Acta Appl Math (2020) 166:29–72
https://doi.org/10.1007/s10440-019-00253-5

Ergodic Behavior of Non-conservative Semigroups via Generalized Doeblin's Conditions

Vincent Bansaye[1] **· Bertrand Cloez**[2] **· Pierre Gabriel**[3]

Received: 7 November 2018 / Accepted: 29 March 2019 / Published online: 4 April 2019

Abstract We provide quantitative estimates in total variation distance for positive semigroups, which can be non-conservative and non-homogeneous. The techniques relies on a family of conservative semigroups that describes a typical particle and Doeblin's type conditions inherited from Champagnat and Villemonais (Probab. Theory Relat. Fields 164(1–2):243–283, 2016) for coupling the associated process. Our aim is to provide quantitative estimates for linear partial differential equations and we develop several applications for population dynamics in varying environment. We start with the asymptotic profile for a growth diffusion model with time and space non-homogeneity. Moreover we provide general estimates for semigroups which become asymptotically homogeneous, which are applied to an age-structured population model. Finally, we obtain a speed of convergence for periodic semigroups and new bounds in the homogeneous setting. They are illustrated on the renewal equation.

Keywords Positive semigroups · Non-autonomous linear evolution equations · Measure solutions · Ergodicity · Krein-Rutman theorem · Floquet theory · Branching processes · Population dynamics

Mathematics Subject Classification (2010) Primary 35B40 · Secondary 47A35 · 47D06 · 60J80 · 92D25

✉ P. Gabriel
pierre.gabriel@uvsq.fr

V. Bansaye
vincent.bansaye@polytechnique.edu

B. Cloez
bertrand.cloez@inra.fr

[1] CMAP, École Polytechnique, Route de Saclay, 91128 Palaiseau cedex, France

[2] MISTEA, INRA, Montpellier SupAgro, Univ. Montpellier, 2 place Pierre Viala, 34060 Montpellier, France

[3] Laboratoire de Mathématiques de Versailles, UVSQ, CNRS, Université Paris-Saclay, 45 Avenue des États-Unis, 78035 Versailles cedex, France

1 Introduction

The solutions of the Cauchy problem associated to a linear Partial Differential Equation (PDE) can be expressed through a semigroup of linear operators. In the present work, we are interested in the ergodic properties of positive semigroups $(M_{s,t})_{t\geq s\geq 0}$ acting on measures, and their application to the study of the asymptotic profile of populations evolving in varying environment, which can be described by linear (nonautonomous) PDEs. Roughly speaking, for any $t \geq s \geq 0$, $M_{s,t}$ is both a positive linear operator on a space of measures ($\mu \mapsto \mu M_{s,t}$) and on a space of measurable functions ($f \mapsto M_{s,t} f$), and the family $(M_{s,t})_{t\geq s\geq 0}$ satisfies the semigroup property

$$\forall s \leq u \leq t, \quad M_{s,t} = M_{s,u} M_{u,t}.$$

For a measure μ and a measurable function f, we denote by $\mu(f)$ the integral of f against μ. We establish ergodic approximations of the following form

$$\mu M_{s,t} \approx \mu(h_s)\, r_{s,t}\, \gamma_t$$

when $t \to \infty$, for a fixed initial time s. The first term in this long-time decomposition is a linear form $\mu \mapsto \mu(h_s)$ on the space of measures, independent of t, which provides the long term impact of the initial distribution μ through the function h_s. The second term is a family $(r_{s,t})_{t\geq s}$ of positive real numbers, independent of μ, describing the evolution of the "mass". Finally γ_t is the asymptotic probability distribution, which does not depend on s nor μ. The harmonic function h_s is unique up to normalization, but the two families $(r_{s,t})_{t\geq s}$ and $(\gamma_t)_{t\geq 0}$ are not. Nevertheless in particular situations, they can be chosen in certain relevant classes in which they are unique. In Sect. 3 we detail and illustrate such cases, briefly presented here:

Homogeneous semigroups. In the homogeneous setting $M_{s,t} = M_{t-s}$, and provided a topology on the space of measures, spectral theorems suggest the behavior

$$\mu M_t = \mu(h) \mathrm{e}^{\lambda t} \gamma + \mathcal{O}\big(\mathrm{e}^{(\lambda-\varepsilon)t}\big),$$

where λ is the dominant eigenvalue of the infinitesimal generator of the semigroup, γ and h are the associated eigenvectors, and ε is the spectral gap. This is an immediate consequence of the Perron Frobenius Theorem [23, 42] in finite state space setting. In a general Banach lattice the existence of the eigentriplet (λ, γ, h) is ensured by the Krein-Rutman Theorem [33] when the semigroup (or the resolvent of its generator) is positive, irreducible, and compact. A refined variant of the Krein-Rutman theorem, with spectral gap, is proved in [40] in the setting of a Banach lattice of functions. The proof relies on a spectral analysis and applies to positive semigroups with a generator which satisfies a strong maximum principle and admits a decomposition verifying a power compactness condition. In contrast with these approaches, our method is based on a contraction argument and can be efficiently applied to time-inhomogeneous semigroups.

Asymptotically homogeneous semigroups. In the case where there exists a homogeneous semigroup $(N_t)_{t\geq 0}$ such that $M_{s,s+t} \approx N_t$ for s large, we prove that the principal eigenvector γ of $(N_t)_{t\geq 0}$ provides a stationary asymptotic profile

$$\mu M_{s,t} \approx \mu(h_s)\, r_{s,t}\, \gamma.$$

But $r_{s,t}$ is not necessarily an exponential growth provided by the associated eigenvalue, and h_s is not the associated eigenfunction.

Periodic semigroups. When there exists $T > 0$ such that $M_{s+T,t+T} = M_{s,t}$ for all $s \le t$, the semigroup is said to be periodic. In this case we can choose for $(\gamma_t)_{t\ge 0}$ a T-periodic family, and similarly as in the homogeneous case, the evolution of the mass is exponential. More precisely there exist a real number λ_F, named Floquet eigenvalue after the work of G. Floquet [22], and a periodic family $(\eta_t)_{t\ge 0}$ bounded from above and below such that

$$\mu M_{s,t} \approx \mu(h_s)\, \mathrm{e}^{\lambda_F t+\eta_t}\, \gamma_t.$$

In all cases, the bound for the speed of convergence is expressed in the total variation norm (see Sect. 2.1 for the definition), which is the natural distance for coupling processes in probability. The proof relies on an auxiliary conservative semigroup $P^{(t)}$, defined for every bounded function f and any times $0 \le s \le u \le t$ by

$$P^{(t)}_{s,u} f = \frac{M_{s,u}(f m_{u,t})}{m_{s,t}}, \quad \text{where } m_{s,t} = M_{s,t}\mathbf{1},$$

for which ergodic behavior can be obtained through coupling arguments. This auxiliary semigroup describes the trajectory of a typical particle and has been used recently for the study of branching Markov processes in discrete and continuous time [2, 4, 5, 34] and processes killed at a boundary [11, 18, 35]. We come back in Appendix A on the link between these topics in probability and ergodic estimates for semigroups.

Doeblin and Lyapounov techniques (or petite sets) [19, 38] provide then a powerful tool to control the ergodic behavior of this auxiliary Markov process. More generally, the constructions of auxiliary Markov processes derived from a typical or tagged particle have been well developed in probability and play a key role in the asymptotic study of stochastic processes. They appear in Feynman-Kac formula [16] and in spine technics via many-to-one formulae [30] for the probabilistic study of branching processes [7, 15, 20] and fragmentation processes [6], to name but a few.

When working on a compact state space or benefiting from an atom or a compact set uniformly accessible for the whole state space, one can hope to check Doeblin conditions on the auxiliary semigroup. Recall that a conservative, positive and homogeneous semigroup $(Q_t)_{t\ge 0}$ satisfies the Doeblin condition if there exist a constant $c > 0$, a coupling probability measure ν and a time $t_0 > 0$ such that for all positive and bounded function f,

$$Q_{t_0} f \ge c\, \nu(f).$$

This condition is equivalent to a contraction in total variation distance and then provides a convenient tool of analysis for non-homogenous models. Sharp assumptions expressed in function of M have recently been obtained in [11] to get a Doeblin condition for the auxiliary semigroup in a context of absorbed Markov process. These conditions are weaker than the classical conditions using Birkhoff contraction [8, 25, 41] and equivalent to uniform exponential convergence.

