline{g_2}_{\widetilde{\Phi}}) \right\|_{\widetilde{L}^2_{T_2, f_2(t)}(B_{1,\alpha})} \leq 0,$$

for all $t \in [0, T_2]$. By Gronwall's inequality, we get that $(g_1^{(1)}, g_2^{(1)}) = (g_1^{(2)}, g_2^{(2)})$ for all $t \in [0, T_2]$. The uniqueness for whole time of existence can be deduced by a continuous argument. Thus, we finish the proof of the uniqueness part of Theorem 2.1. □

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Appendix

5.1 The Tool Box

The result presented in the paper rely on a Littlewood-Paley decomposition in the horizontal variable (x, y). For the convenience of the readers, we recall the following anisotropic Bernstein type inequalities in the 3D case from [44].

Lemma 5.1 *Let $\mathfrak{B}_{\mathfrak{h}}$ be a ball of $\mathbb{R}^2_h$, and $\mathfrak{C}_{\mathfrak{h}}$ be a ring of $\mathbb{R}^2_h$, $1 \leq p_2 \leq p_1 \leq \infty$. Then there hold*:

1. *If the support of $\widehat{a}$ is included in $2^k \mathfrak{B}_{\mathfrak{h}}$, then*

$$\left\| \nabla_h^m a \right\|_{L_h^{p_1}(L_v^q)} \lesssim 2^{k(|m| + 2(\frac{1}{p_2} - \frac{1}{p_1}))} \|a\|_{L_h^{p_2}(L_v^q)}.$$

2. *If the support of $\widehat{a}$ is included in $2^k \mathfrak{C}_{\mathfrak{h}}$, then*

$$\|a\|_{L_h^{p_1}(L_v^q)} \lesssim 2^{-k|m|} \sup_{|m|} \left\| \nabla_h^m a \right\|_{L_h^{p_1}(L_v^q)}.$$

We shall try often use the Bony's decomposition for the horizontal variables:

$$fg = T^h_f g + T^h_g f + R^h(f, g), \tag{5.1}$$

where

$$T^h_f g = \sum_k S^h_{k-1} f \Delta^h_k g, \quad R^h(f, g) = \sum_k \widetilde{\Delta}^h_k f \Delta^h_k g,$$

with $\widetilde{\Delta}^h_k f \doteq \sum_{|k-k'|\le 1} \Delta^h_{k'} f$, and denote $T'_f g \doteq T^h_f g + R^h(f, g)$.

5.2 Perturbed Equations Around the Shear Flow

We choose the boundary condition lift $\varphi(t, z)$ satisfying $\varphi(t, 0) = 0$ and

$$\partial_z \varphi(t, z) = \frac{1}{\sqrt{\pi \langle t\rangle}} \exp\Big(-\frac{z^2}{4\langle t\rangle}\Big), \tag{5.2}$$

where the normalization ensures that $\varphi \to 1$ as $z \to \infty$. It means that

$$\varphi(t, z) = \frac{1}{\sqrt{\pi}} \int_0^{z/\sqrt{\langle t\rangle}} \exp\Big(-\frac{s^2}{4}\Big) ds = \mathrm{erf}\Big(\frac{z}{\sqrt{4\langle t\rangle}}\Big), \tag{5.3}$$

such that

$$\partial_t \varphi - \partial_z^2 \varphi = 0,$$

where erf is the Gauss function, $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-\eta^2) d\eta$.

Then the perturbations satisfy

$$\begin{cases} \partial_t \mathbf{u} - \partial_z^2 \mathbf{u} + \kappa_1 \varphi \partial_x \mathbf{u} + u \partial_x \mathbf{u} + \kappa_2 \varphi \partial_y \mathbf{u} + v \partial_y \mathbf{u} + \kappa w \partial_z \varphi + w \partial_z \mathbf{u} = 0, \\ w = -\int_0^z [\partial_x u(t, x, y, \widetilde{z}) + \partial_y v(t, x, y, \widetilde{z})] d\widetilde{z}, \\ \mathbf{u}|_{z=0} = \mathbf{0}, \qquad \lim\limits_{z\to+\infty} \mathbf{u} = \mathbf{0}, \\ (u, v)|_{t=0} = (u_0 - \kappa_1 \varphi(0, z), v_0 - \kappa_2 \varphi(0, z)), \end{cases} \tag{5.4}$$

where $\mathbf{u} = (u, v)^\top$, $\kappa = (\kappa_1, \kappa_2)^\top$, $\mathbf{0} = (0, 0)^\top$.

And the equations for the vorticity $h_1 = \partial_z u$ and $h_2 = \partial_z v$ are:

$$\begin{cases} \partial_t h - \partial_z^2 h + \kappa_1 \partial_z \varphi \nabla_h^\perp v + \kappa_1 \varphi \partial_x h + h_1 \nabla_h^\perp v + u \partial_x h - \kappa_2 \partial_z \varphi \nabla_h^\perp u + \kappa_2 \varphi \partial_y h \\ \quad - h_2 \nabla_h^\perp u + v \partial_y h + \kappa w \partial_z^2 \varphi + w \partial_z h = 0, \\ \partial_z h|_{z=0} = \mathbf{0}, \qquad \lim\limits_{z\to+\infty} h = \mathbf{0}, \\ (h_1, h_2)|_{t=0} = (\partial_z u_0 - \kappa_1 \partial_z \varphi(0, z), \partial_z v_0 - \kappa_2 \partial_z \varphi(0, z)), \end{cases} \tag{5.5}$$

where $h = (h_1, h_2)^\top$.

