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Transportation Cost-Information Inequality for Stochastic Wave Equation

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Abstract In this paper, we prove a Talagrand's T_2 transportation cost-information inequality for the law of stochastic wave equation in spatial dimension $d = 3$ driven by the Gaussian random field, white in time and correlated in space, on the continuous path space with respect to the weighted L^2-norm on $\mathbb{R}^3$.

Keywords Stochastic wave equation · Girsanov's transformation · Transportation cost-information inequality

Mathematics Subject Classification 60H15 · 60H20

1 Introduction

The purpose of this paper is to study Talagrand's T_2 transportation cost-information inequality for the following stochastic wave equation in spatial dimension $d = 3$:

$$\begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta)u(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x); \\ u(0,x) = \nu_1(x); \\ \frac{\partial}{\partial t}u(0,x) = \nu_2(x), \end{cases} \tag{1}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^3$, where the coefficients $b, \sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous, the term Δu denotes the Laplacian of u in the x-variable and the process $\dot{W}$ is the formal derivative of a Gaussian random field, white in time and correlated in space, ν_1 and ν_2 are

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some measurable functions from $\mathbb{R}^3$ to $\mathbb{R}$. We recall that a random field solution to (1) is a family of random variables $\{u(t,x), t \in \mathbb{R}_+, x \in \mathbb{R}^3\}$ such that $(t,x) \mapsto u(t,x)$ from $\mathbb{R}_+ \times \mathbb{R}^3$ into $L^2(\Omega)$ is continuous and solves an integral form of (1), see Sect. 2 for details.

It is known that random field solutions have been shown to exist when $d \in \{1,2,3\}$, see [9]. In spatial dimension 1, a solution to the non-linear wave equation driven by space-time white noise was given in [34] by using Walsh's martingale measure stochastic integral. In dimensions 2 or higher, there is no function-valued solution with space-time white noise, some spatial correlation is needed. A necessary and sufficient condition on the spatial correlation for the existence of a random field solutions was given in [11]. Since the fundamental solution in spatial dimension $d = 3$ is not a function, this required an extension of Walsh's martingale measure stochastic integral to integrands that are Schwartz distributions, the existence of a random field solution to (1) is given in [9]. Hölder continuity of the solution was established in [13]. The large deviation principle and moderate deviation principle were established in Ortiz-López and Sanz-Solé [22] and Chen et al. [8]. In spatial dimensional $d \geqslant 4$, since the fundamental solution of the wave equation is not a measure, but a Schwarz distribution that is a derivative of some order of a measure, the methods used in dimension 3 do not apply to higher dimensions, see [11] for the study of the solutions.

Transportation cost-information inequalities have been recently deeply studied, especially for their connection with the concentration of measure phenomenon, log-Sobolev inequality, Poincaré inequality and Hamilton-Jacobi's equation, see [1, 3–5, 7, 16, 19–21, 23, 26, 31, 33] and so on.

Let us recall the transportation inequality. Let (E,d) be a metric space equipped with σ-field $\mathcal{B}$ such that $d(\cdot,\cdot)$ is $\mathcal{B} \times \mathcal{B}$ measurable. Given $p \geqslant 1$ and two probability measures μ and μ' on E, the Wasserstein distance is defined by

$$W_{p,d}(\mu,\mu') := \inf_{\pi}\left[\int_{E\times E} d^p(x,y)\pi(dx,dy)\right]^{\frac{1}{p}},$$

where the infimum is taken over all the probability measures π on $E \times E$ with marginal distributions μ and μ'. The relative entropy of μ' with respect to (w.r.t. for short) μ is defined as

$$\mathbf{H}(\mu'|\mu) := \begin{cases} \int_E \log \frac{d\mu'}{d\mu} d\nu, & \text{if } \mu' \ll \mu; \\ +\infty, & \text{otherwise.} \end{cases} \tag{2}$$

