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Finite-Time Blow-up in a Quasilinear Degenerate Chemotaxis System with Flux Limitation

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Abstract This paper deals with the quasilinear degenerate chemotaxis system with flux limitation

$$\begin{cases} u_t = \nabla \cdot \left(\dfrac{u^p \nabla u}{\sqrt{u^2 + |\nabla u|^2}} \right) - \chi \nabla \cdot \left(\dfrac{u^q \nabla v}{\sqrt{1 + |\nabla v|^2}} \right), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu + u, & x \in \Omega,\ t > 0, \end{cases}$$

where $\Omega := B_R(0) \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ is a ball with some $R > 0$, and $\chi > 0$, $p, q \ge 1$, $\mu := \frac{1}{|\Omega|} \int_\Omega u_0$ and u_0 is an initial data of an unknown function u. Bellomo–Winkler (Trans. Am. Math. Soc. Ser. B **4**, 31–67, 2017) established existence of an initial data such that the corresponding solution blows up in finite time when $p = q = 1$. This paper gives existence of blow-up solutions under some condition for χ and u_0 when $1 \le p \le q$.

Keywords Degenerate chemotaxis system · Flux limitation · Finite-time blow-up

Mathematics Subject Classification (2010) Primary 35B44 · Secondary 35K65 · 92C17

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1 Introduction

In this paper we consider the quasilinear degenerate chemotaxis system with flux limitation:

$$\begin{cases} u_t = \nabla \cdot \left(\dfrac{u^p \nabla u}{\sqrt{u^2 + |\nabla u|^2}} \right) - \chi \nabla \cdot \left(\dfrac{u^q \nabla v}{\sqrt{1 + |\nabla v|^2}} \right), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu + u, & x \in \Omega,\ t > 0, \\ \left(\dfrac{u^p \nabla u}{\sqrt{u^2 + |\nabla u|^2}} - \chi \dfrac{u^q \nabla v}{\sqrt{1 + |\nabla v|^2}} \right) \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega = B_R(0) \subset \mathbb{R}^n (n \in \mathbb{N})$ is a ball with some $R > 0$, $\chi > 0$, $p, q \geq 1$ and the initial data u_0 is a function fulfilling that

$$u_0 \in C^3(\overline{\Omega}) \text{ is radially symmetric and positive in } \overline{\Omega} \text{ with } \frac{\partial u_0}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{1.2}$$

and where

$$\mu := \frac{1}{|\Omega|} \int_\Omega u_0(x)\, dx. \tag{1.3}$$

The system (1.1) represents the situation such that a cellular slime moves towards higher concentrations of the chemical substance, and the unknown function $u = u(x,t)$ describes the density of cell and the unknown function $v = v(x,t)$ denotes the concentration of chemoattractant at $x \in \Omega$ and $t \geq 0$. This model is development of the chemotaxis system

$$u_t = \Delta u - \nabla \cdot (u \nabla v), \quad v_t = \Delta v - v + u; \tag{1.4}$$

thanks to effect of the flux limitation, the system (1.1) describes the case that cell diffusivity is suppressed; therefore the system (1.1) is innovative and important because it is considered a sensitive dynamics in aggregation phenomena.

Before we introduce previous works about the system (1.1), we will recall known results about the chemotaxis system (1.4):

The system (1.4) is proposed by Keller–Segel [14] and is called a Keller–Segel system. About the Keller–Segel system it was known that the size of the initial data in some Lebesgue norm determines behaviour of solutions; in the case that $n = 1$, Osaki–Yagi [21] obtained global existence and boundedness of classical solutions of (1.4); in the case that $n = 2$, it is shown that there is a critical value $C > 0$ ($C = 8\pi$ in the radial setting and $C = 4\pi$ in the other setting) such that, if $\|u_0\|_{L^1(\Omega)} < C$ then global solutions exist ([20]), and if $m > C$ then there is an initial data satisfying that $\|u_0\|_{L^1(\Omega)} = m$ and the corresponding solution blows up in finite time ([10, 18]); in the case that $n \geq 3$, Horstmann–Winkler [11] asserted possibility of existence of unbounded solutions; Winkler [23] showed that for all $m > 0$ there exists an initial data such that $\|u_0\|_{L^1(\Omega)} = m$ and the corresponding solution blows up in finite time; also in the case that $n = 3$, Cao [3] established global existence and boundedness under the condition that $\|u_0\|_{L^{\frac{n}{2}}(\Omega)}$ and $\|\nabla v_0\|_{L^n(\Omega)}$ are sufficiently small.

The Keller–Segel system (1.4) is now studied by many mathematicians intensively. Moreover, many variations of generalizations of the Keller–Segel system (1.4) are also sprightly studied. Here one of the important generalized problems is the quasilinear chemotaxis system

$$u_t = \nabla \cdot \big(D(u)\nabla u\big) - \nabla \cdot \big(S(u)\nabla v\big), \quad v_t = \Delta v - v + u. \tag{1.5}$$

This problem is one of the model which has a nonlinear diffusion suggested by Hillen–Painter [9]. In the nondegenerate chemotaxis system which is the system (1.5) with $D(u) = (u+1)^{p-1}$ and $S(u) = u(u+1)^{q-2}$ with $p, q \in \mathbb{R}$, it is known that the relation between p and q determines the properties of solutions to the system; Tao–Winkler [22] established global existence and boundedness under the conditions that $q < p + \frac{2}{n}$ and that Ω is a convex domain, and the convexity of Ω was removed in [13]; Cieślak–Stinner obtained finite-time blow-up in the case that $q > p + \frac{2}{n}$ (the 2-dimensional setting can be found in [5], and the 3-dimensional setting is in [4, 6]); In the parabolic–elliptic setting Lankeit [15] recently showed several results including existence of infinite-time blow-up solutions in the case that $q \le 1$ and $q > p + \frac{2}{n}$. In the degenerate chemotaxis system which is the system (1.5) with $D(u) = u^{p-1}$ and $S(u) = u^{q-1}$ with $p, q \ge 1$, similarly, it is shown that the relation between p and q determines behaviour of solutions; in the case $q < p + \frac{2}{n}$, global solutions are obtained when Ω is a bounded domain (see [12]); in the case $q = p + \frac{2}{n}$, the result is divided by the size of the initial data with some critical mass $m_c = m_c(n)$; when $\|u_0\|_{L^1(\Omega)} = m < m_c$ and $q = 2$, under the Dirichlet–Neumann boundary condition, Mimura [17] showed that there are global solutions when Ω is a bounded domain; on the other hand, if $m > m_c$, then Laurençot–Mizoguchi [16] established that existence of an initial data such that $\|u_0\|_{L^1(\Omega)} = m$ and the corresponding solution blows up in finite time when $q = 2$, $\Omega = \mathbb{R}^n$ and $n = 3, 4$; in the case $q > p + \frac{2}{n}$, it is known that there exists an initial data such that the corresponding solution blows up in finite time (see [8]). Moreover, some simplification of the system (1.5) which is the system that the second equation is $0 = \Delta v - \mu + u$ instead of $v_t = \Delta v - v + u$ in (1.5) with $D(u) = (u+1)^{p-1}$ and $S(u) = u(u+1)^{q-2}$ with $p, q \in \mathbb{R}$ was studied ([7, 24]); Cieślak–Winkler [7] established global existence and boundedness in the case that $q = 2$ and $2 < p + \frac{2}{n}$, and existence of finite-time blow-up solutions in the case that $q = 2$ and $2 > p + \frac{2}{n}$; Winkler–Djie [24] dealt with the case that $p \le 1$ and $q \in \mathbb{R}$ and showed global existence and boundedness in the case that $q < p + \frac{2}{n}$, and existence of blow-up solutions in the case that $q > p + \frac{2}{n}$. Thus, it is clear that the relation between p, q and n strongly affects behaviour of solutions.

On the other hand, the system which describes the situation such that the movement of the species is suppressed, that is, the chemotaxis system with flux limitation

$$\begin{cases} u_t = \nabla \cdot \left(D_u(u,v) \dfrac{u\nabla u}{\sqrt{u^2 + |\nabla u|^2}} \right) - \nabla \cdot \left(S(u,v) \dfrac{u \nabla v}{\sqrt{1 + |\nabla v|^2}} \right) + H_1(u,v), \\ v_t = D_v \Delta v + H_2(u,v), \qquad x \in \Omega, \ t > 0 \end{cases} \tag{1.6}$$

is proposed by Bellomo–Winkler [1], where D_u and D_v show properties of diffusion of the species and the chemoattractant, respectively, and S represents the chemotactic interaction, and H_1 and H_2 are mechanisms of propagation, degeneration, and interaction. Since it has not been known whether there exist valid functions like an energy function and a Lyapunov function yet, the system (1.6) seems to be difficult. Therefore, Bellomo–Winkler [1, 2] have considered the following simplified system

$$u_t = \nabla \cdot \left(\frac{u\nabla u}{\sqrt{u^2 + |\nabla u|^2}} \right) - \chi \nabla \cdot \left(\frac{u\nabla v}{\sqrt{1 + |\nabla v|^2}} \right), \quad 0 = \Delta v - \mu + u. \tag{1.7}$$

In this system, Bellomo–Winkler [1] overcame the difficulty, and they showed global existence when $\chi < 1$. On the other hand, if $\chi > 1$ and

$$\begin{cases} m > \dfrac{1}{\sqrt{\chi^2 - 1}} & \text{if } n = 1, \\ m > 0 \text{ is arbitrary} & \text{if } n \geq 2, \end{cases} \tag{1.8}$$

Bellomo–Winkler [2] found an initial data such that the corresponding solution of (1.7) blows up in finite time. However, the problem (1.6) has not been studied yet when D_u and S are general; in view of the study of the Keller–Segel system, the case that $D_u(u, v) = u^{p-1}$ and $S(u, v) = u^{q-1}$ in (1.6), i.e., the system (1.1) seems to be one of important problems. Recently global existence of solutions to system (1.1) was shown when $p > q + 1 - \frac{1}{n}$ (see [19]). From the results in the degenerate chemotaxis system we can expect that some largeness condition for q derives existence of blow-up solutions.

The purpose of this paper is to determine the condition for p and q such that the corresponding solution blows up in finite time. Here we need to establish different methods because we cannot adopt the same argument as in [2] when $p < q$ holds.

Now main results read as follows.

Theorem 1.1 *Let $\Omega := B_R(0) \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ with $R > 0$ and suppose that $1 \leq p \leq q$.*
(i) *If $n = 1$, then for all $\chi > 1$ ($\chi > 0$ when $q > p$), there exists $m_c = m_c(\chi, p, q, R) > 0$ with the following property: If*

$$m > m_c, \tag{1.9}$$

then there exists a nondecreasing function $M_m \in C^0([0, R])$ satisfying $\sup_{r\in(0,R)} \frac{M_m(r)}{|B_r(0)|} < \infty$, $M_m(R) \leq m$, and that for all $u_0 \in C^3(\overline{\Omega})$ with $\int_\Omega u_0(x)\,dx = m$ and

$$\int_{B_r(0)} u_0(x)\,dx \geq M_m(r) \tag{1.10}$$

for all $r \in [0, R]$, there exists $T^* \in (0, \infty)$ such that a corresponding solution (u, v) of (1.1) blows up in finite time T^* in the sense that

$$\limsup_{t \nearrow T^*} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{1.11}$$

(ii) *If $n \geq 2$ and $m > 0$, then for all $\chi > 0$ satisfying*

$$\chi > \left(\frac{mn}{\omega_n R^n}\right)^{p-q}, \tag{1.12}$$

where ω_n defines the $(n-1)$-dimensional measure of the unit sphere in $\mathbb{R}^n$, there exists a nondecreasing function $M_m \in C^0([0, R])$ satisfying $\sup_{r\in(0,R)} \frac{M_m(r)}{|B_r(0)|} < \infty$, and $M_m(R) \leq m$, such that for all $u_0 \in C^3(\overline{\Omega})$ with $\int_\Omega u_0(x)\,dx = m$ and (1.10) for all $r \in [0, R]$, there exists $T^* \in (0, \infty)$ such that the corresponding solution (u, v) of (1.1) blows up in finite time T^* in the sense of (1.11).

