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On the Uniqueness of Minimizers for a Class of Variational Problems with Polyconvex Integrand

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Abstract We prove existence and uniqueness of minimizers for a family of energy functionals that arises in Elasticity and involves polyconvex integrands over a certain subset of displacement maps. This work extends previous results by Awi and Gangbo to a larger class of integrands. We are interested in Lagrangians of the form $L(A,u) = f(A) + H(\det A) - F \cdot u$. Here the strict convexity condition on f and H have been relaxed to a convexity condition. Meanwhile, we have allowed the map F to be non-degenerate. First, we study these variational problems over displacements for which the determinant is positive. Second, we consider a limit case in which the functionals are degenerate. In that case, the set of admissible displacements reduces to that of incompressible displacements which are measure preserving maps. Finally, we establish that the minimizer over the set of incompressible maps may be obtained as a limit of minimizers corresponding to a sequence of minimization problems over general displacements provided we have enough regularity on the dual problems. We point out that these results do not rely on the direct methods of the calculus of variations.

Keywords Duality · Euler-Lagrange equations · Elasticity theory · Pseudo-projected gradient · Relaxations · Polyconvexity

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1 Introduction

We are interested in Euler-Lagrange equations, existence and uniqueness of minimizers for some problems in the vectorial calculus of variations emanating from elasticity theory. These variational problems are related to an open problem in Partial Differential Equations that we describe as follows: let $T > 0$ and let Ω and Λ be two open subsets of $\mathbb{R}^d$; suppose that $\mathbf{u}_0$ is a diffeomorphism between Ω and Λ; we seek $\mathbf{u} : \Omega \times (0, T) \longrightarrow \mathbb{R}^d$ such that $\mathbf{u}(\cdot, t)(\Omega) = \Lambda$ for each t and

$$\begin{cases} \mathbf{u}_t = \operatorname{div}_x D_\xi L(\nabla \mathbf{u}) & \text{on } \Omega \times (0, T), \\ \mathbf{u}(0, \cdot) = \mathbf{u}_0 & \text{on } \Omega, \end{cases} \tag{1.1}$$

in the sense of distributions. In (1.1), we assume that the map $\mathbb{R}^{d\times d} \ni \xi \mapsto L(\xi)$ is quasi-convex. We refer the reader to [2], [7], [5], [11], and [12] for further details on these gradient flows. Understanding variational problems associated to the time-discretization of (1.1) is arguably an important step toward the construction of a solution. In that regard, several partial results are available in the literature (see for instance [7] and [5]).

In [2], the authors have focused on a class of Lagrangians that arises in elastic materials. More precisely, they have considered polyconvex Lagrangians of the form $\xi \mapsto L(\xi) = f(\xi) + H(\det \xi)$. Here f is a $C^1(\mathbb{R}^d)$ strictly convex function with p-th order growth, and the map H is a $C^1(0, \infty)$ convex function that satisfies

$$\lim_{t\to 0^+} H(t) = \lim_{t\to\infty} \frac{H(t)}{t} = +\infty. \tag{1.2}$$

As a result, a variational problem emerges from the time discretization and has a relaxation that takes the general form:

$$\min\left\{ \int_\Omega \big(f(\nabla u) + H(\beta) - F \cdot u\big)\, dx;\ (u, \beta) \in \mathscr{U} \right\} \tag{1.3}$$

where $F \in L^1(\Omega, \mathbb{R}^d)$ and

$$\mathscr{U} = \Bigg\{ (u, \beta) : u \in W^{1,p}(\Omega, \bar{\Lambda}),\ \beta : \Omega \to [0, \infty); \\ \int_\Omega l(u)\beta\, dx = \int_\Lambda l(y)\, dy; \forall l \in C_c\big(\mathbb{R}^d\big) \Bigg\}. \tag{1.4}$$

Although the existence of minimizers in (1.3) follows from the direct methods in the calculus of variations, the uniqueness is a rather challenging problem. Indeed, because of (1.2) and the non-convexity of the integrand, standard techniques in calculus of variations do not apply.

To bypass these difficulties, the authors of [2] have introduced a pseudo-projected gradient operator $\mathscr{U}_{\mathcal{S}} \ni u \mapsto \nabla_{\mathcal{S}} u$ defined as follows: for a given $u \in \mathscr{U}_{\mathcal{S}}$, the map $\nabla_{\mathcal{S}} u$ is the unique minimizer of

$$\int_\Omega f(G)\, dx$$

over

$$\mathcal{G}_{\mathcal{S}}(u) := \left\{ G \in L^p\big(\Omega, \mathbb{R}^{d\times d}\big) : \int_\Omega u \operatorname{div} \varphi\, dx = -\int_\Omega \langle G, \varphi \rangle\, dx\ \forall \varphi \in \mathcal{S} \right\}.$$

Here, $\mathcal{S}$ is a finite-dimensional subspace of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$, q is the conjugate of p, $\mathcal{U}_{\mathcal{S}}$ is the set of all $u : \Omega \to \bar{\Lambda}$ measurable such that there exists a $c = c(u, \Omega, \Lambda) > 0$ satisfying:

$$\left|\int_\Omega u \cdot \operatorname{div} \varphi \, dx\right| \le c\|\varphi\|_{L^q(\Omega,\mathbb{R}^{d\times d})}, \qquad \forall \varphi \in \mathcal{S}. \tag{1.5}$$

We point out that the pseudo-projected gradient operator depends also on f, though the dependence is not exhibited in its notation. As a first step to approaching (1.3), they have considered the following perturbed problem:

$$\inf\left\{\int_\Omega \big(f(\nabla_{\mathcal{S}} u) + H(\beta) - F \cdot u\big)\, dx;\ (u, \beta) \in \mathcal{U}\right\}. \tag{1.6}$$

The choice of problem (1.6) is justified by the construction of a family of finite dimensional subspaces $\{\mathcal{S}_n\}_n$ dense in $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ such that for $u \in W^{1,p}(\Omega, \mathbb{R}^d)$, one has

$$\lim_{n\to\infty} \int_\Omega f(\nabla_{\mathcal{S}_n} u)\, dx = \int_\Omega f(\nabla u)\, dx. \tag{1.7}$$

We note that a $L^p(\Omega,\ \mathbb{R}^d)$-bounded subset of $\mathcal{U}_{\mathcal{S}}$ whose image by the operator $\nabla_{\mathcal{S}}$ is bounded in $L^p(\Omega,\ \mathbb{R}^{d\times d})$ is not in general strongly pre-compact with respect to the $L^p(\Omega,\ \mathbb{R}^d)$ topology. As a result, compactness of level subsets of the functional in (1.6) cannot be guaranteed. Nevertheless, the authors of [2] have successfully shown existence and, more importantly, uniqueness in (1.6) under the assumption that F is non-degenerate (see definition below). This condition of non-degeneracy for uniqueness is crucial in a similar problem, the so-called Brenier polar factorization, and more generally, in optimal transport problems. Confer [1], [3], [9], [8], [10] and [15].

In this paper, we investigate the respective roles played by the strict convexity of f, the convexity and smoothness of H, and the non-degeneracy of F in problem (1.6). More precisely, we impose less stringent conditions so that either the map F is allowed to be degenerate or f is allowed to be merely convex or H is neither smooth nor strictly-convex. These considerations are not just technicalities. Indeed we note that a prominent case of mere convexity, $f(\xi) = |\xi|$, is typical for the study of minimal surfaces as well as for the study of functionals involving the total variation (see for instance [4]). Furthermore, we observe that cases where H is taken to be the characteristic function of a singleton of $\mathbb{R}$ arise in the study of incompressible deformations in Elasticity theory (see for instance [12] and [15]). Finally, the non-degeneracy condition tests the extent to which one can hope for uniqueness in the variational problem we considered. To deal with these weaker assumptions, we introduce a family of operators $\{V_{\mathcal{S}}^f : \mathcal{S} \subset W_0^{1,q}(\Omega, \mathbb{R}^d),\ f \text{ convex}\}$ defined by

$$W^{1,p}\big(\Omega, \mathbb{R}^d\big) \ni u \mapsto V_{\mathcal{S}}^f[u] := \sup_{\varphi\in\mathcal{S}} \int_\Omega \big(-u \operatorname{div} \varphi - f^*(\varphi)\big)\, dx. \tag{1.8}$$

We note that the operator $V_{\mathcal{S}}^f$ is actually well defined on the set of measurable functions u defined from Ω to $\bar{\Lambda}$ when the set $\mathcal{S}$ is a finite dimensional nonempty set and the function f satisfies appropriate growth conditions. As a family, these operators extend the pseudo-projected gradient operators and the distributional gradient. Indeed, $V_{\mathcal{S}}^f[u] = \int_\Omega f(\nabla_{\mathcal{S}} u)\, dx$ if $\mathcal{S}$ is a finite dimensional subspace of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and $u \in \mathcal{U}_{\mathcal{S}}$ and furthermore $V_{\mathcal{S}}^f[u] = \int_\Omega f(\nabla u)\, dx$ if $\mathcal{S} = W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and $u \in W^{1,p}(\Omega, \mathbb{R}^d)$. These extensions are only valid under appropriate conditions on f. It is worth pointing out that if $f(\xi) = |\xi|$

and $\mathcal{S} = W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ then $V_{\mathcal{S}}^f(u)$ is nothing but the total variation of u on the set Ω. We show that for a collection of sets $\{\mathcal{S}_n\}_{n=1}^{\infty}$ of $W_0^{1,q}(\Omega, \mathbb{R}^d)$ satisfying *Hypothesis (H1)* or *Hypothesis (H2)* (see Sect. 2), we have a convergence result in the same spirit as (1.7):

$$\lim_{n\to\infty} V_{\mathcal{S}_n}^f[u] = V^f_{W_0^{1,q}(\Omega,\mathbb{R}^d)}[u]\Big(= \int_\Omega f(\nabla u)\, dx\Big) \tag{1.9}$$

for any $u \in W^{1,p}(\Omega, \mathbb{R}^{d\times d})$ and appropriate conditions on f. We thus proceed to study a more general problem:

$$\inf_{(u,\beta)\in \mathscr{U}_{\mathcal{S}}^*} \Big\{ V_{\mathcal{S}}^f[u] + \int_\Omega (H(\beta) - F\cdot u)\; dx \Big\} \tag{1.10}$$

where $\mathcal{S}$ is an element of a collection of sets satisfying *Hypothesis (H1)* or *Hypothesis (H2)*, and

$$\mathscr{U}_{\mathcal{S}}^* = \Big\{ (u,\beta) : u \in \mathscr{U}_{\mathcal{S}};\ \beta : \Omega \to [0,\infty);$$
$$\int_\Omega l\big(u(x)\big)\beta(x)\, dx = \int_\Omega l(y)\, dy\ \forall l \in C_c\big(\mathbb{R}^d\big) \Big\}. \tag{1.11}$$

Sublevel sets of the integrand in (1.10) are not compact. Nor is f necessarily strictly convex. However, we show existence and uniqueness in problem (1.10). In fact, this result holds for F non-degenerate as well as for a class of degenerate F provided that the set $\mathcal{S}$ is chosen accordingly (see Corollaries 3.6 and 3.7). Unlike optimal transport theory, this analysis suggests that the non-degeneracy condition is not essential for a uniqueness result in (1.3).

Existence and uniqueness results for problem (1.10) are established thanks to the discovery of suitable dual problems. Indeed, call $\mathcal{C}$ the set of all functions (k,l) with $k,\ l : \mathbb{R}^d \to \mathbb{R}\cup\{\infty\}$ Borel measurable, finite at least at one point, and satisfying the relation $l \equiv \infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$ and such that

$$k(v) + tl(u) + H(t) \ge u\cdot v \quad \forall u, v \in \mathbb{R}^d, t > 0.$$

Let $\mathscr{A}$ be the set of (k,l,φ) such that $(k,l) \in \mathcal{C}$ and $\varphi \in \mathcal{S}$. Define the following functional over the set $\mathscr{A}$:

$$J(k,l,\varphi) := \int_\Omega k(F + \operatorname{div}\varphi)\, dx + \int_\Lambda l\, dy + \int_\Omega f^*(\varphi)\, dx.$$

Next, assume that the map F and the set $\mathcal{S}$ are such that for all $\varphi \in \mathcal{S}$,

$$F + \operatorname{div}\varphi \text{ is non-degenerate.} \tag{1.12}$$

Then $-J$ admits a maximizer (k_0, l_0, φ_0) with k_0 convex and $\operatorname{diam}(\Lambda)$-Lipschitz. As a consequence, problem (1.10) admits a unique minimizer (u_0, β_0) and u_0 satisfies

$$\begin{cases} u_0 = \nabla k_0(F + \operatorname{div}\varphi_0) \\ \varphi_0 \in \Phi_{\mathcal{S}}(u_0). \end{cases} \tag{1.13}$$

Here, we have denoted by $\Phi_{\mathcal{S}}(u_0)$, the non-empty set of maximizers of problem (1.8) (see Proposition 2.8). In order to obtain condition (1.12), we consider two distinct situations.

First, we assume that F has a countable range, thus degenerate. If $\mathcal{S}$ is an element of a collection of sets satisfying hypothesis (H2) then it holds that $F + \operatorname{div} \varphi$ is non-degenerate.

Second, we assume F non-degenerate and $\mathcal{S}$ is a finite dimensional vector space, as in [2]. It holds again that $F + \operatorname{div} \varphi$ is non-degenerate. However, unlike the hypotheses in [2], we have allowed the map f to be as singular as the map $\mathbb{R}^{d\times d} \ni \xi \mapsto |\xi|$.

We have also studied (1.10) when H is replaced by $H_0 : (0,\infty) \to \mathbb{R} \cup \{\infty\}$ defined by $H_0(1) = 0$ and $H_0(t) = \infty$ if $t \neq 1$. This case corresponds to the case of measure preserving maps. Note that H_0 is not even continuous. However, it may be obtained as a limit of functions H_n which are $C^1(0,\infty)$ convex functions and satisfy (1.2). We show that for such singular H_0, the corresponding problem

$$\inf_{u\in\mathscr{U}^1_{\mathcal{S}}} \left\{ V^f_{\mathcal{S}}[u] - \int_\Omega F \cdot u \, dx \right\} \tag{1.14}$$

with

$$\mathscr{U}^1_{\mathcal{S}} = \left\{ u \in \mathscr{U}_{\mathcal{S}} : \int_\Omega l\big(u(x)\big)\, dx = \int_\Omega l(y)\, dy \; \forall l \in C_c\big(\mathbb{R}^d\big) \right\} \tag{1.15}$$

admits a unique minimizer. (See Theorem 4.3.)