Definition 1.1 The probability measure μ is said to satisfy the transportation cost-information inequality $\mathbf{T_p}(C)$ on (E,d) if there exists a constant $C > 0$ such that for any probability measure μ' on E,

$$W_{p,d}(\mu,\mu') \leqslant \sqrt{2C\mathbf{H}(\mu'|\mu)}.$$

Recently, the problem of transportation inequalities and their applications to diffusion processes have been widely studied. The $\mathbf{T_2}(C)$ inequality, first established by M. Talagrand [31] for the Gaussian measure with the sharp constant $C = 2$. The approach of M. Talagrand is generalized by D. Feyel and A.S. Üstünel [15] on the abstract Wiener space with respect to Cameron-Martin distance using the Girsanov theorem. With regard to the path of finite-dimensional stochastic differential equation (SDE for short), by means of Girsanov's transformation and the martingale representation theorem, the $\mathbf{T_2}(C)$ w.r.t. the L^2 and the Cameron-Martin distances were established by H. Djellout et al. [14]; the $\mathbf{T_2}(C)$ w.r.t. the uniform metric was obtained by [32, 36]. J. Bao et al. [2] established the $\mathbf{T_2}(C)$ w.r.t. both the uniform and the L^2 distances on the path space for the segment process associated to a class of neutral function stochastic differential equations. B. Saussereau [28] studied the $\mathbf{T_2}(C)$ for SDE driven by a fractional Brownian motion, and S. Riedel [27] extended this result to the law of SDE driven by general Gaussian processes by using Lyons' rough path theory. S. Pal [24] proved that probability laws of certain multidimensional semimartingales which includes time-inhomogenous diffusions, satisfy quadratic transportation cost inequality under the uniform metric. Those, in particular, imply some results about concentration of boundary local time of reflected Brownian motions.

Motivated by the source of the noise modeled by the random terms in partial differential equations, which include physical noise (such as thermal noise), the stochastic partial differential equations have been studied in many literatures in past thirty years. For the stochastic reaction-diffusion equation, L. Wu and Z. Zhang [37] studied the $\mathbf{T_2}(C)$ w.r.t. L^2-norm by Galerkin's approximation. By Girsanov's transformation, B. Boufoussi and S. Hajji [6] obtained the $\mathbf{T_2}(C)$ w.r.t. L^2-metric for the stochastic heat equations driven by space-time white noise and driven by fractional noise. We particularly like to mention the papers by [18] and [30], where the authors established the transportation inequalities under L^2-distance and the uniform distance for stochastic reaction diffusion equations driven by space-time white noise. Se also [25] and [35] about the transportation inequalities results for SPDEs with reflection or with the random initial values. Those results are established for the stochastic parabolic equations. However, the hyperbolic case is much more complicated, one difficulty comes from the more complicated stochastic integral, another one comes from the lack of good regularity properties of the fundamental solutions. See [10] for the study of the stochastic wave equation.

In this paper, we shall study Talagrand's T_2-transportation inequality for the law of a stochastic wave equation (1) on the continuous path space with respect to the weighted L^2-norm on $\mathbb{R}^3$.

The rest of this paper is organized as follows. In Sect. 2, we first give the properties of Eq. (1), and then state the main result of this paper. In Sect. 3, we shall prove the main result.

2 Background and Results

2.1 Framework and Notation

For any $d \geqslant 1$ and any domain $O \subset \mathbb{R}^d$, let $\mathcal{S}(O)$ denote the Schwartz space of rapidly decreasing C^∞ test functions on the domain O, see [29, p. 244]. $W = (W(\varphi), \varphi \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^3))$ is a Gaussian process defined on some probability space with zero mean and covariance functional

$$\mathbb{E}\big(W(\varphi)W(\psi)\big) = J(\varphi, \psi) := \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \varphi(s, x) f(x-y) \psi(s, y), \tag{3}$$

where $f : \mathbb{R}^3 \to \mathbb{R}_+$ continuous on $\mathbb{R}^3 \setminus \{0\}$.

According to [9], there are some requirements on f. As a covariance functional of a Gaussian process, the function $J(\cdot, \cdot)$ should be non-negative definite, this implies that f is symmetric ($f(x) = f(-x)$ for all $x \in \mathbb{R}^3$), and is equivalent to the existence of a non-negative tempered measure μ on $\mathbb{R}^3$, whose Fourier transform is f. More precisely, for any

$\varphi \in \mathcal{S}(\mathbb{R}^3)$, let $\mathcal{F}\varphi$ be the Fourier transform of φ:

$$\mathcal{F}\varphi(\xi) := \int_{\mathbb{R}^3} \exp(-2i\pi\xi \cdot x)\varphi(x)dx.$$

The relationship between μ and f is that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^3} f(x)\varphi(x)dx = \int_{\mathbb{R}^3} \mathcal{F}\varphi(\xi)\mu(d\xi).$$