In Lemma 2.5 we prove that Doeblin's condition hold for the semigroup $P^{(t)}$, which in turn provides an explicit bound for the decrease of

$$P^{(t)}_{s,t} f(x) - P^{(t)}_{s,t} f(y) = \frac{M_{s,t} f(x)}{m_{s,t}(x)} - \frac{M_{s,t} f(y)}{m_{s,t}(y)}$$

as $t \to \infty$ and the ergodic behavior of the auxiliary semigroup. The proof of this Lemma is essentially an adaptation of the method in [11, 12] that we extend to general semigroups in non-homogeneous environment, while they restrict their study to absorbed Markov processes. This more general semigroup setting allows us to capture a wider range of applications, like the renewal equation we consider in Sect. 3. Moreover, we go beyond the contraction of the auxiliary semigroup $P^{(t)}$ and characterize the asymptotic behavior of $(M_{0,t})_{t\geq 0}$, which is a novelty compared to the previous results.

More precisely, for any initial time $s \geq 0$, we propose conditions involving a coupling probability measure ν which guarantee the existence of a positive bounded function h_s and a family of probabilities $(\gamma_t)_{t\geq 0}$ such that when $t \to \infty$

$$\sup_{\|\mu\|_{\mathrm{TV}}\leq 1} \left\| \mu M_{s,t} - \mu(h_s)\nu(m_{s,t})\gamma_t \right\|_{\mathrm{TV}} = o\big(\nu(m_{s,t})\big).$$

These conditions are stated in Sect. 2 and a quantified version of above convergence is proved. In Sect. 3 the general result is declined in several applications, which are illustrated by concrete and intentionally simple examples of linear PDE issued from population dynamics. We avoid too much technicality but provide some new estimates and explain the way assumptions can be checked. We first consider in Sect. 3.1 a model of population growing in a non-homogeneous and diffusing in a varying environment, which is illustrated by ergodic random environment. Intuitively, if the variation of parameters in the model is not vanishing in large times, one does not expect the convergence of γ_t. In the case of homogeneous or asymptotically homogeneous semigroups, we prove that the asymptotic profile is given by a constant probability measure γ; see Sect. 3.2 and Sect. 3.3 respectively. Finally, when the semigroup evolves periodically we prove that the asymptotic profile γ_t is periodic; see Sect. 3.4. Results of Sect. 3.2 (homogeneous semigroups), Sect. 3.3 (asymptotically homogeneous semigroups) and Sect. 3.4 (periodic semigroups) are illustrated on the renewal equation. In these three settings, we obtain new sharp conditions for convergence with explicit rate of convergence.

2 General Statement and Proof

2.1 Preliminaries on Measures and Semigroups

We start by recalling some definitions and results about measure theory, and we refer to [45] for more details and proofs.

Let $\mathcal{X}$ be a locally compact Hausdorff space and denote by $\mathcal{B}_b(\mathcal{X})$ the space of bounded Borel functions $f : \mathcal{X} \to \mathbb{R}$ endowed with the supremum norm $\|f\|_\infty = \sup_{\mathcal{X}} |f|$. We denote by $\mathcal{M}(\mathcal{X})$ the space of regular signed Borel measures on $\mathcal{X}$,[1] by $\mathcal{M}_+(\mathcal{X})$ its positive cone (*i.e.* the set of regular finite positive Borel measures), and by $\mathcal{P}(\mathcal{X})$ the subset of probability measures. For two measures $\mu, \widetilde{\mu} \in \mathcal{M}(\mathcal{X})$, we say that μ is larger than $\widetilde{\mu}$, and write $\mu \geq \widetilde{\mu}$, if $\mu - \widetilde{\mu} \in \mathcal{M}_+(\mathcal{X})$. The Jordan decomposition theorem ensures that for any $\mu \in \mathcal{M}(\mathcal{X})$ there exists a unique decomposition $\mu = \mu_+ - \mu_-$ with μ_+ and μ_- positive and mutually singular. The positive measure $|\mu| = \mu_+ + \mu_-$ is called the total variation measure of the measure μ, and its mass is the total variation norm of μ

$$\|\mu\|_{\mathrm{TV}} := |\mu|(\mathcal{X}) = \mu_+(\mathcal{X}) + \mu_-(\mathcal{X}).$$

[1] Notice that if $\mathcal{X} \subset \mathbb{R}^n$ is equipped with the induced topology, any signed Borel measure on $\mathcal{X}$ is regular.

Clearly we have the identity[2]

$$\|\mu\|_{\mathrm{TV}} = \sup_{\|f\|_\infty \le 1} \big|\mu(f)\big|,$$

where the supremum is taken over measurable functions. By virtue of the Riesz representation theorem, this supremum can be restricted to the continuous functions vanishing at infinity,[3] *i.e.* $f \in C_0(\mathcal{X}) = \overline{C_c(\mathcal{X})}$. The Riesz representation theorem also ensures that $(\mathcal{M}(\mathcal{X}), \|\cdot\|_{\mathrm{TV}})$ is a Banach space, as a topological dual space. It is worth noticing that the inequality $|\mu(f)| \le \|\mu\|_{\mathrm{TV}}\|f\|_\infty$ which is valid for any $\mu \in \mathcal{M}(\mathcal{X})$ and $f \in \mathcal{B}_b(\mathcal{X})$ can be strengthened into $|\mu(f)| \le \frac{1}{2}\|\mu\|_{\mathrm{TV}}\|f\|_\infty$ when $\mu(\mathcal{X}) = 0$ and $f \ge 0$.

For any $\Omega \subset \mathcal{X}$ we denote by $\mathbf{1}_\Omega$ the indicator function of the subset Ω. And we denote by $\mathbf{1}$ the constant function equal to 1 on $\mathcal{X}$, *i.e.* $\mathbf{1} = \mathbf{1}_{\mathcal{X}}$.

Now we turn to the definition of the (time-inhomogeneous) semigroups we are interested in. Let $(\mathcal{X}_t)_{t\ge 0}$ be a family of locally compact Hausdorff spaces. A semigroup $M = (M_{s,t})_{0\le s\le t}$ is a family of linear operators defined as follows. For any $t \ge s \ge 0$, $M_{s,t}$ is a bounded linear operator from $\mathcal{M}(\mathcal{X}_s)$ to $\mathcal{M}(\mathcal{X}_t)$ through the left action

$$M_{s,t} : \begin{array}{ccc} \mathcal{M}(\mathcal{X}_s) & \to & \mathcal{M}(\mathcal{X}_t) \\ \mu & \mapsto & \mu M_{s,t} \end{array},$$

and a bounded linear operator from $\mathcal{B}_b(\mathcal{X}_t)$ to $\mathcal{B}_b(\mathcal{X}_s)$ through the right action

$$M_{s,t} : \begin{array}{ccc} \mathcal{B}_b(\mathcal{X}_t) & \to & \mathcal{B}_b(\mathcal{X}_s) \\ f & \mapsto & M_{s,t} f \end{array}.$$

The semigroup property means here that for all $s \le u \le t$ and $f \in \mathcal{B}_b(\mathcal{X}_t)$

$$M_{s,t} f = M_{s,u}(M_{u,t} f).$$

Moreover, we make the following assumptions.

Assumption 2.1 We assume that for all $t \ge s \ge 0$ we have

$(f \in \mathcal{B}_b(\mathcal{X}_t),\ f \ge 0) \implies M_{s,t} f \ge 0,$	(positivity)
$\forall x \in \mathcal{X}_s,\ m_{s,t}(x) := (M_{s,t}\mathbf{1})(x) > 0,$	(strong positivity)
$\forall (\mu, f) \in \mathcal{M}(\mathcal{X}_s) \times \mathcal{B}_b(\mathcal{X}_t),\ (\mu M_{s,t})(f) = \mu(M_{s,t} f).$	(left-right compatibility)

Due to the compatibility condition, we can denote without ambiguity $\mu M_{s,t} f = (\mu M_{s,t})(f) = \mu(M_{s,t} f)$, and $(\mu, f) \mapsto \mu M_{s,t} f$ is a bilinear form on $\mathcal{M}(\mathcal{X}_s) \times \mathcal{B}_b(\mathcal{X}_t)$. Notice additionally that the compatibility condition allows to transfer the semigroup property and the positivity to the left action, *i.e.* for all $t \ge s \ge 0$, we have

$$\forall u \in [s,t],\ \forall \mu \in \mathcal{M}(\mathcal{X}_s), \quad \mu M_{s,t} = (\mu M_{s,u}) M_{u,t},$$

$$\mu \in \mathcal{M}_+(\mathcal{X}_s) \quad \implies \quad \mu M_{s,t} \in \mathcal{M}_+(\mathcal{X}_t).$$

[2] We see here that the definition we use for the total variation norm differs from the usual probabilistic definition of a factor $1/2$.