Remark 1.1 This theorem shows existence of blow-up solutions to (1.1) when $q \geq p$. On the other hand, in [19] global existence of solutions is shown when $p > q + 1 - \frac{1}{n}$. Thus we

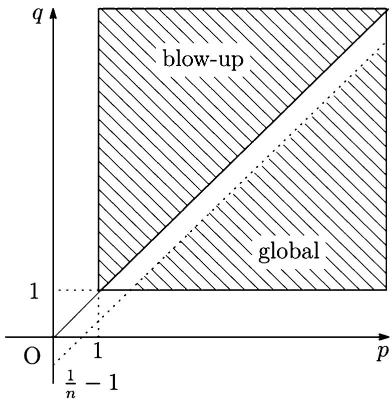

Fig. 1 Classification of behaviour

can observe that the case $p = q$ is critical when $n = 1$. However, except for the case $n = 1$, behaviour of solutions in the case $q < p \le q + 1 - \frac{1}{n}$ is still an open problem (see Fig. 1).

Remark 1.2 If $p = q = 1$, then the condition for m connects with that in [2]; indeed, the constant $m_c(\chi, p, q, R)$ in (1.9) is given by

$$m_c = \inf\left\{ m \;\middle|\; \exists \lambda \in \left(\frac{5-\sqrt{17}}{2}, 1 \right];\ \frac{\{1 - \frac{(1-\lambda)^2}{3\lambda-1}\} m\chi}{\sqrt{\frac{1}{\lambda^2} + \frac{(1-\lambda)^2}{\lambda^2(3\lambda-1)} + m^2}} = \left(\frac{m}{\omega_n} \cdot \frac{(1-\lambda)^2}{\lambda(\lambda+1)R} \right)^{p-q} \right\}$$

(see (3.1) and (3.54)). Thus, in the case $p = q$, since

$$\begin{aligned} m_c(\chi, p, p, R) &= \inf\left\{ m \;\middle|\; \exists \lambda \in \left(\frac{5-\sqrt{17}}{2}, 1 \right];\ \frac{\{1 - \frac{(1-\lambda)^2}{3\lambda-1}\} m\chi}{\sqrt{\frac{1}{\lambda^2} + \frac{(1-\lambda)^2}{\lambda^2(3\lambda-1)} + m^2}} = 1 \right\} \\ &= \inf\left\{ m \;\middle|\; \frac{m\chi}{\sqrt{1+m^2}} = 1 \right\} = \frac{1}{\sqrt{\chi^2 - 1}} \end{aligned}$$

and moreover

$$\left(\frac{mn}{\omega_n R^n} \right)^{p-p} = 1,$$

the conditions (1.9) and (1.12) are reduced to (1.8). Thus Theorem 1.1 is a generalization of the previous work [2]. However, we note that the constants m_c (when $n = 1$) and $(\frac{mn}{\omega_n R^n})^{p-q}$ (when $n \ge 2$) might not be optimal constants but be ones required technically except for the case that $p = q = 1$ ([1, 2]); the opposite cases are open problems.

In view of Remark 1.2 we have the following corollary.

Corollary 1.2 *Let* $\Omega := B_R(0) \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ *with* $R > 0$ *and let* $p = q \ge 1$, *and suppose that* $\chi > 1$ *and* (1.8). *Then there exists a nondecreasing function* $M_m \in C^0([0, R])$ *fulfilling* $\sup_{r \in (0,R)} \frac{M_m(R)}{|B_r(0)|} < \infty$ *and* $M_m(R) \le m$, *which is such that whenever* u_0 *satisfies* (1.2), *as well as* (1.10) *for all* $r \in [0, R]$, *the solution* (u, v) *of* (1.1) *blows up in finite time in the sense of* (1.11).

Theorem 1.1 gives the following byproduct; since arguments similar to those in the proof of [2, Proposition 1.2] enable us to see this proposition, we only write the statement.

Proposition 1.3 *Let $n \in \mathbb{N}$, $R > 0$, $\Omega := B_R(0) \subset \mathbb{R}^n$ and $\chi > 1$.*
(i) *Let $m > 0$ satisfy (1.9). Then there exists a radially symmetric positive $u_m \in C^\infty(\overline{\Omega})$ which is such that*

$$\frac{\partial u_m}{\partial \nu} = 0 \text{ on } \partial\Omega \quad \text{and} \quad \int_\Omega u_m = m,$$

and for which it is possible to choose $\varepsilon > 0$ with the property that whenever u_0 satisfies (1.2) as well as

$$\|u_0 - u_m\|_{L^\infty(\Omega)} \le \varepsilon,$$

the corresponding solution of (1.1) blows up in finite time.
(ii) *Given any u_0 fulfilling (1.2), one can find functions u_{0k}, $k \in \mathbb{N}$, which satisfy (1.2) and*

$$u_{0k} \to u_0 \quad \text{in } L^p(\Omega) \quad \text{as } k \to \infty$$

for all $p \in (0, 1)$, and which are such that for all $k \in \mathbb{N}$ the solution of (1.1) emanating from u_{0k} blows up in finite time.

The proof of the main result is based on that of [2, Theorem 1.1]. Thus in the same way, we introduce $s := r^n$ for $r \in [0, R]$ and the mass accumulation function $w = w(s,t)$ defined as

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} r^{n-1} u(r,t)\, dr,$$

and then, a combination of the fact $u(r_*, t) \ge \frac{w(s_*,t)}{s_*}$ given by using $u \ge w_s$ and the mean value theorem and that w is the solution of a scalar parabolic equation yields that the core of the proof is to find a suitable subsolution $\underline{w}$ such that for some $T > 0$ and some $s_* \in (0, R^n)$,

$$\frac{\underline{w}(s_*, t)}{s_*} \to \infty \quad \text{as } t \nearrow T.$$

In the previous study [2], the interval $(0, R^n)$ is divided three parts, and in very inner region thanks to construction of the subsolution $\underline{w}$ using the structure of a quadratic function, we obtain a suitable estimate. However, in this paper we cannot establish it from the same argument when $p < q$. Therefore, adopting a new subsolution $\underline{w}$ consisted by an exponential function, we can prove existence of an initial data such that the corresponding solution blows up in finite time. In this proof, the key idea is to employ a new viewpoint in the proof of some suitable estimate; by establishing a new estimate where the effect of the aggregation come from chemotactic interaction works adequately (see Lemma 3.5), we can attain a useful estimate.

This paper is organized as follows. In Sect. 2 we recall local existence in (1.1) and we consider the mass accumulation function and a scalar parabolic equation. In order to use the comparison argument as in [2, Lemma 5.1] we construct subsolutions and confirm properties in Sect. 3. Finally, we prove existence of blow-up solutions in Sect. 4.

2 Local Existence and a Parabolic Problem Satisfied by the Mass Accumulation

In this section we provide a local existence result and a mass accumulation function satisfying some parabolic problem. First, we recall a local existence result; the following result was shown in [19, Theorem 1.1].

Lemma 2.1 *Suppose that u_0 complies with (1.2). Then there exist $T_{\max} \in (0, \infty]$ and a pair (u, v) of positive radially symmetric functions $u \in C^{2,1}(\overline{\Omega} \times [0, T_{\max}))$ and $v \in C^{2,0}(\overline{\Omega} \times [0, T_{\max}))$ which solve (1.1) classically in $\Omega \times (0, T_{\max})$, and which are such that*

$$\textit{if} \quad T_{\max} < \infty, \quad \textit{then} \quad \limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.1}$$

In the following let $\Omega := B_R(0) \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ with some $R > 0$ and let u_0 satisfy (1.2), and denote by $(u, v) = (u(r, t), v(r, t))$ the radially symmetric local solution of (1.1) and by $T_{\max}$ its maximal existence time obtained in Lemma 2.1. Moreover, we introduce the mass accumulation function w and the parabolic operator $\mathcal{P}$ such that

$$w(s, t) := \int_0^{s^{\frac{1}{n}}} r^{n-1} u(r, t)\, dr \tag{2.2}$$

for $s \in [0, R^n]$ and $t \in [0, T)$, and

$$(\mathcal{P}\widetilde{w})(s, t) := \widetilde{w}_t - n^{p+1} \cdot \frac{s^{2-\frac{2}{n}} \widetilde{w}_s^p \widetilde{w}_{ss}}{\sqrt{\widetilde{w}_s^2 + n^2 s^{2-\frac{2}{n}} \widetilde{w}_{ss}^2}} - n^q \chi \cdot \frac{(\widetilde{w} - \frac{\mu}{n} s)\widetilde{w}_s^q}{\sqrt{1 + s^{\frac{2}{n}-2}(\widetilde{w} - \frac{\mu}{n} s)^2}}. \tag{2.3}$$

If $\widetilde{w} \in C^1((0, R^n) \times (0, T))$ is such that $\widetilde{w}_s > 0$ and $\widetilde{w}(\cdot, t) \in W^{2,\infty}(0, R^n)$ for all $t \in (0, T)$, then the expression $\mathcal{P}\widetilde{w}$ is well-defined. Now we show that the function w defined as (2.2) fulfills these corresponding condition. Thus, the following lemma yields that the function w satisfies some parabolic problem (see [2, Lemma 2.1]).

Lemma 2.2 *Let $n \in \mathbb{N}$, $\chi > 0$. Then for $T > 0$ and some nonnegative radially symmetric $u_0 \in C^0(\overline{\Omega})$, whenever (u, v) is a positive radially symmetric classical solution of (1.1) in $\Omega \times [0, T)$, the function w defined as (2.2) satisfies*

$$\begin{cases} (\mathcal{P}w)(s, t) = 0, & s \in \big(0, R^n\big),\ t \in (0, T), \\ w(0, t) = 0, \quad w\big(R^n, t\big) = \dfrac{m}{\omega_n}, & t \in (0, T), \\ w(s, 0) = \displaystyle\int_0^{s^{\frac{1}{n}}} r^{n-1} u_0(r)\, dr, & s \in \big[0, R^n\big], \end{cases} \tag{2.4}$$

where $m := \int_\Omega u_0(x)\, dx$ and where ω_n denotes the $(n-1)$-dimensional measure of the unit sphere in $\mathbb{R}^n$.

Proof An argument similar to that in the proof of [2, Lemma 2.1] implies the conclusion of this lemma. □

3 Construction of Subsolutions for (2.4)

In this section we construct a subsolution $\underline{w}$ of (2.4). Then, using a suitable comparison principle (see [2, Lemma 5.1]) to obtain $w \geq \underline{w}$ in $[0, R^n] \times [0, T)$, we derive that $u(r, t)$ blows up in finite time $T > 0$ (fixed later). In Sect. 3.1 we prepare a family of functions, and define $\underline{w}$. In Sects. 3.2, 3.3, 3.4 and 3.5 we divide $[0, R^n]$ into three parts and show properties of a subsolution $\underline{w}$ in respective regions.