Let $\mathcal{H}$ be the Hilbert space obtained by the completion of $\mathcal{S}(\mathbb{R}^3)$ with the inner product

$$\begin{aligned}\langle \varphi, \psi \rangle_{\mathcal{H}} &:= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^3} dy \varphi(x) f(x-y)\psi(y) \\ &= \int_{\mathbb{R}^3} \mu(d\xi)\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)} \quad \forall \varphi, \psi \in \mathcal{S}\big(\mathbb{R}^3\big).\end{aligned} \tag{4}$$

Here $\bar{z}$ is the complex conjugate of z. Denote $\|\varphi\|_{\mathcal{H}} = \sqrt{\langle \varphi, \varphi \rangle_{\mathcal{H}}}$. Let $\mathcal{H}_T := L^2([0,T]; \mathcal{H})$ and consider the usual L^2-norm $\|\cdot\|_{\mathcal{H}_T}$ on this space. Then $\mathcal{H}_T$ is a Hilbert space with the inner product

$$\langle \psi_1, \psi_2 \rangle_{\mathcal{H}_T} := \int_0^T \big\langle \psi_1(t), \psi_2(t) \big\rangle_{\mathcal{H}} dt, \quad \varphi, \psi \in \mathcal{H}_T. \tag{5}$$

According to [12], the Gaussian process W with covariance (3) can be extended to a martingale measure

$$M = \big\{ M_t(A) = W\big([0,t] \times A\big),\ t \geqslant 0,\ A \in \mathcal{B}_b\big(\mathbb{R}^3\big) \big\},$$

where $\mathcal{B}_b(\mathbb{R}^3)$ denotes the collection of all bounded Borel measurable sets in $\mathbb{R}^3$. For each $t \geqslant 0$, denote by $\mathcal{F}_t$ the σ-field generated by the random variables $\{M_s(A), s \in [0,t], A \in \mathcal{B}(\mathbb{R}^3)\}$.

Let $\mathcal{C}([0,T] \times \mathbb{R}^3)$ be the space of all continuous functions from $[0,T] \times \mathbb{R}^3$ to $\mathbb{R}$. For any integer $n \geqslant 1$, let $\mathcal{C}^n(\mathbb{R}^3)$ be the space of all continuous functions from $\mathbb{R}^3$ to $\mathbb{R}$, whose derivatives up to order n are also continuous. For any $\delta \in (0,1)$, let $\mathcal{C}_\delta(\mathbb{R}^3)$ be the space of all Hölder continuous functions of order δ, with the Hölder semi-norm

$$\|g\|_\delta := \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x-y|^\delta} < \infty, \quad \forall g \in \mathcal{C}_\delta\big(\mathbb{R}^3\big);$$

and let $\mathcal{C}_{Lip}(\mathbb{R}^3)$ be the space of all Lipschitz continuous functions, with the semi-norm

$$\|f\|_{Lip} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|} < \infty, \quad \forall f \in \mathcal{C}_{Lip}\big(\mathbb{R}^3\big).$$

Assume the following Hypotheses **H** hold:

(H.1) The coefficients σ and b are real Lipschitz continuous, i.e., there exist some constants $K_b > 0$ and K_σ such that

$$\big|b(x) - b(y)\big| \leqslant K_b|x-y|, \quad \big|\sigma(x) - \sigma(y)\big| \leqslant K_\sigma|x-y|, \quad \forall x, y \in \mathbb{R}. \tag{6}$$

(H.2) The function f given in (3) can be expressed by $f(x)=\varphi(x)|x|^{-\beta}$, $x\in\mathbb{R}^3\backslash\{0\}$, with $\beta\in(0,2)$. Here the functions φ and $\nabla\varphi$ are bounded, $0<\varphi\in\mathcal{C}^1(\mathbb{R}^3)$, $\nabla\varphi\in\mathcal{C}_\delta(\mathbb{R}^3)$ with $\delta\in(0,1]$.

(H.3) The initial values ν_1, ν_2 are bounded, $\nu_1\in\mathcal{C}^2(\mathbb{R}^3)$, $\nabla\nu_1$ is bounded, $\triangle\nu_1$ and ν_2 are Hölder continuous with orders $\gamma_1,\gamma_2\in(0,1]$, respectively.

(H.4) The function σ is bounded, i.e., there exists a constant $\bar{\sigma}$ such that $|\sigma(x)|\leqslant\bar{\sigma}$ for all $x\in\mathbb{R}$.