To obtain existence and uniqueness results in problem (1.14), we exploit a dual formulation and maximize $-J$ over the set that consists of (k,l,φ) such that $\varphi \in \mathcal{S}$ and $k,\ l : \mathbb{R}^d \to \mathbb{R}\cup\{\infty\}$ are Borel measurable, finite at least at one point, and satisfy the relations $l \equiv \infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$ and

$$k(v) + l(u) \geq u \cdot v \quad \forall u, v \in \mathbb{R}^d.$$

One shows that $-J$ admits a maximizer (k_0, l_0, φ_0) with k_0 convex and Lipschitz and the unique minimizer of problem (1.14) is u_0 given by

$$u_0 = \nabla k_0(F + \operatorname{div} \varphi_0).$$

Finally, we show convergence of a sequence of problems of the form (1.10) to (1.14). More precisely, we show that the minimizer of problem (1.14) may be obtained as limit of minimizers of problems of the form (1.10) provided that the dual problems admit regular enough maximizers. In fact, suppose the map F and the set $\mathcal{S}$ are such that for all $\varphi \in \mathcal{S}$, the map $F + \operatorname{div} \varphi$ is non-degenerate. For $(u,\beta) \in \mathscr{U}_{\mathcal{S}}$, define

$$I_n(u,\beta) = V^f_{\mathcal{S}}[u] + \int_\Omega \big(H_n(\beta) - u \cdot F\big)\, dx$$

and set

$$I_0(u) = V^f_{\mathcal{S}}[u] - \int_\Omega u \cdot F \, dx.$$

Thanks to Theorem 3.5, the problem

$$\inf_{(u,\beta)\in\mathscr{U}^*_{\mathcal{S}}} I_n(u,\beta) \tag{1.16}$$

admits a unique minimizer that we denote (u_n, β_n) with $u_n = \nabla k_n(F + \operatorname{div}\varphi_n)$ for some $k_n : \mathbb{R}^d \to \mathbb{R}$ convex and $\varphi_n \in \mathcal{S}$. Denote u_0 the unique minimizer of (1.14). If for all $n \in \mathbb{N}^*$ the map k_n is differentiable then the sequence $\{u_n\}_{n\in\mathbb{N}^*}$ converges almost everywhere to u_0 and in addition, the minima $\{I_n(u_n,\beta_n)\}_{n\in\mathbb{N}^*}$ converge to $I_0(u_0)$ (cf. Theorem 4.7).

2 Preliminaries

2.1 Notation and Definitions

- Throughout this manuscript, Ω and $\Lambda \subset \mathbb{R}^d$ are two bounded open convex sets; $r^* > 1$ is such that $B(0, 1/r^*) \subset \Lambda \subset B(0, r^*/2)$; $p \in (1, \infty)$ and q is its conjugate, that is, $p^{-1} + q^{-1} = 1$.
- Given $A \subset \mathbb{R}^d$, the indicator function of A is defined as

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \in A, \\ \infty & \text{otherwise.} \end{cases}$$

- For any subset $\mathcal{S}$ of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$, we denote by span$(\mathcal{S})$ the linear subspace of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ generated by $\mathcal{S}$.
- We denote by f^* the Legendre transform of a map $f : \mathbb{R}^{d\times d} \longrightarrow \mathbb{R}$ so that

$$f^*\big(\xi^*\big) = \sup_{\xi \in \mathbb{R}^{d\times d}} \big\{\xi \cdot \xi^* - f(\xi)\big\}.$$

- If $h : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{\infty\}$ is convex then the subdifferential $\partial h(x)$ of h at $x \in \mathrm{Dom}(h)$ is closed and convex. If $\partial h(x)$ is non-empty we denote by grad$[h](x)$ the element of $\partial h(x)$ with minimum norm:

$$\big|\mathrm{grad}[h](x)\big| = \min\big\{|y| : y \in \partial h(x)\big\}; \quad x \in \mathrm{Dom}(h).$$

- Let $\mathcal{S} \subset W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. We denote by $\mathscr{S}_f$ the set

$$\mathscr{S}_f := \left\{\varphi \in \mathcal{S} : \int_\Omega f^*(\varphi) \text{ is finite}\right\}. \tag{2.1}$$

- Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be measurable. We say that F is non-degenerate if for any $N \subset \mathbb{R}^d$ such that $\mathcal{L}^d(N) = 0$ we have $\mathcal{L}^d(F^{-1}(N)) = 0$.

2.2 Assumptions

(A0) We additionally assume that there exists a strictly convex function that is $C^1(\bar{\Omega})$ and vanishes on the boundary of Ω.

(A1) The set $\mathcal{S}$ is a subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. In addition, the map $f : \mathbb{R}^{d\times d} \to \mathbb{R}$ is convex and satisfies the following three properties:

(i) There exist $a, b, c > 0$ such that for all $\xi \in \mathbb{R}^{d\times d}$,

$$c\frac{|\xi|^p}{p} + b \geq f(\xi) \geq a|\xi| - b \tag{2.2}$$

and for all $\xi^* \in \partial f(\xi)$,

$$\big|\xi^*\big|^q \leq c|\xi|^p + b. \tag{2.3}$$

(ii) The set $\mathscr{S}_f$ is non-empty.

(iii) One of the following two conditions holds:

(a) The map f is such that $\partial f^*(x^*)$ is non-empty and $\mathrm{grad}[f^*](x^*) = 0$ for each $x^* \in \mathrm{Dom}\, f^*$.

(b) The map f is strictly convex and there exist $\bar{a}, \bar{b} > 0$ such that for all $\xi^* \in \mathbb{R}^{d\times d}$, one has

$$f^*(\xi^*) \leq \bar{a} + \bar{b}|\xi^*|^q \quad \text{and} \quad |\nabla f^*(\xi^*)| \leq \bar{a} + \bar{b}|\xi^*|^{q-1}. \tag{2.4}$$

(A2) The map H is $C^1(0,\infty)$, strictly convex, and such that

$$\lim_{t\to 0^+} H(t) = \lim_{t\to\infty} \frac{H(t)}{t} = +\infty.$$

(A3) The function F is measurable and belongs to $L^1(\Omega)$.
Let $\mathcal{S}$ be a subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. We say that F satisfies the condition $\mathbf{(ND)}_{\mathcal{S}}$ if

$$\mathrm{div}(\varphi) + F \text{ is non-degenerate}$$

for all $\varphi \in \mathcal{S}$.

Remark 2.1

(i) *As f satisfies* (2.2), *we have*

$$-b + c^p \frac{|\xi^*|^q}{q} \leq f^*(\xi^*) \leq \chi_{\bar{B}(0,a)}(\xi^*) + b \tag{2.5}$$

for all $\xi^* \in \mathbb{R}^{d\times d}$.

(ii) *If f satisfies case* (b) *in* (iii) *of Assumption* (**A1**), *then f^* is continuously differentiable. In that case*, $\mathrm{grad}[f^*] = \nabla f^*$.

(iii) *If f satisfies case* (a) *of Assumption* (**A1**)(iii) *then $0 \in \partial f^*(x^*)$ for every element $x^* \in$* $\mathrm{Dom}(f^*)$. *Consequently, the map f^* is constant on* $\mathrm{Dom}(f^*)$ *and the following equalities are satisfied for all x^* and y^* in* $\mathrm{Dom}(f^*)$:

$$f^*(x^*) - f^*(y^*) = \mathrm{grad}[f^*](x^*) = \mathrm{grad}[f^*](y^*) = 0. \tag{2.6}$$

(iv) *Assumption* (**A0**) *is satisfied by $\Omega = B(0,1) \subset \mathbb{R}^d$ with the strictly convex function being the map $\mathbb{R}^d \ni x \mapsto |x|^2 - 1$.*

(v) *The map $f : \mathbb{R}^{d\times d} \to \mathbb{R}$ defined by $f(\xi) = |\xi|$ satisfies case* (a) *in* (iii) *of Assumption* (**A1**). *The map $f : \mathbb{R}^{d\times d} \to \mathbb{R}$ defined by $f(\xi) = |\xi|^p$ satisfies case* (b) *in* (iii) *of Assumption* (**A1**).

The following lemma summarizes some elementary properties of H. We refer the reader to Remark 2.1 in [2].

Lemma 2.2 *Assume* (**A2**) *holds. Then,*

(i) *The map $H' : (0,\infty) \to \mathbb{R}$ is a strictly increasing bijection.*

(ii) *The Legendre transform H^* of H is a strictly increasing bijection from $\mathbb{R}$ to $\mathbb{R}$.*

(iii) *Let $g : \mathbb{R} \to \bar{\mathbb{R}}$ be defined by $g(s) = \alpha s - \beta H^*(s)$, with $\alpha, \beta > 0$. Then*

$$\lim_{s\to-\infty} g(s) = \lim_{s\to\infty} g(s) = -\infty.$$

Define H_0 by

$$H_0(t) = \begin{cases} 0 & t = 1 \\ \infty & t \neq 1 \end{cases} \tag{2.7}$$

and, for $n \geq 1$,

$$H_n(t) = H(t) - H(1) + n(t-1)^2. \tag{2.8}$$

The following lemma is straightforward.

Lemma 2.3 *Assume* (**A2**) *holds. Then,*

(i) *There exists* $\bar{H} \in \mathbb{R}$ *such that*

$$\bar{H} = \min_{t \in [0,\infty)} H(t).$$

(ii) *The collection* $\{H_n\}_{n=1}^{\infty}$ *is a non-decreasing sequence of functions that converges pointwise to* H_0. *In addition, for all* $n \in \mathbb{N}^*$, *the map* H_n *is a* $C^1(0,\infty)$ *strictly convex function that satisfies*

$$\lim_{t \to 0^+} H_n(t) = \lim_{t \to \infty} \frac{H_n(t)}{t} = +\infty.$$

(iii) *Let* $t > 0$. *If* $\{H_n(t)\}_{n=1}^{\infty}$ *is uniformly bounded above by a constant* c_0 *then*

$$n(t-1)^2 \leq c_0 + H(1) - \bar{H}$$

and $t = 1$.

2.3 Hypothesis on the Underlying Sets of Pseudo-Gradients

We recall that in [2], the construction of $\nabla_{\mathcal{S}^\tau} u$ has relied on hypothesis on the underlying sets $\mathcal{S}^\tau$ that we summarize in *Hypothesis (H1)* below.

Hypothesis (H1).
A collection $\{\mathfrak{A}_n\}_{n=1}^{\infty}$ of subsets of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ satisfies *Hypothesis (H1)* if

(i) $\mathfrak{A}_n$ of a finite dimensional subspace of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ for each $n \in \mathbb{N}^*$.
(ii) The map $\nabla\varphi$ has a countable range whenever $\varphi \in \mathfrak{A}_n$, for any $n \in \mathbb{N}^*$.
(iii) The set $\bigcup_{n\in\mathbb{N}^*} \mathfrak{A}_n$ is dense in $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$.
(iv) For $i \leq j$, we have the inclusion $\mathfrak{A}_i \subset \mathfrak{A}_j$.

An explicit construction of sets satisfying *Hypothesis (H1)* is provided in [2]. Here, we build on the conditions of *Hypothesis (H1)* and we relax conditions on the underlying sets:

Hypothesis (H2).
A collection $\{\mathfrak{Q}_n\}_{n=1}^{\infty}$ of subsets of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ satisfies *Hypothesis (H2)* if

(i) $\mathrm{Span}(\mathfrak{Q}_n)$ is of finite dimension and $\mathfrak{Q}_n$ is a non-empty closed and convex subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$.
(ii) The map $\mathrm{div}\,\varphi$ is non-degenerate whenever $\varphi \in \mathfrak{Q}_n$, for any $n \in \mathbb{N}^*$.
(iii) The set $\bigcup_{n\in\mathbb{N}^*} \mathfrak{Q}_n$ is dense in $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$.
(iv) For $i \leq j$, the inclusion $\mathfrak{Q}_i \subset \mathfrak{Q}_j$ holds.

The next lemma asserts that a collection of sets can be constructed to satisfy *Hypothesis (H2)*.

Lemma 2.4 *Assume* (**A0**) *holds. Then, there exists a collection of sets* $\{\mathfrak{Q}_n\}_{n=1}^{\infty}$ *satisfying the requirements of Hypothesis (H2).*

Remark 2.5 *The condition* (**A0**) *in Lemma* 2.4 *is only needed for requirement* (ii) *of Hypothesis (H2).*

Proof Suppose that ψ is a strictly convex function that is $C^1(\bar{\Omega})$ and vanishes on the boundary of Ω as given by Assumption **(A0)**. Let $\varphi_0 : \Omega \to \mathbb{R}^{d\times d}$ be defined by

$$\varphi_0 = \begin{pmatrix} \psi & 0 & \cdots & 0 \\ 0 & \psi & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi \end{pmatrix}.$$

As ψ is $C^1(\bar{\Omega})$, we have $\varphi_0 \in W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and it follows that $\operatorname{div}\varphi_0 = \nabla\psi$. Thus, for almost every x in Ω, we have

$$\det\bigl(\nabla(\operatorname{div}\varphi_0)(x)\bigr) = \det\bigl(\nabla^2\psi(x)\bigr) > 0.$$

Thanks to Lemma 5.5.3 in [1], the map $\operatorname{div}\varphi_0$ is non-degenerate. Let $\{\mathfrak{A}_n\}_{n=1}^{\infty}$ be a collection of sets satisfying Hypothesis (H1). One readily checks that the family of sets defined by

$$\mathfrak{Q}_n = \left\{\varphi + \epsilon\varphi_0 : \varphi \in \mathfrak{A}_n; \epsilon \geq \frac{1}{n}\right\}$$

for $n \in \mathbb{N}^*$, satisfies hypothesis (H2). □

2.4 Special Displacements

To $\mathcal{S} \subset W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ we associate $\mathscr{U}_{\mathcal{S}}$, the set of all $u : \Omega \to \bar{\Lambda}$ measurable such that there exists $\bar{c} = \bar{c}(u, \Omega, \Lambda) > 0$ satisfying:

$$\left|\int_\Omega u \cdot \operatorname{div}\varphi \, dx\right| \leq \bar{c}\|\varphi\|_{L^q(\Omega,\mathbb{R}^{d\times d})} \quad \forall \varphi \in \mathcal{S}. \tag{2.9}$$

Remark that if $u \in \mathscr{U}_{\mathcal{S}}$, then u belongs to $L^\infty(\Omega, \mathbb{R}^d)$ since u has values in $\bar{\Lambda}$ which is bounded. If $\operatorname{span}(\mathcal{S})$ is of finite dimension then $\mathscr{U}_{\mathcal{S}}$ is the set of all measurable maps $u : \Omega \to \bar{\Lambda}$. In fact, the linear map $\operatorname{span}(\mathcal{S}) \ni \varphi \mapsto \int_\Omega u \operatorname{div}\varphi$ is continuous with respect to the L^q-norm as in finite dimension, all norms are equivalent. Therefore, we may find c for which inequality (2.9) holds for all $\varphi \in \operatorname{span}(\mathcal{S})$ and in particular for all $\varphi \in \mathcal{S}$.

