Acta Appl Math (2020) 166:73–83
https://doi.org/10.1007/s10440-019-00255-3

A Blow-up Criterion for the Density-Dependent Navier–Stokes–Korteweg Equations in Dimension Two

Huanyuan Li[1]

Received: 20 July 2018 / Accepted: 8 April 2019 / Published online: 17 April 2019

Abstract This paper proves a blow-up criterion for the strong solutions with vacuum to the density-dependent Navier–Stokes–Korteweg equations over a bounded smooth domain in $\mathbb{R}^2$, which only in terms of the density.

Keywords Navier–Stokes–Korteweg · Blow-up criterion · Vacuum

1 Introduction and Main Result

The motion of a general viscous capillary fluid is governed by the nonhomogeneous incompressible Navier–Stokes–Korteweg equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\big(2\mu(\rho) d\big) + \nabla P + \operatorname{div}\big(\kappa(\rho)\nabla\rho \otimes \nabla\rho\big) = 0, \\ \operatorname{div} u = 0, \end{cases} \tag{1.1}$$

in $\Omega \times (0, \infty)$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^2$. Here ρ, u and P denote the density, velocity field and pressure of the fluid, respectively.

$$d = \frac{1}{2}\big[\nabla u + (\nabla u)^T\big]$$

is the deformation tensor, where ∇u is the gradient matrix $(\partial u_i / \partial x_j)$ and $(\nabla u)^T$ is its transpose. $\kappa = \kappa(\rho)$ and $\mu = \mu(\rho)$ stand for the capillary and viscosity coefficients of the fluid respectively, and are both functions of density ρ. In this paper, they are assumed to satisfy

$$\kappa, \mu \in C^1[0, \infty), \quad \text{and} \quad \kappa \geq 0,\ \mu \geq \underline{\mu} > 0 \text{ on } [0, \infty) \tag{1.2}$$

for some positive constant $\underline{\mu}$.

✉ H. Li
lhytu@ms.u-tokyo.ac.jp; lihuanyuan1111@163.com

1 Graduate School of Mathematical Sciences, University of Tokyo, Komaba 3-8-1, Meguro-Ku, Tokyo, 153-8914, Japan

We focus on the system (1.1)–(1.2) with the initial and boundary conditions:

$$(\rho, u)|_{t=0} = (\rho_0, u_0) \quad \text{in } \Omega,\ u = 0,\ \text{on } \partial\Omega \times [0, T). \tag{1.3}$$

The Navier–Stokes–Korteweg system is usually used to model the dynamics of a fluid endowed with internal capillarity (in the diffuse interface setting), and in general, the capillary tensor is written as

$$\mathrm{div} K = \nabla\left(\rho\kappa(\rho)\triangle\rho + \frac{1}{2}\big(\kappa(\rho) + \rho\kappa'(\rho)\big)|\nabla\rho|^2\right) - \mathrm{div}\big(\kappa(\rho)\nabla\rho \otimes \nabla\rho\big). \tag{1.4}$$

In the case of the nonhomogeneous incompressible Korteweg, the first term in the capillary tensor (1.4) can be absorbed by the pressure term due to the incompressibility condition, thus we directly write the capillary term as a general divergence term (see the Remark 1.1 in [1]), it is exactly the equations (1.1). See more physical background and mathematical modelling in [10, 11].

Now we recall some mathematical results on the nonhomogeneous fluid mechanics. When $\kappa \equiv 0$, the system (1.1)–(1.2) reduces to the nonhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity. Cho and Kim [2] proved the local existence of unique strong solution for all initial data satisfying a compatibility condition. And later Huang and Wang [9] proved the strong solution exists globally in time when the initial gradient of the velocity is suitably small in some Sobolev space. For the related progress, see the references [7–9] and therein.

Let us come back to the fluids with capillary effect, that is, $\kappa(\rho)$ depends on the density ρ. The Navier–Stokes–Korteweg equations are widely studied by many mathematicians since of its physical importance and mathematical complexity, especially a great of efforts have been devoted to the mathematical theory for compressible capillary fluids, see the references [4–6] and therein. To our best knowledge, there are few results on the system (1.1). As far as I know, the first local existence of unique strong solution was obtained by Tan and Wang [12] when the capillary coefficients κ is a nonnegative constant. And very recently, Wang [13] extended their result to the case when $\kappa(\rho)$ is a C^1 function of the density.