[3] A function f on a locally compact Hausdorff space $\mathcal{X}$ is said to vanish at infinity if to every $\varepsilon > 0$, there exists a compact set $K \subset \mathcal{X}$ such that $|f(x)| < \varepsilon$ for all $x \in \mathcal{X} \setminus K$.

2.2 Coupling Constants

Let $\alpha, \beta > 0$ and $\nu \in \mathcal{P}(\mathcal{X}_s)$.

Definition 2.2 (Admissible coupling constants) For any $N \geq 1$, we say that $(c_i, d_i)_{1\leq i\leq N} \in [0, 1]^{2N}$ are (α, β, ν)-admissible coupling constants for M on $[s, t]$ if there exist real numbers $(t_i)_{0\leq i\leq N}$ satisfying $s \leq t_0 \leq \cdots \leq t_N \leq t$ and probability measures ν_i on $\mathcal{X}_{t_i}$ such that for all $i = 1, \ldots, N$ and $x \in \mathcal{X}_{t_{i-1}}$,

$$\delta_x M_{t_{i-1},t_i} \geq c_i m_{t_{i-1},t_i}(x)\nu_i, \tag{A1}$$

and for all $i \in \{1, \ldots, N-1\}$, $\tau \geq t_N$ and $x \in \mathcal{X}_{t_i}$,

$$d_i m_{t_i,\tau}(x) \leq \nu_i(m_{t_i,\tau}) \tag{A2}$$

and for all $\tau \geq t_N$ and $x \in \mathcal{X}_{t_N}$,

$$m_{t_N,\tau}(x) \leq \alpha\, c_N\, \nu_N(m_{t_N,\tau}) \tag{A3}$$

and for all $\tau \geq t_N$ and $x \in \mathcal{X}_s$,

$$m_{s,\tau}(x) \leq \beta\, \nu(m_{s,\tau}). \tag{A4}$$

In the conservative case, Assumption (A1) is the classical Doeblin assumption. It is a strong irreducibility property: whatever the initial distribution is, the semigroup between the times t_{i-1} and t_i is lowerbounded by a fixed measure ν_i. This condition is then sufficient (and even necessary) for uniform exponential convergence.

But this condition is no longer sufficient for non-conservative semi-group. First, the mass of the process has to be added in (A1), as will be seen in examples when the mass vanishes. Moreover the mass of the semi-group has to be essentially the same for any starting distribution. This is the meaning of Assumptions (A2), (A3) and (A4).

The two first assumptions allow to get the contraction of the auxiliary semigroup $P^{(t)}$ in the total variation norm following [11, 12], see Lemma 2.5. The two additional assumptions are needed to prove the existence of harmonic-type functions and control the speed of convergence in the general result, see forthcoming Lemma 2.7. Assumptions (A2), (A3), (A4) all involve the control of the mass m for large times and will be proved in the same time by a coupling argument in applications of Sect. 3. The associated constants may change in varying environment, see Sect. 3.1.

We denote by $\mathcal{H}_{\alpha,\beta,\nu}(s, t)$ the set of (α, β, ν)-admissible coupling constants $(c_i, d_i)_{1\leq i\leq N}$ for M on $[s, t]$. It can be easily seen from Definition 2.2 that for this set to be nonempty, the constants α and β have to be at least greater than or equal to 1. We are interested in the optimal admissible coupling and we set

$$C_{\alpha,\beta,\nu}(s, t) = \sup_{\mathcal{H}_{\alpha,\beta,\nu}(s,t)} \left\{ -\sum_{i=1}^{N} \log(1 - c_i d_i) \right\}, \tag{2.1}$$

where by convention $\sup \varnothing = 0$. We observe that $t \mapsto C_{\alpha,\beta,\nu}(s, t)$ is positive and non-decreasing.

2.3 General Result

Here we state the general result we obtain about the ergodicity of semigroups M which satisfy Assumption 2.1.

Theorem 2.3 *Let $s \geq 0$ and assume that there exist $\alpha, \beta \geq 1$, and ν a probability measure on $\mathcal{X}_s$ such that $C_{\alpha,\beta,\nu}(s,t) \to \infty$ as $t \to \infty$. Then there exists a unique function $h_s : \mathcal{X}_s \to [0,\infty)$ such that for any $\mu \in \mathcal{M}(\mathcal{X}_s)$, $\gamma \in \mathcal{M}(\mathcal{X}_{s_0})$ with $s_0 \in [0,s]$, and for any t such that $C_{\alpha,\beta,\nu}(s,t) \geq \log(4\alpha)$,*

$$\left\| \mu M_{s,t} - \mu(h_s)\nu(m_{s,t}) \frac{\gamma M_{s_0,t}}{\gamma(m_{s_0,t})} \right\|_{\mathrm{TV}} \leq 8(2+\alpha)|\mu|(h_s)\, \nu(m_{s,t})\, \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}.$$

Moreover $h_s(x) \in (0,\beta]$ for any $x \in \mathcal{X}_s$ and $\nu(h_s) = 1$.

Before the proof, let us make two remarks. First, under the assumption of Theorem 2.3, we also prove that for all $t \geq s$,

$$\left\| \mu M_{s,t} - \mu(h_s)\nu(m_{s,t}) \frac{\gamma M_{s_0,t}}{\gamma(m_{s_0,t})} \right\|_{\mathrm{TV}} \leq 2(2+\alpha)\beta \|\mu\|_{\mathrm{TV}}\, \nu(m_{s,t})\, \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}.$$

This bound is thus valid for any time. Second, we can change the measure ν as follows.

Remark 2.4 Suppose that for all $\mu \in \mathcal{M}(\mathcal{X}_s)$ the function $t \mapsto \mu(m_{s,t})$ is continuous and the Assumptions of Theorem 2.3 hold. Then for all $\widetilde{\nu} \in \mathcal{P}(\mathcal{X}_s)$ there exists a constant $\widetilde{\beta}$ such that (A4) is still valid if we replace ν and β by $\widetilde{\nu}$ and $\widetilde{\beta}$. Indeed Theorem 2.3 applied to $\mu = \widetilde{\nu}$ ensures that $\widetilde{\nu}(m_{s,t})/\nu(m_{s,t}) \to \widetilde{\nu}(h_s) > 0$ when $t \to \infty$. Since $t \mapsto \nu(m_{s,t})/\widetilde{\nu}(m_{s,t})$ is continuous, it is bounded on $[s,+\infty)$. Using (A4), we deduce that for all $T \geq t_N$,

$$\|m_{s,T}\|_\infty \leq \beta\nu(m_{s,T}) \leq \beta \sup_{t\geq s}\left(\frac{\nu(m_{s,t})}{\widetilde{\nu}(m_{s,t})}\right)\widetilde{\nu}(m_{s,T}) = \widetilde{\beta}\,\widetilde{\nu}(m_{s,T}).$$

2.4 Proof of Theorem 2.3

We recall from the introduction the definition of $P^{(t)}$. For any $t \geq u \geq s \geq 0$, the linear operator $P^{(t)}_{s,u} : \mathcal{B}_b(\mathcal{X}_u) \to \mathcal{B}_b(\mathcal{X}_s)$ is defined by

$$P^{(t)}_{s,u} f = \frac{M_{s,u}(f m_{u,t})}{m_{s,t}}.$$

By duality we define a left action $P^{(t)}_{s,u} : \mathcal{M}(\mathcal{X}_s) \to \mathcal{M}(\mathcal{X}_u)$ by

$$\forall f \in \mathcal{B}_b(\mathcal{X}_u), \quad \big(\mu P^{(t)}_{s,u}\big)(f) := \mu\big(P^{(t)}_{s,u} f\big) = \int_{\mathcal{X}_s} \mu(dx) \frac{M_{s,u}(f m_{u,t})(x)}{m_{s,t}(x)}. \tag{2.2}$$