Inspired by [16], we take this issue by considering the linearly-good unknown

$$g(t, x, y, z) = h(t, x, y, z) - \mathbf{u}(t, x, y, z) a(t, z), \tag{5.6}$$

where

$$a(t,z)=\frac{\partial_z^2\varphi(t,z)}{\partial_z\varphi(t,z)}=-\frac{z}{2\langle t\rangle}.$$

We solve the first order linear equations (5.6) to compute explicit u and v from g_1 and g_2 respectively, w can be computed from the divergence free condition,

$$\mathbf{u}(t,x,y,z)=U(g)\doteq\exp\left(\frac{-z^2}{4\langle t\rangle}\right)\int_0^z g(t,x,y,\tilde{z})\exp\left(\frac{\tilde{z}^2}{4\langle t\rangle}\right)d\tilde{z}, \tag{5.7}$$

$$\begin{aligned} w(t,x,y,z)&=W(g_1,g_2)\\ &\doteq-\int_0^z\exp\left(-\frac{s^2}{4\langle t\rangle}\right)\int_0^s \operatorname{div}_h g(t,x,y,\tilde{z})\exp\left(\frac{\tilde{z}^2}{4\langle t\rangle}\right)d\tilde{z}ds, \end{aligned} \tag{5.8}$$

where $\operatorname{div}_h g(t,x,y,z)\doteq\partial_x g_1+\partial_y g_2$, and we have used the boundary conditions, $u|_{z=0}=v|_{z=0}=0$. The formulas (5.7)–(5.8) are useful when performing weighted estimates for u and v in terms of weighted norms of g_1 and g_2 respectively.

Subtracting $a\times$ (5.4) from (5.5), we get the system for (g_1,g_2),

$$\begin{cases} \partial_t g-\partial_z^2 g+\frac{1}{\langle t\rangle}g+\kappa_1\partial_z\varphi\nabla_h^\perp v+\kappa_1\varphi\partial_x g+g_1\nabla_h^\perp v-\frac{z}{2\langle t\rangle}u\nabla_h^\perp v+u\partial_x g\\ \quad-\kappa_2\partial_z\varphi\nabla_h^\perp u+\kappa_2\varphi\partial_y g-g_2\nabla_h^\perp u+\frac{z}{2\langle t\rangle}v\nabla_h^\perp u+v\partial_y g+w\partial_z g-\frac{1}{2\langle t\rangle}w\mathbf{u}=0,\\ \mathbf{u}=U(g),\quad \text{and}\quad w=W(g_1,g_2),\\ \partial_z g|_{z=0}=\lim_{z\to+\infty}g=0,\\ g_1|_{t=0}=g_{10},\qquad g_2|_{t=0}=g_{20}. \end{cases} \tag{5.9}$$

5.3 The Key Estimates

We need to deal with other type linear terms.

Lemma 5.2 *Define the lift* $\varphi(t,z)=\frac{1}{\sqrt{\pi}}\int_0^{z/\sqrt{\langle t\rangle}}\exp(-\frac{\tilde{z}^2}{4})d\tilde{z}$, $f_1(t)=\langle t\rangle^{\frac{5}{2}-2\delta}\dot{\theta}(t)$, *and the weight function* $\psi(t,z)=\frac{\alpha z^2}{4\langle t\rangle}$. *Under the conditions in Proposition 3.1, we have*

$$|\kappa_i|\left|\left(e^\psi\partial_z\varphi\Delta_k^h U(\nabla_h g_j)_\Phi\;\middle|\; e^\psi\Delta_k^h g_{q\Phi}\right)_{L^2_+}\right|\le 2^k\dot{\theta}\left\|e^\psi\Delta_k^h g_{j\Phi}\right\|_{L^2_+}\left\|e^\psi\Delta_k^h g_{q\Phi}\right\|_{L^2_+}, \tag{5.10}$$

where $t\in[0,T^*]$, *for any* $k\in\mathbb{Z}$, $i,j,q=1,2$. *Moreover,*

$$\begin{aligned} &\int_0^t\langle t'\rangle^{\frac{5}{2}-2\delta}|\kappa_i|\left|\left(e^\psi\partial_z\varphi\Delta_k^h\nabla_h U(g_{j\Phi})\;\middle|\; e^\psi\Delta_k^h g_{q\Phi}\right)_{L^2_+}\right|dt'\\ &\quad\le Cd_k^2 2^{-2k}\|g_{j\Phi}\|_{\tilde{L}^2_{t,f_1(t)}(X_{\frac{3}{2},\alpha})}\|g_{q\Phi}\|_{\tilde{L}^2_{t,f_1(t)}(X_{\frac{3}{2},\alpha})}, \end{aligned} \tag{5.11}$$

where $i,j,q=1,2$.

Proof Using (2.4), (5.2), the anisotropic Bernstein type inequality in Lemma 5.1 and (3.11) in Lemma 3.2, we get

$$\begin{aligned} &|\kappa_i|\left|\left(\left(e^\psi\partial_z\varphi U(\nabla_h g_j)\right)_\Phi\;\middle|\; e^\psi\Delta_k^h g_{q\Phi}\right)_{L^2_+}\right|\\ &\quad\le|\kappa_i|\left\|e^\psi\partial_z\varphi\right\|_{L^2_z}\left\|\Delta_k^h U(\nabla_h g_{j\Phi})\right\|_{L^\infty_z L^2_{x,y}}\left\|e^\psi\Delta_k^h g_{q\Phi}\right\|_{L^2_+} \end{aligned}$$

$$
\begin{aligned}
&\leq 2^k|\kappa_i|\|e^{\psi}\partial_z\varphi\|_{L_z^2}\|\Delta_k^h U(g_{j\Phi})\|_{L_z^\infty L_{x,y}^2}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L_+^2}\\
&\leq C2^k\langle t\rangle^{\frac14}|\kappa_i|\|e^{\psi}\partial_z\varphi\|_{L_z^2}\|e^{\psi}\Delta_k^h g_{j\Phi}\|_{L_+^2}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L_+^2}\\
&\leq C2^k\dot\theta\|e^{\psi}\Delta_k^h g_{j\Phi}\|_{L_+^2}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L_+^2},
\end{aligned}
$$