Let

$$G(t,x,y):=\frac{1}{4\pi t}\sigma_t(x-y),$$

where σ_t is the uniform surface measure (with total mass $4\pi t^2$) on the sphere of radius t. Then the hypothesis (H.2) implies that, for any $T>0$,

$$M(T):=\sup_{t\in[0,T]}\int_{\mathbb{R}^3}\left|\mathcal{F}G(t)(\xi)\right|^2\mu(d\xi)<\infty, \tag{7}$$

see [10].

By Walsh's theory of stochastic integration with respect to martingale measures, for any $t\geqslant 0$ and $h\in\mathcal{H}$, the stochastic integral

$$B_t(h):=\int_0^t\int_{\mathbb{R}^3}h(y)M(ds,dy)$$

is well defined, and

$$\left\{B_t^k:=\int_0^t\int_{\mathbb{R}^3}e_k(y)M(ds,dy);\ k\geqslant 1\right\}$$

defines a sequence of independent standard Wiener processes, where $\{e_k\}_{k\geqslant 1}$ is a complete orthonormal system of the Hilbert space $\mathcal{H}$. Thus, $B_t:=\sum_{k\geqslant 1}B_t^k e_k$ is a cylindrical Wiener process on $\mathcal{H}$. See [34].

According to R. Dalang and M. Sanz-Solé [13], under hypotheses (H.1)–(H.3), Eq. (1) admits a unique solution u in $\mathcal{C}([0,T]\times\mathbb{R}^3)$:

$$\begin{aligned} u(t,x)=w(t,x)&+\sum_{k\geqslant 1}\int_0^t\left\langle G(t-s,x-\cdot)\sigma\big(u(s,\cdot)\big),e_k(\cdot)\right\rangle_{\mathcal{H}}dB_s^k\\ &+\int_0^t\int_{\mathbb{R}^3}\big[G(t-s,x,y)b\big(u(s,y)\big)dyds, \end{aligned} \tag{8}$$

where

$$w(t,x):=\frac{d}{dt}\int_{\mathbb{R}^3}G(t,x,y)\nu_1(y)dy+\int_{\mathbb{R}^3}G(t,x,y)\nu_2(y)dy.$$

Furthermore, for any $p\in[2,\infty)$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3}\mathbb{E}\big[\left|u(t,x)\right|^p\big]<+\infty. \tag{9}$$

See R. Dalang and M. Sanz-Solé [13] or Y. Hu et al. [17] for details.

2.2 Main Results

Given any positive bounded function Φ from $\mathbb{R}^3$ to $\mathbb{R}_+$ satisfying that

$$\int_{\mathbb{R}^3} \Phi(x)dx = 1, \tag{10}$$

define the weighted L^2-norm on $\mathcal{C}([0,T]\times\mathbb{R}^3)$

$$\|f\|_{\Phi,2} := \left(\int_0^T \int_{\mathbb{R}^3} \Phi(x)\big|f(t,x)\big|^2 dtdx\right)^{\frac{1}{2}}. \tag{11}$$

For example, $\Phi(x) = \frac{(1+|x|^{3+\delta})^{-1}}{\int_{\mathbb{R}^3}(1+|x|^{3+\delta})^{-1}dx}$ or $\Phi(x) = \frac{e^{-\delta|x|}}{\int_{\mathbb{R}^3} e^{-\delta|x|}dx}$ for any $\delta > 0$ satisfy (10).

For any initial function $\nu := (\nu_1, \nu_2)$ satisfying (H.3), let $\mathbb{P}_\nu$ be the law of $\{u(t,x), (t,x) \in [0,T]\times\mathbb{R}^3\}$ on $\mathcal{C}([0,T]\times\mathbb{R}^3)$ with initial value $u(0,x) = \nu_1(x)$ and $\frac{\partial}{\partial t}u(0,x) = \nu_2(x)$.

In this paper, we establish the following results:

Theorem 2.1 *Under assumptions* **H**, *there exists a constant C given by*

$$C = 3\bar{\sigma}^2 T^2 M(T) \exp\left(12K_\sigma^2 M(T)T + \frac{3T^4K_b^2}{2}\right), \tag{12}$$

such that the probability measure $\mathbb{P}_\nu$ satisfies $\mathbf{T_2}(C)$ on the space $\mathcal{C}([0,T]\times\mathbb{R}^3)$ endowed with the weighted L^2-norm $\|\cdot\|_{\Phi,2}$.