We recall that this is a positive conservative semigroup. Indeed we readily check that $P^{(t)}_{s,u}\mathbf{1} = \mathbf{1}$ and $P^{(t)}_{s,u} f \geq 0$ if $f \geq 0$. Moreover

$$P^{(t)}_{s,u}\big(P^{(t)}_{u,v} f\big) = \frac{M_{s,u}((P^{(t)}_{u,v} f)m_{u,t})}{m_{s,t}} = \frac{M_{s,u}\big(\frac{M_{u,v}(f m_{v,t})}{m_{u,t}} m_{u,t}\big)}{m_{s,t}} = P^{(t)}_{s,v} f.$$

It is also worth noticing that for all $t \geq s \geq 0$ and all $x \in \mathcal{X}_s$

$$\delta_x P^{(t)}_{s,t} = \frac{\delta_x M_{s,t}}{m_{s,t}(x)}. \tag{2.3}$$

The first key ingredient is the following lemma, which gives the ergodic behavior of the auxiliary conservative semigroup under assumptions (A1) and (A2). This is an almost direct generalization of [11], which holds for homogeneous and sub-conservative (or sub-Markov) semigroups; namely $M_{s,t} = M_{t-s}$ and $M_t\mathbf{1} \leq \mathbf{1}$, for all $t \geq s \geq 0$. These semigroups are associated to the evolution of absorbed (or killed) Markov processes (see Sect. A). This is also related to [17, Chap. 12] or [16, Chap. 4.3.2]. The proof is given here for the sake of completeness.

Lemma 2.5 (Doeblin contraction) *Let* $0 \leq s \leq t$ *and* $(c_i, d_i)_{1\leq i\leq N}$ *satisfying* (A1) *and* (A2) *for the time subdivision* $s \leq t_0 \leq \cdots \leq t_N \leq t$. *Let* $\tau \geq t_N$.

(i) *For any* $i = 1, \ldots, N$, *there exists* $\mu_i \in \mathcal{P}(\mathcal{X}_{t_i})$ *such that for all* $x \in \mathcal{X}_{t_{i-1}}$

$$\delta_x P^{(\tau)}_{t_{i-1},t_i} \geq c_i d_i \mu_i.$$

(ii) *For any* $\mu, \widetilde{\mu}$ *finite measures on* $\mathcal{X}_s$,

$$\left\| \mu P^{(\tau)}_{s,\tau} - \widetilde{\mu} P^{(\tau)}_{s,\tau} \right\|_{\mathrm{TV}} \leq \prod_{i\leq N}(1 - c_i d_i)\|\mu - \widetilde{\mu}\|_{\mathrm{TV}}.$$

(iii) *For any non-zero* $\mu, \widetilde{\mu} \in \mathcal{M}_+(\mathcal{X}_s)$,

$$\left\| \frac{\mu M_{s,\tau}}{\mu(m_{s,\tau})} - \frac{\widetilde{\mu} M_{s,\tau}}{\widetilde{\mu}(m_{s,\tau})} \right\|_{\mathrm{TV}} \leq 2 \prod_{i\leq N}(1 - c_i d_i).$$

Remark 2.6 (Sharper bound) In view of the proof below, one can replace Lemma 2.5 (iii) by

$$\left\| \frac{\mu M_{s,\tau}}{\mu(m_{s,\tau})} - \frac{\widetilde{\mu} M_{s,\tau}}{\widetilde{\mu}(m_{s,\tau})} \right\|_{\mathrm{TV}} \leq 2 \prod_{i\leq N}(1 - c_i d_i)\, \mathcal{W}_{s,t_N}(\mu, \widetilde{\mu}), \tag{2.4}$$

where $\mathcal{W}_{s,t_N}$ is a Wasserstein distance (see for instance [49]) defined by

$$\mathcal{W}_{s,t_N}(\mu, \widetilde{\mu}) = \inf_{\Pi} \frac{1}{\mu(m_{s,t_N})\widetilde{\mu}(m_{s,t_N})} \int_{\mathcal{X}_s} m_{s,t_N}(y) m_{s,t_N}(x) \mathbf{1}_{x\neq y} \Pi(dx, dy),$$

and the infimum runs over all coupling measures Π of μ and $\widetilde{\mu}$; a coupling measure is a positive measure on $\mathcal{X}_s^2$ whose marginals are given by μ and $\widetilde{\mu}$. Even if the right-hand side of (2.4) vanishes now when $\mu = \widetilde{\mu}$, this bound depends on t_N and incalculable quantities. However if there exists $A_s, B_s > 0$ such that $A_s \leq \sup_{\tau\geq s} \nu m_{s,\tau}/\mu m_{s,\tau} \leq B_s$ then Eq. (2.4) entails that

$$\left\| \frac{\mu M_{s,\tau}}{\mu(m_{s,\tau})} - \frac{\widetilde{\mu} M_{s,\tau}}{\widetilde{\mu}(m_{s,\tau})} \right\|_{\mathrm{TV}} \leq 2 \frac{B_s^2}{A_s^2} \prod_{i\leq N}(1 - c_i d_i)\, \|\mu - \widetilde{\mu}\|_{\mathrm{TV}}.$$

See Sect. 3.1 and Inequality (3.6) for an example.

Proof of Lemma 2.5 *Proof of (i).* Let $i \leq N$ and f be a positive function of $\mathcal{B}_b(\mathcal{X}_{t_i})$. Using (A1), we have

$$\delta_x M_{t_{i-1},t_i}(f m_{t_i,\tau}) \geq c_i \nu_i(f m_{t_i,\tau}) m_{t_{i-1},t_i}(x) = c_i \nu_i(f m_{t_i,\tau}) \frac{m_{t_{i-1},t_i}(x)}{m_{t_{i-1},\tau}(x)} m_{t_{i-1},\tau}(x). \quad (2.5)$$

Let us find μ_i satisfying

$$\nu_i(f m_{t_i,\tau}) \frac{m_{t_{i-1},t_i}(x)}{m_{t_{i-1},\tau}(x)} \geq d_i \mu_i(f). \quad (2.6)$$

Using (A2), the semigroup and positivity properties ensure that

$$d_i m_{t_{i-1},\tau}(x) = d_i \delta_x M_{t_{i-1},t_i}(m_{t_i,\tau}) \leq m_{t_{i-1},t_i}(x) \nu_i(m_{t_i,\tau}).$$

Thus

$$\nu_i(f m_{t_i,\tau}) \frac{m_{t_{i-1},t_i}(x)}{m_{t_{i-1},\tau}(x)} \geq d_i \mu_i(f),$$

where μ_i defined by

$$\mu_i(f) = \frac{\nu_i(f m_{t_i,\tau})}{\nu_i(m_{t_i,\tau})}$$

is a probability measure. Recalling (2.2), (i) follows from (2.5) and (2.6).

Proof of (ii). We consider the conservative linear operator U_i on $\mathcal{B}_b(\mathcal{X}_{t_i})$, defined by

$$U_i f(x) := \frac{P^{(\tau)}_{t_{i-1},t_i} f(x) - c_i d_i \mu_i(f)}{1 - c_i d_i},$$

for $f \in \mathcal{B}_b(\mathcal{X}_{t_i})$ and $x \in \mathcal{X}_{t_{i-1}}$. It is positive by ($i$) and $\|U_i f\|_\infty \leq \|f\|_\infty$. Using

$$\begin{aligned}\delta_x P^{(\tau)}_{s,t_i} f - \delta_y P^{(\tau)}_{s,t_i} f &= \delta_x P^{(\tau)}_{s,t_{i-1}} P^{(\tau)}_{t_{i-1},t_i} f - \delta_y P^{(\tau)}_{s,t_{i-1}} P^{(\tau)}_{t_{i-1},t_i} f \\ &= (1 - c_i d_i)\big(\delta_x P^{(\tau)}_{s,t_{i-1}}(U_i f) - \delta_y P^{(\tau)}_{s,t_{i-1}}(U_i f)\big),\end{aligned}$$

we get

$$\big\|\delta_x P^{(\tau)}_{s,t_i} - \delta_y P^{(\tau)}_{s,t_i}\big\|_{\mathrm{TV}} \leq (1 - c_i d_i)\big\|\delta_x P^{(\tau)}_{s,t_{i-1}} - \delta_y P^{(\tau)}_{s,t_{i-1}}\big\|_{\mathrm{TV}}$$

since $\|U_i f\|_\infty \leq \|f\|_\infty$. Using that $P^{(\tau)}_{t_N,\tau}$ is also contraction since it is conservative, we obtain

$$\big\|\delta_x P^{(\tau)}_{s,\tau} - \delta_y P^{(\tau)}_{s,\tau}\big\|_{\mathrm{TV}} \leq 2 \prod_{i \leq N} (1 - c_i d_i). \quad (2.7)$$