which immediately implies (5.10). Multiplying upon $\langle t\rangle^{\frac52-2\delta}$ on both hand sides of (5.10), and integrating over $[0,t]$, we obtain

$$
\begin{aligned}
&\int_0^t\langle t'\rangle^{\frac52-2\delta}|\kappa_i|\left|\left(\left(e^{\psi}\partial_z\varphi U(\nabla_h g_j)\right)_{\Phi}\mid e^{\psi}\Delta_k^h g_{q\Phi}\right)_{L_+^2}\right|dt'\\
&\leq C2^k\int_0^t\langle t'\rangle^{\frac52-2\delta}\dot\theta(t')\|e^{\psi}\Delta_k^h g_{j\Phi}\|_{L_+^2}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L_+^2}dt'\\
&\lesssim d_k^2 2^{-2k}\|g_{j\Phi}\|_{\widetilde L^2_{t,f_1(t)}(X_{\frac32,\alpha})}\|g_{q\Phi}\|_{\widetilde L^2_{t,f_1(t)}(X_{\frac32,\alpha})},
\end{aligned}
$$

which implies (5.11). □

We should list an easy result before proving Lemma 3.2.

Lemma 5.3 *When ψ is defined as* (2.4), *$u=U(g_1)$ and $v=U(g_2)$ defined by* (2.2), *under the conditions in Proposition* 3.1, *for $i=1,2$ we have*

$$
\|e^{\psi}\Delta_k^h U(g_{i\Phi})\|_{L_z^\infty}\leq\|e^{\psi}\Delta_k^h g_{i\Phi}\|_{L_z^1},\tag{5.12}
$$

for all $k\in\mathbb{Z}$. For $p\in[1,2]$ and $\alpha\in[\frac14,\frac12]$, we get

$$
\left|e^{\psi}\Delta_k^h U(g_{i\Phi})(t,x,y,z)\right|\leq C\langle t\rangle^{\frac{1}{2p}}\|e^{\psi}\Delta_k^h g_{i\Phi}\|_{L_z^{\frac{p}{p-1}}}\frac{1}{(1+\zeta(t,z))^{\frac1p}}.\tag{5.13}
$$

Lemma 5.3 is trivial, we omit the proof for simplicity, cf. (3.3) in [16]. Now we can prove Lemma 3.2 as follows.

Proof of Lemma 3.2 From (5.12), we obtain

$$
\begin{aligned}
\left|\Delta_k^h U(g_{i\Phi})(z)\right|&\leq\frac{1}{e^{\psi(z)}}\int_0^z\left|e^{\psi(\tilde z)}\Delta_k^h g_{i\Phi}(\tilde z)\right|\exp\left(\frac{(1-\alpha)}{4\langle t\rangle}(\tilde z^2-z^2)\right)d\tilde z\\
&\leq\|e^{\psi}\Delta_k^h g_{i\Phi}\|_{L_z^2}\frac{\sqrt z}{e^{\psi(z)}}\\
&=\langle t\rangle^{\frac14}\|e^{\psi}\Delta_k^h g_{i\Phi}\|_{L_z^2}\sqrt\zeta\exp\left(\frac{-\alpha\zeta^2}{4}\right)\\
&\leq C\langle t\rangle^{\frac14}\|e^{\Psi}\Delta_k^h g_{i\Phi}\|_{L_z^2}.
\end{aligned}
$$

Taking the L^2 norm in x and y variables of the above inequality, we obtain (3.11) in Lemma 3.2. For the w bounds, taking L^∞ norm in z, we obtain

$$
\begin{aligned}
\|\Delta_k^h W(g_{1\Phi},g_{2\Phi})\|_{L_z^\infty}&\leq\|\Delta_k^h[\partial_x U(g_{1\Phi})+\partial_y U(g_{2\Phi})]\|_{L_z^1}\\
&\leq\|e^{\psi}\Delta_k^h\partial_x U(g_{1\Phi})\|_{L_z^\infty}\|e^{-\psi}\|_{L_z^1}+\|e^{\psi}\Delta_k^h\partial_y U(g_{2\Phi})\|_{L_z^\infty}\|e^{-\psi}\|_{L_z^1}
\end{aligned}
$$

$$\leq C\langle t\rangle^{\frac{1}{2}}\left\|e^{\Psi}\Delta_k^h\partial_x U(g_{1\Phi})\right\|_{L_z^\infty}+C\langle t\rangle^{\frac{1}{2}}\left\|e^{\Psi}\Delta_k^h\partial_y U(g_{2\Phi})\right\|_{L_z^\infty}$$
$$\leq C\langle t\rangle^{\frac{3}{4}}2^k\left\|e^{\Psi}\Delta_k^h g_{1\Phi}\right\|_{L_z^2}+C\langle t\rangle^{\frac{3}{4}}2^k\left\|e^{\Psi}\Delta_k^h g_{2\Phi}\right\|_{L_z^2}.$$

For the last inequality, we use the Bernstein type lemma and (5.13) when $p=2$. Taking the L^2 norm in x and y variables of the above inequality, we get (3.12).