The purpose of this paper is to prove a blow-up criterion for the strong solutions to the problem (1.1)–(1.3). First we give the definition of strong solution to the initial and boundary problem (1.1)–(1.3) as follows (two dimensional version).

Definition 1.1 (Strong solution) A pair of functions $(\rho \geq 0, u, P)$ is called a strong solution to the problem (1.1)–(1.3) in $\Omega \times (0, T)$, if for some $q_0 \in (2, \infty)$,

$$\begin{aligned} &\rho \in C\big([0, T]; W^{2,q_0}\big), \quad u \in C\big([0, T]; H_0^1 \cap H^2\big) \cap L^2\big(0, T; W^{2,q_0}\big), \\ &\rho_t \in C\big([0, T]; W^{1,q_0}\big), \quad \nabla P \in C\big([0, T]; L^2\big) \cap L^2\big(0, T; L^{q_0}\big), \quad u_t \in L^2\big(0, T; H_0^1\big), \end{aligned} \tag{1.5}$$

and (ρ, u, P) satisfies (1.1) a.e. in $\Omega \times (0, T)$.

In the case when the initial data may vanish in an open subset of Ω, that is, the initial vacuum is allowed, the following local well-posedness of strong solution to (1.1)–(1.3) was obtained by Wang [13] in a three dimensional bounded domain. In fact, the local existence of unique strong solution with vacuum to the system (1.1) in a two dimensional bounded domain can be established in the same manner as Wang [13] and Cho and Kim [2], also see the Remark 2 in Tan and Wang [12].

Theorem 1.2 *Assume that the initial data* (ρ_0, u_0) *satisfies the regularity condition*

$$0 \le \rho_0 \in W^{2,q}, \quad 2 < q < \infty, \; u_0 \in H^1_{0,\sigma} \cap H^2, \tag{1.6}$$

and the compatibility condition

$$-\mathrm{div}\big(\mu(\rho_0)\big(\nabla u_0 + (\nabla u_0)^T\big)\big) + \nabla P_0 + \mathrm{div}\big(\kappa(\rho_0)\nabla\rho_0 \otimes \nabla\rho_0\big) = \rho_0^{1/2} g, \tag{1.7}$$

for some $(P_0, g) \in H^1 \times L^2$. *Then there exist a small time* T *and a unique strong solution* (ρ, u, P) *to the initial boundary value problem* (1.1)–(1.3).

Motivated by the work of Huang and Wang [8], which proved a new type blow-up criterion for the 2D nonhomogeneous incompressible Navier–Stokes flow only involving the density. The main purpose is to derive a similar blow-up criterion for the nonhomogeneous Navier–Stokes–Korteweg equations with density-dependent viscosity and capillary coefficients. More precisely, our main result can be stated as follows.

Theorem 1.3 *Assume that the initial data* (ρ_0, u_0) *satisfies the regularity condition* (1.6) *and the compatibility condition* (1.7), *as in Theorem* 1.2. *Let* (ρ, u, P) *be a strong solution of the problem* (1.1)–(1.3) *satisfying* (1.5). *If* $0 < T^* < \infty$ *is the maximal time of existence, then*

$$\lim_{T \to T*} \|\nabla\rho\|_{L^\infty(0,T;W^{1,q})} = \infty. \tag{1.8}$$

Remark 1 It is still unknown that if we can extend the local strong solution to a global one for any arbitrary large initial data when the viscosity and capillary coefficients are constants, since our blow-up criterion involves the gradient of density but not the gradient of viscosity or capillary. We will consider the problem whether we can replace the density with viscosity or capillary in our blow-up criterion in the future work.

The proof of Theorem 1.3 is based on the contradiction argument. In view of the local existence result, to prove Theorem 1.3, it suffices to verify that (ρ, u) satisfy (1.6) and (1.7) at the time T^* under the assumption of the left hand side of (1.8) is finite.

The remainder of this paper is arranged as follows. In Sect. 2, we give some auxiliary lemmas which is useful in our later analysis. The proof of Theorem 1.3 will be done by combining the contradiction argument with the estimates derived in Sect. 3.