To conclude, we now check that for any conservative positive kernel P on some $\mathcal{X}$, any $\mu, \widetilde{\mu} \in \mathcal{M}(\mathcal{X})$ such that $\mu(\mathcal{X}) = \widetilde{\mu}(\mathcal{X}) < \infty$,

$$\|\mu P - \widetilde{\mu} P\|_{\mathrm{TV}} \leq \frac{1}{2} \sup_{x,y \in \mathcal{X}} \|\delta_x P - \delta_y P\|_{\mathrm{TV}} \|\mu - \widetilde{\mu}\|_{\mathrm{TV}} \leq \sup_{x,y \in \mathcal{X}} \|\delta_x P - \delta_y P\|_{\mathrm{TV}}. \quad (2.8)$$

Indeed, $\mu P - \widetilde{\mu} P = (\mu - \widetilde{\mu}) P = (\mu - \widetilde{\mu})_+ P - (\widetilde{\mu} - \mu)_+ P$ and $(\mu - \widetilde{\mu})_+(\mathcal{X}) = (\widetilde{\mu} - \mu)_+(\mathcal{X})$,

$$(\mu P - \widetilde{\mu} P)(f) = \frac{1}{(\mu - \widetilde{\mu})_+(\mathcal{X})} \int_{\mathcal{X}^2} (\mu - \widetilde{\mu})_+(dx)(\widetilde{\mu} - \mu)_+(dy)(\delta_x P f - \delta_y P f),$$

so we get

$$\|\mu P - \widetilde{\mu} P\|_{\mathrm{TV}} \le \sup_{x,y\in\mathcal{X}} \|\delta_x P - \delta_y P\|_{\mathrm{TV}} (\mu - \widetilde{\mu})_+(\mathcal{X}).$$

This proves (2.8) since, by definition, $\|\mu - \widetilde{\mu}\|_{\mathrm{TV}} = (\mu - \widetilde{\mu})_+(\mathcal{X}) + (\widetilde{\mu} - \mu)_+(\mathcal{X}) = 2(\mu - \widetilde{\mu})_+(\mathcal{X})$ and yields (*ii*).

Proof of (iii). Using now (2.7) and recalling (2.3), for any $x, y \in \mathcal{X}_s$, we have

$$\left\| \frac{\delta_x M_{s,\tau}}{m_{s,\tau}(x)} - \frac{\delta_y M_{s,\tau}}{m_{s,\tau}(y)} \right\|_{\mathrm{TV}} \le 2 \prod_{i\le N} (1 - c_i d_i).$$

Then for any nonzero $\mu \in \mathcal{M}_+(\mathcal{X}_s)$,

$$\begin{aligned}
\left\| \frac{\mu M_{s,\tau}}{\mu(m_{s,\tau})} - \frac{\delta_y M_{s,\tau}}{m_{s,\tau}(y)} \right\|_{\mathrm{TV}} &= \frac{1}{\mu m_{s,\tau}} \left\| \mu M_{s,\tau} - \frac{\mu m_{s,\tau}}{m_{s,\tau}(y)} \delta_y M_{s,\tau} \right\|_{\mathrm{TV}} \\
&\le \frac{1}{\mu(m_{s,\tau})} \int_{\mathcal{X}_s} \mu(dx) \left\| \delta_x M_{s,\tau} - \frac{m_{s,\tau}(x)}{m_{s,\tau}(y)} \delta_y M_{s,\tau} \right\|_{\mathrm{TV}} \\
&= \frac{1}{\mu(m_{s,\tau})} \int_{\mathcal{X}_s} \mu(dx) m_{s,\tau}(x) \left\| \frac{\delta_x M_{s,\tau}}{m_{s,\tau}(x)} - \frac{\delta_y M_{s,\tau}}{m_{s,\tau}(y)} \right\|_{\mathrm{TV}} \\
&\le 2 \prod_{i\le N} (1 - c_i d_i).
\end{aligned}$$

The inequality can be similarly extended from δ_y to a finite measure ν, which proves (*iii*). □

Now (A3) is involved to get the following non-degenerate bound for the mass.

Lemma 2.7 *Let* $0 \le s \le t$ *and* $(c_i, d_i)_{1\le i\le N}$ *satisfying* (A1), (A2) *and* (A3) *for the time subdivision* $s \le t_0 \le \cdots \le t_N \le t$. *For any* $\tau \ge t_N$ *and any measure* $\mu \in \mathcal{M}_+(\mathcal{X}_s)$, *we have*

$$\frac{\mu M_{s,t_N}}{\mu m_{s,t_N}} \left(\frac{m_{t_N,\tau}}{\|m_{t_N,\tau}\|_\infty} \right) \ge \frac{1}{\alpha} \tag{2.9}$$

and for any $x \in \mathcal{X}_s$,

$$\left| \frac{m_{s,\tau}(x)}{\mu(m_{s,\tau})} - \frac{m_{s,t}(x)}{\mu(m_{s,t})} \right| \le 2\alpha \frac{m_{s,t_N}(x)}{\mu(m_{s,t_N})} \prod_{i\le N} (1 - c_i d_i). \tag{2.10}$$

If furthermore $2\alpha \prod_{i\le N}(1 - c_i d_i) < 1$, *then for any* $x \in \mathcal{X}_s$,

$$\left| \frac{m_{s,\tau}(x)}{\mu(m_{s,\tau})} - \frac{m_{s,t}(x)}{\mu(m_{s,t})} \right| \le \frac{m_{s,t}(x)}{\mu(m_{s,t})} \frac{2\alpha \prod_{i\le N}(1 - c_i d_i)}{1 - \alpha \prod_{i\le N}(1 - c_i d_i)}. \tag{2.11}$$

Proof First, using (A1),

$$\frac{\mu M_{s,t_N}}{\mu(m_{s,t_N})} = \frac{\mu M_{s,t_{N-1}} M_{t_{N-1},t_N}}{\mu(m_{s,t_N})} \ge c_N \frac{\mu M_{s,t_{N-1}}(m_{t_{N-1},t_N})}{\mu(m_{s,t_N})} \nu_N = c_N \nu_N.$$

Moreover (A3) ensures that for any $\tau \geq t_N$,

$$\|m_{t_N,\tau}\|_\infty \leq \alpha c_N \nu_N(m_{t_N,\tau}).$$

This proves (2.9). Now, the semigroup property yields

$$\frac{m_{s,\tau}(x)}{\mu(m_{s,\tau})} = \frac{\delta_x M_{s,t_N} M_{t_N,\tau}\mathbf{1}}{\mu M_{s,t_N} M_{t_N,\tau}\mathbf{1}} = \frac{m_{s,t_N}(x)}{\mu(m_{s,t_N})} \frac{\frac{\delta_x M_{s,t_N}}{m_{s,t_N}(x)}(m_{t_N,\tau})}{\frac{\mu M_{s,t_N}}{\mu(m_{s,t_N})}(m_{t_N,\tau})}.$$

Then

$$\frac{m_{s,\tau}(x)}{\mu(m_{s,\tau})} - \frac{m_{s,t_N}(x)}{\mu(m_{s,t_N})} = \frac{m_{s,t_N}(x)}{\mu(m_{s,t_N})} \frac{\left[\frac{\delta_x M_{s,t_N}}{m_{s,t_N}(x)} - \frac{\mu M_{s,t_N}}{\mu(m_{s,t_N})}\right](m_{t_N,\tau})}{\frac{\mu M_{s,t_N}}{\mu(m_{s,t_N})}(m_{t_N,\tau})}.$$