As indicated in [3], many interesting consequences can be derived from Theorem 2.1, see also Corollary 5.11 of [14]. For example, we give the following application of Theorem 2.1.

Corollary 2.2 *Under assumptions* **H**, *the following statements hold for the constant C given by* (12):

(a) *For any Lipschitz function U on $\mathcal{C}([0,T]\times\mathbb{R}^3)$ with respect to the weighted L^2-norm $\|\cdot\|_{\Phi,2}$ and with $\|U\|_{\mathrm{Lip}} := \sup_{u_1,u_2\in\mathcal{C}([0,T]\times\mathbb{R}^3),u_1\neq u_2} \frac{|U(u_1)-U(u_2)|}{\|u_1-u_2\|_{\Phi,2}} < \infty$, we have*

$$\mathbb{E}^{\mathbb{P}_\nu}\left[\exp\left(U - \mathbb{E}^{\mathbb{P}_\nu}U\right)\right] \leqslant e^{\frac{C}{2}\|U\|_{\mathrm{Lip}}^2}.$$

(b) (*Hoeffding-type inequality*) *For any Lipschitz function $V : \mathbb{R} \to \mathbb{R}$ with $\|V\|_{\mathrm{Lip}} := \sup_{x,y\in\mathbb{R},x\neq y} \frac{|V(x)-V(y)|}{|x-y|} < \infty$, we have that for any $r \geqslant 0$,*

$$\mathbb{P}\left(\frac{1}{T}\int_0^T\int_{\mathbb{R}^3} V\big(u(t,x)\big)\Phi(x)dtdx - \mathbb{E}\left[\frac{1}{T}\int_0^T\int_{\mathbb{R}^3} V\big(u(t,x)\big)\Phi(x)dtdx\right] > r\right) \leqslant \exp\left(-\frac{r^2T}{2C\|V\|_{\mathrm{Lip}}^2}\right).$$

3 The Proof

3.1 An Important Lemma

We will apply Girsanov's theorem to prove Theorem 2.1. To do this, we need the following lemma describing all probability measures which are absolutely continuous with respect to $\mathbb{P}_\nu$. It is analogous to [14, Theorem 5.6] in the setting of finite-dimensional Brownian motion and [18, Lemma 3.1] in the setting of space-time white noise. Its proof is given in the first version of [18, Lemma 6.2].

Lemma 3.1 [18] *For every probability measure $\mathbb{Q} \ll \mathbb{P}_\nu$ on the space $L^2([0,T]\times\mathbb{R}^3;\mathbb{R})$, there exists an adapted $\mathbb{Q}$-a.s. $h = \{h(s,x),(s,x)\in[0,T]\times\mathbb{R}^3\}$ such that $\|h\|_{\mathcal{H}_T} < \infty$, $\mathbb{Q}$-a.s., and the function $\widetilde{W} : L^2([0,T]\times\mathbb{R}^3;\mathbb{R}) \to L^2(\Omega)$ defined by*

$$\tilde{W}(\phi) := W(\phi) - \int_0^t \langle \phi(s), h(s)\rangle_{\mathcal{H}} ds, \tag{13}$$

is a space-colored time-white noise with the spectral density f with respect to the measure $\mathbb{Q}$. The Randon-Nikodym derivative is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}_\nu}\bigg|_{\mathcal{F}_T}\big(u_\cdot(\nu)\big) = \exp\bigg(\int_0^T \int_{\mathbb{R}^3} h(s,x) W(ds,dx) - \frac{1}{2}\int_0^T \big\|h(s)\big\|_{\mathcal{H}}^2 ds\bigg), \tag{14}$$

and the relative entropy is given by

$$\mathbf{H}(\mathbb{Q}|\mathbb{P}_\nu) = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\big[\|h\|_{\mathcal{H}_T}^2\big]. \tag{15}$$