At any rate, $\mathscr{U}_{\mathcal{S}}$ contains $W^{1,p}(\Omega, \mathbb{R}^d)$. Indeed, notice that for a fixed $u \in W^{1,p}(\Omega, \mathbb{R}^d)$, we have, for all $\varphi \in \mathcal{S}$:

$$\left|\int_\Omega u \cdot \operatorname{div}\varphi \, dx\right| = \left|-\int_\Omega \langle\nabla u, \varphi\rangle \, dx\right| \leq \|\nabla u\|_{L^p(\Omega,\mathbb{R}^{d\times d})}\|\varphi\|_{L^q(\Omega,\mathbb{R}^{d\times d})}.$$

We introduce the following set

$$\mathscr{U}_{\mathcal{S}}^1 = \left\{ u \in \mathscr{U}_{\mathcal{S}} : \int_\Omega l\big(u(x)\big)\,dx = \int_\Lambda l(y)\,dy\ \forall l \in C_c\big(\mathbb{R}^d\big) \right\}$$

and

$$\mathscr{U}_{\mathcal{S}}^* = \Big\{ (u,\beta) : u \in \mathscr{U}_{\mathcal{S}};\ \beta : \Omega \to [0,\infty);$$
$$\int_\Omega l\big(u(x)\big)\beta(x)\,dx = \int_\Lambda l(y)\,dy\ \forall l \in C_c\big(\mathbb{R}^d\big) \Big\}.$$

Notice that $\mathscr{U}_{\mathcal{S}}^1 = \{u \in \mathscr{U}_{\mathcal{S}} : (u,1) \in \mathscr{U}_{\mathcal{S}}^*\}$. This corresponds to measure preserving displacements.

2.5 Extended Pseudo-Projected Gradient

Let $\mathcal{S} \subset W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and $u \in \mathscr{U}_{\mathcal{S}}$. Define

$$\mathcal{G}_{\mathcal{S}}(u) := \left\{ G \in L^p\big(\Omega, \mathbb{R}^{d\times d}\big) : \int_\Omega u \operatorname{div}\varphi\,dx = -\int_\Omega \langle G, \varphi\rangle\,dx\ \forall \varphi \in \mathcal{S} \right\}.$$

Consider the operator

$$V_{\mathcal{S}}^f(u) := \sup_{\varphi\in\mathcal{S}} \int_\Omega \big(-u \operatorname{div}\varphi - f^*(\varphi)\big)\,dx = \sup_{\varphi\in\mathscr{S}_f} \int_\Omega \big(-u \operatorname{div}\varphi - f^*(\varphi)\big)\,dx. \tag{2.10}$$

We denote by $\Phi_S(u)$ the set of maximizers of problem (2.10).

Lemma 2.6 *Let $\mathcal{S} \subset W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and $u \in \mathscr{U}_{\mathcal{S}}$.*

1. *We have*

$$\mathcal{G}_{\mathcal{S}}(u) = \left\{ G \in L^p\big(\Omega, \mathbb{R}^{d\times d}\big) : \int_\Omega u \operatorname{div}\varphi\,dx = -\int_\Omega \langle G, \varphi\rangle\,dx;\ \ \forall \varphi \in \operatorname{span}(\mathcal{S}) \right\}.$$

2. *If* $\operatorname{span}(\mathcal{S})$ *is finite dimensional, then* $\mathcal{G}_{\mathcal{S}}(u)$ *is nonempty.*

Proof Set

$$\bar{\mathcal{G}}_{\mathcal{S}}(u) = \left\{ G \in L^p\big(\Omega, \mathbb{R}^{d\times d}\big) : \int_\Omega u \operatorname{div}\varphi\,dx = -\int_\Omega \langle G, \varphi\rangle\,dx\ \forall \varphi \in \operatorname{span}(\mathcal{S}) \right\}.$$

As $\mathcal{S} \subset \operatorname{span}(\mathcal{S})$, we have $\bar{\mathcal{G}}_{\mathcal{S}}(u) \subset \mathcal{G}_{\mathcal{S}}(u)$. Next, let $G \in \mathcal{G}_{\mathcal{S}}(u)$. Assume that $\varphi \in \operatorname{span}(\mathcal{S})$. We may find $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{S}$ such that $\varphi = \sum_{i=1}^n \lambda_i \varphi_i$. Then

$$\int_\Omega u \operatorname{div}\varphi\,dx = \int_\Omega u \operatorname{div}\sum_{i=1}^n \lambda_i\varphi_i\,dx = \sum_{i=1}^n \lambda_i \int_\Omega u \operatorname{div}\varphi_i\,dx = \sum_{i=1}^n -\lambda_i \int_\Omega \langle G, \varphi_i\rangle\,dx$$

and

$$-\int_\Omega \langle G, \varphi\rangle\,dx = -\int_\Omega \left\langle G, \sum_{i=1}^n \lambda_i\varphi_i \right\rangle dx = \sum_{i=1}^n -\lambda_i \int_\Omega \langle G, \varphi_i\rangle\,dx.$$

Thus $G \in \bar{\mathcal{G}}_{\mathcal{S}}(u)$. We deduce that $\mathcal{G}_{\mathcal{S}}(u) \subset \bar{\mathcal{G}}_{\mathcal{S}}(u)$. It follows that part (1.) holds. To obtain part (2.), we use part (1.) and the Riesz Representation Theorem. □

The following results are essentially found in Proposition 3.1 in [2].

Proposition 2.7 *Suppose that the set $\mathcal{S}$ is a finite dimensional subspace of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and f is C^1 and strictly convex. Suppose, in addition that there exist constants $c_1, c_2, c_3 > 0$ such that*

$$-c_3 + c_2|\xi|^p \leq f(\xi) \leq c_3 + c_1|\xi|^p$$
$$\left|Df(\xi)\right| \leq c_3 + c_1|\xi|^{p-1}$$
$$\left|Df^*(\xi)\right| \leq c_3 + c_1|\xi|^{q-1}$$

for all $\xi \in \mathbb{R}^{d\times d}$. Then, there exists a unique map denoted $\nabla_{\mathcal{S}} u$ that minimizes

$$\inf_{G\in\mathcal{G}_{\mathcal{S}}(u)} \int_\Omega f(G)\, dx.$$

Moreover, $\nabla_{\mathcal{S}} u$ is the unique map $G \in \mathcal{G}_{\mathcal{S}}(u)$ that satisfies $Df(G) \in \mathcal{S}$.

In the next proposition, we establish similar results as in Proposition 2.7 but under weaker assumptions on $\mathcal{S}$ and f (except in part 4).

Proposition 2.8 *Assume* (**A1**) *holds. Assume $\mathcal{S}$ is a finite dimensional non-empty closed and convex subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and let $u \in \mathcal{U}_{\mathcal{S}}$.*

1. *For all $G \in \mathcal{G}_{\mathcal{S}}(u)$, $\varphi \in \mathcal{S}$, we have*

$$\int_\Omega f(G)\, dx \geq \int_\Omega \big(-u \operatorname{div}\varphi - f^*(\varphi)\big)\, dx.$$

2. *The supremum in problem (2.10) is attained.*
3. *A map $\bar{\varphi}$ belongs to $\Phi_{\mathcal{S}}(u)$ if and only if $\bar{\varphi}$ belongs to $\mathscr{S}_f$ and*

$$\int_\Omega \big(\operatorname{grad}\big[f^*\big](\bar{\varphi}) \cdot (\varphi - \bar{\varphi}) + u \cdot (\operatorname{div}\varphi - \operatorname{div}\bar{\varphi})\big)\, dx \geq 0$$

   for all $\varphi \in \mathscr{S}_f$.
4. *Suppose that the hypotheses of Proposition 2.7 are satisfied. Then we have*

$$\int_\Omega f(\nabla_{\mathcal{S}} u)\, dx = V_{\mathcal{S}}^f(u)$$

   and $\Phi_{\mathcal{S}}(u) = \{Df(\nabla_{\mathcal{S}} u)\}$.

Proof (1.) Let $\varphi \in \mathcal{S}$ and $G \in \mathcal{G}_{\mathcal{S}}(u)$. By using the Legendre transformation,

$$\int_\Omega f(G)\, dx \geq \int_\Omega \big(G \cdot \varphi - f^*(\varphi)\big)\, dx = \int_\Omega \big(-u \cdot \operatorname{div}\varphi - f^*(\varphi)\big)\, dx.$$

(2.) Let $\varphi \in \mathcal{S}$. We use (2.9) and (2.5) to get

$$\int_\Omega \big(u \operatorname{div}\varphi + f^*(\varphi)\big)\,dx \geq -\bar{c}\|\varphi\|_{L^q(\Omega,\mathbb{R}^{d\times d})} + \int_\Omega f^*(\varphi)\,dx$$
$$\geq -\bar{c}\|\varphi\|_{L^q(\Omega,\mathbb{R}^{d\times d})} + q^{-1}c^{-q}\|\varphi\|^q_{L^q(\Omega,\mathbb{R}^{d\times d})}. \tag{2.11}$$

In light of (2.11), $q > 1$ implies that the map

$$\mathscr{S}_f \ni \varphi \mapsto T(\varphi) := \int_\Omega \big(u \operatorname{div}\varphi + f^*(\varphi)\big)\,dx$$

is L^q-coercive. Moreover, the convexity of f^* guarantees that T is lower semi-continuous. The direct methods of the calculus of variations thus yield the existence of a maximizer in problem (2.10).

(3.) Let $\bar{\varphi} \in \Phi_{\mathcal{S}}(u)$ so that $\bar{\varphi} \in \mathscr{S}_f$. Let $\varphi \in \mathscr{S}_f$ and $\epsilon \in (0,1)$. The convexity of f^* ensures that $\bar{\varphi} + \epsilon(\varphi - \bar{\varphi}) \in \mathscr{S}_f$ and the maximality property of $\bar{\varphi}$ implies that

$$\int_\Omega u \cdot \operatorname{div}\bar{\varphi} + f^*(\bar{\varphi})\,dx \leq \int_\Omega u \cdot \big(\operatorname{div}\bar{\varphi} + \epsilon \operatorname{div}(\varphi - \bar{\varphi})\big)\,dx + f^*\big(\bar{\varphi} + \epsilon(\varphi - \bar{\varphi})\big)\,dx. \tag{2.12}$$

We rewrite (2.12), in turn, as

$$\int_\Omega \frac{f^*(\bar{\varphi} + \epsilon(\varphi - \bar{\varphi})) - f^*(\bar{\varphi})}{\epsilon} + u \cdot \operatorname{div}(\varphi - \bar{\varphi})\,dx \geq 0. \tag{2.13}$$

Note that $\operatorname{grad}[f^*](\bar{\varphi} + \epsilon(\varphi - \bar{\varphi}))$ belongs to the set $\partial f^*((\bar{\varphi} + \epsilon(\varphi - \bar{\varphi})))$ whenever $(\bar{\varphi} + \epsilon(\varphi - \bar{\varphi}))$ is in the domain of f^*. It follows that

$$\int_\Omega \big(\operatorname{grad}\big[f^*\big]\big((\bar{\varphi} + \epsilon(\varphi - \bar{\varphi}))\big) \cdot \big(-\epsilon(\varphi - \bar{\varphi})\big)\big)\,dx \leq \int_\Omega \big(f^*(\bar{\varphi}) - f^*\big(\bar{\varphi} + \epsilon(\varphi - \bar{\varphi})\big)\big)\,dx$$

that is,

$$\int_\Omega \big(\operatorname{grad}\big[f^*\big]\big(\bar{\varphi} + \epsilon(\varphi - \bar{\varphi})\big) \cdot (\varphi - \bar{\varphi})\big)\,dx \geq \int_\Omega \frac{f^*(\bar{\varphi} + \epsilon(\varphi - \bar{\varphi})) - f^*(\bar{\varphi})}{\epsilon}\,dx. \tag{2.14}$$

We combine (2.13) and (2.14) to get

$$\int_\Omega \big(\operatorname{grad}\big[f^*\big]\big(\bar{\varphi} + \epsilon(\varphi - \bar{\varphi})\big) \cdot (\varphi - \bar{\varphi}) + u \operatorname{div}(\varphi - \bar{\varphi})\big)\,dx \geq 0. \tag{2.15}$$

First, we assume that (**A1**)(iii)(a) holds. In light of (2.6), we have $\operatorname{grad}[f^*](\bar{\varphi} + \epsilon(\varphi - \bar{\varphi})) = \operatorname{grad}[f^*](\bar{\varphi})$. Equation (2.15) becomes

$$\int_\Omega \big(\operatorname{grad}\big[f^*\big](\bar{\varphi}) \cdot (\varphi - \bar{\varphi}) + u \operatorname{div}(\varphi - \bar{\varphi})\big)\,dx \geq 0.$$

Second, we assume that (**A1**)(iii)(b) holds. In light of Remark 2.1(ii), we use the growth condition on ∇f^* in (2.4), the Lebesgue dominated convergence theorem and let ϵ go to 0 in (2.15) to obtain that:

$$\int_\Omega \big(\operatorname{grad}\big[f^*\big](\bar{\varphi}) \cdot (\varphi - \bar{\varphi}) + u \operatorname{div}(\varphi - \bar{\varphi})\big)\,dx \geq 0.$$

We next show the converse implication. Let $\varphi \in \mathscr{S}_f$ such that

$$0 \leq \int_\Omega \big(u \operatorname{div}(\varphi - \bar{\varphi}) + \operatorname{grad}\big[f^*\big](\bar{\varphi}) \cdot (\varphi - \bar{\varphi})\big)\, dx, \tag{2.16}$$

for all $\varphi \in \mathscr{S}_f$. We notice that, as f^* is convex, the range of the map $\operatorname{grad}[f^*](\bar{\varphi})$ lies in the sub-differential of f^* so that $f^*(\varphi) - f^*(\bar{\varphi}) \geq \operatorname{grad}[f^*](\bar{\varphi})(\varphi - \bar{\varphi})$ for all $\varphi \in \mathscr{S}_f$. Then, the inequality (2.16) implies that

$$0 \leq \int_\Omega \big(u \operatorname{div}(\varphi - \bar{\varphi}) + \big(f^*(\varphi) - f^*(\bar{\varphi})\big)\big)\, dx$$

for all $\varphi \in \mathscr{S}_f$, that is,

$$\int_\Omega \big(u \operatorname{div} \bar{\varphi} + f^*(\bar{\varphi})\big)\, dx \leq \int_\Omega \big(u \operatorname{div} \varphi + f^*(\varphi)\big)\, dx$$

for all $\varphi \in \mathscr{S}_f$. We conclude that $\bar{\varphi} \in \Phi_{\mathcal{S}}(u)$.