3.1 Constructing a Family of Candidates

In order to construct subsolutions $\underline{w}$ for (2.4) we first provide some parameter and some function; for $\lambda \in [\frac{1}{3}, 1]$ we put

$$a_\lambda := \frac{(1-\lambda)^2}{2\lambda} \geq 0 \quad \text{and} \quad b_\lambda := \frac{3\lambda - 1}{2\lambda} \geq 0 \tag{3.1}$$

and define

$$\varphi(\xi) := \begin{cases} \frac{2\lambda}{de^d}(e^{d\xi} - 1) & \text{if } \xi \in [0, 1), \\ 1 - \frac{a_\lambda}{\xi - b_\lambda} & \text{if } \xi \geq 1, \end{cases} \tag{3.2}$$

where $1 < d < 2$ is such that

$$(2-d)e^d - 2 = 0. \tag{3.3}$$

Here we note that there is a solution $d \in (1, 2)$ of (3.3); indeed, since

$$\big((2-d)e^d - 2\big)|_{d=1} = e - 2 > 0 \quad \text{and} \quad \big((2-d)e^d - 2\big)|_{d=2} = -2 < 0$$

hold, the intermediate value theorem enables us to find $d \in (1, 2)$ satisfying (3.3). Then we can show that $\varphi \in C^1([0, \infty)) \cap W^{2,\infty}(0, \infty) \cap C^2([0, \infty) \setminus \{1\})$ with

$$\varphi'(\xi) = \begin{cases} 2\lambda e^{d(\xi-1)} & \text{if } \xi \in [0, 1), \\ \frac{a_\lambda}{(\xi - b_\lambda)^2} & \text{if } \xi \geq 1 \end{cases} \tag{3.4}$$

and

$$\varphi''(\xi) = \begin{cases} 2d\lambda e^{d(\xi-1)} & \text{if } \xi \in [0, 1), \\ -\frac{2a_\lambda}{(\xi - b_\lambda)^3} & \text{if } \xi \geq 1. \end{cases} \tag{3.5}$$

In particular, $\varphi'(\xi) > 0$ for all $\xi \geq 0$. Then we have to choose $\lambda \in (\frac{5-\sqrt{17}}{2}, 1]$ suitably in the case that $n = 1$ (see Lemma 3.12), and we must fix $\lambda = \frac{1}{3}$ in the case $n \geq 2$ (see Lemma 3.13). The following lemma has already been proved in the proof of [2, Lemma 3.1]. Thus we recall only the statement of the lemma.

Lemma 3.1 *Let $n \in \mathbb{N}$, $m > 0$, $\lambda \in [\frac{1}{3}, 1]$, $K > 0$, $T > 0$, and suppose that $B \in C^1([0, T))$ satisfies that $B(t) \in (0, 1)$, $K\sqrt{B(t)} < R^n$ for all $t \in [0, T)$ and*

$$B(t) \leq \frac{K^2}{4(a_\lambda + b_\lambda)^2} \tag{3.6}$$

for all $t \in [0, T)$, *where* a_λ *and* b_λ *are given by* (3.1). *Then the following functions are well-defined*:

$$A(t) := \frac{m}{\omega_n} \cdot \frac{K^2 - 2b_\lambda K\sqrt{B(t)} + b_\lambda^2 B(t)}{N(t)} \quad \textit{for } t \in [0, T) \tag{3.7}$$

and

$$D(t) := \frac{m}{\omega_n} \cdot \frac{a_\lambda}{N(t)} \quad \textit{for } t \in [0, T) \tag{3.8}$$

as well as

$$E(t) := \frac{m}{\omega_n} - R^n D(t) = \frac{m}{\omega_n} \cdot \frac{K^2 - (a_\lambda + b_\lambda)(2K\sqrt{B(t)} - b_\lambda B(t))}{N(t)} \quad \textit{for } t \in [0, T) \tag{3.9}$$

with

$$N(t) := K^2 + a_\lambda R^n - (a_\lambda + b_\lambda)\big(2K\sqrt{B(t)} - b_\lambda B(t)\big) \quad \textit{for } t \in [0, T). \tag{3.10}$$

Furthermore, we have

$$A'(t) = \frac{m}{\omega_n} \cdot \frac{(\frac{K}{\sqrt{B(t)}} - b_\lambda) \cdot (a_\lambda K^2 - a_\lambda b_\lambda R^n) \cdot B'(t)}{N^2(t)}$$

as well as

$$D'(t) = \frac{m}{\omega_n} \cdot \frac{a_\lambda(a_\lambda + b_\lambda) \cdot (\frac{K}{\sqrt{B(t)}} - b_\lambda) \cdot B'(t)}{N^2(t)} \tag{3.11}$$

for all $t \in (0, T)$.

Using these definitions, we can express clearly our comparison function $\underline{w}$. Letting $K > 0$ be a constant fixed later and letting B be a function chosen suitably later, we will give a composite structure of $\underline{w}$ by separating $[0, R^n]$ into two parts (an inner region and an outer region).

Lemma 3.2 *Let* $n \in \mathbb{N}$, $m > 0$, $\lambda \in [\frac{1}{3}, 1]$, $K > 0$ *and* $T > 0$, *and let* $B \in C^1([0, T))$ *satisfy that* $B(t) \in (0, 1)$, $K\sqrt{B(t)} < R^n$ *and* (3.6) *hold for all* $t \in [0, T)$. *Suppose that*

$$\underline{w}(s, t) := \begin{cases} w_{\text{in}}(s, t) & \textit{if } t \in [0, T) \textit{ and } s \in [0, K\sqrt{B(t)}], \\ w_{\text{out}}(s, t) & \textit{if } t \in [0, T) \textit{ and } s \in (K\sqrt{B(t)}, R^n], \end{cases}$$

where

$$w_{\text{in}}(s, t) := A(t)\varphi(\xi), \quad \xi = \xi(s, t) := \frac{s}{B(t)} \tag{3.12}$$

for $t \in [0, T)$, $s \in [0, K\sqrt{B(t)}]$ *with* φ *and* A *introduced as* (3.2) *and* (3.7), *respectively, and where*

$$w_{\text{out}}(s, t) := D(t)s + E(t) \tag{3.13}$$

for $t \in [0, T)$ and $s \in [K\sqrt{B(t)}, R^n]$ with D and E as in (3.8) and (3.9), respectively. Then $\underline{w}$ is well-defined and satisfies

$$\underline{w} \in C^1\big(\big[0, R^n\big] \times [0, T)\big)$$

and $\underline{w}(\cdot, t) \in W^{2,\infty}(0, R^n) \cap C^2([0, R^n] \setminus \{B(t), K\sqrt{B(t)}\})$ for all $t \in [0, T)$ as well as

$$\underline{w}(0, t) = 0 \quad \text{and} \quad \underline{w}\big(R^n, t\big) = \frac{m}{\omega_n}$$

for all $t \in (0, T)$.

Proof An argument similar to that in the proof of [2, Lemma 3.2] implies the conclusion of this lemma. □

3.2 Subsolution Properties: Outer Region

First, we will consider in the outer region. In the following lemma, we show that if the function B constructing $\underline{w}$ is suitably small and fulfilling a differential inequality, then $\underline{w}$ is a subsolution of (2.4) in the region.

Lemma 3.3 *Let $n \in \mathbb{N}$, $\chi > 0$, $m > 0$, $\lambda \in [\frac{1}{3}, 1]$, $K > 0$, $T > 0$ and $B_0 \in (0, 1)$ fulfill $K\sqrt{B_0} < R^n$ and*

$$B_0 \le \frac{K^2}{16(a_\lambda + b_\lambda)^2} \tag{3.14}$$

with a_λ and b_λ taken from (3.1). Then if $B \in C^1([0, T))$ is positive and nonincreasing and satisfies that

$$\begin{cases} B'(t) \ge -\dfrac{a_\lambda^{q-1}(nm)^q \chi K}{2(a_\lambda+b_\lambda)(K^2+a_\lambda R^n)^{q-1}\omega_n^q R^n\sqrt{1+K^{\frac{2}{n}-2}\cdot\frac{m^2}{\omega_n^2}}} \cdot B^{1-\frac{1}{2n}}(t), \\ B(0) \le B_0 \end{cases} \tag{3.15}$$

for all $t \in (0, T)$, then the function w_{out} given by (3.13) fulfills that

$$(\mathcal{P}w_{\text{out}})(s, t) \le 0 \tag{3.16}$$

for all $t \in (0, T)$ and all $s \in (K\sqrt{B(t)}, R^n)$ with $\mathcal{P}$ defined in (2.3).

Proof The proof is based on that of [2, Lemma 3.3]. Recalling that $E(t) = \frac{m}{\omega_n} - R^n D(t)$ for all $t \in (0, T)$ by (3.9), we have

$$w_{\text{out}}(s, t) = D(t)s + E(t) = \frac{m}{\omega_n} - D(t) \cdot \big(R^n - s\big) \tag{3.17}$$

for all $t \in (0, T)$ and all $s \in (K\sqrt{B(t)}, R^n)$. Straightforward calculations together with (3.11) yield that

$$\begin{aligned} (w_{\text{out}})_t(s, t) &= -D'(t) \cdot \big(R^n - s\big) \\ &= -\frac{m}{\omega_n} \cdot \frac{a_\lambda(a_\lambda + b_\lambda)(\frac{K}{\sqrt{B(t)}} - b_\lambda)}{N^2(t)} \cdot B'(t) \cdot \big(R^n - s\big) \end{aligned} \tag{3.18}$$

for all $t \in (0, T)$ and all $s \in (K\sqrt{B(t)}, R^n)$ with N as in (3.10). In order to obtain an estimate for $\mathcal{P}w_{\mathrm{out}}$, noting from (3.18) and the fact $(w_{\mathrm{out}})_{ss} \equiv 0$ that

$$(\mathcal{P}w_{\mathrm{out}})(s,t) = (w_{\mathrm{out}})_t(s,t) - n^{p+1} \cdot \frac{s^{2-\frac{2}{n}}\{(w_{\mathrm{out}})_s\}^p (w_{\mathrm{out}})_{ss}}{\sqrt{(w_{\mathrm{out}})_s{}^2 + n^2 s^{2-\frac{2}{n}} (w_{\mathrm{out}})_{ss}{}^2}} + I(s,t)$$
$$= -\frac{m}{\omega_n} \cdot \frac{a_\lambda(a_\lambda + b_\lambda)(\frac{K}{\sqrt{B(t)}} - b_\lambda)}{N^2(t)} \cdot B'(t) \cdot \big(R^n - s\big) + I(s,t), \qquad (3.19)$$

where

$$I(s,t) := -n^q \chi \cdot \frac{(w_{\mathrm{out}} - \frac{\mu}{n}s) \cdot \{(w_{\mathrm{out}})_s\}^q}{\sqrt{1 + s^{\frac{2}{n}-2}(w_{\mathrm{out}} - \frac{\mu}{n}s)^2}} \qquad (3.20)$$

for $t \in (0, T)$ and $s \in (K\sqrt{B(t)}, R^n)$, we shall derive an estimate for $I(s,t)$. Since (3.17) holds, we first obtain from the identity $\frac{\mu}{n} = \frac{m}{\omega_n R^n}$ that

$$w_{\mathrm{out}} - \frac{\mu}{n}s = \frac{m}{\omega_n} - D(t) \cdot \big(R^n - s\big) - \frac{m}{\omega_n R^n} \cdot s$$
$$= \left(\frac{m}{\omega_n R^n} - D(t)\right) \cdot \big(R^n - s\big) \qquad (3.21)$$

for all $t \in (0, T)$ and all $s \in (K\sqrt{B(t)}, R^n)$. Noticing from arguments similar to those in the proof of [2, Lemma 3.3] that

$$\sqrt{1 + s^{\frac{2}{n}-2}\left(w_{\mathrm{out}}(s,t) - \frac{\mu}{n}s\right)^2} \le \sqrt{1 + K^{\frac{2}{n}-2} \cdot \frac{m^2}{\omega_n^2}} \cdot B^{\frac{1}{2n}-\frac{1}{2}}(t)$$

for all $t \in (0, T)$ and all $s \in (K\sqrt{B(t)}, R^n)$, we infer from (3.20) and (3.21) that