3.2 The Proof of Theorem 2.1

It is enough to prove the result for any probability measure $\mathbb{Q}$ on $\mathcal{C}([0,T]\times\mathbb{R}^3)$ such that $\mathbb{Q} \ll \mathbb{P}_\nu$ and $\mathbf{H}(\mathbb{Q}|\mathbb{P}_\nu) < \infty$. By Lemma 3.1, we may assume that the Randon-Nikodym derivative is

$$\frac{d\mathbb{Q}}{d\mathbb{P}_\nu}\bigg|_{\mathcal{F}_T} = \exp\bigg(\int_0^T \int_{\mathbb{R}^3} h(s,x) W(ds,dx) - \frac{1}{2}\int_0^T \big\|h(s)\big\|_{\mathcal{H}}^2 ds\bigg), \tag{16}$$

and the relative entropy is

$$\mathbf{H}(\mathbb{Q}|\mathbb{P}_\nu) = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\big[\|h\|_{\mathcal{H}_T}^2\big]. \tag{17}$$

For any $n \geqslant 1$, let $G_n = n \wedge \frac{d\mathbb{Q}}{d\mathbb{P}_\nu}|_{\mathcal{F}_T}$, and $d\mathbb{Q}_n = G_n d\mathbb{P}_\nu / \mathbb{E}_\nu[G_n]$. Then $\mathbb{Q}_n \to \mathbb{Q}$ in total variation norm as $n \to \infty$. Recall at first two facts (e.g., see [14, 33]):

(1) If $\mathbb{Q}_n \to \mathbb{Q}$ weakly, then $W_{\Phi,2}^2(\mathbb{Q},\mathbb{P}_\nu) \leqslant \liminf_{n\to\infty} W_{\Phi,2}^2(\mathbb{Q}_n,\mathbb{P}_\nu)$.
(2) If $\mathbb{Q}_n \to \mathbb{Q}$ weakly and $\mathbf{H}(\mathbb{Q}|\mathbb{P}_\nu) < \infty$, then $\mathbf{H}(\mathbb{Q}|\mathbb{P}_\nu) = \lim_{n\to\infty} \mathbf{H}(\mathbb{Q}_n|\mathbb{P}_\nu)$.

Thus, to prove the main result, it is enough to prove it for $\mathbb{Q}_n$. Without loss of generality, we assume that $\frac{d\mathbb{Q}}{d\mathbb{P}_\nu}|_{\mathcal{F}_T}$ is bounded.

Let $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$ be a complete probability space on which W is space-colored time-white noise with the spectral density f with respect to the measure $\mathbb{Q}$. Let

$$\mathcal{F}_t = \mathcal{F}_t^W = \sigma\big(W\big([0,s]\times A\big), s \leqslant t,\ \forall A \subset \mathbb{R}^3\big)^{\widetilde{\mathbb{P}}} \quad \text{completion by } \widetilde{\mathbb{P}}.$$

Let $u_t(\nu)$ be the unique solution of (1) with initial condition $\nu = (\nu_1, \nu_2)$. Then the law of $u_\cdot(\nu)$ is $\mathbb{P}_\nu$. Consider

$$\widetilde{\mathbb{Q}} := \frac{d\mathbb{Q}}{d\mathbb{P}_\nu}\big(u_\cdot(\nu)\big) \cdot \widetilde{\mathbb{P}}.$$

Then

$$\mathbf{H}(\mathbb{Q}|\mathbb{P}_\nu)=\mathbf{H}(\widetilde{\mathbb{Q}}|\widetilde{\mathbb{P}})=\frac{1}{2}\mathbb{E}^{\mathbb{Q}}\big[\|h\|^2_{\mathcal{H}_T}\big]$$

For a complete orthonormal system $\{e_k\}_{k\geqslant 1}$ of the Hilbert space $\mathcal{H}$, let

$$\left\{B_t^k:=\int_0^t\int_{\mathbb{R}^3}e_k(y)W(ds,dy);\ k\geqslant 1\right\}.$$

Then $B_t:=\sum_{k\geqslant 1}B_t^ke_k$ is a cylindrical Wiener process on $\mathcal{H}$ under $\widetilde{\mathbb{P}}$, and $\sum_{k\geqslant 1}(B_t^k+\langle h,e_k\rangle_{\mathcal{H}})e_k$ is a cylindrical Wiener process on $\mathcal{H}$ under $\widetilde{\mathbb{Q}}$ by Lemma 3.1.