(4.) Thanks to Proposition 2.7, $Df(\nabla_{\mathcal{S}} u) \in \mathcal{S}$. Next, we set $\varphi_0 := Df(\nabla_{\mathcal{S}} u)$. By definition of f^*,

$$f(\nabla_{\mathcal{S}} u) + f^*(\varphi) \geq \varphi \cdot \nabla_{\mathcal{S}} u$$

for all $\varphi \in \mathcal{S}$. As f is convex and $\varphi_0 = Df(\nabla_{\mathcal{S}} u)$, we have

$$f(\nabla_{\mathcal{S}} u) + f^*(\varphi_0) = \varphi_0 \cdot \nabla_{\mathcal{S}} u.$$

Thus,

$$\int_\Omega f(\nabla_{\mathcal{S}} u)\, dx \geq \int_\Omega \varphi \cdot \nabla_{\mathcal{S}} u\, dx - \int_\Omega f^*(\varphi)\, dx = \int_\Omega -u \operatorname{div} \varphi\, dx - \int_\Omega f^*(\varphi)\, dx$$

and

$$\int_\Omega f(\nabla_{\mathcal{S}} u)\, dx = \int_\Omega \varphi_0 \cdot \nabla_{\mathcal{S}} u\, dx - \int_\Omega f^*(\varphi_0)\, dx = \int_\Omega -u \operatorname{div} \varphi_0\, dx - \int_\Omega f^*(\varphi_0)\, dx.$$

We deduce that $\varphi_0 \in \Phi_{\mathcal{S}}(u)$. Since f^* is strictly convex, we conclude that $\Phi_{\mathcal{S}}(u) = \{Df(\nabla_{\mathcal{S}} u)\}$ and moreover, $\int_\Omega f(\nabla_{\mathcal{S}} u) = V_{\mathcal{S}}^f(u)$, see (2.10). □

In the next proposition, we establish a convergence result in the spirit of (1.7). We also connect the operator $V_{\mathcal{S}}^f$ with the usual notions of gradient and total variation.

Proposition 2.9 *Assume* (**A1**) *holds. Assume that $\mathcal{S}_n$ is a finite dimensional non-empty closed and convex subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ for each $n \geq 1$. The following holds.*

1. *If $\{\mathcal{S}_n\}_{n=1}^\infty$ is a monotonically increasing family of subsets of some set $\mathcal{S}_0$ and $\bigcup_{n\in\mathbb{N}^*} \mathcal{S}_n$ is dense in $\mathcal{S}_0$ with respect to the $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ norm then*

$$\lim_{n\to\infty} V_{\mathcal{S}_n}^f[u] = V_{\mathcal{S}_0}^f[u]$$

   for any $u \in \mathscr{U}_{\mathcal{S}_0}$.
2. *If $\mathcal{S} = W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ then $V_{\mathcal{S}}^f[u] = \int_\Omega f(\nabla u)\, dx$.*

3. *Assume* $u \in BV(\Omega, \mathbb{R}^{d\times d})$ *and* $f(\xi) = |\xi|$ *for all* $\xi \in \mathbb{R}^{d\times d}$. *If* $\mathcal{S} = W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ *then* $V_{\mathcal{S}}^f[u]$ *is the total variation of* u.

Remark 2.10 *A consequence of Proposition 2.9 is the following: If the sequence of sets* $\{\mathcal{S}_n\}_{n\in\mathbb{N}^*}$ *is monotonically increasing to* $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ *and* $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ *we have*

$$\lim_{n\to\infty} V_{\mathcal{S}_n}^f[u] = \int_\Omega f(\nabla u)\, dx.$$

Proof (1.) Recall that

$$V_{\mathcal{S}_n}^f[u] = \sup_{\varphi\in\mathcal{S}_n}\left\{\int_\Omega \left(-u\cdot \operatorname{div}\varphi - f^*(\varphi)\right) dx\right\}.$$

As $\{\mathcal{S}_n\}_{n=1}^\infty$ is a monotonically increasing, $\lim_{n\to\infty} V_{\mathcal{S}_n}^f[u]$ exists. Moreover, since $\mathcal{S}_n \subset \mathcal{S}_0$ for all $n \geq 1$,

$$\lim_{n\to\infty} V_{\mathcal{S}_n}^f[u] \leq V_{\mathcal{S}_0}^f[u]. \tag{2.17}$$

Let $\epsilon > 0$ and choose $\varphi^\epsilon \in \mathcal{S}_0$ such that

$$V_{\mathcal{S}_0}^f[u] \leq \epsilon + \int_\Omega \left(-u\cdot \operatorname{div}\varphi^\epsilon - f^*\left(\varphi^\epsilon\right)\right) dx.$$

Let $\{\varphi_n^\epsilon\}_{n\in\mathbb{N}^*}$ be a sequence converging to φ^ϵ in $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and such that $\varphi_n^\epsilon \in \mathcal{S}_n$ for all $n \in \mathbb{N}^*$. Then, we use the growth conditions on f^* in (2.4) and (2.5), the continuity of f^* on its domain and the Lebesgue dominated convergence theorem to obtain that

$$\int_\Omega -f^*\left(\varphi^\epsilon\right) dx = \lim_{n\to\infty}\int_\Omega -f^*\left(\varphi_n^\epsilon\right) dx.$$

It follows that

$$\begin{aligned} V_{\mathcal{S}_0}^f[u] &\leq \epsilon + \int_\Omega \left(-u\cdot \operatorname{div}\varphi^\epsilon - f^*\left(\varphi^\epsilon\right)\right) dx \\ &= \epsilon + \lim_{n\to\infty}\int_\Omega \left(-u\cdot \operatorname{div}\varphi_n^\epsilon - f^*\left(\varphi_n^\epsilon\right)\right) dx \\ &\leq \epsilon + \limsup_{n\to\infty} V_{\mathcal{S}_n}^f[u] \\ &= \epsilon + \lim_{n\to\infty} V_{\mathcal{S}_n}^f[u]. \end{aligned}$$

As ϵ is arbitrary, we have

$$\lim_{n\to\infty} V_{\mathcal{S}_n}^f[u] \geq V_{\mathcal{S}_0}^f[u]. \tag{2.18}$$

From (2.17) and (2.18), we conclude that $\lim_{n\to\infty} V_{\mathcal{S}_n}^f[u] = V_{\mathcal{S}_0}^f[u]$.

(2.) One has

$$\begin{aligned} V_{\mathcal{S}}^{f}[u] &= \sup_{\varphi \in W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})} \left\{ \int_\Omega \big(-u\cdot \operatorname{div}\varphi - f^*(\varphi)\big)\,dx \right\} \\ &= \sup_{\varphi \in W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})} \left\{ \int_\Omega \big(\nabla u\cdot \varphi - f^*(\varphi)\big)\,dx \right\} \\ &\le \int_\Omega f(\nabla u)\,dx. \end{aligned}$$

The inequality above is obtained by using the definition of the Legendre transform f^* of f. Let $\bar\varphi \in \partial f(\nabla u)$. Then $f^*(\bar\varphi) + f(\nabla u) = \nabla u \cdot \bar\varphi$. Thanks to the growth conditions (2.2) and (2.3) on f, it holds that $\bar\varphi \in L^q(\Omega,\mathbb{R}^{d\times d})$. Since $W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})$ is dense in $L^q(\Omega,\mathbb{R}^{d\times d})$ for the $L^q(\Omega,\mathbb{R}^{d\times d})$ norm, we get

$$\begin{aligned} \int_\Omega f(\nabla u)\,dx &= \int_\Omega \big(\nabla u\cdot\bar\varphi - f^*(\bar\varphi)\big)\,dx \\ &\le \sup_{\varphi \in W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})} \left\{ \int_\Omega \big(\nabla u\cdot \varphi - f^*(\varphi)\big)\,dx \right\} \\ &= V_{\mathcal{S}}^{f}[u]. \end{aligned}$$

We conclude that $V_{\mathcal{S}}^{f}[u] = \int_\Omega f(\nabla u)\,dx$.

(3.) The total variation of $u \in BV(\Omega,\mathbb{R}^{d\times d})$ is

$$\|Du\|(\Omega) = \sup\left\{ \int_\Omega u\cdot \operatorname{div}\varphi\,dx : \varphi \in C_c^1\big(\Omega,\mathbb{R}^{d\times d}\big);\ |\varphi| \le 1 \right\} \tag{2.19}$$

while, using the Legendre transform of $f(\xi) = |\xi|$, we obtain for every $q > 1$

$$V_{\mathcal{S}}^{f}(u) = \sup\left\{ \int_\Omega u\cdot \operatorname{div}\varphi\,dx : \varphi \in W_0^{1,q}\big(\Omega,\mathbb{R}^{d\times d}\big);\ |\varphi| \le 1 \right\}. \tag{2.20}$$

It follows directly from (2.19) and (2.20) that $\|Du\|(\Omega) \le V_{\mathcal{S}}^{f}[u]$. The converse inequality $\|Du\|(\Omega) \ge V_{\mathcal{S}}^{f}[u]$ follows from the density of $C_c^1(\Omega,\mathbb{R}^{d\times d})$ in $W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})$ and an argument similar to the one made in the proof of (2) in the proposition. □

3 Minimization with General Displacements

We consider the following:

$$\inf_{(u,\beta)\in\mathscr{U}_{\mathcal{S}}^*} \left\{ I(u,\beta) = V_{\mathcal{S}}^{f}(u) + \int_\Omega \big(H(\beta) - F\cdot u\big)\,dx \right\}. \tag{3.1}$$

This problem will be studied via a dual problem that we will formulate next. We assume in this section that Assumption **(A2)** holds.

3.1 An Auxiliary Problem

For $l, k : \mathbb{R}^d \to (-\infty, \infty]$, define for $u, v \in \mathbb{R}^d$

$$l^{\#}(v) := \sup_{u \in \bar{\Lambda}, t > 0} \{u \cdot v - l(u)t - H(t)\} \tag{3.2}$$

and

$$k_{\#}(u) := \sup_{v \in \mathbb{R}^d, t > 0} \{(1/t)(u \cdot v - k(v) - H(t))\}. \tag{3.3}$$

Under Assumption **(A2)**, it is known that $((l^{\#})_{\#})^{\#} = l^{\#}$ and $((k_{\#})^{\#})_{\#} = k_{\#}$ (see for instance Lemma A1 of [11]). Call $\mathcal{C}$ the set of all functions (k, l) with $k,\ l : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ Borel measurable, finite at least at one point, and satisfying $l \equiv \infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$ and such that

$$k(v) + t \cdot l(u) + H(t) \geq u \cdot v \quad \forall u, v \in \mathbb{R}^d, t > 0. \tag{3.4}$$

Call $\mathcal{C}'$ the set of all functions $(k, l) \in \mathcal{C}$ such that $l = k_{\#}$ and $k = l^{\#}$. The set $\mathcal{C}'$ is nonempty. Indeed, $(\chi^{\#}_{\bar{\Lambda}}, (\chi^{\#}_{\bar{\Lambda}})_{\#}) \in \mathcal{C}'$ as $((\chi^{\#}_{\bar{\Lambda}})_{\#})^{\#} = \chi^{\#}_{\bar{\Lambda}}$.

Let $\mathscr{A}$ be the set of (k, l, φ) such that $(k, l) \in \mathcal{C}$ and $\varphi \in \mathcal{S}$. Consider the following functional defined on $\mathscr{A}$:

$$J(k, l, \varphi) := \int_{\Omega} k(F + \operatorname{div} \varphi)\, dx + \int_{\Lambda} l\, dy + \int_{\Omega} f^*(\varphi)\, dx.$$

The following problem will play an important role in this section:

$$\inf\{J(k, l, \varphi) : (k, l, \varphi) \in \mathscr{A}\}. \tag{3.5}$$

The value of the expression (3.5) is the opposite of the value of the following expression:

$$\sup\{-J(k, l, \varphi) : (k, l, \varphi) \in \mathscr{A}\}. \tag{3.6}$$

Let $\mathscr{A}'$ denote the subset of $\mathscr{A}$ consisting of all $(k, l, \varphi) \in \mathscr{A}$ that satisfy $(k, l) \in \mathcal{C}'$. It holds that

$$\inf\{J(k, l, \varphi) : (k, l, \varphi) \in \mathscr{A}\} = \inf\{J(k, l, \varphi) : (k, l, \varphi) \in \mathscr{A}'\}. \tag{3.7}$$

Indeed, the key observation to this end is that for $(k, l, \varphi) \in \mathscr{A}$, one has $l \geq k_{\#}$ and $k \geq (k_{\#})^{\#}$ so that

$$J(k, l, \varphi) \geq J((k_{\#})^{\#}, k_{\#}, \varphi) \quad \text{and} \quad ((k_{\#})^{\#}, k_{\#}, \varphi) \in \mathscr{A}'.$$

For $R > 0$, we set

$$\mathscr{A}_R = \{(k, l, \varphi) \in \mathscr{A}' : J(k, l, \varphi) \leq R\}.$$

Lemma 3.1 *Assume that* **(A1)**, **(A2)** *and* **(A3)** *hold. Let* $(k, l, \varphi) \in \mathscr{A}_R$. *Set* $s_l := -\inf_{u \in \bar{\Lambda}} l(u)$. *Then,*

$$\int_{\Omega} k(F + \operatorname{div} \varphi)\, dx \geq \mathcal{L}^d(\Omega) H^*(s_l) - r^* \|F\|_{L^1(\Omega)}.$$

Moreover, there exists $M := M(R, F, f, \Omega, \Lambda) > 0$ *such that*

$$|s_l| \leq M. \tag{3.8}$$

Proof As Λ is bounded and l is convex, we choose $u_l \in \overline{\Lambda}$ such that $-l(u_l) = s_l$. Since $k := l^\#$, in view of (3.2), we have

$$-tl(u_l) - H(t) + u_l \cdot v = ts_l - H(t) + u_l \cdot v \leq H^*(s_l) + u_l \cdot v \leq k(v). \tag{3.9}$$

Using the last inequality in (3.9), one gets

$$\int_\Omega k(F + \operatorname{div}\varphi)\,dx \geq \int_\Omega \big(H^*(s_l) + u_l \cdot (F + \operatorname{div}\varphi)\big)\,dx \tag{3.10}$$

$$= H^*(s_l)\mathcal{L}^d(\Omega) + \int_\Omega u_l \cdot F\,dx. \tag{3.11}$$

We have used the fact that u_l is a constant vector and $\varphi \in W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ to obtain the equality in (3.11). Hence,

$$\int_\Omega k(F + \operatorname{div}\varphi)\,dx \geq \mathcal{L}^d(\Omega)H^*(s_l) - r^*\|F\|_{L^1(\Omega)}.$$

Thus,

$$R \geq J(k, l, \varphi) \geq -s_l\mathcal{L}^d(\Lambda) + \mathcal{L}^d(\Omega)H^*(s_l) - r^*\|F\|_{L^1(\Omega)} + \inf f^*.$$