2 Preliminaries

2.1 Notations and General Inequalities

Ω is a smooth bounded domain in $\mathbb{R}^2$. For notations simplicity below, we omit the integration domain Ω. And for $1 \le r \le \infty$ and $k \in \mathbb{N}$, the Sobolev spaces are defined in a standard way,

$$L^r = L^r(\Omega), \qquad W^{k,r} = \big\{f \in L^r : \nabla^k f \in L^r\big\},$$

$$H^k = W^{k,2}, \qquad C^\infty_{0,\sigma} = \big\{f \in \big(C^\infty_0\big)^3 : \mathrm{div} f = 0\big\}.$$

$$H^1_0 = \overline{C^\infty_0}, \qquad H^1_{0,\sigma} = \overline{C^\infty_{0,\sigma}}, \text{ closure in the norm of } H^1.$$

The following Ladyzhenskaya inequality in 2D case will be often used.

$$\|u\|_{L^4}^2 \le C\|u\|_{L^2}\|\nabla u\|_{L^2}. \tag{2.1}$$

However, to deal with a nonhomogeneous problem with vacuum, some interpolation inequality for u with degenerate weight like $\sqrt{\rho}$ is required. We look for a similar estimate for $\sqrt{\rho}u$ as in (2.1). This technique can be found in the paper of Desjardin [3].

Lemma 2.1 *Assume that* $0 \le \rho \le \bar{\rho}$, $u \in H_0^1$; *then*

$$\|\sqrt{\rho}u\|_{L^4}^2 \le C\big(1+\|\rho u\|_{L^2}\big)\|\nabla u\|_{L^2}\sqrt{\log\big(2+\|\nabla u\|_{L^2}^2\big)} \tag{2.2}$$

where C *is a positive constant depending only on* $\bar{\rho}$ *and the domain* Ω.

2.2 Higher Order Estimates on *u*

High-order a priori estimates of velocity field u rely on the following regularity results for the stationary density-dependent Stokes equations.

Lemma 2.2 *Assume that* $\rho \in W^{2,q}$, $2 < q < \infty$, *and* $0 \le \rho \le \bar{\rho}$. *Let* $(u, P) \in H_{0,\sigma}^1 \times L^2$ *be the unique weak solution to the boundary value problem*

$$-\operatorname{div}\big(2\mu(\rho)d\big) + \nabla P = F, \quad \operatorname{div} u = 0 \text{ in } \Omega, \quad \text{and} \quad \int P dx = 0, \tag{2.3}$$

where $d = \frac{1}{2}[\nabla u + (\nabla u)^T]$ *and*

$$\mu \in C^1[0,\infty), \quad \underline{\mu} \le \mu(\rho) \le \bar{\mu} \text{ on } [0,\bar{\rho}].$$

Then we have the following regularity results:

(1) *If* $F \in L^2$, *then* $(u, P) \in H^2 \times H^1$ *and*

$$\|u\|_{H^2} + \|P\|_{H^1} \le C\big(1+\|\nabla\rho\|_{L^\infty}\big)\|F\|_{L^2}, \tag{2.4}$$

(2) *If* $F \in L^r$ *for some* $r \in (2,\infty)$, *then* $(u, P) \in W^{2,r} \times W^{1,r}$ *and*

$$\|u\|_{W^{2,r}} + \|P\|_{W^{1,r}} \le C\big(1+\|\nabla\rho\|_{L^\infty}\big)\|F\|_{L^r}. \tag{2.5}$$

The proof of Lemma 2.2 has been given by Wang [13]. And refer to Lemma 2.1 in his paper.

3 Proof of Theorem 1.3

Let (ρ, u, P) be a strong solution to the initial and boundary value problem (1.1)–(1.3) as derived in Theorem 1.2. Then it follows from the standard energy estimate that

Lemma 3.1 *For any* $0 < T < T^*$, *it holds that for any* $p \in [1,\infty]$,

$$\sup_{0\le t\le T}\Big(\|\rho\|_{L^p} + \big\|\sqrt{\rho}u\big\|_{L^2}^2 + \big\|\sqrt{\kappa(\rho)}\nabla\rho\big\|_{L^2}^2 + \iint_0^T |\nabla u|^2 dxds \le C. \tag{3.1}$$

As mentioned in the Sect. 1, the main theorem will be proved by using a contradiction argument. Denote $0 < T^* < \infty$ the maximal existence time for the strong solution to the initial and boundary value problem (1.1)–(1.3). Suppose that (1.8) were false, that is

$$M_0 := \lim_{T \to T^*} \|\nabla \rho\|_{L^\infty(0,T;W^{1,q})} < \infty. \tag{3.2}$$

Under the condition (3.2), one will extend the existence time of the strong solutions to (1.1)–(1.3) beyond T^*, which contradicts the definition of maximum of T^*.

The following estimate can be derived quickly from the Lemma 2.2, which is used later.