For (3.13), using the inequality (5.13) in the case $p=1$ and take L^2 norm in z, we get

$$\begin{aligned}\left\|e^{\Psi}\Delta_k^h U(g_{i\Phi})\right\|_{L_z^2} &\leq C\langle t\rangle^{\frac{1}{2}}\left\|e^{\Psi}\Delta_k^h g_{i\Phi}\right\|_{L_z^\infty}\left\|\left(1+\zeta(t,z)\right)^{-1}\right\|_{L_z^2}\\ &\leq C\langle t\rangle^{\frac{3}{4}}\left\|e^{\Psi}\Delta_k^h g_{i\Phi}\right\|_{L_z^2}^{\frac{1}{2}}\left(\left\|e^{\Psi}\Delta_k^h\partial_z g_{i\Phi}\right\|_{L_z^2}+\left\|\partial_z e^{\Psi}\Delta_k^h g_{i\Phi}\right\|_{L_z^2}\right)^{\frac{1}{2}}\\ &\leq C\langle t\rangle^{\frac{3}{4}}\left\|e^{\Psi}\Delta_k^h g_{i\Phi}\right\|_{L_z^2}^{\frac{1}{2}}\left\|e^{\Psi}\Delta_k^h\partial_z g_{i\Phi}\right\|_{L_z^2}^{\frac{1}{2}}\\ &\quad+C\langle t\rangle^{\frac{1}{2}}\left\|e^{\Psi}\Delta_k^h g_{i\Phi}\right\|_{L_z^2}^{\frac{1}{2}}\left\|\zeta e^{\Psi}\Delta_k^h g_{i\Phi}\right\|_{L_z^2}^{\frac{1}{2}},\end{aligned}$$

where we use 1D Agmon's inequality in the second inequality. Then taking the L^2 norm in x and y variables of the above inequality, we finish the proof of (3.13).

For (3.14), using the inequality (5.13) in the case $p=2$ and take L^2 norm in x and y, we get the bound (3.14). □

5.4 The Detail Proof of (3.17)–(3.20) and Lemma 4.1

At this section, we prove (3.17)–(3.20) in Lemma 3.3. Follow the same line to prove (3.16), i.e. applying Bony's decomposition (5.1) for the (x,y) variable, the property of $\Phi(t,\xi)$ in (3.15), the support properties of the Fourier transform, Hölder's inequality, (3.13), (3.14) and the anisotropic Bernstein type inequality in Lemma 5.1, we have

$$\begin{aligned}&\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta}\left|\frac{1}{\langle t'\rangle}\left(e^{\Psi}z\Delta_k^h\big(U(g_i)U(\nabla_h g_j)\big)_\Phi\,\middle|\,e^{\Psi}\Delta_k^h g_{q\Phi}\right)_{L_+^2}\right|dt'\\ &\leq\sum_{|k-k'|\leq 4}\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta-\frac{1}{2}}\left\|e^{\Psi}S_{k'-1}^h\widetilde{U(g_{i\Phi})}\right\|_{L_z^2L_{x,y}^\infty}\\ &\quad\times\left\|e^{\Psi}\Delta_{k'}^h U(\nabla_h g_{j\Phi})\right\|_{L_z^\infty L_{x,y}^2}\left\|e^{\Psi}\Delta_k^h g_{q\Phi}\right\|_{L_+^2}dt'\\ &\quad+\sum_{k'\geq k-3}\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta-\frac{1}{2}}\left\|e^{\Psi}S_{k'+2}^h\widetilde{U(\nabla_h g_{j\Phi})}\right\|_{L_z^2L_{x,y}^\infty}\\ &\quad\times\left\|e^{\Psi}\Delta_{k'}^h U(g_{i\Phi})\right\|_{L_z^\infty L_{x,y}^2}\left\|e^{\Psi}\Delta_k^h g_{q\Phi}\right\|_{L_+^2}dt'\\ &\leq C\sum_{\substack{|k-k'|\leq 4\\ l\leq k'-2}}2^{k'+l}\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta-\frac{1}{4}}\left\|e^{\Psi}\Delta_l^h g_{i\Phi}\right\|_{L_+^2}^{\frac{1}{2}}\left\|e^{\Psi}\Delta_{k'}^h g_{j\Phi}\right\|_{L_+^2}\left\|e^{\Psi}\Delta_k^h g_{q\Phi}\right\|_{L_+^2}\\ &\quad\times\left(\langle t'\rangle^{\frac{3}{4}}\left\|e^{\Psi}\Delta_l^h\partial_z g_{i\Phi}\right\|_{L_+^2}^{\frac{1}{2}}+\langle t'\rangle^{\frac{1}{2}}\left\|\zeta e^{\Psi}\Delta_l^h g_{i\Phi}\right\|_{L_+^2}^{\frac{1}{2}}\right)dt'\end{aligned}$$

$$+C\sum_{\substack{k'\geq k-3\\ l\leq k'+1}}2^{2l}\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta-\frac{1}{4}}\|e^{\psi}\Delta_l^h g_{j\Phi}\|_{L^2_+}^{\frac{1}{2}}\|e^{\psi}\Delta_{k'}^h g_{i\Phi}\|_{L^2_+}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L^2_+}$$

$$\times\big(\langle t'\rangle^{\frac{3}{4}}\|e^{\psi}\Delta_l^h\partial_z g_{j\Phi}\|_{L^2_+}^{\frac{1}{2}}+\langle t'\rangle^{\frac{1}{2}}\|\zeta e^{\psi}\Delta_l^h g_{j\Phi}\|_{L^2_+}^{\frac{1}{2}}\big)dt'$$

$$\leq C\sum_{|k-k'|\leq 4}2^{k'}\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta}\|e^{\psi}\Delta_{k'}^h g_{j\Phi}\|_{L^2_+}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L^2_+}\|g_{i\Phi}\|_{X_{1,\alpha}}^{\frac{1}{2}}$$

$$\times\big(\langle t'\rangle^{\frac{1}{2}}\|\partial_z g_{i\Phi}\|_{X_{1,\alpha}}^{\frac{1}{2}}+\langle t'\rangle^{\frac{1}{4}}\|\zeta g_{i\Phi}\|_{X_{1,\alpha}}^{\frac{1}{2}}\big)dt'$$

$$+C\sum_{k'\geq k-3}2^{k'}\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta}\|e^{\psi}\Delta_{k'}^h g_{i\Phi}\|_{L^2_+}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L^2_+}\|g_{j\Phi}\|_{X_{1,\alpha}}^{\frac{1}{2}}$$