$$-I(s,t) \ge n^q \chi \frac{(\frac{m}{\omega_n R^n} - D(t)) \cdot D^q(t)}{\sqrt{1 + K^{\frac{2}{n}-2} \cdot \frac{m^2}{\omega_n^2}}} \cdot B^{\frac{1}{2}-\frac{1}{2n}}(t) \cdot \big(R^n - s\big) \qquad (3.22)$$

holds for all $t \in (0, T)$ and all $s \in (K\sqrt{B(t)}, R^n)$. Here in light of the definitions of D and N (see (3.8) and (3.10), respectively), we can see that

$$\left(\frac{m}{\omega_n R^n} - D(t)\right) \cdot D^q(t) = \left(\frac{m}{\omega_n R^n} - \frac{m}{\omega_n} \cdot \frac{a_\lambda}{N(t)}\right)\left(\frac{m}{\omega_n} \cdot \frac{a_\lambda}{N(t)}\right)^q$$
$$= \frac{m^{q+1} a_\lambda^q}{\omega_n^{q+1} N^q(t)} \cdot \frac{N(t) - a_\lambda R^n}{R^n N(t)}$$
$$= \frac{m^{q+1} a_\lambda^q}{\omega_n^{q+1} N^2(t)} \cdot \frac{K^2 - 2(a_\lambda + b_\lambda)K\sqrt{B(t)} + (a_\lambda + b_\lambda)b_\lambda B(t)}{R^n N(t)^{q-1}} \qquad (3.23)$$

for all $t \in (0, T)$ and all $s \in (K\sqrt{B(t)}, R^n)$. Moreover, the fact $B_0 < \frac{4K^2}{b_\lambda^2}$ by (3.14) leads to that

$$N(t) = K^2 + a_\lambda R^n - (a_\lambda + b_\lambda)\big(2K\sqrt{B(t)} - b_\lambda B(t)\big) \le K^2 + a_\lambda R^n \tag{3.24}$$

for all $t \in (0, T)$, a combination of the relation

$$K^2 - 2(a_\lambda + b_\lambda)K\sqrt{B(t)} \ge \frac{1}{2}K^2$$

(by (3.6)), (3.23), (3.24) and the fact $(a_\lambda + b_\lambda)b_\lambda B(t) \ge 0$ yields that

$$\begin{aligned}\left(\frac{m}{\omega_n R^n} - D(t)\right) \cdot D^q(t) &\ge \frac{m^{q+1}a_\lambda^q}{\omega_n^{q+1}N^2(t)} \cdot \frac{K^2 - 2(a_\lambda + b_\lambda)K\sqrt{B(t)}}{R^n(K^2 + a_\lambda R^n)^{q-1}} \\ &\ge \frac{m^{q+1}a_\lambda^q}{\omega_n^{q+1}N^2(t)} \cdot \frac{K^2}{2R^n(K^2 + a_\lambda R^n)^{q-1}} \\ &= \frac{m}{\omega_n} \cdot \frac{a_\lambda}{N^2(t)} \cdot \frac{m^q a_\lambda^{q-1}}{\omega_n^q} \cdot \frac{K^2}{2R^n(K^2 + a_\lambda R^n)^{q-1}}\end{aligned}$$

for all $t \in (0, T)$. Therefore we verify from (3.22) that

$$-I(s,t) \ge n^q \chi \cdot \frac{\frac{m}{\omega_n} \cdot \frac{a_\lambda}{N^2(t)} \cdot \frac{m^q a_\lambda^{q-1}}{\omega_n^q} \cdot \frac{K^2}{2R^n(K^2 + a_\lambda R^n)^{q-1}}}{\sqrt{1 + K^{\frac{2}{n}-2} \cdot \frac{m^2}{\omega_n^2}}} \cdot B^{\frac{1}{2}-\frac{1}{2n}}(t) \cdot \big(R^n - s\big) \tag{3.25}$$

for all $t \in (0, T)$ and all $s \in (K\sqrt{B(t)}, R^n)$. Here putting

$$c_1 := \frac{a_\lambda^{q-1}(nm)^q \chi}{2(K^2 + a_\lambda R^n)^{q-1}\omega_n^q R^n \sqrt{1 + K^{\frac{2}{n}-2} \cdot \frac{m^2}{\omega_n^2}}}$$

and using the fact $(a_\lambda + b_\lambda)b_\lambda B'(t) \le 0$ for all $t \in (0, T)$, from (3.19) and (3.25) we can confirm that

$$\begin{aligned}(\mathcal{P}w_{\text{out}})(s,t) &\le \frac{m}{\omega_n} \cdot \frac{a_\lambda}{N^2(t)} \cdot \big(R^n - s\big) \cdot \Bigg\{\left(-\frac{(a_\lambda + b_\lambda)K}{\sqrt{B(t)}} + (a_\lambda + b_\lambda)b_\lambda\right) \cdot B'(t) \\ &\quad - n^q \chi \cdot \frac{\frac{m^q a_\lambda^{q-1}}{\omega_n^q} \cdot \frac{K^2}{2R^n(K^2 + a_\lambda R^n)^{q-1}}}{\sqrt{1 + K^{\frac{2}{n}-2} \cdot \frac{m^2}{\omega_n^2}}} \cdot B^{\frac{1}{2}-\frac{1}{2n}}(t)\Bigg\} \\ &\le \frac{m}{\omega_n} \cdot \frac{a_\lambda}{N^2(t)} \cdot \big(R^n - s\big) \cdot \Bigg\{-\frac{(a_\lambda + b_\lambda)K}{\sqrt{B(t)}} \cdot B'(t) \\ &\quad - \frac{a_\lambda^{q-1}(nm)^q \chi K^2}{2(K^2 + a_\lambda R^n)^{q-1}\omega_n^q R^n \sqrt{1 + K^{\frac{2}{n}-2} \cdot \frac{m^2}{\omega_n^2}}} \cdot B^{\frac{1}{2}-\frac{1}{2n}}(t)\Bigg\} \\ &= \frac{m}{\omega_n} \cdot \frac{a_\lambda}{N^2(t)} \cdot \big(R^n - s\big) \cdot \left\{-\frac{(a_\lambda + b_\lambda)K}{\sqrt{B(t)}} \cdot B'(t) - c_1 K^2 B^{\frac{1}{2}-\frac{1}{2n}}(t)\right\}\end{aligned} \tag{3.26}$$

for all $t \in (0, T)$ and all $s \in (K\sqrt{B(t)}, R^n)$. Thanks to (3.15), we finally derive that

$$-\frac{(a_\lambda + b_\lambda)K}{\sqrt{B(t)}} \cdot B'(t) - c_1 K^2 B^{\frac{1}{2}-\frac{1}{2n}}(t) = \frac{(a_\lambda + b_\lambda)K}{\sqrt{B(t)}} \cdot \left\{ -B'(t) - \frac{c_1 K}{(a_\lambda + b_\lambda)} B^{1-\frac{1}{2n}}(t) \right\} \le 0$$

for all $t \in (0, T)$ and all $s \in (K\sqrt{B(t)}, R^n)$, which together with (3.26) implies (3.16) holds. □

3.3 Subsolution Properties: Inner Region

In this subsection we will consider in the inner region. In the following lemma we provide calculations of $\mathcal{P}w_{\text{in}}$ and properties of the function A defined as (3.7) constructing $\underline{w}$ in the corresponding region.

Lemma 3.4 *Let $n \in \mathbb{N}$, $m > 0$, $\lambda \in [\frac{1}{3}, 1]$, $K > 0$ be such that $K \ge \sqrt{b_\lambda R^n}$, and $T > 0$, and let $B \in C^1([0, T))$ be positive and fulfill (3.6) and $K\sqrt{B(t)} < R^n$ for all $t \in [0, T)$. Then the function w_{in} defined as (3.12) satisfies that*

$$(\mathcal{P}w_{\text{in}})(s, t) = A'(t)\varphi(\xi) + \frac{A(t)\varphi'(\xi)}{B(t)} \cdot \left\{ -\xi B'(t) + J_1(s, t) + J_2(s, t) \right\} \tag{3.27}$$

for all $t \in (0, T)$ and all $s \in (0, K\sqrt{B(t)}) \setminus \{B(t)\}$, where $\xi = \xi(s, t) = \frac{s}{B(t)}$, $\mathcal{P}$ is given by (2.3), and

$$J_1(s, t) := -n^{p+1} \cdot \frac{\xi^{2-\frac{2}{n}}\varphi''(\xi)}{\sqrt{B^{\frac{4}{n}-2}(t)\varphi'(\xi) + n^2 B^{\frac{2}{n}-2}(t)\xi^{2-\frac{2}{n}}\varphi''^2(\xi)}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{p-1} \tag{3.28}$$

as well as

$$J_2(s, t) := -n^q \chi \cdot \frac{A(t)\varphi(\xi) - \frac{\mu}{n}B(t)\xi}{\sqrt{1 + B^{\frac{2}{n}-2}(t)\xi^{\frac{2}{n}-2}(A(t)\varphi(\xi) - \frac{\mu}{n}B(t)\xi)^2}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \tag{3.29}$$

for $t \in (0, T)$ and $s \in (0, K\sqrt{B(t)}) \setminus \{B(t)\}$. Moreover, the function A defined as (3.7) fulfills

$$A'(t) \le 0 \tag{3.30}$$

for all $t \in (0, T)$; in particular,

$$A(t) \ge A_T := \frac{m}{\omega_n} \cdot \frac{1}{1 + \frac{a_\lambda R^n}{K^2}} \tag{3.31}$$

holds for all $t \in (0, T)$.

Proof Aided by arguments similar to those in the proofs of [2, Lemmas 3.4 and 3.5], from straightforward calculations we can attain the conclusion of this lemma. □

3.4 Subsolution Properties: Very Inner Region

In this subsection we will consider the case that $\xi = \frac{s}{B(t)} \in (0, 1)$ which means $0 < s < B(t)$. In order to show $\mathcal{P}w_{\text{in}} \leq 0$ we have to see that for some $C > 0$ and some $\beta \geq 0$,

$$J_1(s,t) + J_2(s,t) \leq -CB^{\beta}(t) \tag{3.32}$$

holds for all $t \in (0, T)$ and all $s \in (0, B(t))$ with J_1 and J_2 as in (3.28) and (3.29), respectively. Thanks to the convexity of φ for $\xi \in (0, 1)$, we obtain the term J_1 is negative in this region. However, it seems to be difficult to show (3.32) in the case that $p < q$. Indeed, when $\varphi(\xi) = \lambda\xi^2$ which is used in [2], if $B(t)$ is close to 0, then we obtain from an arguments that

$$J_1(s,t) + J_2(s,t) \geq -C_1 B^{1-\frac{1}{n}}(t)\xi^{1-\frac{1}{n}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{p-1} + C_2 B(t)\xi \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \geq 0$$

with some $C_1, C_2 > 0$. Thus, we modify a function φ on $(0, 1)$ from [2] to infer that even though $B(t)$ is suitably small, the term $A(t)\varphi(\xi) - \frac{\mu}{n}B(t)\xi$ is positive which means $J_2 < 0$ for $t \in (0, T)$ and $\xi \in (0, B(t))$. Furthermore, we divide the estimate for $\mathcal{P}w_{\text{in}}$ into the case $n = 1$ and the case $n \geq 2$ to achieve our purpose.