Lemma 3.2 *Under the assumption* (3.2), *it holds for all* $0 < T < T^*$,

$$\|\nabla u\|_{H^1} \le C\|\rho u_t\|_{L^2} + C\|\rho u\|_{L^4}^2 \|\nabla u\|_{L^2} + C, \tag{3.3}$$

and consequently by Sobolev embedding,

$$\|\nabla u\|_{H^1} \le C\|\rho u_t\|_{L^2} + C\|\nabla u\|_{L^2}^3 + C. \tag{3.4}$$

Proof According to the Lemma 2.2 and the Gagliardo–Nirenberg inequality,

$$\begin{aligned}
\|\nabla u\|_{H^1} &\le C\big(1 + \|\nabla \rho\|_{L^\infty}\big)\big(\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} + 1\big) \\
&\le C\|\rho u_t\|_{L^2} + C\|\rho u\|_{L^4} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{H^1}^{\frac{1}{2}} + C \\
&\le C\|\rho u_t\|_{L^2} + C\|\rho u\|_{L^4}^2 \|\nabla u\|_{L^2} + C + \frac{1}{2}\|\nabla u\|_{H^1},
\end{aligned}$$

which complete the proof of (3.3). □

The key step is to derive the L^2-norm of the first order spatial derivatives of u under the assumption of initial data and (3.2).

Lemma 3.3 *Under the condition* (3.2), *it holds that for any* $0 < T < T^*$,

$$\sup_{0 \le t \le T} \|\nabla u\|_{L^2}^2 + \int_0^T \big\|\sqrt{\rho} u_t\big\|_{L^2}^2 dt \le C. \tag{3.5}$$

Proof Multiplying the momentum equations $(1.1)_2$ by u_t, and integrating the resulting equations over Ω, we have

$$\begin{aligned}
&\int \rho |u_t|^2 dx + \frac{d}{dt}\int \mu(\rho)|d|^2 dx \\
&= -\int (\rho u \cdot \nabla u) \cdot u_t dx - \int u \cdot \nabla \mu(\rho)|d|^2 dx + \int \kappa(\rho)\nabla\rho \otimes \nabla\rho : \nabla u_t dx \\
&= \frac{d}{dt}\int \kappa(\rho)\nabla\rho \otimes \nabla\rho : \nabla u dx + \int \kappa'(\rho)(u \cdot \nabla\rho)\nabla\rho \otimes \nabla\rho : \nabla u dx \\
&\quad + 2\int \kappa(\rho)\nabla(u \cdot \nabla\rho) \otimes \nabla\rho : \nabla u dx - \int (\rho u \cdot \nabla u) \cdot u_t dx - \int u \cdot \nabla\mu(\rho)|d|^2 dx \\
&= \frac{d}{dt}\int \kappa(\rho)\nabla\rho \otimes \nabla\rho : \nabla u dx + \sum_{k=1}^4 I_k.
\end{aligned} \tag{3.6}$$

Now let us estimate these terms one by one, by use of the Poincaré inequality, we get

$$\begin{aligned} I_1 &= \int \kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx \\ &\le \left\|\kappa'(\rho)\right\|_{L^\infty}\|\nabla\rho\|_{L^\infty}^3\|u\|_{L^2}\|\nabla u\|_{L^2} \\ &\le C\|\nabla u\|_{L^2}^2. \end{aligned} \tag{3.7}$$

Similarly, dividing I_2 into two parts,

$$\begin{aligned} I_2 &= \int \kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho : \nabla u dx \\ &\le \left\|\kappa(\rho)\right\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\|\nabla^2\rho\|_{L^q}\|u\|_{L^{q^*}}\|\nabla u\|_{L^2} \\ &\quad + \left\|\kappa(\rho)\right\|_{L^\infty}\|\nabla\rho\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 \\ &\le C\|\nabla u\|_{L^2}^2, \end{aligned} \tag{3.8}$$

here $\frac{1}{q}+\frac{1}{q^*}=\frac{1}{2}$. and $q^*>2$. For the term I_3, using Cauchy-Schwarz inequality to get