Thanks to Lemma 2.2(iii), s_l is bounded uniformly in l. □

Lemma 3.2 *Assume that* (**A1**), (**A2**) *and* (**A3**) *hold.*

1. *There exists $M > 0$ such that for all $(k, l, \varphi) \in \mathscr{A}_R$ one has*

$$\int_\Lambda \big|l(y)\big|\,dy \leq M. \tag{3.12}$$

2. *There exist $a_0, b_0, c_0 > 0$ such that for all $(k, l, \varphi) \in \mathscr{A}_R$, the map k is r^*-Lipschitz, and one has for all $v \in \mathbb{R}^d$*

$$-c_0 + a_0|v| \leq k(v) \leq b_0 + r^*|v|. \tag{3.13}$$

Proof (1.) Recall that for $(k, l, \varphi) \in \mathcal{A}_R$, one has

$$J(k, l, \varphi) = \int_\Omega k(F + \operatorname{div}\varphi)\,dx + \int_\Lambda l\,dy + \int_\Omega f^*(\varphi)\,dx.$$

By Lemma 3.1, for all $(k, l, \varphi) \in \mathcal{A}_R$, if we define $s_l := -\inf_{u\in\bar{\Lambda}} l(u)$, we get

$$R \geq J(k, l, \varphi) \geq \mathcal{L}^d(\Omega)H^*(s_l) - r^*\|F\|_{L^1(\Omega)} + \int_\Lambda l(y)\,dy + \mathcal{L}^d(\Omega)\inf f^*.$$

Rearranging the terms, we get:

$$\int_\Lambda l(y)\,dy \leq R - \mathcal{L}^d(\Omega)H^*(s_l) + r^*\|F\|_{L^1(\Omega)} - \inf f^*\mathcal{L}^d(\Omega).$$

By definition of s_l we also have $-s_l\mathcal{L}^d(\Omega) \leq \int_\Lambda l(y)\,dy$ and thus

$$-s_l\mathcal{L}^d(\Omega) \leq \int_\Lambda l(y)\,dy \leq R - \mathcal{L}^d(\Omega)H^*(s_l) + r^*\|F\|_{L^1(\Omega)} - \inf f^*\mathcal{L}^d(\Omega). \tag{3.14}$$

We consider the negative part of l defined by $l^- := \max\{-l, 0\}$ and note that

$$\int_\Lambda \big|l(y)\big|\, dy = \int_\Lambda l(y)\, dy + 2\int_\Lambda l^-(y)\, dy. \tag{3.15}$$

Observe that, by the definition of s_l, we have $l^- \le |s_l|$. This, combined with (3.14), (3.15) and (3.8) yields (3.12).

(2.) Let $(k, l, \varphi) \in \mathscr{A}_R$. Since $k = l^\#$, by Eq. (3.2), k is a r^*-Lipschitz as Λ has diameter less or equal to r^*. Next, we have

$$\begin{aligned} k(0) &= \sup_{u\in\bar{\Lambda}, t>0} \big\{-tl(u) - H(t)\big\} \\ &= \sup_{t>0}\big\{-ts_l - H(t)\big\}. \end{aligned}$$

As s_l is uniformly bounded, the growth condition on H ensures that $|k(0)|$ is uniformly bounded say by some $b_0 > 0$. We get then the inequality $k(v) \le b_0 + r^*|v|$ for all $v \in \mathbb{R}^d$.

Because of the hypothesis on the domain Λ, we take $a_0 > 0$ such that $B(0, a_0) \subset \Lambda$. As $(k, l, \varphi) \in \mathcal{A}_R$, we use relation (3.4) to obtain for $v \neq 0$

$$k(v) \ge v \cdot \left(a_0 \frac{v}{|v|}\right) - l\left(a_0 \frac{v}{|v|}\right) - H(1). \tag{3.16}$$

Thanks to inequality (3.12), $\int_\Lambda |l|\, dy$ is uniformly bounded in l. We use in addition the fact that l is bounded to deduce that $\sup_{y\in\bar{B}(0,a_0)} |l|(y)$ is bounded by a constant independent of l (see for instance Theorem 1, p. 236 in [6]). Thus Eq. (3.16) implies that there exists $c_0 > 0$ such that $k(v) \ge a_0|v| - c_0$ for all $v \in \mathbb{R}^d$. □

Proposition 3.3 *Assume that* (**A1**), (**A2**), *and* (**A3**) *hold. Assume* $\mathcal{S}$ *is a finite dimensional non-empty closed and convex subset of* $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. *Then, the functional* J *admits a minimizer* (k_0, l_0, φ_0) *in* $\mathscr{A}'$.

Proof Let $(\bar{k}, \bar{l}, \bar{\varphi}) \in \mathscr{A}$. Set $R = J(\bar{k}, \bar{l}, \bar{\varphi})$. Take a minimizing sequence $\{(k_n, l_n, \varphi_n)\}_{n\in\mathbb{N}^*}$ of problem (3.5) that is in $\mathscr{A}_R$. By Lemma 3.1 and the growth condition on f^* we may assume without loss of generality that $\{\varphi_n\}_{n=1}^\infty$ converges to some $\varphi_0 \in \mathcal{S}$ weakly in $L^q(\Omega, \mathbb{R}^{d\times d})$. Since Span$(\mathcal{S})$ is finite dimensional, $\{\varphi_n\}_{n=1}^\infty$ converges to some $\varphi_0 \in \mathcal{S}$ strongly in the $L^q(\Omega, \mathbb{R}^{d\times d})$ norm. We deduce

$$\int_\Omega f^*(\varphi_0)\, dx \le \liminf_{n\to\infty} \int_\Omega f^*(\varphi_n)\, dx. \tag{3.17}$$

From Lemma 3.2, as l_n is convex, we use Ascoli-Arzelà Theorem together with Theorem 1, p. 236 in [6] to deduce that up to a subsequence, we may assume that (k_n, l_n) converges locally uniformly $\mathbb{R}^d \times \Lambda$ to $(k_0, l_0) \in \mathcal{C}'$. The Lebesgue dominated convergence together with inequality (3.13) yield

$$\int_\Omega k(F + \operatorname{div}\varphi_0)\, dx \le \liminf_{n\to\infty} \int_\Omega k_n(F + \operatorname{div}\varphi_n)\, dx. \tag{3.18}$$

Since $\{l_n\}_{n=1}^\infty$ is uniformly bounded below (thanks to Lemma 3.1), by Fatou's Lemma we get

$$\int_\Lambda l_0\, dy \le \liminf_{n\to\infty} \int_\Lambda l_n\, dy. \tag{3.19}$$

By inequalities (3.17), (3.18) and (3.19), we get

$$J(k_0, l_0, \varphi_0) \le \liminf_{n\to\infty} J(k_n, l_n, \varphi_n)$$

and (k_0, l_0, φ_0) is a minimizer of J over $\mathscr{A}'$. □

3.2 A Uniqueness Result

Here, we prove the main result of this section. We will need the following lemma which is in the spirit of Lemma 4.3 and Lemma 4.4 in [2]. A proof of Lemma 3.4 is given in Sect. A.1.

Lemma 3.4 *Assume that* **(A2)** *holds. Consider a lower semicontinuous function* $l_0 : \mathbb{R}^d \to \bar{\mathbb{R}}$ *such that* $\inf_{\bar{\Lambda}} l_0 > -\infty$; l_0 *is finite on* Λ *and* $l_0 \equiv +\infty$ *on* $\mathbb{R}^d \setminus \bar{\Lambda}$. *Set* $k_0 = (l_0)^{\#}$. *Let* $v \in \mathbb{R}^d$ *be such that* k_0 *is differentiable at* v.

1. *There exist unique* $u_0 \in \bar{\Lambda}$ *and* $t_0 > 0$ *such that* $k_0(v) = -t_0 l_0(u_0) - H(t_0) - u_0 \cdot v$. *In addition,* u_0 *and* t_0 *are characterized by* $u_0 = \nabla k_0(v)$ *and* $H'(t_0) + l(u_0) = 0$.
2. *Let* $\hat{l} \in C_b(\mathbb{R}^d)$ *and let* $1 \ge \epsilon > 0$. *Define* $l_\epsilon = l_0 + \epsilon \hat{l}$ *and* $k_\epsilon = (l_\epsilon)^{\#}$.
   (a) *There exists a constant* M *independent of* v *and* ϵ *such that*

$$\left| \frac{k_\epsilon(v) - k_0(v)}{\epsilon} \right| \le M.$$

   (b) *We have*

$$\lim_{\epsilon\to 0} \frac{k_\epsilon(v) - k_0(v)}{\epsilon} = -t_0 \hat{l}(u_0).$$

Next, we give the main result of this section.

Theorem 3.5 *Assume that* (**A1**), (**A2**), *and* (**A3**) *hold. Assume* $\mathcal{S}$ *is a finite dimensional non-empty closed and convex subset of* $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. *Assume* F *satisfies the condition* $(\boldsymbol{ND})_{\mathcal{S}}$. *Then, problems* (3.1) *and* (3.6) *are dual. Problem* (3.6) *admits a maximizer* (k_0, l_0, φ_0) *with* $k_0 = l_0^{\#}$ *and* $l_0 = (k_0)_{\#}$. *Problem* (3.1) *admits a unique minimizer* (u_0, β_0). *Moreover* u_0 *satisfies*

$$\begin{cases} u_0 = \nabla k_0(F + \operatorname{div} \varphi_0) \\ \varphi_0 \in \Phi_S(u_0). \end{cases}$$

Proof **Step 1**. For $(u, \beta) \in \mathscr{U}^*_{\mathcal{S}}$ and $(k, l, \varphi) \in \mathscr{A}$, one has

$$\begin{aligned} I(u,\beta) &= V^f_{\mathcal{S}}(u) + \int_\Omega \big(H(\beta) - F\cdot u\big)\,dx \\ &\ge \int_\Omega \big(-u\cdot(div\,\varphi + F)\big)\,dx - \int_\Omega f^*(\varphi)\,dx \\ &\quad + \int_\Omega H(\beta)\,dx + \int_\Omega \beta l(u)\,dx - \int_\Lambda l(y)\,dy \\ &\ge \int_\Omega -k(div\,\varphi + F)\,dx - \int_\Omega f^*(\varphi)\,dx - \int_\Lambda l(y)\,dy. \end{aligned}$$

Thus $I(u,\beta) \geq -J(k,l,\varphi)$ with equality if and only if $\varphi \in \Phi_{\mathcal{S}}(u)$ and

$$k(F+\operatorname{div}\varphi) + \beta l(u) + H(\beta) = u \cdot (F+\operatorname{div}\varphi).$$

Note that if k is convex, the map $\nabla k(F+\operatorname{div}\varphi)$ is well defined as the map $F+\operatorname{div}\varphi$ is non-degenerate. Using Lemma 3.4(i), it follows that if k is convex, then $I(u,\beta) = -J(k,l,\varphi)$ if and only if

$$\begin{cases} \varphi \in \Phi_{\mathcal{S}}(u) \\ u = \nabla k(F+\operatorname{div}\varphi) \\ \beta = (H')^{-1}(-l(u)). \end{cases} \tag{3.20}$$

Step 2. Thanks to Eq. (3.7), we may find a maximizer (k_0, l_0, φ_0) of problem (3.5) satisfying $k_0 = l_0^{\#}$ and $l_0 = (k_0)_{\#}$. The function $u_0 = \nabla k_0(F+\operatorname{div}\varphi_0)$ is well defined as k_0 is convex and we set $\beta_0 = (H')^{-1}(-l(u_0))$. We have to show that $(u_0,\beta_0) \in \mathscr{U}_{\mathcal{S}}^*$ and $\varphi_0 \in \Phi_{\mathcal{S}}(u_0)$.

Step 3. Let $\bar{l} \in C_c(\mathbb{R}^d)$. For $\epsilon \in (0,1)$, define $l_\epsilon = l_0 + \epsilon\bar{l}$ and $k_\epsilon = (l_\epsilon)^{\#}$. Using Lemma 3.4, one has

$$\begin{aligned} &\lim_{\epsilon\to 0^+} \int_\Omega (1/\epsilon)\big(k_0(F+\operatorname{div}\varphi_0) - k_\epsilon(F+\operatorname{div}\varphi_0)\big)\,dx \\ &\quad = \int_\Omega \beta_0\bar{l}\big(\nabla k_0(F+\operatorname{div}\varphi_0)\big)\,dx = \int_\Omega \beta_0\bar{l}(u_0)\,dx. \end{aligned} \tag{3.21}$$

Since $J(k_0,l_0,\varphi_0) \leq J(k_\epsilon,l_\epsilon,\varphi_0)$, we deduce that $-\int_\Lambda \bar{l}\,dy + \int_\Omega \beta_0\bar{l}(u_0)\,dx \leq 0$. As we can replace $\bar{l}$ by $-\bar{l}$, one deduces that $\int_\Lambda \bar{l}\,dy = \int_\Omega \beta_0\bar{l}(u_0)\,dx$. Therefore $(u_0,\beta_0) \in \mathscr{U}_{\mathcal{S}}^*$.

Step 4. Let $\varphi \in \mathcal{S}$. For $\epsilon \in (0,1)$, set $\varphi_\epsilon = \epsilon\varphi + (1-\epsilon)\varphi_0$. By the convexity of $\mathcal{S}$, the map φ_ϵ belongs to $\mathcal{S}$. As $J(k_0,l_0,\varphi_0) \leq J(k_0,l_0,\varphi_\epsilon)$, we have

$$\begin{aligned} 0 \leq &\int_\Omega (1/\epsilon)\big(k_0\big(F+\operatorname{div}\varphi_0 + \epsilon\operatorname{div}(\varphi-\varphi_0)\big) - k_0(F+\operatorname{div}\varphi_0)\big)\,dx \\ &+ (1/\epsilon)\int_\Omega \big(f^*\big(\varphi_0 + \epsilon(\varphi-\varphi_0)\big) - f^*(\varphi_0)\big)\,dx \end{aligned} \tag{3.22}$$

Thanks to Lemma 3.4, Inequality (3.22) implies

$$\begin{aligned} &\int_\Omega \big(u_0\cdot\operatorname{div}(\varphi-\varphi_0) + \operatorname{grad}\big[f^*\big](\varphi_0)\cdot(\varphi-\varphi_0)\big)\,dx \\ &\quad = \int_\Omega \big(\nabla k_0(F+\operatorname{div}\varphi_0)\cdot\operatorname{div}(\varphi-\varphi_0) + \operatorname{grad}\big[f^*\big](\varphi_0)\cdot(\varphi-\varphi_0)\big)\,dx \\ &\quad \geq 0. \end{aligned}$$

It follows from Proposition 2.8 that $\varphi_0 \in \Phi_{\mathcal{S}}(u_0)$.