Lemma 3.5 *Let $n \in \mathbb{N}$, $m > 0$, $\lambda \in [\frac{1}{3}, 1]$, and $K > 0$ be such that $K \geq \sqrt{b_\lambda R^n}$, and let $B_0 \in (0, 1)$ be such that $K\sqrt{B_0} < R^n$ and*

$$B_0 \leq \frac{K^2}{4(a_\lambda + b_\lambda)^2} \tag{3.33}$$

as well as

$$B_0 < \frac{2\lambda n A_T}{e^d \mu} \tag{3.34}$$

with μ, d and A_T given by (1.3), (3.3) and (3.31), respectively. Then, under the condition that for some $T > 0$, $B \in C^1([0, T))$ is a positive and nonincreasing function satisfying $B(0) \leq B_0$, the inequality

$$A(t)\varphi(\xi) - \frac{\mu}{n}B(t)\xi > 0 \tag{3.35}$$

holds for all $t \in (0, T)$ and all $s \in (0, B(t))$ with φ and A as in (3.2) and (3.7), respectively.

Proof We write $\xi = \frac{s}{B(t)}$ for $t \in (0, T)$ and $s \in (0, B(t))$. According to (3.31) and $B(t) \leq B(0) \leq B_0$, we obtain from (3.2) that

$$\begin{aligned} A(t)\varphi(\xi) - \frac{\mu}{n}B(t)\xi &\geq A_T \cdot \varphi(\xi) - \frac{\mu}{n}B_0 \cdot \xi \\ &= \frac{\mu}{n}\xi\left\{\frac{nA_T}{\mu} \cdot \frac{\varphi(\xi)}{\xi} - B_0\right\} \\ &= \frac{\mu}{n}\xi\left\{\frac{2\lambda n A_T}{e^d\mu} \cdot \frac{e^{d\xi} - 1}{d\xi} - B_0\right\} \end{aligned} \tag{3.36}$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$. Thanks to that

$$\frac{e^{d\xi} - 1}{d\xi} \geq 1 \tag{3.37}$$

for all $\xi \in (0, 1)$, we infer from (3.34) and (3.36) that

$$\begin{aligned} A(t)\varphi(\xi) - \frac{\mu}{n} B(t)\xi &\geq \frac{\mu}{n}\xi \left\{ \frac{2\lambda n A_T}{e^d \mu} \cdot \frac{e^{d\xi} - 1}{d\xi} - B_0 \right\} \\ &\geq \frac{\mu}{n}\xi \left\{ \frac{2\lambda n A_T}{e^d \mu} - B_0 \right\} > 0 \end{aligned}$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$. □

Invoking Lemma 3.5, under the assumption that the function B is small and satisfies a suitable inequality, we derive that $\underline{w}$ becomes a subsolution of (2.4). First, we will note when $n = 1$. In this case, thanks to the definition of φ on $(0, 1)$, we can establish that $J_1 \leq -C$ holds with some $C > 0$ and show the purpose of this subsection.

Lemma 3.6 *Let $n = 1$, $m > 0$, $\lambda \in [\frac{1}{3}, 1]$ and $K > 0$ be such that $K \geq \sqrt{b_\lambda R^n}$, and let $B_0 \in (0, 1)$ be such that $K\sqrt{B_0} \leq R^n$, (3.33) and (3.34) hold. For some $T > 0$, if $B \in C^1([0, T))$ satisfies that*

$$\begin{cases} B'(t) \geq -\dfrac{d}{\sqrt{d^2 + 1}} \cdot \left(\dfrac{2\lambda A_T}{e^d} \right)^{p-1}, \\ B(0) \leq B_0 \end{cases} \tag{3.38}$$

for all $t \in (0, T)$ with d and A_T as in (3.3) and (3.31), respectively, then the function w_{in} defined as (3.12) has the property that

$$(\mathcal{P} w_{\text{in}})(s, t) \leq 0$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$.

Proof Writing $\xi = \frac{s}{B(t)}$ for $t \in (0, T)$ and $s \in (0, B(t))$, we establish that

$$\frac{B^2(t)\varphi'^2(\xi)}{\varphi''^2(\xi)} = B^2(t) \cdot \frac{(2\lambda e^{d(\xi-1)})^2}{(2d\lambda e^{d(\xi-1)})^2} = B^2(t) \cdot \frac{1}{d^2} \leq \frac{1}{d^2}$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$. The inequality leads to that

$$\begin{aligned} -J_1(s, t) &= \frac{\varphi''(\xi)}{\sqrt{B^2(t)\varphi'^2(\xi) + \varphi''^2(\xi)}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)} \right)^{p-1} \\ &\geq \frac{\varphi''(\xi)}{\sqrt{(\frac{1}{d^2} + 1)\varphi''^2(\xi)}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)} \right)^{p-1} \\ &= \frac{d}{\sqrt{d^2 + 1}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)} \right)^{p-1} \end{aligned} \tag{3.39}$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$. Thanks to (3.31) and $\varphi'(\xi) \geq \varphi'(0) = \frac{2\lambda}{e^d}$, we infer from $B(t) < 1$ that

$$\left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{p-1} \geq \left(\frac{A_T \cdot \frac{2\lambda}{e^d}}{1}\right)^{p-1} = \left(\frac{2\lambda A_T}{e^d}\right)^{p-1} \tag{3.40}$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$. On the other hand, using (3.35), we have

$$J_2 < 0 \tag{3.41}$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$. Thus, plugging (3.39) and (3.41) into (3.27), we derive from (3.38), (3.40) and the inequality $\xi \leq 1$ that

$$\begin{aligned}
\frac{B(t)}{A(t)\varphi'(\xi)}(\mathcal{P}w_{\text{in}})(s,t) &\leq -\xi B'(t) + J_1(s,t) + J_2(s,t) \\
&\leq -\xi B'(t) - \frac{d}{\sqrt{d^2+1}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{p-1} \\
&\leq -\xi B'(t) - \frac{d}{\sqrt{d^2+1}} \cdot \left(\frac{2\lambda A_T}{e^d}\right)^{p-1} \\
&\leq \xi \left\{-B'(t) - \frac{d}{\sqrt{d^2+1}} \cdot \left(\frac{2\lambda A_T}{e^d}\right)^{p-1}\right\} \\
&\leq 0
\end{aligned}$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$, and it concludes the proof. □

In the case $n \geq 2$, if the function B is suitably small, then we can obtain that $J_2 \leq -C$ holds with some $C > 0$, which leads to achievement of the purpose.

Lemma 3.7 *Let $n \geq 2$, $m > 0$, $\lambda \in [\frac{1}{3}, 1]$, and $K > 0$ be such that $K \geq \sqrt{b_\lambda R^n}$, and let $B_0 \in (0, 1)$ be such that $K\sqrt{B_0} \leq R^n$, (3.33), (3.34) and*

$$B_0^{2-\frac{2}{n}} \leq \left(\frac{2\lambda A_T}{e^d} - \frac{\mu}{n} B_0\right)^2 \tag{3.42}$$

hold. For some $T > 0$, if $B \in C^1([0, T))$ satisfies that

$$\begin{cases} B'(t) \geq -\dfrac{n^q \chi}{\sqrt{2}} \cdot \left(\dfrac{2\lambda A_T}{e^d}\right)^{q-1} B^{1-\frac{1}{n}}(t), \\ B(0) \leq B_0 \end{cases} \tag{3.43}$$

for all $t \in (0, T)$ with d and A_T as in (3.3) and (3.31), respectively, then the function w_{in} defined as (3.12) has the property that

$$(\mathcal{P}w_{\text{in}})(s,t) \leq 0$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$.

Proof We write $\xi = \frac{s}{B(t)}$ for $t \in (0, T)$ and $s \in (0, B(t))$. Thanks to (3.42) and

$$B(t) \le B(0) \le B_0,$$

we obtain that

$$\begin{aligned} B^{2-\frac{2}{n}}(t) \le B_0^{2-\frac{2}{n}} &\le \left(\frac{2\lambda A_T}{e^d} - \frac{\mu}{n} B_0\right)^2 \\ &\le \left(\frac{2\lambda A_T}{e^d} - \frac{\mu}{n} B(t)\right)^2 \end{aligned}$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$. Moreover, since (3.31) and (3.37) hold, we can see from (3.2), (3.37) and the inequality $\xi < 1$ that

$$\begin{aligned} B^{2-\frac{2}{n}}(t) &\le \left(\frac{2\lambda A_T}{e^d} \cdot 1 - \frac{\mu}{n} B(t)\right)^2 \\ &\le \left(A(t) \cdot \frac{2\lambda}{e^d} \cdot \frac{e^d \xi - 1}{d\xi} - \frac{\mu}{n} B(t)\right)^2 \\ &= \xi^{-2}\left(A(t)\varphi(\xi) - \frac{\mu}{n} B(t)\xi\right)^2 \end{aligned}$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$. Therefore, we derive that

$$1 \le B^{\frac{2}{n}-2}\xi^{-2}\left(A(t)\varphi(\xi) - \frac{\mu}{n} B(t)\xi\right)^2 \tag{3.44}$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$. Thanks to (3.44), we infer from $\xi < 1$ that

$$\begin{aligned} -J_2(s,t) &= n^q \chi \cdot \frac{A(t)\varphi(\xi) - \frac{\mu}{n} B(t)\xi}{\sqrt{1 + B^{\frac{2}{n}-2}(t)\xi^{\frac{2}{n}-2}(A(t)\varphi(\xi) - \frac{\mu}{n} B(t)\xi)^2}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \\ &\ge n^q \chi \cdot \frac{A(t)\varphi(\xi) - \frac{\mu}{n} B(t)\xi}{\sqrt{(1 + \xi^{\frac{2}{n}})B^{\frac{2}{n}-2}(t)\xi^{-2}(A(t)\varphi(\xi) - \frac{\mu}{n} B(t)\xi)^2}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \\ &\ge n^q \chi \cdot \frac{1}{\sqrt{(1 + 1^{\frac{2}{n}})B^{\frac{2}{n}-2}(t)\xi^{-2}}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \\ &= \frac{n^q \chi}{\sqrt{2}} B^{1-\frac{1}{n}}(t)\xi\left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \end{aligned} \tag{3.45}$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$. Moreover, using (3.31) and $\varphi'(\xi) \ge \varphi'(0) = \frac{2\lambda}{e^d}$, we obtain from $B(t) < 1$ that

$$\left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \ge \left(\frac{A_T \cdot \frac{2\lambda}{e^d}}{1}\right)^{q-1} = \left(\frac{2\lambda A_T}{e^d}\right)^{q-1}. \tag{3.46}$$

On the other hand, from the fact $\varphi', \varphi'' > 0$ (given by (3.4) and (3.5), respectively), we verify that

$$J_1 < 0 \tag{3.47}$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$. Recalling (3.27) and (3.30) (see Lemma 3.4), we can show that a combination of (3.45) and (3.47) yields that by (3.43) and (3.46),

$$\begin{aligned}\frac{B(t)}{A(t)\varphi'(\xi)}(\mathcal{P}w_{\text{in}})(s,t) &\le -\xi B'(t) - \frac{n^q\chi}{\sqrt{2}}\cdot\left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} B^{1-\frac{1}{n}}(t)\xi \\ &\le \xi\left\{-B'(t) - \frac{n^q\chi}{\sqrt{2}}\cdot\left(\frac{2\lambda A_T}{e^d}\right)^{q-1} B^{1-\frac{1}{n}}(t)\right\} \\ &\le 0\end{aligned}$$

for all $t \in (0, T)$ and all $s \in (0, B(t))$, which means the end of the proof. □

3.5 Subsolution Properties: Intermediate Region

In this subsection we will consider the case that $s \in (B(t), K\sqrt{B(t)})$. Here the term J_1 is positive due to the definition of φ in the region. Therefore, an estimate for J_2 is important in this part. The following arguments are based on these of [2].

Lemma 3.8 *Let $n \in \mathbb{N}$, $m > 0$, $K > 0$ and $T > 0$, and let $B \in C^1([0, T))$ be positive and satisfy that (3.6) and $K\sqrt{B(t)} < R^n$ for all $t \in [0, T)$. Then the function J_1 defined as (3.28) satisfies*

$$J_1(s,t) \le n^p B^{1-\frac{1}{n}}(t)\xi^{1-\frac{1}{n}}\left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{p-1} \tag{3.48}$$

for all $t \in (0, T)$ and all $s \in (B(t), K\sqrt{B(t)})$ with $\xi = \frac{s}{B(t)}$.