$$\begin{aligned} I_3 &= \int \rho u_t\cdot(u\cdot\nabla u)dx \\ &\le \frac{1}{8}\left\|\sqrt{\rho}u_t\right\|_{L^2}^2 + C\left\|\sqrt{\rho}u\right\|_{L^4}^2\|\nabla u\|_{L^4}^2 \\ &\le \frac{1}{8}\left\|\sqrt{\rho}u_t\right\|_{L^2}^2 + C\left\|\sqrt{\rho}u\right\|_{L^4}^2\|\nabla u\|_{L^2}\|\nabla u\|_{H^1} \\ &\le \frac{1}{4}\left\|\sqrt{\rho}u_t\right\|_{L^2}^2 + C\left\|\sqrt{\rho}u\right\|_{L^4}^4\|\nabla u\|_{L^2}^2, \end{aligned} \tag{3.9}$$

and finally

$$\begin{aligned} I_4 &= \int u\cdot\nabla\mu(\rho)|d|^2 dx \\ &\le \left\|\mu'(\rho)\right\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\|u\|_{L^2}\|\nabla u\|_{L^4}^2 \\ &\le C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{H^1} \\ &\le C\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1} \\ &\le C\|\nabla u\|_{L^2}^2\|\rho u_t\|_{L^2} + C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2}^3 + C\|\nabla u\|_{L^2}^2 \\ &\le \frac{1}{4}\left\|\sqrt{\rho}u_t\right\|_{L^2}^2 + C\|\rho u\|_{L^4}^4\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2. \end{aligned} \tag{3.10}$$

Note that Lemma 2.1 tells us that

$$\begin{aligned} \left\|\sqrt{\rho}u\right\|_{L^4}^4 &\le C\big(1+\|\rho u\|_{L^2}^2\big)\|\nabla u\|_{L^2}^2\cdot\log\big(2+\|\nabla u\|_{L^2}^2\big) \\ &\le C\|\nabla u\|_{L^2}^2\cdot\log\big(2+\|\nabla u\|_{L^2}^2\big). \end{aligned} \tag{3.11}$$

Insert the estimates (3.7)–(3.10) into (3.6) to obtain

$$\frac{1}{2}\int \rho|u_t|^2dx + \frac{d}{dt}\int \big(\mu(\rho)|d|^2 - \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u\big)dx$$
$$\leq C\|\nabla u\|_{L^2}^2\big(1+\|\nabla u\|_{L^2}^2\big)\big(1+\log\big(2+\|\nabla u\|_{L^2}^2\big)\big) \quad (3.12)$$

and we know that

$$\frac{3}{4}\underline{\mu}\|\nabla u\|_{L^2}^2 - C_0 \leq \int \big(\mu(\rho)|d|^2 - \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u\big)dx \leq \frac{5}{4}\underline{\mu}\|\nabla u\|_{L^2}^2 + C_0, \quad (3.13)$$

owing to the following estimate

$$\int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx \leq \big\|\sqrt{\kappa(\rho)}\nabla\rho\big\|_{L^\infty}\big\|\sqrt{\kappa(\rho)}\nabla\rho\big\|_{L^2}\|\nabla u\|_{L^2}$$
$$\leq \frac{1}{4}\underline{\mu}\|\nabla u\|_{L^2}^2 + C\big\|\sqrt{\kappa(\rho)}\nabla\rho\big\|_{L^\infty}^2\big\|\sqrt{\kappa(\rho)}\nabla\rho\big\|_{L^2}^2$$
$$\leq \frac{1}{4}\underline{\mu}\|\nabla u\|_{L^2}^2 + C_0.$$

Taking this into account, we can conclude from (3.12) and the logarithmic type Gronwall inequality that (3.5) holds for all $0 \leq T < T^*$. Therefore we complete the proof of Lemma 3.3. □

Before we prove the boundedness of $\|\sqrt{\rho}u_t\|_{L^2}$, we insert the following estimate on the L^∞-norm of u.

Lemma 3.4 *Under the condition* (3.2), *it holds that for any* $0 < T < T^*$,

$$\sup_{0\leq t\leq T}\big(\|u\|_{L^2(0,T;L^\infty)} + \|u\|_{L^4(0,T;L^\infty)}\big) \leq C. \quad (3.14)$$

Proof By the Gagliardo–Nirenberg inequality and Lemma 3.2, we have

$$\int_0^T \|u\|_{L^\infty}^4 dt \leq C\int_0^T \|u\|_{L^2}^2\|\nabla u\|_{H^1}^2 dt$$
$$\leq C\int_0^T \big(\|\nabla u\|_{L^2}^2\|\rho u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^8 + \|\nabla u\|_{L^2}^2\big)dt, \quad (3.15)$$

which completes the proof of (3.14), owing to the Lemma 3.3. □

Now we can give the proof of the boundedness of $\|\sqrt{\rho}u_t\|_{L^2}$, by use of the compatibility condition (1.7) on the initial data.