Dividing by $\|m_{t_N,\tau}\|_\infty$ and using $m_{t_N,\tau} \geq 0$ and recalling that $\gamma(\mathcal{X}) = 0$ and $f \geq 0$ implies that $|\gamma(f)| \leq \frac{1}{2}\|\gamma\|_{\mathrm{TV}}\|f\|_\infty$, we get

$$\left|\frac{m_{s,\tau}(x)}{\mu(m_{s,\tau})} - \frac{m_{s,t_N}(x)}{\mu(m_{s,t_N})}\right| \leq \frac{m_{s,t_N}(x)}{\mu(m_{s,t_N})} \frac{\frac{1}{2}\left\|\frac{\delta_x M_{s,t_N}}{m_{s,t_N}(x)} - \frac{\mu M_{s,t_N}}{\mu(m_{s,t_N})}\right\|_{\mathrm{TV}}}{\frac{\mu M_{s,t_N}}{\mu(m_{s,t_N})}\left(\frac{m_{t_N,\tau}}{\|m_{t_N,\tau}\|_\infty}\right)}.$$

Now combining Lemma 2.5 (*iii*) and (2.9) yields

$$\left|\frac{m_{s,\tau}(x)}{\mu(m_{s,\tau})} - \frac{m_{s,t_N}(x)}{\mu(m_{s,t_N})}\right| \leq \alpha \frac{m_{s,t_N}(x)}{\mu(m_{s,t_N})} \prod_{i \leq N}(1 - c_i d_i) \tag{2.12}$$

and using twice this bound proves (2.10) by triangular inequality.

Finally, (2.12) also gives, for $\tau = t$,

$$\frac{m_{s,t_N}(x)}{\mu(m_{s,t_N})} \leq \frac{m_{s,t}(x)}{\mu(m_{s,t})} \frac{1}{1 - \alpha \prod_{i \leq N}(1 - c_i d_i)}.$$

Then (2.10) implies (2.11). □

Using the previous results, we now prove the existence of harmonic functions and Theorem 2.3.

Proof of Theorem 2.3 We fix $s \geq 0$, $\nu \in \mathcal{P}(\mathcal{X}_s)$ and $\beta > 0$. We begin by proving that there exists a function h_s positive and bounded such that for any $x \in \mathcal{X}_s$ and any $t \geq s$,

$$\left|\frac{m_{s,t}(x)}{\nu(m_{s,t})} - h_s(x)\right| \leq 2\alpha e^{-C_{\alpha,\beta,\nu}(s,t)} \min\left\{\beta, \frac{m_{s,t}(x)}{\nu(m_{s,t})} \frac{1}{(1 - \alpha e^{-C_{\alpha,\beta,\nu}(s,t)})_+}\right\}. \tag{2.13}$$

First, optimizing Inequality (2.10) over all the admissible coupling constants yields

$$\left|\frac{m_{s,\tau}(x)}{\nu(m_{s,\tau})} - \frac{m_{s,t}(x)}{\nu(m_{s,t})}\right| \leq 2\beta\alpha e^{-C_{\alpha,\beta,\nu}(s,t)} \tag{2.14}$$

by recalling Definition 2.1 and that (A4) guarantees $m_{s,t_N}(x)/\nu(m_{s,t_N}) \leq \beta$.

Using that $C_{\alpha,\beta,\nu}(s,t) \to \infty$ as $t \to \infty$, Cauchy criterion ensures that the following limit exists

$$h_s(x) = \lim_{\tau\to\infty} \frac{m_{s,\tau}(x)}{\nu(m_{s,\tau})}. \tag{2.15}$$

Moreover, letting $\tau \to \infty$ in (2.14) shows that

$$\left|\frac{m_{s,t}(x)}{\nu(m_{s,t})} - h_s(x)\right| \le 2\beta\alpha \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}.$$

Optimizing now similarly over coupling constants in (2.11) and letting $\tau \to \infty$ yields

$$\left|\frac{m_{s,t}(x)}{\nu(m_{s,t})} - h_s(x)\right| \le \frac{m_{s,t}(x)}{\nu(m_{s,t})} \frac{2\alpha \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}}{1-\alpha \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}},$$

for any t such that $\alpha \exp(-C_{\alpha,\beta,\nu}(s,t)) < 1$. Combining these two bounds proves (2.13).

Integrating (2.13) over some $\mu \in \mathcal{M}_+(\mathcal{X}_s)$, we get

$$\left|\mu(m_{s,t}) - \mu(h_s)\nu(m_{s,t})\right| \le 2\alpha \min\left\{\beta\mu(\mathcal{X})\nu(m_{s,t}), \frac{\mu(m_{s,t})}{(1-\alpha \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)})_+}\right\} \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}. \tag{2.16}$$

Moreover Lemma 2.5 (*iii*) yields (after optimization over coupling constants) for non-zero μ,

$$\left\|\frac{\mu M_{s,t}}{\mu(m_{s,t})} - \frac{\nu M_{s,t}}{\nu(m_{s,t})}\right\|_{\mathrm{TV}} \le 2\mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)} \tag{2.17}$$

and combining the two previous inequalities gives

$$\begin{aligned}
&\left\|\mu M_{s,t} - \mu(h_s)\nu(m_{s,t})\frac{\nu M_{s,t}}{\nu(m_{s,t})}\right\|_{\mathrm{TV}} \\
&\le \left\|\mu M_{s,t} - \mu(m_{s,t})\frac{\nu M_{s,t}}{\nu(m_{s,t})}\right\|_{\mathrm{TV}} + \left|\mu(m_{s,t}) - \mu(h_s)\nu(m_{s,t})\right| \left\|\frac{\nu M_{s,t}}{\nu(m_{s,t})}\right\|_{\mathrm{TV}} \\
&\le 2\left(\mu(m_{s,t}) + \alpha \min\left\{\beta\mu(\mathcal{X})\nu(m_{s,t}), \frac{\mu(m_{s,t})}{(1-\alpha \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)})_+}\right\}\right) \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}.
\end{aligned}$$

Using again Inequality (2.17), with $\mu = \gamma M_{s_0,s}$, we obtain

$$\begin{aligned}
&\left\|\mu M_{s,t} - \mu(h_s)\nu(m_{s,t})\frac{\gamma M_{s_0,t}}{\gamma(m_{s_0,t})}\right\|_{\mathrm{TV}} \qquad (2.18)\\
&\le 2\left(\mu(m_{s,t}) + \mu(h_s)\nu(m_{s,t}) + \alpha \min\left\{\beta\mu(\mathcal{X})\nu(m_{s,t}), \frac{\mu(m_{s,t})}{(1-\alpha \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)})_+}\right\}\right) \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}.
\end{aligned}$$

To conclude it remains to control $\mu(m_{s,t})$ and $\mu(h_s)$. First, we notice that h_s is bounded by β using (A4) and (2.15). Using again (A4), we have

$$\mu(m_{s,t}) \le \beta\mu(\mathcal{X})\nu(m_{s,t})$$

and the first bound of (2.18) yields

$$\left\|\mu M_{s,t} - \mu(h_s)\nu(m_{s,t})\frac{\gamma M_{s_0,t}}{\gamma(m_{s_0,t})}\right\|_{\mathrm{TV}} \le 2(2+\alpha)\beta\mu(\mathcal{X})\nu(m_{s,t})\mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}.$$

Moreover, if $C_{\alpha,\beta,\nu}(s,t) > \log(3\alpha)$, using the second part of (2.16) and the fact that $|a-b| \leq \eta|b|$ and $\eta \in [0,1)$ imply that $|b| \leq |a|/(1-\eta)$ ensures that

$$\mu(m_{s,t}) \leq \frac{1-\alpha \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}}{1-3\alpha \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}}\mu(h_s)\nu(m_{s,t}),$$

so that the second part of (2.18) becomes

$$\left\| \mu M_{s,t} - \mu(h_s)\nu(m_{s,t})\frac{\gamma M_{s_0,t}}{\gamma(m_{s_0,t})}\right\|_{\mathrm{TV}} \leq 2\frac{2+\alpha-4\alpha \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}}{1-3\alpha \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}}\mu(h_s)\nu(m_{s,t})\mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}.$$

This proves the estimate stated in Theorem 2.3 when $C_{\alpha,\beta,\nu}(s,t) \geq \log(4\alpha)$. Finally, this estimate applied to $\mu = \delta_x$ ensures that

$$\left|m_{s,t}(x) - h_s(x)\nu(m_{s,t})\right| \leq 8(2+\alpha)h_s(x)\,\nu(m_{s,t})\,\mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}$$

and then

$$\left(1+8(2+\alpha)\mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}\right)h_s(x) \geq \frac{m_{s,t}(x)}{\nu(m_{s,t})} > 0,$$

so that $h_s > 0$. The fact that $\nu(h_s) = 1$ follows directly from (2.15) and dominated convergence theorem, while uniqueness of h_s is derived letting t go to infinity. □

3 Applications

In the present section, we develop different applications of Theorem 2.3. We aim at illustrating the main result and show how to check the required assumptions. Yet we also obtain new estimates and mention that the models can be made more complex.