Proof An argument similar to that in the proof of [2, Lemma 3.7] implies the conclusion this lemma. □

The following two lemmas have already been proved in proofs of [2, Lemmas 3.8 and 3.10]. Thus we only recall statements of lemmas.

Lemma 3.9 *Let $n \in \mathbb{N}$, $m > 0$, $\lambda \in [\frac{1}{3}, 1]$, $K > 0$ be such that $K \ge \sqrt{b_\lambda R^n}$, and let $B_0 \in (0, 1)$ satisfy $K\sqrt{B_0} < R^n$ and (3.33), as well as for some $T > 0$ let $B \in C^1([0, T))$ be positive and nonincreasing and be such that $B(0) \le B_0$. Then the inequality*

$$\frac{1}{A^2(t)B^{\frac{2}{n}-2}(t)\xi^{\frac{2}{n}-2}\varphi^2(\xi)} \le \frac{\omega_n^2}{\lambda^2 m^2}\cdot\left(1 + \frac{a_\lambda R^n}{K^2}\right)\cdot K^{2-\frac{2}{n}} B_0^{3-\frac{3}{n}} \tag{3.49}$$

holds for all $t \in (0, T)$ and all $s \in (B(t), K\sqrt{B(t)})$ with $\xi = \frac{s}{B(t)}$.

Lemma 3.10 *Let $n \in \mathbb{N}$, $m > 0$, $\lambda \in [\frac{1}{3}, 1]$, $K > 0$, $\delta \in (0, 1)$, and let $B_0 \in (0, 1)$ fulfill $K\sqrt{B_0} < R^n$, (3.33) and*

$$\frac{\mu}{nA_T} \cdot \max\left\{\frac{B_0}{\lambda},\ 2K\sqrt{B_0}\right\} \leq \delta$$

with μ and A_T introduced in (1.3) and (3.31), respectively, and suppose that $T > 0$ and $B \in C^1([0, T))$ is positive and such that

$$B(t) \leq B_0$$

for all $t \in (0, T)$. Then the inequality

$$A(t)\varphi(\xi) - \frac{\mu}{n}B(t)\xi \geq (1 - \delta)A(t)\varphi(\xi) \tag{3.50}$$

holds for all $t \in (0, T)$ and all $s \in ((B(t), K\sqrt{B(t)})$ with $\xi = \frac{s}{B(t)}$ for $s \in (B(t), K\sqrt{B(t)})$ and $t \in (0, T)$.

In order to have the estimate for $\mathcal{P}w_{\text{in}}$ in the intermediate region when $p < q$, we will provide the following lemma.

Lemma 3.11 *Let $n \in \mathbb{N}$, $m > 0$, $\lambda \in [\frac{1}{3}, 1]$, $K > 0$, and let $B \in C^1([0, T))$ be such that $K\sqrt{B(t)} < R^n$ and be a positive and nonincreasing function and put*

$$\sigma := \frac{m}{\omega_n} \cdot \frac{a_\lambda}{K^2 + a_\lambda R^n}. \tag{3.51}$$

Then the inequality

$$\left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{-k} \leq \sigma^{-k} \tag{3.52}$$

holds for all $k \geq 0$ and all $s \in (B(t), K\sqrt{B(t)})$, $t \in (0, T)$ with $\xi = \frac{s}{B(t)}$, where A and φ' are as in (3.7) and (3.4), respectively.

Proof We write $\xi = \frac{s}{B(t)}$ for all $t \in (0, T)$ and all $s \in (B(t), K\sqrt{B(t)})$. Using that

$$\varphi'(\xi) = \frac{a_\lambda}{(\xi - b_\lambda)^2}$$

for $1 < \xi < \frac{K}{\sqrt{B(t)}}$, we infer from $B(t) < 1$ and $0 \leq b_\lambda \leq 1$ for $\lambda \in [\frac{1}{3}, 1]$ that

$$\begin{aligned}
\varphi'(\xi) = \frac{a_\lambda}{(\xi - b_\lambda)^2} &= \frac{a_\lambda}{\xi^2 - 2\xi b_\lambda + b_\lambda^2} \\
&\geq \frac{a_\lambda}{(\frac{K}{\sqrt{B(t)}})^2 - 2 \cdot 1 \cdot b_\lambda + b_\lambda^2} \\
&= \frac{a_\lambda B(t)}{K^2 + (b_\lambda^2 - 2b_\lambda)B(t)} \geq \frac{a_\lambda}{K^2}B(t).
\end{aligned} \tag{3.53}$$

A combination of (3.31) and (3.53) yields that

$$\left(\frac{A(t)\varphi'(\xi)}{B(t)}\right) \geq \frac{m}{\omega_n} \cdot \frac{1}{1+\frac{a_\lambda R^n}{K^2}} \cdot \frac{a_\lambda}{K^2} = \frac{m}{\omega_n} \cdot \frac{a_\lambda}{K^2 + a_\lambda R^n} = \sigma$$

for all $t \in (0,T)$ and all $s \in (B(t), K\sqrt{B(t)})$, which means the end of proof. □

Thanks to above lemmas, we can see that if a function B is suitably small and nonincreasing and fulfills some differential inequality, then $\underline{w}$ becomes a subsolution of (2.4) in the intermediate region $(B(t), K\sqrt{B(t)})$.

First, we will consider the case $n = 1$. The largeness condition for m (see (1.9)) will lead to the following lemma.

Lemma 3.12 *Let $p \leq q$, $n = 1$, $\chi > 0$ and $\lambda \in (\frac{5-\sqrt{17}}{2}, 1]$, and put $\delta_\lambda := \frac{a_\lambda}{b_\lambda}$ with a_λ and b_λ as in* (3.1), *and let $m_c(p,q,\chi,\lambda,R)$ be such that*

$$m_c := \inf\left\{ m \;\middle|\; \exists\lambda \in \left(\frac{5-\sqrt{17}}{2}, 1\right];\ \frac{(1-\delta_\lambda)m\chi}{\sqrt{\frac{1+\delta_\lambda}{\lambda^2}+m^2}} - \left(\frac{m}{\omega_n}\cdot\frac{a_\lambda}{(a_\lambda+b_\lambda)R}\right)^{p-q} = 0 \right\}. \tag{3.54}$$

Then for all $m > m_c$ there exist $K > 0$, $\kappa_1 > 0$, and B_{01} such that $K\sqrt{B_{01}} < R^n$, and for some $T > 0$, if $B \in C^1([0,T))$ is a positive and nonincreasing function fulfilling (3.6) *as well as*

$$\begin{cases} B'(t) \geq -\kappa_1\sqrt{B(t)}, \\ B(0) \leq B_{01} \end{cases} \tag{3.55}$$

for all $t \in (0,T)$, then w_{in} as in (3.12) *satisfies*

$$(\mathcal{P}w_{\text{in}})(s,t) \leq 0$$

for all $t \in (0,T)$ and all $s \in (B(t), K\sqrt{B(t)})$.

Proof The proof is based on that of [2, Lemma 3.11]. Since $\delta_\lambda = \frac{a_\lambda}{b_\lambda}$ holds, we obtain $b_\lambda R = \frac{a_\lambda R}{\delta_\lambda}$ with a_λ and b_λ as in (3.1), which enables us to pick $K > 0$ such that

$$K \geq \sqrt{b_\lambda R} \tag{3.56}$$

and

$$\frac{a_\lambda R}{K^2} \leq \delta_\lambda \tag{3.57}$$

as well as

$$c_1 := \frac{(1-\delta_\lambda)m\chi}{\sqrt{\frac{1+\delta_\lambda}{\lambda^2}+m^2}} - \left(\frac{m}{\omega_n}\cdot\frac{a_\lambda}{K^2+a_\lambda R}\right)^{p-q}$$

$$= \frac{(1-\delta_\lambda)m\chi}{\sqrt{\frac{1+\delta_\lambda}{\lambda^2}+m^2}} - \sigma^{p-q} \tag{3.58}$$

is positive (by (3.54) and $m > m_c$) with σ as in (3.51). Furthermore, it is possible to fix $B_{01} \in (0,1)$ fulfilling $K\sqrt{B_{01}} < R$ and

$$B_{01} \le \frac{K^2}{4(a_\lambda + b_\lambda)^2} \tag{3.59}$$

as well as

$$\frac{\mu}{nA_T} \cdot \max\left\{ \frac{B_{01}}{\lambda}, 2K\sqrt{B_{01}} \right\} \le \delta_\lambda \tag{3.60}$$

with A_T as in (3.31), and put

$$\kappa_1 := \frac{\sigma^{q-1}c_1}{K}, \tag{3.61}$$

and let $T > 0$ and $B \in C^1([0,T))$ be positive and nonincreasing and such that (3.55) holds. Then, recalling $A' \le 0$ on $(0,T)$ (see Lemma 3.4), we infer from (3.27) that

$$\frac{B(t)}{A(t)\varphi'(\xi)} \cdot (\mathcal{P}w_{\mathrm{in}})(s,t) \le -\xi B'(t) + J_1(s,t) + J_2(s,t) \tag{3.62}$$

for all $t \in (0,T)$ and all $s \in (B(t), K\sqrt{B(t)})$ with $\xi = \frac{s}{B(t)}$ and J_1 and J_2 as in (3.28) and (3.29). Here, according to Lemma 3.8, we obtain from (3.48) that

$$J_1(s,t) \le \left(\frac{A(t)\varphi'(\xi)}{B(t)} \right)^{p-1} \tag{3.63}$$

for all $t \in (0,T)$ and all $s \in (B(t), K\sqrt{B(t)})$. Since (3.56), (3.57), (3.59) and (3.60) hold, a combination of Lemmas 3.9 and 3.10 with an argument similar to that in the proof of [2, Lemma 3.11] leads to that

$$\sqrt{1 + \big(A(t)\varphi(\xi) - \mu B(t)\xi\big)^2} \le \sqrt{\frac{1+\delta_\lambda}{\lambda^2 m^2} + 1} \cdot A(t)\varphi(\xi)$$

for all $t \in (0,T)$ and all $s \in (B(t), K\sqrt{B(t)})$. Therefore, we can see from (3.50) that

$$\begin{aligned} -J_2(s,t) &= \chi \cdot \frac{A(t)\varphi(\xi) - \mu B(t)\xi}{\sqrt{1 + (A(t)\varphi(\xi) - \mu B(t)\xi)^2}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)} \right)^{q-1} \\ &\ge \chi \cdot \frac{(1-\delta_\lambda)A(t)\varphi(\xi)}{\sqrt{\frac{1+\delta_\lambda}{\lambda^2 m^2} + 1} \cdot A(t)\varphi(\xi)} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)} \right)^{q-1} \\ &= \frac{(1-\delta_\lambda)m\chi}{\sqrt{\frac{1+\delta_\lambda}{\lambda^2} + m^2}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)} \right)^{q-1} \end{aligned} \tag{3.64}$$

for all $t \in (0, T)$ and all $s \in (B(t), K\sqrt{B(t)})$. From the relation $p \le q$, a combination of (3.52) and (3.61)–(3.64), along with the definition of c_1 (see (3.58)) implies that

$$
\begin{aligned}
&\frac{B(t)}{A(t)\varphi'(\xi)} \cdot (\mathcal{P}w_{\mathrm{in}})(s,t) \\
&\le -\xi B'(t) + \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{p-1} - \frac{(1-\delta_\lambda)m\chi}{\sqrt{\frac{1+\delta_\lambda}{\lambda^2} + m^2}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \\
&= \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \left\{ -\xi B'(t) \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{1-q} + \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{p-q} - \frac{(1-\delta_\lambda)m\chi}{\sqrt{\frac{1+\delta_\lambda}{\lambda^2} + m^2}} \right\} \\
&\le \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \left\{ -\xi B'(t)\sigma^{1-q} + \sigma^{p-q} - \frac{(1-\delta_\lambda)m\chi}{\sqrt{\frac{1+\delta_\lambda}{\lambda^2} + m^2}} \right\} \\
&= \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \sigma^{1-q}\left(-\xi B'(t) - \sigma^{q-1}c_1\right) \qquad (3.65)
\end{aligned}
$$

for all $t \in (0, T)$ and all $s \in (B(t), K\sqrt{B(t)})$. Recalling that $\xi \le \frac{K}{\sqrt{B(t)}}$ holds in the region, we infer from the definition of κ_1 (see (3.61)) that

$$
\begin{aligned}
-\xi B'(t) - \sigma^{q-1}c_1 &= \xi \cdot \left(-B'(t) - \frac{\sigma^{q-1}c_1}{\xi}\right) \\
&\le \xi \cdot \left(-B'(t) - \frac{\sigma^{q-1}c_1\sqrt{B(t)}}{K}\right) \\
&= \xi \cdot \left(-B'(t) - \kappa_1\sqrt{B(t)}\right) \\
&\le 0 \qquad (3.66)
\end{aligned}
$$

for all $t \in (0, T)$ and all $s \in (B(t), K\sqrt{B(t)})$. A combination of (3.65) and (3.66) leads to that we can attain the conclusion of the proof. □