Step 5. Since $(u_0,\beta_0) \in \mathscr{U}_{\mathcal{S}}^*$, $\varphi_0 \in \Phi_{\mathcal{S}}(u_0)$, $u_0 = \nabla k_0(F+\operatorname{div}\varphi_0)$, and $\beta_0 = (H')^{-1}(-l(u_0))$, we deduce that $I(u_0,\beta_0) = J(k_0,l_0,\varphi_0)$ and u_0 is a minimizer of problem (3.1) thanks to relation (3.20). Suppose $(u_1,\beta_1) \in \mathscr{U}_{\mathcal{S}}^*$ is another minimizer of problem (3.1). Then we have $I(u_1,\beta_1) = J(k_0,l_0,\varphi_0)$ and by relation (3.20), we get $u_1 = \nabla k_0(F+\operatorname{div}\varphi_0)$ which implies $u_1 = u_0$. Next the strict convexity of H yields that $\beta_0 = \beta_1$. We conclude that (u_0,β_0) is the unique minimizer of problem (3.1) and u_0 is characterized by

$$\begin{cases} u_0 = \nabla k_0(F + \operatorname{div} \varphi_0) \\ \varphi_0 \in \Phi_{\mathcal{S}}(u_0). \end{cases}$$

□

Corollary 3.6 *Assume that* (**A0**), (**A1**), (**A2**), *and* (**A3**) *hold. Assume $\mathcal{S}$ is a finite dimensional non-empty closed and convex subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and $\nabla\varphi$ is non-degenerate whenever $\varphi \in \mathcal{S}$. Suppose F has a countable range* (*thus degenerate*)*. Then, F satisfies the condition* **(*ND*)**$_{\mathcal{S}}$ *and problem* (3.1) *admits a unique solution.*

Corollary 3.7 *Assume that* (**A1**), (**A2**), *and* (**A3**) *hold. Assume $\mathcal{S}$ is a finite dimensional subspace of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and $\nabla\varphi$ has a countable range whenever $\varphi \in \mathcal{S}$. Suppose F is non-degenerate. Then, F satisfies the condition* **(*ND*)**$_{\mathcal{S}}$ *and problem* (3.1) *admits a unique solution.*

4 The Incompressible Case

Throughout this section, we assume that $\mathcal{S}$ is a subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. We consider the following problem:

$$\inf_{u\in\mathcal{U}_{\mathcal{S}}^1} \left\{ I_0(u) := V_{\mathcal{S}}^f(u) - \int_\Omega F\cdot u\, dx \right\} \tag{4.1}$$

and we recall that the set $\mathcal{U}_{\mathcal{S}}^1$ is defined as

$$\mathscr{U}_{\mathcal{S}}^1 = \left\{ u \in \mathscr{U}_{\mathcal{S}} : \int_\Omega l\big(u(x)\big)\, dx = \int_\Lambda l(y)\, dy\ \forall l \in C_c\big(\mathbb{R}^d\big) \right\}.$$

We assume $\mathcal{L}^d(\Omega) = \mathcal{L}^d(\Lambda)$ so that $\mathscr{U}_{\mathcal{S}}^1$ is non-empty.

4.1 Existence and Uniqueness via Duality

We study problem (4.1) via duality. Let $u \in \mathscr{U}_{\mathcal{S}}^1$, $\varphi \in \mathcal{S}$, $l \in C(\Lambda)$ and $k : \mathbb{R}^d \to \mathbb{R}$ satisfy $k(v) + l(u) \geq u\cdot v$ for all $u \in \Lambda$ and all $v \in \mathbb{R}^d$. One has

$$V_{\mathcal{S}}^f(u) - \int_\Omega F\cdot u\, dx \tag{4.2}$$

$$= -\int_\Omega u\cdot(F + \operatorname{div}\varphi)\, dx + \int_\Omega l(u)\, dx - \int_\Lambda l(y)\, dy - \int_\Omega f^*(\varphi)\, dx \tag{4.3}$$

$$\geq -\int_\Omega k(F + \operatorname{div}\varphi)\, dx - \int_\Lambda l(y)\, dy - \int_\Omega f^*(\varphi)\, dx. \tag{4.4}$$

This suggests that we consider the dual problem

$$M_0 := \inf_{(k,l,\varphi)\in A_0} \left\{ J(k,l,\varphi) := \int_\Omega k(F + \operatorname{div}\varphi)\, dx + \int_\Lambda l(y)\, dy + \int_\Omega f^*(\varphi)\, dx \right\} \tag{4.5}$$

with A_0 being the set of all (k,l,φ) such that $\varphi \in \mathcal{S}$, $l \in C(\Lambda)$, $\inf_\Lambda l = 0$ and $k : \mathbb{R}^d \to \mathbb{R}$ satisfies $k(v) + l(u) \geq u \cdot v$ for all $u \in \bar{\Lambda}$, and all $v \in \mathbb{R}^d$. Remark that we have

$$-M_0 = \sup_{(k,l,\varphi)\in A_0} \big\{-J(k,l,\varphi)\big\}. \tag{4.6}$$

4.1.1 Existence and Regularity of Minimizers of Problem (4.5)

Denote by $\mathcal{C}$ the set of all (k,l) such that $k : \mathbb{R}^d \to \mathbb{R}$ and $l : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ satisfy

$$k(v) + l(u) \geq u \cdot v; \quad \forall u \in \bar{\Lambda};\ \forall v \in \mathbb{R}^d \tag{4.7}$$

and $l \equiv \infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$. Consider the subset $\mathcal{C}_0$ of $\mathcal{C}$ consisting of $(k,l) \in \mathcal{C}$ such that $l \in C(\Lambda)$ and $\inf_\Lambda l = 0$. The following lemma is standard:

Lemma 4.1 *Let $(k,l) \in \mathcal{C}$. It holds that $(l^*, l^{**}) \in \mathcal{C}$, $l^* \leq k$, $0 \leq l^{**} \leq l$ and $l^{***} = l^*$. If $(k,l) \in \mathcal{C}_0$ then $l^*(0) = 0$.*

Let us denote by C_0' the set of all $(k,l) \in C_0$ such that $l^* = k$, $k^* = l$, $k(0) = 0$, and $l \geq 0$, and by A_0' the set of all (k,l,φ) with $(k,l) \in C_0'$ and $\varphi \in \mathcal{S}$. Remark that an element in C_0' is the couple $(\chi_{\bar{\Lambda}}, (\chi_{\bar{\Lambda}})^*)$. Hence A_0' is nonempty when $\mathcal{S}$ is nonempty. One readily checks that, in light of Lemma 4.1, problem (4.5) has the same infimum value as

$$\inf_{(k,l,\varphi)\in A_0'} \left\{ J(k,l,\varphi) := \int_\Omega k(F + \operatorname{div}\varphi)\, dx + \int_\Omega f^*(\varphi)\, dx + \int_\Lambda l(y)\, dy \right\}. \tag{4.8}$$

We recall that r^* is such that $B(0, 1/r^*) \subset \Lambda \subset B(0, r^*/2)$.

Lemma 4.2 *Assume that* (**A1**) *and* (**A3**) *hold. Assume that the set $\mathcal{S}$ is a finite dimensional non-empty closed and convex subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. Then, problem* (4.8) *admits a minimizer $(k_0, l_0, \varphi_0) \in A_0'$ with k_0 convex and r^*-Lipschitz and $k_0(0) = 0$.*

Proof Consider a minimizing sequence $\{(k_n, l_n, \varphi_n)\}_{n=1}^\infty$ of problem (4.8). Since $k_n = l_n^*$ and $l_n = (k_n)^*$, k_n is r^*-Lipschitz. As $k_n(0) = 0$, we use Ascoli-Arzelà theorem to deduce that a subsequence of $\{k_n\}_{n=1}^\infty$ converges locally uniformly to some k_0. Next, using the growth condition (2.5) on f^* as well as the facts that k_n is r^*-Lipschitz, $k_n(0) = 0$, we establish the following estimate:

$$J(k_n, l_n, \varphi_n) \geq \int_\Omega \left(-r^* |F + \operatorname{div}\varphi_n| + c^p \frac{|\varphi_n|^q}{q} - b \right) dx + \int_\Lambda l_n(y)\, dy. \tag{4.9}$$

As the left hand side of (4.9) is bounded, $l_n \geq 0$ and $\mathcal{S}$ is finite dimensional, we deduce from (4.9) that a subsequence of $\{\varphi_n\}_{n=1}^\infty$ converges strongly to some φ_0 in $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. Invoking (4.9) again, we show that $\{\int_\Lambda l_n(y)\, dy\}_{n=1}^\infty$ is bounded. This, combined with the fact that l_n is non-negative and convex, yields the existence of a subsequence of $\{l_n\}_{n=1}^\infty$ that converges locally uniformly to some l_0 (see for instance Theorem 1, p. 236 in [6]). One readily checks that $(k_0, l_0, \varphi_0) \in A_0'$. We next exploit lower semi-continuity properties of the functional J to conclude that (k_0, l_0, φ_0) is a minimizer of J over A_0'. □

4.1.2 A Duality Result

We have the following theorem.

Theorem 4.3 *Assume that* (**A1**) *and* (**A3**) *hold. Assume $\mathcal{S}$ is a finite dimensional non-empty closed and convex subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. Suppose that the map F satisfies the condition* $(\boldsymbol{ND})_{\mathcal{S}}$. *Then problems* (4.1) *and* (4.6) *are dual. Problem* (4.6) *admits a maximizer (k_0, l_0, φ_0) with $k_0 = l_0^*$ and $l_0 = (k_0)^*$. Problem* (4.1) *admits a unique minimizer u_0. Moreover u_0 satisfies*

$$\begin{cases} u_0 = \nabla k_0(F + \operatorname{div}\varphi) \\ \varphi_0 \in \Phi_S(u_0). \end{cases}$$

Proof Suppose $u \in \mathcal{U}_S^1$ and $(k, l, \varphi) \in A_0$. Using (4.3) and (4.4), we see that $I_0(u) \geq -J(k, l, \varphi)$ with equality if and only if $\varphi \in \Phi_S(u)$ and $l(u) + k(F + \operatorname{div}\varphi) = u \cdot (F + \operatorname{div}\varphi)$ for almost every $x \in \Omega$. The latter condition reduces to $u(x) = \nabla k(F(x) + \operatorname{div}\varphi(x))$ if k is convex, under the assumption $F + \operatorname{div}\varphi$ is non-degenerate. Now, let $(k_0, l_0, \varphi_0) \in A_0'$ be a minimizer of J over A_0. Since $F + \operatorname{div}\varphi_0$ is non-degenerate and k_0 is convex, the map $u_0 = \nabla k_0(F + \operatorname{div}\varphi_0)$ is well defined.

Variation around l_0. Let $\bar{l} \in C_c(\mathbb{R}^d)$. For $\epsilon \in (0, 1)$, set $l_\epsilon = l_0 + \epsilon \bar{l}$ and $k_\epsilon = (l_\epsilon)^*$. Let $v \in \mathbb{R}^d$ be a point where k_0 is differentiable. Using the measurable selection theorem, one deduces that there exists $T_\epsilon : \mathbb{R}^d \to \mathbb{R}^d$ measurable such that for all $\epsilon \in [0, 1)$

$$k_\epsilon(v) = T_\epsilon(v) \cdot v - l_\epsilon\big(T_\epsilon(v)\big).$$

Then, for $\epsilon \in (0, 1)$, we have

$$\bar{l}\big(T_\epsilon(v)\big) \leq -(1/\epsilon)\big(k_\epsilon(v) - k_0(v)\big) \leq \bar{l}\big(T_0(v)\big) \tag{4.10}$$

and

$$\big|(1/\epsilon)\big(k_\epsilon(v) - k_0(v)\big)\big| \leq \|\bar{l}\|_{L^\infty(\mathbb{R}^d)}. \tag{4.11}$$

Moreover,

$$\lim_{\epsilon \to 0^+} -(1/\epsilon)\big(k_\epsilon(v) - k_0(v)\big) = \bar{l}\big(T_0(v)\big). \tag{4.12}$$

We refer the reader to Lemma A.3 for (4.10)–(4.12). Hence, as

$$T_0(F + \operatorname{div}\psi_0) = \nabla k_0(F + \operatorname{div}\psi_0) = u_0 \quad \text{a.e.}$$

using again (4.12), one has

$$\begin{aligned} &\lim_{\epsilon \to 0^+} \int_\Omega (1/\epsilon)\big(k_0(F + \operatorname{div}\psi_0) - k_\epsilon(F + \operatorname{div}\psi_0)\big)\,dx \\ &\quad = \int_\Omega \bar{l}\big(T_0(F + \operatorname{div}\psi_0)\big)\,dx = \int_\Omega \bar{l}(u_0)\,dx. \end{aligned} \tag{4.13}$$

Since $J(k_0, l_0, \varphi_0) \leq J(k_\epsilon, l_\epsilon, \varphi_0)$, we deduce from (4.13) that $-\int_\Lambda \bar{l} + \int_\Omega \bar{l}(u_0) \leq 0$. By replacing l by $-l$ in the above argument, one deduces that $\int_\Lambda \bar{l} = \int_\Omega \bar{l}(u_0)$. As a result, $u_0 \in \mathcal{U}_S^1$.