We first consider the heat equation with growth and reflecting boundary on the compact set $[0,1]$ and time space inhomogeneity. The coupling capacity is then expressed in terms of the function describing the diffusion coefficient.

Then we prove general statements when the semigroup is homogeneous, asymptotically homogeneous, and periodic. The results are illustrated by asymptotic estimates for the renewal equation.

3.1 A Growth-Diffusion Equation with Reflecting Boundary and Varying Environment

In this section $\mathcal{X}_t = \mathcal{X} = [0,1]$ for every $t \geq 0$. We consider a population of particles which reproduce and move following a diffusion varying in time. The evolution of its density is prescribed by the following PDE

$$\begin{cases} \partial_t u_{s,t}(x) = \frac{1}{2}\sigma_t \Delta u_{s,t}(x) + r(x)u_{s,t}(x), \quad 0 < x < 1, \\ \partial_x u_{s,t}(0) = \partial_x u_{s,t}(1) = 0, \\ \quad u_{s,s}(x) = \phi(x), \end{cases} \tag{3.1}$$

for some $\phi \in L^1([0,1])$. As usual, we do not stress the dependence on ϕ of u. This equation is the nonautonomous Heat Equation with growth under Neumann boundary conditions.

More precisely, particles diffuse with coefficients $(\sigma_t)_{t\geq 0}$ on the space $[0, 1]$. The growth rate $r(x)$ is the difference between birth and death rate at position $x \in [0, 1]$.

In this example, σ is time-dependent but not space-dependent and conversely for r. Our coupling methods provide a relevant approach for estimating the speed of convergence in this varying environment case. This analysis could be easily generalized for both time-dependant and space-dependant parameters but would provide tedious computations, so it is left for future works.

Let us work with another representation of the solution of (3.1). Let $(X^x_{s,t})_{t\geq s}$ be a reflected Brownian motion on $[0, 1]$ starting from x at time s, with diffusion coefficient σ_t at time t, see (3.3) below for a construction. We define the positive semigroup M by

$$M_{s,t} f = \mathbb{E}_x\left[f\left(X^x_{s,t}\right)e^{\int_s^t r(X^x_{s,u})\,du}\right] \tag{3.2}$$

for every bounded Borel function f on $[0, 1]$, and $t \geq s \geq 0$. Then $\mu M_{s,t}$ is defined by setting for all $f \in \mathcal{B}_b([0, 1])$

$$(\mu M_{s,t})(f) = \mu(M_{s,t} f).$$

Feynman-Kac formula [44, Chapter VII Proposition (3.10) p. 358] states the duality relation of this semigroup with the solution u of (3.1):

$$\int_0^1 \phi(x) M_{s,t} f(x)\,dx = \int_0^1 u_{s,t}(x) f(x)\,dx$$

for every bounded measurable function f. This property allows to see the mapping $t \to \mu M_{s,t}$ as the unique solution to Eq. (3.1) when the initial density ϕ is replaced by a measure μ.

3.1.1 Statements

We assume that $t \mapsto \sigma_t$ is a non-negative and càdlàg function, r is continuous and

$$-\infty < \underline{r} := \inf_{x\in[0,1]} r(x); \qquad \overline{r} := \sup_{x\in[0,1]} r(x) < +\infty.$$

Introduce the function g with value in $\mathbb{R} \cup \{+\infty\}$ to measure the coupling capacity in function of the parameters

$$g : (s, t) \mapsto (\overline{r} - \underline{r})(t - s) - \log\big((1 - 4/\sigma_{s,t})_+\big),$$

where

$$\sigma_{s,t} := \sqrt{2\pi \int_s^t \sigma_u^2\,du}.$$

These functions allow to control the coupling capacity in this model by considering

$$\mathfrak{C}_{\tau,\rho}(s, t) = \sup_{\mathcal{T}_{\tau,\rho}(s,t)} \left\{-\sum_{i=1}^{N} \log\Big(1 - \exp\big(-\big(g(t_{i-1}, t_i) + g(t_i, t_{i+1})\big)\big)\Big)\right\},$$

where $\mathcal{T}_{\tau,\rho}(s, t)$ is the set of subdivisions $(t_i)_{i=0}^{N+1}$ such that $N \geq 1$, $s = t_0 \leq \cdots \leq t_{N+1} \leq t$ and

$$t_1 - t_0 \leq \rho,\ t_N - t_{N-1} \leq \tau,\ t_{N+1} - t_N \leq \tau \quad \text{and} \quad \text{for } i \in \{0, N-1, N\},\ \sigma_{t_i,t_{i+1}} \geq 5.$$

Indeed, $\mathfrak{C}_{\tau,\rho}(s,t)$ is a lower bound of (2.1). The constant 5 may be improved and replaced for instance by $4+\varepsilon$, but we restrict ourselves here to this value which allows to get a simple expression of the coupling constants α and β, namely $\alpha = \gamma_\tau^2$, $\beta = \gamma_\rho\gamma_\tau$ where

$$\gamma_s = 5\exp\big((\bar{r} - \underline{r})s\big) \in [5, \infty).$$

The first time interval of size ρ is involved in the control of the mass and the expression of β. A general quantitative bound can now be given as follows, writing λ the Lebesgue measure on [0, 1].

Theorem 3.1 *Let $s \geq 0$ and $\tau > 0$. Assume that $\mathfrak{C}_{\tau,\rho}(s,t) \to \infty$ as $t \to \infty$. Then there exists a function $h_s : [0,1] \to (0, \gamma_\rho\gamma_\tau]$ and probabilities $(\pi_t)_{t\geq 0}$ such that*

$$\big\|\mu M_{s,t} - \mu(h_s)\lambda(m_{s,t})\pi_t\big\|_{\mathrm{TV}} \leq 8\big(2+\gamma_\tau^2\big)|\mu|(h_s)\lambda(m_{s,t})\mathrm{e}^{-\mathfrak{C}_{\tau,\rho}(s,t)},$$

for any $\mu \in \mathcal{M}([0,1])$ and t such that $\mathfrak{C}_{\tau,\rho}(s,t) \geq 2\log(2) + 2\log(\gamma_\tau)$.

The proof of Theorem 3.1 is postponed to Sect. 3.1.2. Let us now illustrate this result by constructing a relevant lower bound of $\mathfrak{C}_{\tau,\rho}(s,t)$. Let $s \geq 0$, $\tau > 0$ and set

$$t_1(s,\tau) = \inf\{u \geq s : \sigma_{s,u} \geq 10\},$$

and the sequence $(t_k(s,\tau))_k$ defined by induction:

$$t_{k+1}(s,\tau) = \inf\big\{u \geq t_k(s,\tau) + \tau : \sigma_{u,u+\tau} \geq 10\big\} \quad (k \geq 1),$$

using again the convention $\inf\varnothing = +\infty$. As $(s,t) \mapsto \sigma_{s,t}$ is continuous, there exists $t'_k(s,\tau)$ such that

$$\sigma_{t_k(s,\tau),t'_k(s,\tau)} \geq 5 \quad \text{and} \quad \sigma_{t'_k(s,\tau),t_k(s,\tau)+\tau} \geq 5.$$

Using then the time subdivision $s, t_1(s,\tau), \ldots, t_k(s,\tau), t'_k(s,\tau), t_k(s,\tau)+\tau, \ldots$ $(k \geq 2)$, we get an upper bound for $g(t'_k(s,\tau) - t_k(s,\tau))$ and $g(t_k(s,\tau) + \tau - t'_k(s,\tau))$ and

$$\mathfrak{C}_{\tau,t_1(s,\tau)-s}(s,t) \geq -\log\big(1 - 1/\gamma_\tau^2\big)\max\big\{N : t_{N+1}(s,\tau) \leq t - \tau\big\}.$$

We derive then immediately a speed of convergence from Theorem 3.1.