Secondly, we will consider the case $n \ge 2$. Due to choosing suitably χ as in (1.12), we will infer from the condition for a function B that $\underline{w}$ becomes a subsolution of (2.4).

Lemma 3.13 *Let $p \le q$, $n \ge 2$, $m > 0$, $\chi > 0$ and $\lambda = \frac{1}{3}$. Then for all $\chi > (\frac{mn}{\omega_n R^n})^{p-q}$ there exist $K > 0$, $\kappa_n > 0$, and B_{0n} such that $K\sqrt{B_{0n}} < R^n$, and such that if $T > 0$ and $B \in C^1([0, T))$ is a positive and nonincreasing such that*

$$
\begin{cases}
B'(t) \ge -\kappa_n B^{1-\frac{1}{2n}}(t), \\
B(0) \le B_{0n}
\end{cases} \qquad (3.67)
$$

for all $t \in (0, T)$, then the function w_{in} defined in (3.12) satisfies

$$(\mathcal{P}w_{\mathrm{in}})(s,t) \le 0$$

for all $t \in (0, T)$ and all $s \in (B(t), K\sqrt{B(t)})$.

Proof The proof is based on that of [2, Lemma 3.12]. Thanks to $\lambda = \frac{1}{3}$ and $\chi > (\frac{mn}{\omega_n R^n})^{p-q}$, we have that $b_\lambda = 0$ and

$$\chi > \left(n \cdot \frac{m}{\omega_n} \cdot \frac{a_\lambda}{a_\lambda R^n}\right)^{p-q}$$

with a_λ as in (3.1). Then there exists some $K > 0$ fulfilling

$$\begin{aligned} \chi &> \left(n \cdot \frac{m}{\omega_n} \cdot \frac{a_\lambda}{K^2 + a_\lambda R^n}\right)^{p-q} \\ &= (n\sigma)^{p-q} \end{aligned} \tag{3.68}$$

with σ given by (3.51). Using (3.68), we can choose $\delta \in (0, 1)$ suitably small such that

$$c_1 := n^q \cdot \left\{ \frac{(1-\delta)\chi}{\sqrt{1+\delta}} - (n\sigma)^{p-q} \right\} \tag{3.69}$$

is positive. Finally, we take $B_{0n} \in (0, 1)$ such that $K\sqrt{B_{0n}} < R^n$ and

$$B_{0n} \le \frac{K^2}{4(a_\lambda + b_\lambda)^2} \tag{3.70}$$

as well as

$$\frac{\mu}{nA_T} \cdot \max\left\{ \frac{B_{0n}}{\lambda}, 2K\sqrt{B_{0n}} \right\} \le \delta \tag{3.71}$$

with A_T as in (3.31) and

$$\frac{\omega_n^2}{\lambda^2 m^2} \cdot \left(1 + \frac{a_\lambda R^n}{K^2}\right) \cdot K^{2-\frac{2}{n}} B_{0n}^{3-\frac{3}{n}} \le \delta, \tag{3.72}$$

and we put

$$\kappa_n := \sigma^{q-1} c_1 K^{-\frac{1}{n}}, \tag{3.73}$$

and suppose $T > 0$ and $B \in C^1([0, T))$ is positive and nonincreasing and such that (3.67) holds. Then, from (3.70), Lemma 3.4 implies that $A' \le 0$ on $(0, T)$. Recalling (3.27), we derive that

$$\frac{B(t)}{A(t)\varphi'(\xi)} \cdot (\mathcal{P} w_{\text{in}})(s, t) \le -\xi B'(t) + J_1(s, t) + J_2(s, t) \tag{3.74}$$

for all $t \in (0, T)$ and all $s \in (B(t), K\sqrt{B(t)})$ with $\xi = \frac{s}{B(t)}$, and J_1 and J_2 given by (3.28) and (3.29), respectively. From (3.70) and (3.71), using Lemma 3.10, we obtain that

$$A(t)\varphi(\xi) - \frac{\mu}{n} B(t)\xi \ge (1-\delta)A(t)\varphi(\xi) \tag{3.75}$$

for all $t \in (0, T)$ and all $s \in (B(t), K\sqrt{B(t)})$. Noticing (3.70), we can use Lemma 3.9 to show from (3.72) that

$$\frac{1}{A^2(t)B^{\frac{2}{n}-2}(t)\xi^{\frac{2}{n}-2}\varphi^2(\xi)} \le \frac{\omega_n^2}{\lambda^2 m^2} \cdot \left(1 + \frac{a_\lambda R^n}{K^2}\right) \cdot K^{2-\frac{2}{n}} B_{0n}^{3-\frac{3}{n}} \le \delta \tag{3.76}$$

for all $t \in (0, T)$ and all $s \in (B(t), K\sqrt{B(t)})$. Employing (3.75), (3.76) and the fact that $\delta < 1$, we can estimate that

$$\begin{aligned}
-J_2(s,t) &= n^q \chi \cdot \frac{A(t)\varphi(\xi) - \frac{\mu}{n}B(t)\xi}{\sqrt{1 + B^{\frac{2}{n}-2}(t)\xi^{\frac{2}{n}-2}(A(t)\varphi(\xi) - \frac{\mu}{n}B(t)\xi)^2}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \\
&\ge n^q \chi \cdot \frac{(1-\delta)A(t)\varphi(\xi)}{\sqrt{1 + B^{\frac{2}{n}-2}(t)\xi^{\frac{2}{n}-2}A^2(t)\varphi^2(\xi)}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \\
&\ge n^q \chi \cdot \frac{(1-\delta)A(t)\varphi(\xi)}{\sqrt{(\delta+1)B^{\frac{2}{n}-2}(t)\xi^{\frac{2}{n}-2}A^2(t)\varphi^2(\xi)}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \\
&= \frac{(1-\delta)n^q\chi}{\sqrt{\delta+1}} \cdot B^{1-\frac{1}{n}}(t)\xi^{1-\frac{1}{n}} \cdot \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1}
\end{aligned} \tag{3.77}$$

for all $t \in (0, T)$ and all $s \in (B(t), K\sqrt{B(t)})$. On the other hand, we use Lemma 3.8 to show that

$$J_1(s,t) \le n^p B^{1-\frac{1}{n}}(t)\xi^{1-\frac{1}{n}} \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{p-1} \tag{3.78}$$

for all $t \in (0, T)$ and all $s \in (B(t), K\sqrt{B(t)})$. Thanks to (3.74), (3.77) and (3.78), we derive that

$$\begin{aligned}
\frac{B(t)}{A(t)\varphi'(\xi)} \cdot (\mathcal{P}w_{\text{in}})(s,t) \le -\xi B'(t) + n^p B^{1-\frac{1}{n}}(t)\xi^{1-\frac{1}{n}} \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{p-1} \\
- \frac{(1-\delta)n^q\chi}{\sqrt{\delta+1}} \cdot B^{1-\frac{1}{n}}(t)\xi^{1-\frac{1}{n}} \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1}.
\end{aligned} \tag{3.79}$$

Since

$$\begin{aligned}
&-\xi B'(t) + n^p B^{1-\frac{1}{n}}(t)\xi^{1-\frac{1}{n}} \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{p-1} \\
&\quad - \frac{(1-\delta)n^q\chi}{\sqrt{\delta+1}} \cdot B^{1-\frac{1}{n}}(t)\xi^{1-\frac{1}{n}} \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \\
&= \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1} \left\{ -\xi B'(t) \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{1-q} \right. \\
&\quad \left. + \left(n^p \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{p-q} - \frac{(1-\delta)n^q\chi}{\sqrt{\delta+1}}\right) B^{1-\frac{1}{n}}(t)\xi^{1-\frac{1}{n}} \right\}
\end{aligned}$$

holds, a combination of (3.69), (3.79) and Lemma 3.11, along with the relation $p \le q$ yields that

$$\begin{aligned}
&\frac{B(t)}{A(t)\varphi'(\xi)} \cdot (\mathcal{P}w_{\text{in}})(s,t) \\
&\le \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1}\left\{-\xi B'(t)\sigma^{1-q} + \left(n^p\sigma^{p-q} - \frac{(1-\delta)n^q\chi}{\sqrt{\delta+1}}\right)B^{1-\frac{1}{n}}(t)\xi^{1-\frac{1}{n}}\right\} \\
&= \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1}\sigma^{1-q}\left\{-\xi B'(t) - n^q\sigma^{q-1}\left(\frac{(1-\delta)\chi}{\sqrt{\delta+1}} - (n\sigma)^{p-q}\right)B^{1-\frac{1}{n}}(t)\xi^{1-\frac{1}{n}}\right\} \\
&= \left(\frac{A(t)\varphi'(\xi)}{B(t)}\right)^{q-1}\sigma^{1-q}\left(-\xi B'(t) - \sigma^{q-1}c_1B^{1-\frac{1}{n}}(t)\xi^{1-\frac{1}{n}}\right) \qquad (3.80)
\end{aligned}$$

for all $t \in (0,T)$ and all $s \in (B(t), K\sqrt{B(t)})$. Finally, according to that

$$\xi < \frac{K}{\sqrt{B(t)}}$$

which means that $s < K\sqrt{B(t)}$, we infer from (3.67) and (3.73) that

$$\begin{aligned}
-\xi B'(t) - \sigma^{q-1}c_1B^{1-\frac{1}{n}}(t)\xi^{1-\frac{1}{n}} &= \xi \cdot \left(-B'(t) - \sigma^{q-1}c_1B^{1-\frac{1}{n}}(t)\xi^{-\frac{1}{n}}\right) \\
&\le \xi \cdot \left(-B'(t) - \sigma^{q-1}c_1B^{1-\frac{1}{n}}(t)\cdot K^{-\frac{1}{n}}B^{\frac{1}{2n}}(t)\right) \\
&= \xi \cdot \left(-B'(t) - \kappa_n B^{1-\frac{1}{2n}}(t)\right) \\
&\le 0 \qquad (3.81)
\end{aligned}$$

for all $t \in (0,T)$ and all $s \in (B(t), K\sqrt{B(t)})$. Thus (3.80) and (3.81) lead to the end of the proof. □

4 Blow-up. Proof of Theorem 1.1

In this section, by virtue of a combination of Lemmas 3.3, 3.6, 3.7, 3.12 and 3.13, we can show our main purpose such that there is an initial data satisfying that the corresponding solution blows up by using the comparison argument from [2, Lemma 5.1].