According to Lemma 3.1, we couple $(\mathbb{P},\mathbb{Q})$ as the law of a process (u,v) under $\mathbb{Q}$

$$\begin{aligned}u(t,x)=&w(t,x)+\int_0^t\int_{\mathbb{R}^3}G(t-s,x,y)\sigma\big(u(s,y)\big)\widetilde{W}(ds,dy)\\&+\int_0^t\int_{\mathbb{R}^3}G(t-s,x,y)b\big(u(s,y)\big)dyds\\&+\sum_{k\geqslant 1}\int_0^t\big\langle G(t-s,x-\cdot)\sigma\big(u(s,y)\big)h(s,\cdot),e_k(\cdot)\big\rangle_{\mathcal{H}}ds,\end{aligned}\tag{18}$$

and

$$\begin{aligned}v(t,x)=&w(t,x)+\int_0^t\int_{\mathbb{R}^3}G(t-s,x,y)\sigma\big(v(s,y)\big)\widetilde{W}(ds,dy)\\&+\int_0^t\int_{\mathbb{R}^3}G(t-s,x,y)b\big(v(s,y)\big)dyds.\end{aligned}\tag{19}$$

By the definition of the Wasserstein distance, we have

$$W^2_{\Phi,2}(\mathbb{Q},\mathbb{P})\leqslant\mathbb{E}^{\mathbb{Q}}\left[\int_0^T\int_{\mathbb{R}^3}\Phi(x)\big|u(t,x)-v(t,x)\big|^2dxdt\right].\tag{20}$$

In view of (15) and (20), it remains to prove that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T\int_{\mathbb{R}^3}\Phi(x)\big|u(t,x)-v(t,x)\big|^2dxdt\right]\leqslant C\mathbb{E}^{\mathbb{Q}}\big[\|h\|^2_{\mathcal{H}_T}\big].\tag{21}$$

From (8), (18) and (19), we can represent $u(t,x)-v(t,x)$ as

$$\begin{aligned}u(t,x)-v(t,x)=&\sum_{k\geqslant 1}\int_0^t\big\langle G(t-s,x-\cdot)\big[\sigma\big(u(s,\cdot)\big)-\sigma\big(v(s,\cdot)\big)\big],e_k(\cdot)\big\rangle_{\mathcal{H}}d\widetilde{B}_s^k\\&+\int_0^t\int_{\mathbb{R}^3}G(t-s,x,y)\big[b\big(u(s,y)\big)-b\big(v(s,y)\big)\big]dyds\\&+\sum_{k\geqslant 1}\int_0^t\big\langle G(t-s,x-\cdot)\sigma\big(u(s,x-\cdot)\big)h(s,\cdot),e_k(\cdot)\big\rangle_{\mathcal{H}}ds,\end{aligned}\tag{22}$$

where $\widetilde{B}_t^k = B_t^k - \int_0^t \langle h(s), e_k \rangle_{\mathcal{H}} ds$ is a Brownian motion under $\mathbb{Q}$ for any $k \geqslant 1$. Thus,

$$\begin{aligned}
\left|u(t,x)-v(t,x)\right|^2 \leqslant & 3\left|\sum_{k\geqslant 1}\int_0^t \left\langle G(t-s,x-\cdot)\left[\sigma\left(u(s,\cdot)\right)-\sigma\left(v(s,\cdot)\right)\right], e_k(\cdot)\right\rangle_{\mathcal{H}} d\widetilde{B}_s^k\right|^2 \\
& + 3\left|\int_0^t\int_{\mathbb{R}^3} G(t-s,x,y)\left[b\left(u(s,y)\right)-b\left(v(s,y)\right)\right]dyds\right|^2 \\
& + 3\left|\sum_{k\geqslant 1}\int_0^t \left\langle G(t-s,x-\cdot)\sigma\left(u(s,x-\cdot)\right)h(s,\cdot), e_k(\cdot)\right\rangle_{\mathcal{H}} ds\right|^2 \\
=: & 3I_1(t,x)+3I_2(t,x)+3I_3(t,x).
\end{aligned} \tag{23}$$

For every $t \in [0,T]$, define

$$\eta(t) = \sup_{(s,x)\in[0,t]\times\mathbb{R}^3} \mathbb{E}^{\mathbb{Q}}\left[\left|u(s,x)-v(s,x)\right|^2\right].$$

By (9) and the boundedness of $\frac{d\mathbb{Q}}{d\mathbb{P}_v}$, we know that $\eta(t) < \infty$.