Lemma 3.5 *Under the condition* (3.2), *it holds that for any* $0 < T < T^*$,

$$\sup_{0\leq t\leq T}\big\|\sqrt{\rho}u_t\big\|_{L^2}^2 + \int_0^T \|\nabla u_t\|_{L^2}^2 dt \leq C. \quad (3.16)$$

Proof Differentiating the momentum equations $(1.1)_2$ with respect to t, along with the continuity equation $(1.1)_1$, we get

$$\begin{aligned}&\rho u_{tt}+\rho u\cdot\nabla u_t-\operatorname{div}\left(2\mu(\rho)d_t\right)+\nabla P_t\\&\quad=(u\cdot\nabla\rho)(u_t+u\cdot\nabla u)-\rho u_t\cdot\nabla u-\operatorname{div}\left(2\mu'(\rho)(u\cdot\nabla\rho)d\right)\\&\qquad+\operatorname{div}\left(\kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho\right)+2\operatorname{div}\big(\kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho\big).\end{aligned}\tag{3.17}$$

Multiplying (3.17) by u_t and integrating over Ω, we get after integration by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho|u_t|^2dx+2\int\mu(\rho)|d_t|^2dx\\&\quad=\int-2\rho u\cdot\nabla u_t\cdot u_tdx+\int(u\cdot\nabla\rho)(u\cdot\nabla u)\cdot u_tdx-\int\rho u_t\cdot\nabla u\cdot u_tdx\\&\qquad+\int2\mu'(\rho)(u\cdot\nabla\rho)d:\nabla u_tdx-\int\kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho:\nabla u_tdx\\&\qquad-\int2\kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho:\nabla u_tdx=:\sum_{k=1}^{6}J_k.\end{aligned}\tag{3.18}$$

Now let us estimate the terms on the right hand side one by one. First

$$\begin{aligned}J_1&=\int-2\rho u\cdot\nabla u_t\cdot u_tdx\\&\le C\|\rho\|_{L^\infty}^{\frac{1}{2}}\left\|\sqrt{\rho}u_t\right\|_{L^2}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\\&\le\frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2+C\|u\|_{L^\infty}^2\left\|\sqrt{\rho}u_t\right\|_{L^2}^2.\end{aligned}\tag{3.19}$$

Similarly,

$$\begin{aligned}J_2&=\int(u\cdot\nabla\rho)(u\cdot\nabla u)\cdot u_tdx\\&\le C\|\nabla\rho\|_{L^\infty}\|\nabla u\|_{L^2}\|u\|_{L^\infty}^2\|u_t\|_{L^2}\\&\le C\|\nabla\rho\|_{L^\infty}\|\nabla u\|_{L^2}\|u\|_{L^\infty}^2\|\nabla u_t\|_{L^2}\\&\le\frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2+C\|u\|_{L^\infty}^4\|\nabla u\|_{L^2}^2,\end{aligned}\tag{3.20}$$

$$\begin{aligned}J_3&=-\int\rho u_t\cdot\nabla u\cdot u_tdx\\&\le C\|\rho\|_{L^\infty}^{\frac{1}{2}}\|u_t\|_{L^4}\left\|\sqrt{\rho}u_t\right\|_{L^2}\|\nabla u\|_{L^4}\\&\le C\|\nabla u_t\|_{L^2}\left\|\sqrt{\rho}u_t\right\|_{L^2}\|\nabla u\|_{H^1}\\&\le\frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2+C\left\|\sqrt{\rho}u_t\right\|_{L^2}^2\|\nabla u\|_{H^1}^2,\\&\le\frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2+C\left\|\sqrt{\rho}u_t\right\|_{L^2}^4+C\left\|\sqrt{\rho}u_t\right\|_{L^2}^2+C\left\|\sqrt{\rho}u_t\right\|_{L^2}^2\|\nabla u\|_{L^2}^6,\end{aligned}\tag{3.21}$$

$$\begin{aligned}J_4 &= \int 2\mu'(\rho)(u\cdot\nabla\rho)d:\nabla u_t dx\\&\le C\left\|\mu'(\rho)\right\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}\\&\le \frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2,\end{aligned}\tag{3.22}$$