Proof of Theorem 1.1 If $n = 1$, thanks to (1.9), for all $\chi > 1$ ($\chi > 0$ when $q > p$) and some $\lambda \in (\frac{5-\sqrt{17}}{2}, 1)$, Lemma 3.12 entails that there exist $K > 0$, $\kappa_1 > 0$ and $B_{01} \in (0,1)$ with properties listed there. On the other hand, if $n \ge 2$, $m > 0$ and $\lambda = \frac{1}{3}$, then from the condition (1.12) we can use Lemma 3.13 to see that for all $\chi > 0$ there exist $K > 1$, $\kappa_n > 0$ and $B_{0n} \in (0,1)$ with properties noted there. Moreover, we introduce $\kappa \in (0, \kappa_n]$ given by

$$\kappa \le \frac{a_\lambda^{q-1}(nm)^q\chi K}{2(a_\lambda + b_\lambda)(K^2 + a_\lambda R^n)^{q-1}\omega_n^q R^n\sqrt{1 + K^{\frac{2}{n}-2}\cdot\frac{m^2}{\omega_n^2}}} \qquad (4.1)$$

and

$$\kappa \le \frac{d}{\sqrt{d^2+1}} \cdot \left(\frac{2\lambda A_T}{e^d}\right)^{p-1} \qquad (4.2)$$

as well as

$$\kappa \le \frac{n^q \chi}{\sqrt{2}} \cdot \left(\frac{2\lambda A_T}{e^d}\right)^{q-1} \tag{4.3}$$

with d and A_T as in (3.3) and (3.31), respectively, and we take $B_0 \in (0, B_{0n}]$ satisfying (3.14), (3.34) and (3.42). Here we define

$$B(t) := \left(B_0^{\frac{1}{2n}} - \frac{\kappa}{2n} t\right)^{2n}$$

for $t \in (0, T)$ with

$$T := \frac{2n}{\kappa} \cdot B_0^{\frac{1}{2n}}.$$

Then, $B \in C^1([0, T))$ is the solution of the following initial-value problem:

$$\begin{cases} B'(t) = -\kappa B^{1-\frac{1}{2n}}(t), \\ B(0) = B_0 \end{cases} \tag{4.4}$$

for all $t \in (0, T)$. According to Lemma 3.2, putting

$$\underline{w}(s,t) := \begin{cases} w_{\text{in}}(s,t) & \text{if } t \in [0,T) \text{ and } s \in [0, K\sqrt{B(t)}], \\ w_{\text{out}}(s,t) & \text{if } t \in [0,T) \text{ and } s \in (K\sqrt{B(t)}, R^n] \end{cases}$$

with functions w_{in} and w_{out} as in (3.12) and (3.13), respectively, we see that the function $\underline{w}$ is well-defined and satisfies

$$\underline{w} \in C^1\big([0, R^n] \times [0, T)\big)$$

and $\underline{w}(\cdot, t) \in C^2([0, R^n] \setminus \{B(t), K\sqrt{B(t)}\})$ for all $t \in [0, T)$. Moreover, thanks to (4.1), (4.2), (4.3), (4.4) and the fact that $B(t) \le B_0 < 1$, we can use Lemmas 3.3, 3.6, 3.7, 3.12 and 3.13 to lead to functions w_{out} and w_{in} fulfilling

$$(\mathcal{P} w_{\text{out}})(s,t) \le 0 \tag{4.5}$$

for all $t \in (0, T)$ and all $s \in (K\sqrt{B(t)}, R^n)$ as well as

$$(\mathcal{P} w_{\text{in}})(s,t) \le 0 \tag{4.6}$$

for all $t \in (0, T)$ and all $s \in (0, B(t)) \cup (B(t), K\sqrt{B(t)})$. Therefore, we obtain that

$$(\mathcal{P}\underline{w})(s,t) \le 0$$

for all $t \in (0, T)$ and all $s \in (0, R^n) \setminus \{B(t), K\sqrt{B(t)}\}$ since (4.5) and (4.6) hold. Here we assume that u_0 satisfies (1.2) and also

$$\int_{B_r(0)} u_0(x)\,dx \ge M_m(r) := \omega_n \underline{w}\big(r^n, 0\big)$$

for all $r \in [0, R]$. Then, we can see that

$$w(s, 0) \ge \underline{w}(s, 0)$$

for all $s \in (0, R^n)$ with the solution w of (2.4) defined by (2.2). Moreover, putting $\widetilde{T} := \min\{T_{\max}, T\}$, we derive that

$$w(0,t) = \underline{w}(0,t) = 0 \quad \text{and} \quad w\big(R^n, t\big) = \underline{w}\big(R^n, t\big) = \frac{m}{\omega_n}$$

for all $t \in (0, \widetilde{T})$. In order to use the comparison principle stated in [2, Lemma 5.1] we write $\alpha := 2 - \frac{2}{n} \geq 0$ and let

$$\phi(s,t,y_0,y_1,y_2) := n^{p+1} \cdot \frac{s^\alpha {y_1}^p y_2}{\sqrt{y_1^2 + n^2 s^\alpha y_2^2}} + n^q \chi \cdot \frac{(y_0 - \frac{\mu}{n}s){y_1}^q}{\sqrt{1 + s^{-\alpha}(y_0 - \frac{\mu}{n}s)^2}}$$

for $(s,t,y_0,y_1,y_2) \in G := (0,R^n) \times (0,\infty) \times \mathbb{R} \times (0,\infty) \times \mathbb{R}$. Then $\phi \in C^1(G)$ with

$$\frac{\partial \phi}{\partial y_2}(s,t,y_0,y_1,y_2) = \frac{n^{p+1} s^\alpha y_1^{p+2}}{\sqrt{y_1^2 + n^2 s^\alpha y_2^2}^{\,3}} \geq 0 \tag{4.7}$$

and

$$\frac{\partial \phi}{\partial y_1}(s,t,y_0,y_1,y_2) = n^{p+1} \cdot \frac{p s^\alpha y_1^{p-1} y_2 (y_1^2 + n^2 s^\alpha y_2^2) - s^\alpha y_1^{p+1} y_2}{\sqrt{y_1^2 + n^2 s^\alpha y_2^2}^{\,3}}$$
$$+ n^q \chi \cdot \frac{q(y_0 - \frac{\mu}{n}s) y_1^{q-1}}{\sqrt{1 + s^{-\alpha}(y_0 - \frac{\mu}{n}s)^2}}$$

as well as

$$\frac{\partial \phi}{\partial y_0}(s,t,y_0,y_1,y_2) = n^q \chi \frac{y_1^q}{\sqrt{1 + s^{-\alpha}(y_0 - \frac{\mu}{n}s)^2}^{\,3}}$$

for all $(s,t,y_0,y_1,y_2) \in G$. Here if $S \subset (0,R^n) \times \mathbb{R} \times (0,\infty) \times \mathbb{R}$ is compact, then there exist $a_1, a_2 \in (0,\infty)$ and $b_1, b_2 \in \mathbb{R}$ satisfying that

$$S \subset \big(0, R^n\big) \times \mathbb{R} \times (a_1, a_2) \times (b_1, b_2).$$

Therefore, putting $b := \max\{|b_1|, |b_2|\}$ and $a := \max\{a_1^{p-2}, a_2^{p-2}\}$, we establish that for any $t \in (0, \widetilde{T})$,

$$\left|\frac{\partial \phi}{\partial y_1}(s,t,y_0,y_1,y_2)\right| \leq \left| n^{p+1} \cdot \frac{p s^\alpha y_1^{p-1} y_2 (y_1^2 + n^2 s^\alpha y_2^2) - s^\alpha y_1^{p+1} y_2}{\sqrt{y_1^2 + n^2 s^\alpha y_2^2}^{\,3}} \right|$$
$$+ \left| n^q \chi \cdot \frac{q(y_0 - \frac{\mu}{n}s) y_1^{q-1}}{\sqrt{1 + s^{-\alpha}(y_0 - \frac{\mu}{n}s)^2}} \right|$$

$$\le \left|n^{p+1}\cdot\frac{(p-1)s^{\alpha}y_1^{p+1}y_2}{\sqrt{y_1^2}^{\,3}}\right|+\left|n^{p+3}\cdot\frac{ps^{2\alpha}y_1^{p-1}y_2^3}{\sqrt{n^2s^{\alpha}y_2^2}^{\,3}}\right|$$

$$+\left|n^q\chi q s^{\frac{\alpha}{2}}y_1^{q-1}\cdot\frac{\sqrt{s^{-\alpha}(y_0-\frac{\mu}{n}s)^2}}{\sqrt{1+s^{-\alpha}(y_0-\frac{\mu}{n}s)^2}}\right|$$

$$\le\left|n^{p+1}(p-1)s^{\alpha}y_1^{p-2}y_2\right|+\left|n^p ps^{\alpha}y_1^{p-1}\right|+\left|n^q\chi q s^{\frac{\alpha}{2}}y_1^{q-1}\right|$$

$$\le\left|n^{p+1}(p-1)R^{n\alpha}a^{p-2}b\right|+\left|n^p pR^{n\alpha}a_2^{p-1}\right|+\left|n^q\chi qR^{\frac{n\alpha}{2}}a_2^{q-1}\right| \tag{4.8}$$

for all $(s, y_0, y_1, y_2) \in S$, and thus $|\frac{\partial\phi}{\partial y_1}(\cdot, t, \cdot, \cdot, \cdot)| \in L^\infty_{\mathrm{loc}}((0, R^n)\times\mathbb{R}\times(0,\infty)\times\mathbb{R})$, and for all $T_0 \in (0, \widetilde{T})$ and all $\Lambda > 0$ we derive that

$$\left|\frac{\partial\phi}{\partial y_0}(s,t,y_0,y_1,y_2)\right|\le n^q\chi y_1^q\le n^q\chi\Lambda^q \tag{4.9}$$

for all $(s, t, y_0, y_1, y_2) \in G$ with $t \in (0, T_0)$ and $y_1 \in (0, \Lambda)$. Thanks to (4.7), (4.8) and (4.9), we can use [2, Lemma 5.1] to yield that

$$w(s,t)\ge \underline{w}(s,t)$$

for all $s \in [0, R^n]$ and all $t \in [0, \widetilde{T})$. Since $w(0,t) = \underline{w}(0,t) = 0$ for all $t \in [0, \widetilde{T})$, the mean value theorem implies that for each $t \in [0, \widetilde{T})$ there is some $\theta(t) \in (0, R^n)$ with the property that

$$w_s\big(\theta(t),t\big)=\frac{w(B(t),t)}{B(t)}\ge\frac{\underline{w}(B(t),t)}{B(t)}=\frac{A(t)\varphi(1)}{B(t)}=\lambda\cdot\frac{A(t)}{B(t)}$$

for all $t \in [0, \widetilde{T})$. Noting that $u(r,t) = nw_s(r^n, t)$ for all $r \in (0, R)$ and all $t \in (0, \widetilde{T})$, we can show that

$$\sup_{r\in(0,R)}u(r,t)\ge w_s\big(\theta(t),t\big)=\lambda\cdot\frac{A(t)}{B(t)}$$

for all $t \in (0, \widetilde{T})$. Thanks to the facts that

$$B(t)\searrow 0\quad\text{and}\quad A(t)\to\frac{m}{\omega_n}\cdot\frac{K^2}{K^2+a_\lambda R^n}$$

as $t \nearrow T$ by (3.7), using a consequence of the extensibility criterion (2.1), we can see that (1.11) holds with $T^* = T_{\max} \le T < \infty$, which enables us to attain Theorem 1.1. □

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