$$\times\big(\langle t'\rangle^{\frac{1}{2}}\|\partial_z g_{j\Phi}\|_{X_{1,\alpha}}^{\frac{1}{2}}+\langle t'\rangle^{\frac{1}{4}}\|\zeta g_{j\Phi}\|_{X_{1,\alpha}}^{\frac{1}{2}}\big)dt'$$

$$\leq C\sum_{|k-k'|\leq 4}2^{k'}\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta}\dot{\theta}(t')\|e^{\psi}\Delta_{k'}^h g_{j\Phi}\|_{L^2_+}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L^2_+}dt'$$

$$+C\sum_{k'\geq k-3}2^{k'}\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta}\dot{\theta}(t')\|e^{\psi}\Delta_{k'}^h g_{i\Phi}\|_{L^2_+}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L^2_+}dt'$$

$$\lesssim d_k^2 2^{-2k}\|(g_{i\Phi},g_{j\Phi})\|_{\widetilde{L}^2_{T^*,f_1(t)}(X_{\frac{3}{2},\alpha})}\|g_{q\Phi}\|_{\widetilde{L}^2_{T^*,f_1(t)}(X_{\frac{3}{2},\alpha})},$$

which immediately implies (3.17).

We use the same method in the proof of (3.16)–(3.17) to get (3.18) as follows. Applying Bony's decomposition for $(U(g_i)\nabla_h g_j)_\Phi$ for the (x, y) variable, using Hölder's inequality, Lemma 5.1, (3.11) in Lemma 3.2, we obtain

$$\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta}\big|\big(e^{\psi}\Delta_k^h\big(U(g_i)\nabla_h g_j\big)_\Phi\mid e^{\psi}\Delta_k^h g_{q\Phi}\big)_{L^2_+}\big|dt'$$

$$\leq\sum_{|k-k'|\leq 4}\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta}\|S_{k'-1}^h\widetilde{U(g_{i\Phi})}\|_{L_z^\infty L_{x,y}^\infty}\|e^{\psi}\Delta_{k'}^h\nabla_h g_{j\Phi}\|_{L_z^2L_{x,y}^2}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L^2_+}dt'$$

$$+\sum_{k'\geq k-3}\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta}\|\Delta_{k'}^h U(g_{i\Phi})\|_{L_z^\infty L_{x,y}^2}\|e^{\psi}S_{k'+2}^h\nabla_h\widetilde{g_{j\Phi}}\|_{L_z^2L_{x,y}^\infty}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L^2_+}dt'$$

$$\leq C\sum_{\substack{|k-k'|\leq 4\\ l\leq k'-2}}2^{k'+l}\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta+\frac{1}{4}}\|e^{\psi}\Delta_l^h g_{i\Phi}\|_{L^2_+}\|e^{\psi}\Delta_{k'}^h g_{j\Phi}\|_{L^2_+}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L^2_+}dt'$$

$$+C\sum_{\substack{k'\geq k-3\\ l\leq k'+1}}2^{2l}\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta+\frac{1}{4}}\|e^{\psi}\Delta_{k'}^h g_{i\Phi}\|_{L^2_+}\|e^{\psi}\Delta_l^h g_{j\Phi}\|_{L^2_+}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L^2_+}dt'$$

$$\leq C\sum_{|k-k'|\leq 4}2^{k'}\int_0^{T^*}\langle t'\rangle^{\frac{5}{2}-2\delta+\frac{1}{4}}\|g_{i\Phi}\|_{X_{1,\alpha}}\|e^{\psi}\Delta_{k'}^h g_{j\Phi}\|_{L^2_+}\|e^{\psi}\Delta_k^h g_{q\Phi}\|_{L^2_+}dt'$$

$$+ C \sum_{k' \geq k-3} 2^{k'} \int_0^{T^*} \langle t' \rangle^{\frac{5}{2}-2\delta+\frac{1}{4}} \|g_{j\Phi}\|_{X_{1,\alpha}} \|e^{\psi} \Delta_{k'}^h g_{i\Phi}\|_{L_+^2} \|e^{\psi} \Delta_k^h g_{q\Phi}\|_{L_+^2} dt'$$

$$\leq C \sum_{|k-k'| \leq 4} 2^{k'} \int_0^{T^*} \langle t' \rangle^{\frac{5}{2}-2\delta} \dot{\theta}(t') \|e^{\psi} \Delta_{k'}^h g_{j\Phi}\|_{L_+^2} \|e^{\psi} \Delta_k^h g_{q\Phi}\|_{L_+^2} dt'$$

$$+ C \sum_{k' \geq k-3} 2^{k'} \int_0^{T^*} \langle t' \rangle^{\frac{5}{2}-2\delta} \dot{\theta}(t') \|e^{\psi} \Delta_{k'}^h g_{i\Phi}\|_{L_+^2} \|e^{\psi} \Delta_k^h g_{q\Phi}\|_{L_+^2} dt'$$

$$\lesssim d_k^2 2^{-2k} \|(g_{i\Phi}, g_{j\Phi})\|_{\widetilde{L}^2_{T^*, f_1(t)}(X_{\frac{3}{2},\alpha})} \|g_{q\Phi}\|_{\widetilde{L}^2_{T^*, f_1(t)}(X_{\frac{3}{2},\alpha})},$$

which immediately implies (3.18).