$$\begin{aligned}J_5 &= \int \kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho:\nabla u_t dx\\&\le C\left\|\kappa'(\rho)\right\|_{L^\infty}\|\nabla\rho\|_{L^\infty}^3\|u\|_{L^2}\|\nabla u_t\|_{L^2}\\&\le \frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2,\end{aligned}\tag{3.23}$$

$$\begin{aligned}J_6 &= \int 2\kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho:\nabla u_t dx\\&\le C\left\|\kappa(\rho)\right\|_{L^\infty}\|\nabla\rho\|_{L^\infty}^2\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}\\&\quad + C\left\|\kappa(\rho)\right\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\|\nabla^2\rho\|_{L^q}\|u\|_{L^{q^*}}\|\nabla u_t\|_{L^2}\\&\le \frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2,\end{aligned}\tag{3.24}$$

here note we take $q > 2$. Substituting all the estimates (3.19)–(3.24) into (3.18), we deduce

$$\begin{aligned}&\frac{d}{dt}\int \rho|u_t|^2dx + \int \mu(\rho)|d_t|^2dx\\&\quad\le C\|u\|_{L^\infty}^2\left\|\sqrt{\rho}u_t\right\|_{L^2}^2 + C\left(1+\|\nabla u\|_{L^2}^6\right)\left\|\sqrt{\rho}u_t\right\|_{L^2}^2\\&\qquad + \left\|\sqrt{\rho}u_t\right\|_{L^2}^4 + C\left(1+\|u\|_{L^\infty}^4\right)\|\nabla u\|_{L^2}^2,\end{aligned}\tag{3.25}$$

consequently, it follows from Gronwall inequality and Lemma 3.3, 3.4 that

$$\sup_{0\le t\le T}\left\|\sqrt{\rho}u_t\right\|_{L^2}^2 + \int_0^T\|\nabla u_t\|_{L^2}^2dt \le C. \qquad \square$$

Lemma 3.6 *Under the condition* (3.2), *it holds that for any* $0 < T < T^*$,

$$\sup_{0\le t\le T}\left(\|\rho_t\|_{W^{1,q}} + \|u\|_{H^2} + \|P\|_{H^1}\right) + \int_0^T\left(\|u\|_{W^{2,q}}^2 + \|P\|_{W^{1,q}}^2\right)dt \le C. \tag{3.26}$$

Proof By Lemma 2.2 and (3.4), it is easy to deduce

$$\|u\|_{H^2} + \|P\|_{H^1} \le C\|\rho u_t\|_{L^2} + C\|\nabla u\|_{L^2}^3 + C \le C, \tag{3.27}$$

with the aid of Lemma 3.3 and 3.5.

And, together with $(1.1)_1$, yields

$$\begin{aligned}\|\rho_t\|_{W^{1,q}} &\le C\left(\|\rho_t\|_{L^q} + \|\nabla\rho_t\|_{L^q}\right)\\&\le C\left(\|u\cdot\nabla\rho\|_{L^q} + \left\|\nabla(u\cdot\nabla\rho)\right\|_{L^q}\right)\end{aligned}$$

$$
\begin{aligned}
&\le C\big(\|u\|_{L^\infty}\|\nabla\rho\|_{L^q}+\|u\|_{L^\infty}\left\|\nabla^2\rho\right\|_{L^q}+\|\nabla u\|_{L^2}\|\nabla\rho\|_{L^{\frac{2q}{q-2}}}\big)\\
&\le C\|u\|_{H^2}\|\nabla\rho\|_{W^{1,q}}\le C.
\end{aligned}
\tag{3.28}
$$

Finally, applying (2.5) in Lemma 2.2 with $F=-\rho u_t-\rho u\cdot\nabla u-\operatorname{div}(\kappa(\rho)\nabla\rho\otimes\nabla\rho)$, we get

$$
\begin{aligned}
\|\nabla u\|_{W^{1,q}}+\|P\|_{W^{1,q}} &\le C\big(1+\|\nabla\rho\|_{L^\infty}\big)\big(\|\rho u_t\|_{L^q}+\|\rho u\cdot\nabla u\|_{L^q}\\
&\quad+\left\|\kappa(\rho)\left|\nabla^2\rho\right||\nabla\rho|\right\|_{L^q}+\left\|\kappa'(\rho)|\nabla\rho|^3\right\|_{L^q}\big)\\
&\le C\big(\|\rho u_t\|_{L^q}+\|\rho u\cdot\nabla u\|_{L^q}+1\big)\\
&\le C\big(\|\nabla u_t\|_{L^2}+\|\nabla u\|_{H^1}^2+1\big),
\end{aligned}
\tag{3.29}
$$

hence

$$
\begin{aligned}
\int_0^T\big(\|\nabla u\|_{W^{1,q}}^2+\|P\|_{W^{1,q}}^2\big)dt &\le C\int_0^T\big(\|\nabla u_t\|_{L^2}^2+\|\nabla u\|_{H^1}^4\big)dt+C\\
&\le C.
\end{aligned}
\tag{3.30}
$$