Variation around φ_0. Let $\varphi \in \mathcal{S}$. For $\epsilon \in (0,1)$, by convexity of $\mathcal{S}$, we have $\varphi_\epsilon := \epsilon\varphi + (1-\epsilon)\varphi_0 \in \mathcal{S}$. Then $J(k_0, l_0, \varphi_0) \le J(k_0, l_0, \varphi_\epsilon)$. This implies that

$$0 \ge \int_\Omega (1/\epsilon)\big(k_0(F + \operatorname{div}\varphi_0) - k_0\big(F + \operatorname{div}\varphi_0 + \epsilon \operatorname{div}(\varphi - \varphi_0)\big) \\ + f^*(\varphi_0) - f^*(\varphi_0 + \epsilon(\varphi - \varphi_0)\big)\, dx.$$

As ϵ tends to 0^+, the above equation yields

$$0 \ge -\int_\Omega \nabla k_0(F + \operatorname{div}\varphi_0) \cdot \operatorname{div}(\varphi - \varphi_0) - \operatorname{grad}\big[f^*\big](\varphi_0) \cdot (\varphi - \varphi_0)\, dx$$

It follows from Proposition 2.8 that $\varphi_0 \in \Phi_{\mathcal{S}}(u_0)$. □

Corollary 4.4 *Assume that* (**A0**), (**A1**), *and* (**A3**) *hold. Assume that* $\mathcal{S}$ *is a finite dimensional non-empty closed and convex subset of* $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ *and* $\nabla\varphi$ *is non-degenerate whenever* $\varphi \in \mathcal{S}$. *Suppose* F *has a countable range* (*thus degenerate*). *Then*, F *satisfies the condition* **(*ND*)**$_{\mathcal{S}}$ *and problem* (4.1) *admits a unique solution.*

Corollary 4.5 *Assume that* (**A1**) *and* (**A3**) *hold. Assume that* $\mathcal{S}$ *is a finite dimensional subspace of* $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ *and* $\nabla\varphi$ *has a countable range whenever* $\varphi \in \mathcal{S}$. *Suppose* F *is non-degenerate. Then*, F *satisfies the condition* **(*ND*)**$_{\mathcal{S}}$ *and problem* (4.1) *admits a unique solution.*

4.2 A Link Between Problem (3.1) and Problem (4.1)

Here, we explore the relationships between problem (3.1) and problem (4.1). For this purpose, we make a further assumption of the domains Ω and Λ by requiring that $\Omega = \Lambda$. Assume (**A1**) holds and recall $\{H_n\}_{n=0}^\infty$ as defined in (2.7) and (2.8). Then, Lemma 2.3 ensures that (**A2**) holds for H_n for all $n \ge 1$. Define

$$I_n(u, \beta) := V_{\mathcal{S}}^f(u) + \int_\Omega (H_n(\beta) - u \cdot F)\, dx \quad n \ge 1$$

and

$$I_0(u) := V_{\mathcal{S}}^f(u) - \int_\Omega u \cdot F\, dx.$$

Recall that C_0 is the set of all (k, l) such that $l \in C(\bar{\Lambda})$, $\inf l = 0$ and $k : \mathbb{R}^d \to \mathbb{R}$ satisfies for all $u \in \Lambda$ and all $v \in \mathbb{R}^d$:

$$k(v) + l(u) \ge u \cdot v. \tag{4.14}$$

Let C_n be the set of all (k, l) such that $l \in C(\bar{\Lambda})$ and $k : \mathbb{R}^d \to \mathbb{R}$ satisfy:

$$k(v) + tl(u) + H_n(t) \ge u \cdot v; \quad \forall u \in \Lambda;\ \forall v \in \mathbb{R}^d. \tag{4.15}$$

We denote by $\mathcal{A}_0$ the set of all (k, l, φ) satisfying $(k, l) \in C_0$ and $\varphi \in S$. Similarly $\mathcal{A}_n$ denotes the set of all (k, l, φ) satisfying $(k, l) \in C_n$ and $\varphi \in S$. If $(k, l, \varphi) \in \mathcal{A}_0 \cup \mathcal{A}_n$, we still set

$$J(k, l, \varphi) = \int_\Omega k(F + \operatorname{div}\varphi)\, dx + \int_\Lambda l(y)\, dy + \int_\Omega f^*(\varphi)\, dx.$$

Lemma 4.6 *Assume that* (**A1**), (**A2**), *and* (**A3**) *hold. Assume that* $\mathcal{S}$ *is a finite dimensional non-empty closed and convex subset of* $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. *For each* $n \in \mathbb{N}$, *let* (u_n, β_n) *be the unique minimizer of* I_n *over* $\mathscr{U}_{\mathcal{S}}^*$ *as given by Theorem 3.5 and let* (k_n, l_n, φ_n) *be a minimizer of* J *over* $\mathcal{A}_n$ *with* k_n *convex and* r^**-Lipschitz as ensured by Proposition 3.3 and Lemma 4.2. Then,*

1. *The sequence* $\{I_n(u_n, \beta_n)\}_{n\in\mathbb{N}^*}$ *is bounded.*
2. *The sequence* $\{\beta_n\}_{n\in\mathbb{N}^*}$ *converges to* 1 *in* $L^2(\Omega)$.
3. *The sequence* $\{\varphi_n\}_{n\in\mathbb{N}^*}$ *admits a subsequence that converges to some* $\bar{\varphi}$ *in* S *with respect to the* $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$*-norm.*

Proof **Step 1**. Let $\bar{u} \in \mathscr{U}_{\mathcal{S}}^1$. We have $(\bar{u}, 1) \in \mathscr{U}_{\mathcal{S}}^*$ and thus $I_n(u_n, \beta_n) \le I_n(\bar{u}, 1)$ for all $n \ge 1$. As $H_n(1) = 0$, it holds that $I_n(\bar{u}, 1) = V_{\mathcal{S}}^f(\bar{u}) - \int_\Omega \bar{u}\cdot F\,dx$ which is finite. Hence

$$R_0 := V_{\mathcal{S}}^f(\bar{u}) - \int_\Omega \bar{u}\cdot F\,dx \ge I_n(u_n, \beta_n). \tag{4.16}$$

On the other hand, we use growth condition (2.5) to get

$$I_n(u_n, \beta_n) \ge \int_\Omega (-b + u_n\cdot F)\,dx \ge -b\mathcal{L}^d(\Omega) - r^*\|F\|_{L^1(\Omega,\mathbb{R}^d)} := -R_1. \tag{4.17}$$

Finally, we use (4.16) and (4.17) to prove (1).

Step 2. Let $\varphi_0 \in \mathcal{S}$. As u_n has values in Λ, it holds that

$$V_{\mathcal{S}}^f(u_n) = \sup_{\varphi\in S}\int_\Omega \big(-u_n \operatorname{div}\varphi - f^*(\varphi)\big)\,dx \ge \int_\Omega \big(-r^*|\operatorname{div}\varphi_0| - f^*(\varphi_0)\big)\,dx =: R_2 \tag{4.18}$$

and

$$\int_\Omega -u_n\cdot F\,dx \ge -r^*\|F\|_{L^1(\Omega,\mathbb{R}^d)}. \tag{4.19}$$

We combine (4.16), (4.17), (4.18), (4.19) to get

$$R_2 - r^*\|F\|_{L^1(\Omega,\mathbb{R}^d)} + \int_\Omega H_n(\beta_n)\,dx \le I_n(u_n, \beta_n) \le R_0. \tag{4.20}$$

Setting $c_0\mathcal{L}^d(\Omega) := R_0 - R_2 + r^*\|F\|_{L^1(\Omega,\mathbb{R}^d)}$, we use Lemma 2.3 and (4.20) to obtain

$$\int_\Omega n\big(\beta_n(x) - 1\big)^2\,dx \le \big(c_0 + \bar{H} - H(1)\big)\mathcal{L}^d(\Omega).$$

This establishes (2).

Step 3. As $\{H_n\}_{n=1}^\infty$ is a non-decreasing sequence that converges to H_0, it holds that $C_{n+1} \subset C_n \subset C_0$ for all $n \in \mathbb{N}$. Thus, as $(k_n, l_n) \in C_n$, we have $(k_n, l_n) \in C_0$ so that

$$k_n(F + \operatorname{div}\varphi_n) + l_n(x) \ge x\cdot(F + \operatorname{div}\varphi_n). \tag{4.21}$$

Since $-J(k_n, l_n, \varphi_n) = I_n(u_n, \beta_n)$, we have $J(k_n, l_n, \varphi_n) \leq R_1$ for all $n \in \mathbb{N}^*$. This, combined with $\Omega = \Lambda$, and (4.21) yields

$$R_1 \geq \int_{\Omega} \big(k_n(F + \operatorname{div}\varphi_n) + l_n(x) + f^*(\varphi_n)\big)\,dx \tag{4.22}$$

$$\geq \int_{\Omega} \big(x \cdot (F + \operatorname{div}\varphi_n) + f^*(\varphi_n)\big)\,dx. \tag{4.23}$$

In view of the growth condition (2.5) and boundedness of Ω, (4.22) implies

$$R_1 \geq \int_{\Omega} \left(r^*|F + \operatorname{div}\varphi_n| - b + c^p \frac{|\varphi_n|^q}{q}\right) dx. \tag{4.24}$$

As the space $\mathcal{S}$ is of finite dimension and the div operator is continuous on $\mathcal{S}$, we conclude that $\{\varphi_n\}_{n=1}^{\infty}$ is convergent up to a subsequence in $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ which allows us to conclude (3). □

Theorem 4.7 *Assume that* (**A1**), (**A2**), *and* (**A3**) *hold. Assume that $\mathcal{S}$ is a finite dimensional non-empty closed and convex subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. Assume F satisfies the condition* $(\mathbf{ND})_{\mathcal{S}}$. *For each $n \in \mathbb{N}$, let (u_n, β_n) be the unique minimizer of I_n over $\mathcal{U}_{\mathcal{S}}^*$ as given by Theorem 3.5 and let (k_n, l_n, φ_n) be a minimizer of J over $\mathcal{A}_n$ with k_n convex and r^*-Lipschitz as ensured by Proposition 3.3 and Lemma 4.2. Suppose that k_n is differentiable for all $n \in \mathbb{N}^*$. Then, the sequence $\{u_n\}_{n\in\mathbb{N}^*}$ converges almost everywhere to the unique minimizer u_0 of I_0 over $\mathcal{U}_S^1$. In addition, the minima $\{I_n(u_n, \beta_n)\}_{n=1}^{\infty}$ converge to $I_0(u_0)$.*

Proof **Step 1**. For $n \in \mathbb{N}^*$, set $\bar{k}_n = k_n - k_n(0)$. Note that we have $\bar{k}_n(0) = 0$. Since the functions k_n are r^*-Lipschitz, so are the functions $\bar{k}_n$ and we obtain that, up to a subsequence, the sequence $\{\bar{k}_n\}_{n=1}^{\infty}$ converges locally uniformly to a certain function $\bar{k}$. Since $F + \operatorname{div}\varphi_n$ is non-degenerate, we have that $\nabla\bar{k}_n(F + \operatorname{div}\varphi_n)$ is well-defined. Furthermore, Lemma 4.6 ensures that $\{\varphi_n\}_{n=1}^{\infty}$ converges up to a subsequence to some $\bar{\varphi} \in S$ with respect to the $W^{1,q}(\Omega, \mathbb{R}^d)$-norm. As a result, $\{\operatorname{div}\varphi_n\}_{n=1}^{\infty}$ converges to $\operatorname{div}\bar{\varphi}$ in $L^q(\Omega, \mathbb{R}^d)$. Since $\mathcal{S}$ is of finite dimension, the L^q convergence of $\{\operatorname{div}\varphi_n\}_{n=1}^{\infty}$ reduces to a pointwise convergence. Next, using the convexity of the $\bar{k}_n$ and the pointwise convergence of $\{\operatorname{div}\varphi_n\}_{n=1}^{\infty}$ to $\operatorname{div}\varphi$, we deduce that up to a subsequence $\{\nabla\bar{k}_n(F + \operatorname{div}\varphi_n)\}_{n=1}^{\infty}$ converges a.e to $\nabla\bar{k}(F + \operatorname{div}\bar{\varphi})$ (cf. [13] Theorem 25.7).

As a duality result, Theorem 3.5 ensures that $\nabla\bar{k}_n(F + \operatorname{div}\varphi_n) = u_n$. If we denote $\bar{u} := \nabla\bar{k}(F + \operatorname{div}\bar{\varphi})$, then, up to a subsequence, the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges a.e to $\bar{u}$.

Step 2. Let $l \in C_b(\mathbb{R}^d)$. The strong convergence in $L^2(\Omega)$ of $\{\beta_n\}_{n=1}^{\infty}$ to 1 established in Lemma 4.6 and the almost everywhere convergence of $\{u_n\}_{n\in\mathbb{N}}$ to $\bar{u}$ obtained in Step 1 ensure that $\lim_{n\to\infty}\int_{\Omega}\beta_n l(u_n)\,dx = \int_{\Omega} l(\bar{u}(x))\,dx$. As $(u_n, \beta_n) \in \mathcal{U}_S^*$, $\int_{\Omega}\beta_n(x)l(u_n)\,dx = \int_{\Omega} l(y)\,dy$ for all $l \in C_b(\mathbb{R}^d)$. It follows that in the limit $\int_{\Omega} l(\bar{u})\,dx = \int_{\Omega} l(y)\,dy$ for all $l \in C_b(\mathbb{R}^d)$ and thus $\bar{u} \in \mathcal{U}_S^1$.

Step 3. We recall that

$$I_n(u, \beta) = V_S^f(u) + \int_{\Omega}\big(H_n(\beta) - u \cdot F\big)\,dx.$$

Since $u \mapsto V_S^f(u)$ is lower-semicontinuous as a supremum of affine functions, by applying the Fatou's Lemma, we have

$$\liminf_n I_n(u_n, \beta_n) \geq V_S^f(\bar{u}) + \int_{\Omega} -\bar{u} \cdot F\,dx = I_0(\bar{u}).$$

Let u_0 be the unique minimizer of I_0 over $\mathscr{U}_S^1$ as given by Theorem 4.3. Then,

$$\liminf_n I_n(u_n, \beta_n) \geq I_0(\bar{u}) \geq I_0(u_0). \tag{4.25}$$

Meanwhile, as $C_n \subset C_0$ and (k_0, l_0, φ_0) is a minimizer of J over C_0, we have

$$J(k_0, l_0, \varphi_0) \leq J(k_n, l_n, \varphi_n).$$

This, along with the duality established in Theorem 3.5 imply that

$$\limsup_n I_n(u_n, \beta_n) \leq \limsup_n \big(-J(k_n, l_n, \varphi_n)\big) \leq -J(k_0, l_0, \varphi_0) = I_0(u_0). \tag{4.26}$$

We combine (4.25) and (4.26) to obtain $I_0(\bar{u}) = I_0(u_0)$. As u_0 is the unique minimizer of I_0 over $\mathscr{U}_S^1$ we have $u_0 = \bar{u}$. We note that the limit $\bar{u}$ does not depend on the subsequence of $\{u_n\}_n$ chosen. Thus, the whole sequence $\{u_n\}_n$ converges a.e. to u_0. In addition, $\{I_n(u_n, \beta_n)\}_n$ converges to $I_0(u_0)$. □

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Appendix

A.1 Proof of Lemma 3.4

We will prove Lemma 3.4 through two lemmas. The results of the first lemma can be found in Lemma 4.3 of [2]. We give here a sketch of the proof for the convenience of the reader.

Lemma A.1 *Assume that* (**A2**) *holds. Consider a lower semicontinuous function* $l : \mathbb{R}^d \to \bar{\mathbb{R}}$ *such that* $\inf_{\bar{\Lambda}} l > -\infty$; l *is finite on* Λ *and* $l \equiv +\infty$ *on* $\mathbb{R}^d \setminus \bar{\Lambda}$. *Set* $k = l^\#$ *and let* $w \in \mathbb{R}^d$. *Then*:

1. *There exist* $\bar{u} \in \bar{\Lambda}$ *and* $\bar{t} > 0$ *such that*

$$k(w) = -\bar{t} l(\bar{u}) - H(\bar{t}) - \bar{u} \cdot w. \tag{A.1}$$

   Moreover, $\bar{u}$ *and* $\bar{t}$ *satisfy* $\bar{u} \in \partial k(w)$ *and* $H'(\bar{t}) + l(\bar{u}) = 0$.
2. *If* k *is differentiable at* w *then* $\bar{u}$ *and* $\bar{t}$ *are uniquely determined by* $\bar{u} = \nabla k(w)$ *and* $\bar{t} = (H')^{-1}(-l(\bar{u}))$.