Along the same line of the proof of (3.16)–(3.18), we can easily get (3.19) as follows.

$$\int_0^{T^*} \langle t' \rangle^{\frac{5}{2}-2\delta} \left| \left(e^{\psi} W(g_1, g_2) \partial_z g_j\right)_{\Phi} \mid e^{\psi} \Delta_k^h g_{q\Phi}\right)_{L_+^2} \Big| dt'$$

$$\leq \sum_{|k-k'| \leq 4} \int_0^{T^*} \langle t' \rangle^{\frac{5}{2}-2\delta} \|S_{k'-1}^h W(\widetilde{g_{1\Phi}, g_{2\Phi}})\|_{L_z^\infty L_{x,y}^\infty} \|e^{\psi} \Delta_{k'}^h \partial_z g_{j\Phi}\|_{L_z^2 L_{x,y}^2} \|e^{\psi} \Delta_k^h g_{q\Phi}\|_{L_+^2} dt'$$

$$+ \sum_{k' \geq k-3} \int_0^{T^*} \langle t' \rangle^{\frac{5}{2}-2\delta} \|e^{\psi} S_{k'+2}^h \widetilde{\partial_z g_{j\Phi}}\|_{L_z^2 L_{x,y}^\infty} \|\Delta_{k'}^h W(g_{1\Phi}, g_{2\Phi})\|_{L_z^\infty L_{x,y}^2} \|e^{\psi} \Delta_k^h g_{q\Phi}\|_{L_+^2} dt'$$

$$\leq C \sum_{i=1}^{2} \sum_{\substack{|k-k'| \leq 4 \\ l \leq k'-2}} 2^{2l} \int_0^{T^*} \langle t' \rangle^{\frac{5}{2}-2\delta+\frac{3}{4}} \|e^{\psi} \Delta_l^h g_{i\Phi}\|_{L_+^2} \|e^{\psi} \Delta_{k'}^h \partial_z g_{j\Phi}\|_{L_+^2} \|e^{\psi} \Delta_k^h g_{q\Phi}\|_{L_+^2} dt'$$

$$+ C \sum_{i=1}^{2} \sum_{\substack{k' \geq k-3 \\ l \leq k'+1}} 2^{k'+l} \int_0^{T^*} \langle t' \rangle^{\frac{5}{2}-2\delta+\frac{3}{4}} \|e^{\psi} \Delta_{k'}^h g_{i\Phi}\|_{L_+^2} \|e^{\psi} \Delta_l^h \partial_z g_{j\Phi}\|_{L_+^2} \|e^{\psi} \Delta_k^h g_{q\Phi}\|_{L_+^2} dt'$$

$$\leq C \sum_{i=1}^{2} \sum_{l \leq k-2} 2^{2l-k} \int_0^{T^*} \langle t' \rangle^{\frac{5}{2}-2\delta+\frac{3}{4}} \|\partial_z g_{j\Phi}\|_{X_{1,\alpha}} \|e^{\psi} \Delta_l^h g_{i\Phi}\|_{L_+^2} \|e^{\psi} \Delta_k^h g_{q\Phi}\|_{L_+^2} dt'$$

$$+ C \sum_{i=1}^{2} \sum_{k' \geq k-3} 2^{k'} \int_0^{T^*} \langle t' \rangle^{\frac{5}{2}-2\delta+\frac{3}{4}} \|\partial_z g_{j\Phi}\|_{X_{1,\alpha}} \|e^{\psi} \Delta_{k'}^h g_{i\Phi}\|_{L_+^2} \|e^{\psi} \Delta_k^h g_{q\Phi}\|_{L_+^2} dt'$$

$$\leq C \sum_{i=1}^{2} \sum_{l \leq k-2} 2^{2l-k} \int_0^{T^*} \langle t' \rangle^{\frac{5}{2}-2\delta} \dot{\theta}(t') \|e^{\psi} \Delta_l^h g_{i\Phi}\|_{L_+^2} \|e^{\psi} \Delta_k^h g_{q\Phi}\|_{L_+^2} dt'$$

$$+ C \sum_{i=1}^{2} \sum_{k' \geq k-3} 2^{k'} \int_0^{T^*} \langle t' \rangle^{\frac{5}{2}-2\delta} \dot{\theta}(t') \|e^{\psi} \Delta_{k'}^h g_{i\Phi}\|_{L_+^2} \|e^{\psi} \Delta_k^h g_{q\Phi}\|_{L_+^2} dt'$$

$$\lesssim d_k^2 2^{-2k} \|(g_{1\Phi}, g_{2\Phi})\|_{\widetilde{L}^2_{T^*, f_1(t)}(X_{\frac{3}{2},\alpha})} \|g_{q\Phi}\|_{\widetilde{L}^2_{T^*, f_1(t)}(X_{\frac{3}{2},\alpha})}.$$

We use the same method in the proof of (3.16)–(3.19) to get (3.20) as follows. Applying Bony's decomposition for $\frac{1}{\langle t \rangle} W(g_1, g_2) U(g_j)$ for the (x, y) variable, using Hölder's inequality, Lemma 5.1, (3.12) and (3.13) in Lemma 3.2, we obtain