By Burkholder's inequality, Hölder's inequality, the Lipschitz continuity of σ and (7), we have

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[\left|I_1(t,x)\right|\right] \\
= & \mathbb{E}^{\mathbb{Q}}\left[\left|\sum_{k\geqslant 1}\int_0^t \left\langle G(t-s,x-\cdot)\left[\sigma\left(u(s,\cdot)\right)-\sigma\left(v(s,\cdot)\right)\right], e_k(\cdot)\right\rangle_{\mathcal{H}} d\widetilde{B}_s^k\right|^2\right] \\
\leqslant & 4K_\sigma^2 \mathbb{E}^{\mathbb{Q}}\left[\int_0^t \left\|G(t-s,x-\cdot)\left[u(s,\cdot)-v(s,\cdot)\right]\right\|_{\mathcal{H}}^2 ds\right] \\
\leqslant & 4K_\sigma^2 \int_0^t \left(\int_{\mathbb{R}^3}\left|\mathcal{F}G(t-s)(\xi)\right|^2\mu(d\xi)\right)\left(\sup_{(r,z)\in[0,s]\times\mathbb{R}^3}\mathbb{E}^{\mathbb{Q}}\left[\left|u(r,z)-v(r,z)\right|^2\right]\right)ds \\
\leqslant & 4K_\sigma^2 M(T)\int_0^t \eta(s)ds < \infty.
\end{aligned} \tag{24}$$

By the Cauchy-Schwarz inequality with respect to the finite measure $G(t-s,x,y)dyds$ on $[0,T]\times\mathbb{R}^3$, with total measure $t^2/2$, and by the Lipschitz continuity of b, we obtain that for any $t \leqslant T$,

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[\left|I_2(t,x)\right|\right] \\
\leqslant & K_b^2 \int_0^t\int_{\mathbb{R}^3} G(t-s,x,y)dyds \cdot \int_0^t\int_{\mathbb{R}^3} G(t-s,x,y)\mathbb{E}^{\mathbb{Q}}\left[\left|u(s,y)-v(s,y)\right|^2\right]dyds \\
\leqslant & \frac{t^2K_b^2}{2}\int_0^t\int_{\mathbb{R}^3} G(t-s,x,y)\eta(s)dyds \\
= & \frac{t^2K_b^2}{2}\int_0^t (t-s)\eta(s)ds \\
\leqslant & \frac{T^3K_b^2}{2}\int_0^t \eta(s)ds,
\end{aligned} \tag{25}$$

where $\int_{\mathbb{R}^3} G(t-s,x,y)dy = t-s$ is used in the last second line.

Let us estimate the third term. By the Cauchy-Schwarz inequality, the boundedness of σ and (7), we have for any $t \leqslant T$,

$$\begin{aligned} \left|I_3(t,x)\right| &\leqslant \bar{\sigma}^2 \int_0^t \|G(t-s,x,\cdot)\|^2_{\mathcal{H}} ds \times \int_0^t \|h(s)\|^2_{\mathcal{H}} ds \\ &\leqslant TM(T)\bar{\sigma}^2 \int_0^t \|h(s)\|^2_{\mathcal{H}} ds. \end{aligned} \tag{26}$$

Putting (24)–(26) together, we have

$$\eta(t) \leqslant \left(12K_\sigma^2 M(T) + \frac{3T^3K_b^2}{2}\right)\int_0^t \eta(s)ds + 3\bar{\sigma}^2 TM(T)\mathbb{E}^{\mathbb{Q}}\big[\|h\|^2_{\mathcal{H}_T}\big].$$

Using Gronwall's inequality, we obtain that

$$\eta(T) \leqslant 3\bar{\sigma}^2 TM(T)\exp\left(12K_\sigma^2 M(T)T + \frac{3T^4K_b^2}{2}\right)\mathbb{E}^{\mathbb{Q}}\big[\|h\|^2_{\mathcal{H}_T}\big]. \tag{27}$$

Thus, we have

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \int_{\mathbb{R}^3} \Phi(x)\left|u(s,x)-v(s,x)\right|^2 dsdx\right] \\ \leqslant &\int_0^T \int_{\mathbb{R}^3} \Phi(x) \sup_{(s,x)\in[0,T]\times\mathbb{R}^3} \left[\mathbb{E}^{\mathbb{Q}}\left|u(s,x)-v(s,x)\right|^2\right] dsdx \\ \leqslant &3\bar{\sigma}^2 T^2 M(T)\exp\left(12K_\sigma^2 M(T)T + \frac{3T^4K_b^2}{2}\right)\mathbb{E}^{\mathbb{Q}}\big[\|h\|^2_{\mathcal{H}_T}\big]. \end{aligned}$$

The proof is complete.

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