Proof (1.) We have

$$k(w) = \sup\big\{u \cdot w - l(u)t - H(t) : u \in \bar{\Lambda}, t > 0\big\}. \tag{A.2}$$

Consider a maximizing sequence $\{(u_n, t_n)\}_{n=1}^\infty$ in (A.2). As $0 \in \Lambda$, we may assume without loss of generality that

$$u_n w - l(u_n)t_n - H(t_n) \geq 0 \cdot w - l(0) - H(1)$$

for $n \geq 1$. It follows that

$$|w| r^* + l(0) + H(1) \geq \Big(\inf_{\bar{\Lambda}} l\Big) t_n + H(t_n)$$

for $n \geq 1$. In light of the growth condition on H in **(A2)** there exists a positive real number α such that $\{t_n\}_{n=1}^{\infty} \subset [\alpha, \alpha^{-1}]$. As Λ is bounded, we may assume without loss of generality that the sequence $\{(u_n, t_n)\}_{n=1}^{\infty}$ converges to some $(\bar{u}, \bar{t}) \in \bar{\Lambda} \times [\alpha, \alpha^{-1}]$. We next use the lower semicontinuity of H and l to deduce that

$$k(w) = \bar{u} \cdot w - l(\bar{u})\bar{t} - H(\bar{t}). \tag{A.3}$$

Note that $k(w) \geq \bar{u} \cdot w - l(\bar{u})t - H(t)$ for all $t > 0$. In view of (A.3), it follows that $g : (0, \infty) \to \mathbb{R}$ defined by $g(t) = \bar{u} \cdot w - l(\bar{u})t - H(t)$ admits a maximum at $\bar{t}$. As g is differentiable at $\bar{t}$, we have $g'(\bar{t}) = 0$, that is, $l(\bar{u}) + H'(\bar{t}) = 0$. Next, observe that $k(z) \geq \bar{u} \cdot z - l(\bar{u})\bar{t} - H(\bar{t})$ for all $z \in \mathbb{R}^d$. In light of the convexity of k we have that $\bar{u} \in \partial k(w)$.

(2.) Assume that k is differentiable at w. Then, $\bar{u}$ is uniquely determined as $\bar{u} = \nabla k(w)$. As $H'(\bar{t}) = -l_0(\bar{u})$ and H' is a bijection, we obtain that $\bar{t}$ is also uniquely determined as $\bar{t} = (H')^{-1}(-l(\bar{u}))$. □

The second lemma which is inspired by Lemma 4.4 in [2] is the following:

Lemma A.2 *Assume that* (**A2**) *holds. Consider a lower semicontinuous function* $l_0 : \mathbb{R}^d \to \bar{\mathbb{R}}$ *such that* $\inf_{\bar{\Lambda}} l_0 > -\infty$; l_0 *is finite on* Λ *and* $l_0 \equiv +\infty$ *on* $\mathbb{R}^d \setminus \bar{\Lambda}$. *Set* $k_0 = (l_0)^{\#}$. *Let* $\hat{l} \in C_b(\mathbb{R}^d)$ *and let* $1 \geq \epsilon > 0$. *Define* $l_\epsilon = l_0 + \epsilon \hat{l}$ *and* $k_\epsilon = (l_\epsilon)^{\#}$. *Let* $v \in \mathbb{R}^d$ *be such that* k_0 *is differentiable at* v.

1. *There exists a constant* M *independent of* v *and* ϵ *such that*

$$\left| \frac{k_\epsilon(v) - k_0(v)}{\epsilon} \right| \leq M. \tag{A.4}$$

2. *We have*

$$\lim_{\epsilon \to 0} \frac{k_\epsilon(v) - k_0(v)}{\epsilon} = -t_0 \hat{l}(u_0). \tag{A.5}$$

Proof Note that the map $l_\epsilon = l_0 + \epsilon \hat{l}$ is bounded below by $m - |\hat{l}|_\infty$. As $k_\epsilon = (l_\epsilon)^{\#}$ and $k_0 = (l_0)^{\#}$, Lemma A.1 ensures that there exist $t_0, t_\epsilon > 0$ and $u_0, u_\epsilon \in \bar{\Lambda}$ such that

$$k_\epsilon(v) = u_\epsilon v - l(u_\epsilon)t_\epsilon - H(t_\epsilon)$$

and

$$k_0(v) = u_0 v - l(u_0)t_0 - H(t_0).$$

We then have

$$k_\epsilon(v) = -\epsilon \hat{l}(u_\epsilon)t_\epsilon + u_\epsilon v - l_0(u_\epsilon)t_\epsilon - H(t_\epsilon) \leq -\epsilon \hat{l}(u_\epsilon)t_\epsilon + k_0(v) \tag{A.6}$$

and

$$k_0(v) = \epsilon \hat{l}(u_0)t_0 + u_0 v - l_\epsilon(u_0)t_0 - H(t_0) \leq \epsilon \hat{l}(u_0)t_0 + k_\epsilon(v). \tag{A.7}$$

We combine (A.6) and (A.7) to get

$$-\hat{l}(u_0)t_0 \leq \frac{(k_\epsilon(v) - k_0(v))}{\epsilon} \leq -\hat{l}(u_\epsilon)t_\epsilon. \tag{A.8}$$

Using again Lemma A.1 we have

$$t_\delta = \big(H'\big)^{-1}\big(l_\delta(u_\delta)\big), \quad u_\delta \in \partial k_\delta(v),\ \delta \in \{0, \epsilon\}.$$

As l_δ is bounded below by $m - |l|_\infty$, we use the fact that H' is a continuous and strictly increasing bijection from $(0, \infty)$ to $\mathbb{R}$ to deduce that t_δ is bounded above by $M_1 > 0$ given by $M_1 := (H')^{-1}(-m + |\hat{l}|_\infty)$. This bound on t_δ combined with (A.8) yields a constant $M := |\hat{l}|_\infty (H')^{-1}(-m + |\hat{l}|_\infty)$ such that (A.4) holds. As a result $\lim_{\epsilon\to 0^+} k_\epsilon(v) = k_0(v)$. Next, let $\{e_n\}_{n=1}^\infty \subset (0, 1]$ converging to 0 such that $\limsup_{\epsilon\to 0} \hat{l}(u_\epsilon)t_\epsilon = \lim_{n\to\infty} \hat{l}(u_{e_n})t_{e_n}$. Without loss of generality, we may assume that $\{u_{e_n}\}_{n=1}^\infty$ converges to some $\bar{u} \in \bar{\Lambda}$ and $\{t_{e_n}\}_{n=1}^\infty$ converges to $\bar{t} \in [0, M_1]$. Exploiting the lower semicontinuity of l_0, $\hat{l}$ and H, we get:

$$\begin{aligned} k_0(v) &= \lim_{n\to\infty} k_{e_n}(v) \\ &= \lim_{n\to\infty} u_{e_n} v - l_{e_n}(u_{e_n})t_{e_n} - H(t_{e_n}) \\ &\le \bar{u}v - l_0(\bar{u})\bar{t} - H(\bar{t}) \\ &\le k_0(v). \end{aligned}$$

It follows that $k_0(v) = \bar{u}v - l_0(\bar{u})\bar{t} - H(\bar{t})$. As k_0 is differentiable at v, we have $t_0 = \bar{t}$ and $u_0 = \bar{u}$. We use (A.8), the definition of $\{e_n\}_{n=1}^\infty$, the convergence of $\{u_{e_n}\}_{n=1}^\infty$ and $\{t_{e_n}\}_{n=1}^\infty$ to obtain

$$-t_0\hat{l}(u_0) \le \liminf_{\epsilon\to 0} -t_\epsilon\hat{l}(u_\epsilon) \le \limsup_{\epsilon\to 0} -t_\epsilon\hat{l}(u_\epsilon) = \lim_{n\to\infty} -t_{e_n}\hat{l}(u_{e_n}) = -t_0\hat{l}(u_0). \tag{A.9}$$

As a result, $\lim_{\epsilon\to 0} -t_\epsilon\hat{l}(u_\epsilon) = -t_0\hat{l}(u_0)$. We invoke one more time Eq. (A.8) to obtain (A.5). □

A.2 Some Properties of the Legendre Transform

We have the following lemma which is similar to Lemma 3.4 but uses the Legendre transform instead of the $(\cdot)^\#$ operator.

Lemma A.3 *Consider a lower semicontinuous function $l_0 : \mathbb{R}^d \to \bar{\mathbb{R}}$ such that $\inf_{\bar{\Lambda}} l_0 > -\infty$; l_0 is finite on Λ and $l_0 \equiv +\infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$. Set $k_0 = (l_0)^*$.*

1. *There exists a measurable map $T_0 : \mathbb{R}^d \to \mathbb{R}^d$ such that $k_0(v) = v \cdot T_0(v) - l_0(T_0(v))$ for all $v \in \mathbb{R}^d$ and $T_0(v) = \nabla k_0(v)$ whenever k_0 is differentiable at $v \in \mathbb{R}^d$.*
2. *Let $\hat{l} \in C_b(\mathbb{R}^d)$ and let $1 \ge \epsilon > 0$. Define $l_\epsilon = l_0 + \epsilon\hat{l}$ and $k_\epsilon = (l_\epsilon)^*$.*
   (a) *For all $v \in \mathbb{R}^d$ we have:*

$$\left|\frac{k_\epsilon(v) - k_0(v)}{\epsilon}\right| \le |\hat{l}|_\infty.$$

   (b) *For $\epsilon \in (0, 1)$, there exists a map $T_\epsilon : \mathbb{R}^d \to \mathbb{R}^d$ satisfying for all $v \in \mathbb{R}^d$: $k_\epsilon(v) = vT_\epsilon(v) - l_\epsilon(T_\epsilon(v))$. When k_0 is differentiable at $v \in \mathbb{R}^d$, we have $\lim_{\epsilon\to 0} T_\epsilon(v) = \nabla k_0(v)$ and*

$$\lim_{\epsilon\to 0} \frac{k_\epsilon(v) - k_0(v)}{\epsilon} = -t_0\hat{l}\big(\nabla k_0(v)\big).$$

Proof (1.) Let $v \in \mathbb{R}^d$. We have

$$k_0(v) = \sup\{uv - l_0(u) : u \in \mathbb{R}^d\} = \sup\{uv - l_0(u) : u \in \bar{\Lambda}\}.$$

We use the lower semicontinuity of l_0 and the compactness of $\bar{\Lambda}$ to deduce that there exists $\bar{u} \in \Lambda$ such that $k_0(v) = \bar{u}v - l_0(\bar{u})$. We have $k_0(w) - (\bar{u}w - l_0(\bar{u})) \geq 0$ for all $w \in \mathbb{R}^d$ while $k_0(v) - (\bar{u}v - l_0(\bar{u})) = 0$. Since k_0 is convex, we deduce that $\bar{u} \in \partial k_0(v)$.

Next, for $v \in \mathbb{R}^d$, define

$$\Gamma(v) = \{u \in \bar{\Lambda} : k_0(v) = uv - l_0(u)\}.$$

Assume $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ converges to u; $\{v_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ converges to v and for all $n \in \mathbb{N}$, one has $u_n \in \Gamma(v_n)$. Then $u \in \Gamma(v)$. Indeed, one has

$$k_0(v) \leq \liminf_{n\to\infty} k_0(v_n) = \liminf_{n\to\infty}\big(u_n v_n - l_0(u_n)\big) \leq uv - l_0(u) \leq k_0(v).$$

Therefore, $uv - l_0(u) = k_0(v)$ and $u \in \Gamma(v)$. As a result, the multifunction $\Gamma : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is closed and nonempty valued. By the Measurable Selection Theorem [14, Corollary 14.6], there exists a measurable map $T_0 : \mathbb{R}^d \to \mathbb{R}^d$ such that for all $v \in \mathbb{R}^d$, one has $T_0(v) \in \Gamma(v)$. That is $k_0(v) = vT_0(v) - l_0(T_0(v))$. As $T(v) \in \Gamma(v) \subset \partial k_0(v)$, we also have $T_0 = \nabla k_0$ almost everywhere.

(2.) For $\epsilon > 0$, l_ϵ is bounded below and satisfies the hypothesis on l_0. Let $k_\epsilon = l_\epsilon^*$ and consider a map T_ϵ satisfying for all $v \in \mathbb{R}^d$: $k_\epsilon(v) = vT_\epsilon(v) - l_\epsilon(T_\epsilon(v))$ as given by part 1.). We have for $v \in \mathbb{R}^d$:

$$k_\epsilon(v) = vT_\epsilon(v) - l_\epsilon\big(T_\epsilon(v)\big) = -\epsilon\hat{l}\big(T_\epsilon(v)\big) + vT_\epsilon(v) - l_0\big(T_\epsilon(v)\big) \leq -\epsilon\hat{l}\big(T_\epsilon(v)\big) + k_0(v). \tag{A.10}$$

Similarly, for $v \in \mathbb{R}^d$ we have

$$k_0(v) = vT_0(v) - l_0\big(T_0(v)\big) = \epsilon\hat{l}\big(T_0(v)\big) + vT_0(v) - l_\epsilon\big(T_0(v)\big) \leq \epsilon\hat{l}\big(T_0(v)\big) + k_\epsilon(v). \tag{A.11}$$

We combine (A.10) and (A.11) to get

$$-\hat{l}\big(T_0(v)\big) \leq \frac{k_\epsilon(v) - k_0(v)}{\epsilon} \leq -\hat{l}\big(T_\epsilon(v)\big), \tag{A.12}$$

which leads to

$$\left|\frac{k_\epsilon(v) - k_0(v)}{\epsilon}\right| \leq |\hat{l}|_\infty. \tag{A.13}$$

Consider a sequence $\{\epsilon_n\}_n$ converging to 0. The sequence $\{T_{\epsilon_n}(v)\}_n$ is bounded so we may find a subsequence $\{\epsilon_n'\}_n$ of $\{\epsilon_n\}_n$ such that the sequence $\{T_{\epsilon_n'}(v)\}_n$ converges to $u \in \bar{\Lambda}$. We then have:

$$k_0(v) = \lim_{n\to\infty} k_{\epsilon_n'}(v) = \lim_{n\to\infty}\Big(vT_{\epsilon_n'}(v) - l_{\epsilon_n}\big(T_{\epsilon_n'}(v)\big)\Big) \leq vu - l_0(u) \leq k_0(v). \tag{A.14}$$

We use (A.14) to obtain $k_0(v) = vu - l_0(u)$ and thus $u = \nabla k_0(v)$ as k_0 is differentiable at v. It follows that $\lim_{\epsilon\to 0} T_\epsilon(v) = \nabla k_0(v)$. We use Eq. (A.12) and the continuity of $\hat{l}$ to obtain

$$\lim_{\epsilon\to 0}\frac{k_\epsilon(v) - k_0(v)}{\epsilon} = -t_0\hat{l}\big(\nabla k_0(v)\big).$$

□

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