$$\begin{aligned}
&\int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta} \Big| \frac{1}{\langle t'\rangle} \big((W(g_1,g_2)U(g_j))_\Phi | e^\psi \Delta_k^h g_{q\Phi} \big)_{L^2_+} \Big| dt' \\
&\le \sum_{|k-k'|\le 4} \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta-1} \| S^h_{k'-1} W(\widetilde{g_{1\Phi}, g_{2\Phi}}) \|_{L^\infty_z L^\infty_{x,y}} \\
&\quad \times \| e^\psi \Delta^h_{k'} U(g_{j\Phi}) \|_{L^2_z L^2_{x,y}} \| e^\psi \Delta^h_k g_{q\Phi} \|_{L^2_+} dt' \\
&\quad + \sum_{k'\ge k-3} \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta-1} \| \Delta^h_{k'} W(g_{1\Phi}, g_{2\Phi}) \|_{L^\infty_z L^2_{x,y}} \\
&\quad \times \| e^\psi S^h_{k'+2} \widetilde{U(g_{j\Phi})} \|_{L^2_z L^\infty_{x,y}} \| e^\psi \Delta^h_k g_{q\Phi} \|_{L^2_+} dt' \\
&\le \sum_{i=1}^2 \sum_{\substack{|k-k'|\le 4 \\ l\le k'-2}} 2^{2l} \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta} \| e^\psi \Delta^h_l g_{i\Phi} \|_{L^2_{x,y,z}} \| e^\psi \Delta^h_{k'} g_{j\Phi} \|^{\frac{1}{2}}_{L^2_{x,y,z}} \| e^\psi \Delta^h_k g_{q\Phi} \|_{L^2_+} \\
&\quad \times \big(\langle t'\rangle^{\frac{1}{2}} \| e^\psi \Delta^h_{k'} \partial_z g_{j\Phi} \|^{\frac{1}{2}}_{L^2_{x,y,z}} + \langle t'\rangle^{\frac{1}{4}} \| \zeta e^\psi \Delta^h_{k'} g_{j\Phi} \|^{\frac{1}{2}}_{L^2_{x,y,z}} \big) dt' \\
&\quad + \sum_{i=1}^2 \sum_{\substack{k'\ge k-3 \\ l\le k'+1}} 2^{k'+l} \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta} \| e^\psi \Delta^h_{k'} g_{i\Phi} \|_{L^2_{x,y,z}} \| e^\psi \Delta^h_l g_{j\Phi} \|^{\frac{1}{2}}_{L^2_{x,y,z}} \\
&\quad \times \big(\langle t'\rangle^{\frac{1}{2}} \| e^\psi \Delta^h_l \partial_z g_{j\Phi} \|^{\frac{1}{2}}_{L^2_{x,y,z}} + \langle t'\rangle^{\frac{1}{4}} \| \zeta e^\psi \Delta^h_l g_{j\Phi} \|^{\frac{1}{2}}_{L^2_{x,y,z}} \big) dt' \\
&\le \sum_{i=1}^2 \sum_{l\le k-2} 2^{2l-k} \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta} \| e^\psi \Delta^h_l g_{i\Phi} \|_{L^2_+} \| e^\psi \Delta^h_k g_{q\Phi} \|_{L^2_+} \| g_{j\Phi} \|^{\frac{1}{2}}_{X_{1,\alpha}} \\
&\quad \times \big(\langle t'\rangle^{\frac{1}{2}} \| \partial_z g_{j\Phi} \|^{\frac{1}{2}}_{X_{1,\alpha}} + \langle t'\rangle^{\frac{1}{4}} \| \zeta g_{j\Phi} \|^{\frac{1}{2}}_{X_{1,\alpha}} \big) dt' \\
&\quad + \sum_{i=1}^2 \sum_{k'\ge k-3} 2^{k'} \int_0^{T^*} \langle t'\rangle^{\frac{5}{2}-2\delta} \| e^\psi \Delta^h_{k'} g_{i\Phi} \|_{L^2_+} \| e^\psi \Delta^h_k g_{q\Phi} \|_{L^2_+} \| g_{j\Phi} \|^{\frac{1}{2}}_{X_{1,\alpha}} \\
&\quad \times \big(\langle t'\rangle^{\frac{1}{2}} \| \partial_z g_{j\Phi} \|^{\frac{1}{2}}_{X_{1,\alpha}} + \langle t'\rangle^{\frac{1}{4}} \| \zeta g_{j\Phi} \|^{\frac{1}{2}}_{X_{1,\alpha}} \big) dt' \\
&\le C \sum_{i=1}^2 \sum_{l\le k-2} 2^{2l-k} \int_0^{T^*} \dot\theta(t') \| e^\psi \Delta^h_l g_{i\Phi} \|_{L^2_+} \| e^\psi \Delta^h_k g_{q\Phi} \|_{L^2_+} dt' \\
&\quad + C \sum_{i=1}^2 \sum_{k'\ge k-3} 2^{k'} \int_0^{T^*} \dot\theta(t') \| e^\psi \Delta^h_{k'} g_{i\Phi} \|_{L^2_+} \| e^\psi \Delta^h_k g_{q\Phi} \|_{L^2_+} dt' \\
&\lesssim d_k^2 2^{-2k} \| (g_{1\Phi}, g_{2\Phi}) \|_{\widetilde{L}^2_{T^*, f_1(t)}(X_{\frac{3}{2},\alpha})} \| g_{q\Phi} \|_{\widetilde{L}^2_{T^*, f_1(t)}(X_{\frac{3}{2},\alpha})}.
\end{aligned}$$

□

Proof of Lemma 4.1 Applying Bony's decomposition for $w^{(2)}\partial_z \overline{g_i}$ for the (x, y) variable, using Hölder's inequality, Lemma 5.1, (3.12) in Lemma 3.2, the inequality (4.8) and

Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}
&\int_0^{T_2} \langle t'\rangle^{\frac{5}{2}-2\delta} \big| \big(e^{\psi}\Delta_k^h\big(w^{(2)}\partial_z\overline{g_i}\big)_{\widetilde{\Phi}} \mid e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\big)_{L^2_+}\big| dt' \\
&\quad\le \int_0^{T_2} \langle t'\rangle^{\frac{5}{2}-2\delta} \big(\|e^{\psi}\Delta_k^h\big(T^h_{w^{(2)}}\partial_z\overline{g_i}\big)_{\widetilde{\Phi}}\|_{L^2_+} + \|e^{\psi}\Delta_k^h\big(T'_{\partial_z\overline{g_i}}w^{(2)}\big)_{\widetilde{\Phi}}\|_{L^2_+}\big) \|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|_{L^2_+} dt' \\
&\quad\le \sum_{|k-k'|\le 4} \int_0^{T_2} \langle t'\rangle^{\frac{5}{2}-2\delta} \|S^h_{k'-1}\widetilde{w^{(2)}_{\widetilde{\Phi}}}\|_{L^\infty_z L^\infty_{x,y}} \|e^{\psi}\Delta_{k'}^h\partial_z\overline{g_i}_{\widetilde{\Phi}}\|_{L^2_z L^2_{x,y}} \|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|_{L^2_+} dt' \\
&\qquad + \sum_{k'\ge k-3} \int_0^{T_2} \langle t'\rangle^{\frac{5}{2}-2\delta} \|e^{\psi}S^h_{k'+2}\widetilde{\partial_z\overline{g_i}_{\widetilde{\Phi}}}\|_{L^2_z L^\infty_{x,y}} \|\Delta_{k'}^h w^{(2)}_{\widetilde{\Phi}}\|_{L^\infty_z L^2_{x,y}} \|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|_{L^2_+} dt' \\
&\quad\le \sum_{j=1}^{2} \sum_{\substack{|k-k'|\le 4\\ l\le k'-2}} 2^{2l} \int_0^{T_2} \langle t'\rangle^{\frac{5}{2}-2\delta+\frac{3}{4}} \|e^{\psi}\Delta_l^h g^{(2)}_{j\widetilde{\Phi}}\|_{L^2_+} \|e^{\psi}\Delta_{k'}^h\partial_z\overline{g_i}_{\widetilde{\Phi}}\|_{L^2_+} \|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|_{L^2_+} dt' \\
&\qquad + \sum_{j=1}^{2} \sum_{\substack{k'\ge k-3\\ l\le k'+1}} 2^{l+k'} \int_0^{T_2} \langle t'\rangle^{\frac{5}{2}-2\delta+\frac{3}{4}} \|e^{\psi}\Delta_l^h\partial_z\overline{g_i}_{\widetilde{\Phi}}\|_{L^2_+} \|e^{\psi}\Delta_{k'}^h g^{(2)}_{j\widetilde{\Phi}}\|_{L^2_+} \|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|_{L^2_+} dt' \\
&\quad\le \sum_{j=1}^{2} \sum_{\substack{|k-k'|\le 4\\ l\le k'-2}} 2^{2l} \sup_{t\in[0,T_2]} \langle t\rangle^{\frac{5}{4}-\delta} \|e^{\psi}\Delta_l^h g^{(2)}_{j\widetilde{\Phi}}\|_{L^2_+} \big\|\langle t'\rangle^{\frac{5}{4}-\delta} \|e^{\psi}\Delta_{k'}^h\partial_z\overline{g_i}_{\widetilde{\Phi}}\|_{L^2_+}\big\|_{L^2_{t'}} \\
&\qquad \times \sup_{t\in[0,T_2]} \langle t\rangle^{\frac{5}{4}-\delta} \|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|_{L^2_+} \Big(\int_0^{T_2} \langle t'\rangle^{2\delta-1} dt'\Big)^{\frac{1}{2}} \\
&\qquad + \sum_{j=1}^{2} \sum_{\substack{k'\ge k-3\\ l\le k'+1}} 2^{l+k'} \sup_{t\in[0,T_2]} \langle t\rangle^{\frac{5}{4}-\delta} \|e^{\psi}\Delta_{k'}^h g^{(2)}_{j\widetilde{\Phi}}\|_{L^2_+} \big\|\langle t'\rangle^{\frac{5}{4}-\delta} \|e^{\psi}\Delta_l^h\partial_z\overline{g_i}_{\widetilde{\Phi}}\|_{L^2_+}\big\|_{L^2_{t'}} \\
&\qquad \times \sup_{t\in[0,T_2]} \langle t\rangle^{\frac{5}{4}-\delta} \|e^{\psi}\Delta_k^h\overline{g_i}_{\widetilde{\Phi}}\|_{L^2_+} \Big(\int_0^{T_2} \langle t'\rangle^{2\delta-1} dt'\Big)^{\frac{1}{2}} \\
&\quad\lesssim d_k^2 2^{-2k} \big\|\big(g^{(2)}_{1\widetilde{\Phi}}, g^{(2)}_{2\widetilde{\Phi}}\big)\big\|_{\widetilde{L}^\infty_{T_2,f_3(t)}(X_{2,\alpha})} \|\overline{g_i}_{\widetilde{\Phi}}\|_{\widetilde{L}^2_{T_2,f_2(t)}(B_{1,\alpha})} \|\overline{g_i}_{\widetilde{\Phi}}\|_{\widetilde{L}^\infty_{T_2,f_3(t)}(X_{1,\alpha})} \frac{\langle T_2\rangle^{\delta}-1}{\sqrt{\delta}} \\
&\quad\le \frac{C\sqrt{C_2}}{\tau_0} d_k^2 2^{-2k} \big\|\big(g^{(2)}_{1\Phi^2}, g^{(2)}_{2\Phi^2}\big)\big\|_{\widetilde{L}^\infty_{T_2,f_3(t)}(X_{1,\alpha})} \|\overline{g_i}_{\widetilde{\Phi}}\|_{\widetilde{L}^2_{T_2,f_2(t)}(B_{1,\alpha})} \|\overline{g_i}_{\widetilde{\Phi}}\|_{\widetilde{L}^\infty_{T_2,f_3(t)}(X_{1,\alpha})} \frac{\langle T_2\rangle^{\delta}-1}{\sqrt{\delta}} \\
&\quad\le d_k^2 2^{-2k} \frac{\delta}{4C^2C_0} \|\overline{g_i}_{\widetilde{\Phi}}\|^2_{\widetilde{L}^2_{T_2,f_2(t)}(B_{1,\alpha})} + \frac{C_0C_2C^4}{\delta\tau_0^2} d_k^2 2^{-2k} \Big(\big\|\big(g^{(2)}_{1\Phi^2}, g^{(2)}_{2\Phi^2}\big)\big\|_{\widetilde{L}^\infty_{T_2,f_3(t)}(X_{1,\alpha})}\Big)^2 \\
&\qquad \times \Big(\|\overline{g_i}_{\widetilde{\Phi}}\|_{\widetilde{L}^\infty_{T_2,f_3(t)}(X_{1,\alpha})} \frac{\langle T_2\rangle^{\delta}-1}{\sqrt{\delta}}\Big)^2.
\end{aligned}$$

In the last two inequalities, we use (4.8) and Cauchy-Schwarz's inequality respectively. It immediately implies (4.11). □

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