Therefore we complete the proof of Lemma 3.6. □

Proof of Theorem 1.3 In fact, in view of (3.2) and (3.27), it is easy to see that the functions $(\rho,u)(x,t=T^*)=\lim_{t\to T^*}(\rho,u)$ have the same regularities imposed on the initial data (1.6) at the time $t=T^*$. Furthermore,

$$
\begin{aligned}
&-\operatorname{div}\big(2\mu(\rho)d\big)+\nabla P+\operatorname{div}\big(\kappa(\rho)\nabla\rho\otimes\nabla\rho\big)\big|_{t=T^*}\\
&=\lim_{t\to T^*}\rho^{\frac12}\big(\rho^{\frac12}u_t+\rho^{\frac12}u\cdot\nabla u\big):=\rho^{\frac12}g\big|_{t=T^*}
\end{aligned}
$$

with $g=(\rho^{\frac12}u_t+\rho^{\frac12}u\cdot\nabla u)|_{t=T^*}\in L^2$ due to (3.16). Thus the functions $(\rho,u)|_{t=T^*}$ satisfy the compatibility condition (1.7) at time T^*. Therefore we can take $(\rho,u)|_{t=T^*}$ as the initial data and apply the local existence theorem (Theorem 1.2) to extend the local strong solution beyond T^*. This contradicts the definition of maximal existence time T^*, and thus, the proof of Theorem 1.3 is completed. □

References

1. Burtea, C., Charve, F.: Lagrangian methods for a general inhomogeneous incompressible Navier–Stokes–Korteweg system with variable capillarity and viscosity coefficients. SIAM J. Math. Anal. **49**(5), 3476–3495 (2017)
2. Cho, Y., Kim, H.: Unique solvability for the density-dependent Navier–Stokes equations. Nonlinear Anal. **59**(4), 465–489 (2004)
3. Desjardins, B.: Regularity results for two-dimensional flows of multiphase viscous fluids. Arch. Ration. Mech. Anal. **137**, 135–158 (1997)
4. Danchin, R., Desjardins, B.: Existence of solutions for compressible fluid models of Korteweg type. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **18**, 97–133 (2001)

5. Hattori, H., Li, D.: Solutions for two dimensional system for materials of Korteweg type. SIAM J. Math. Anal. **25**, 85–98 (1994)
6. Hattori, H., Li, D.: Global solutions of a high-dimensional system for Korteweg materials. J. Math. Anal. Appl. **198**, 84–97 (1996)
7. Huang, X.D., Wang, Y.: Global strong solution to the 2D nonhomogeneous incompressible MHD system. J. Differ. Equ. **254**(2), 511–527 (2013)
8. Huang, X.D., Wang, Y.: Global strong solution with vacuum to the two dimensional density-dependent Navier–Stokes system. SIAM J. Math. Anal. **46**(3), 1771–1788 (2014)
9. Huang, X.D., Wang, Y.: Global strong solution of 3D inhomogeneous Navier–Stokes equations with density-dependent viscosity. J. Differ. Equ. **259**(4), 1606–1627 (2015)
10. Kostin, I., Marion, M., Texier-Picard, R., Volpert Vitaly, A.: Modelling of miscible liquids with the Korteweg stress. Math. Model. Numer. Anal. **37**(5), 741–753 (2003)
11. Sy, M., Bresch, D., Guillén-González, F., Lemoine, J., Rodríguez-Bellido, M.A.: Local strong solution for the incompressible Korteweg model. C. R. Math. Acad. Sci. Paris **342**(3), 169–174 (2006)
12. Tan, Z., Wang, Y.J.: Strong solutions for the incompressible fluid models of Korteweg type. Acta Math. Sci. Ser. B Engl. Ed. **30**(3), 799–809 (2010)
13. Wang, T.: Unique solvability for the density-dependent incompressible Navier–Stokes–Korteweg system. J. Math. Anal. Appl. **455**(1), 606–618 (2017)