Corollary 3.2 *Let $s \geq 0$ and $\tau > 0$. Assume that $t_k(s,\tau) < \infty$ for any $k \geq 1$.*

Then there exists a positive bounded function h_s and probabilities $(\pi_t)_{t\geq 0}$ such that

$$\liminf_{t\to\infty} -\frac{1}{t}\log\left\|\frac{\mu M_{s,t}}{\lambda(m_{s,t})} - \mu(h_s)\pi_t\right\|_{\mathrm{TV}} \geq -\log\big(1 - 1/\gamma_\tau^2\big).\liminf_{k\to\infty}\frac{k}{t_k(s,\tau)},$$

uniformly over $\mu \in \mathcal{P}([0,1])$. Moreover h_s is bounded by $\gamma_\tau\gamma_{t_1(s,\tau)-s}$.

As soon as $\limsup_i t_i(s,\tau)/i < \infty$, we obtain an exponential speed. Note also that superexponential or subexponential speed could be obtained by alternative constructions of time sequences.

As an application for exponential convergence in random environment, let us consider a Feller càdlàg Markov process $(t,w) \in [0,\infty) \times \Omega \to \sigma_t(w) \in [0,\infty)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that the process $(\sigma_t)_{t\geq 0}$ is Harris positive recurrent with stationary probability $\pi \neq \delta_0$. For each $w \in \Omega$, we write $M_{s,t} = M_{s,t}(w)$ the semigroup defined by (3.2) and associated to the diffusion coefficient $(\sigma_t(w))_{t\geq 0}$. By a Birkhoff Theorem (see [37, Theorem 8.1] and [37, Theorem 3.2]), we obtain that $\max\{N : t_{N+1}(s,\tau) \leq t-\tau\}$ grows linearly as $t \to \infty$ with a deterministic speed. It yields the following quenched estimate: there exists $v > 0$ such that

$$\mathbb{P}\left(\liminf_{t\to\infty} -\frac{1}{t}\log\left(\sup_{\mu\in\mathcal{P}([0,1])}\left\|\frac{\mu M_{s,t}}{\lambda(m_{s,t})} - \mu(h_s)\pi_t\right\|_{\mathrm{TV}}\right) \geq v\right) = 1,$$

and the convergence is uniform and (at least) exponential. Let us observe that the diffusion may be zero for arbitrarily large time intervals.

3.1.2 *Proof of Theorem 3.1*

We begin by the construction and some useful estimates for process $(X^x_{s,t})_{s,t}$, which is associated to a nonhomogeneous diffusion [0, 1] with Neumann boundary condition. Let $(B_t)_{t\geq 0}$ be a classical Brownian motion on $\mathbb{R}$, and set $(W^x_{s,t})_{t\geq s\geq 0}$ defined for all $x \in [0,1]$ and $t \geq s \geq 0$ by

$$W^x_{s,t} = x + \int_s^t \sigma_u \, dB_u. \tag{3.3}$$

The random variable $W^x_{s,t}$ is distributed according to a Gaussian law $\mathcal{N}(x, \sigma_{s,t})$. The reflected process $(X^x_{s,t})$ can now be defined by

$$\forall t \geq s \geq 0, \quad X^x_{t,s} = \sum_{n\in\mathbb{Z}} \big(W^x_{s,t} - 2n\big)\mathbf{1}_{W^x_{s,t}\in[2n,2n+1]} + \big(2n - W^x_{s,t}\big)\mathbf{1}_{W^x_{s,t}\in[2n-1,2n]}.$$

Lemma 3.3 (Bounds on the density for the diffusion) *For any $t > s \geq 0$, there exists $c_{s,t} \geq 1$ such that for any Borel set A of* $[0,1]$,

$$\left(c_{s,t} - \frac{4}{\sigma_{s,t}}\right)_+ \lambda(A) \leq \mathbb{P}\big(X^x_{s,t} \in A\big) \leq c_{s,t}\lambda(A),$$

with the convention $1/0 = \infty$.

Proof We define

$$\phi^x_{s,t}(y) = \frac{1}{\sqrt{2\pi\sigma_{s,t}}} \exp\left(-\frac{(y-x)^2}{2\sigma_{s,t}}\right)$$

the density of $W^x_{s,t}$. Using that

$$\mathbb{P}_x(X_{s,t} \in A) = \sum_{n\in\mathbb{Z}} \mathbb{P}\big(W^x_t \in (A+2n)\cap[2n,2n+1]\big) + \mathbb{P}\big(W^x_t \in (2n-A)\cap[2n-1,2n]\big),$$

we obtain

$$d^x_{s,t}(A)\lambda(A) \leq \mathbb{P}_x(X_{s,t} \in A) \leq c^x_{s,t}(A)\lambda(A), \tag{3.4}$$

with

$$c^x_{s,t}(A) = \sum_{n\in\mathbb{Z}} \Big(\sup_{A+2n} \phi^x_{s,t} + \sup_{2n-A} \phi^x_{s,t}\Big),$$

$$d_{s,t}^x(A) = \sum_{n\in\mathbb{Z}} \Big(\inf_{A+2n} \phi_{s,t}^x + \inf_{2n-A} \phi_{s,t}^x \Big).$$

Using (3.4), these constants verify

$$c_{s,t} := \sup_{x\in[0,1]} c_{s,t}^x\big([0,1]\big) \geq c_{s,t}^x(A) \geq 1 \geq d_{s,t}^x(A) \geq \inf_{x\in[0,1]} d_{s,t}^x\big([0,1]\big) =: d_{s,t}.$$

It remains to prove an upperbound of the difference $c_{s,t} - d_{s,t}$ and conclude. Decomposing the sum following $n = 0$, $n \geq 1$ and $n \leq -1$ shows on a first hand that

$$c_{s,t} = \sup_{x\in[0,1]} \sum_{n\in\mathbb{Z}} \sup_{[n,n+1]} \phi_{s,t}^x = \sup_{x\in[0,1]} \Big(\phi_{s,t}^0(0) + \sum_{n\in\mathbb{Z}} \phi_{s,t}^0(n-x) \Big),$$

and on the other hand

$$d_{s,t} = \inf_{x\in[0,1]} \sum_{n\in\mathbb{Z}} \inf_{[n,n+1]} \phi_{s,t}^x = \inf_{x\in[0,1]} \Big(\phi_{s,t}^0\big(\max(x,1-x)\big) + \sum_{n\in\mathbb{Z}\setminus\{0,1\}} \phi_{s,t}^0(n-x) \Big).$$

Combining

$$c_{s,t} \leq \phi_{s,t}^0(0) + \sum_{n\in\mathbb{Z}} \sup_{x\in[0,1]} \phi_{s,t}^0(n-x) = 2\phi_{s,t}^0(0) + \sum_{n\in\mathbb{Z}} \phi_{s,t}^0(n),$$

and

$$\begin{aligned} d_{s,t} &\geq \inf_{x\in[0,1]} \phi_{s,t}^0\big(\max(x,1-x)\big) + \sum_{n\in\mathbb{Z}\setminus\{0,1\}} \inf_{x\in[0,1]} \phi_{s,t}^0(n-x) \\ &= \phi_{s,t}^0(1) + \sum_{n\in\mathbb{Z}\setminus\{-1,0,1\}} \phi_{s,t}^0(n) = \sum_{n\in\mathbb{Z}\setminus\{-1,0\}} \phi_{s,t}^0(n), \end{aligned}$$

we obtain $c_{s,t} - d_{s,t} \leq 4\phi_{s,t}^0(0)$. This ends the proof. □

We give now the coupling constants and introduce

$$\overline{\sigma}_{s,t} = \Big(1 - \frac{4}{\sigma_{s,t}}\Big) \mathbf{1}_{\sigma_{s,t}>4}.$$

Lemma 3.4 (Coupling constants) *Let $s = t_0 \leq \cdots \leq t_N \leq t_{N+1} \leq t$.*

(i) *Assumptions (A1) and (A2) are satisfied with the constants*

$$c_i = \overline{\sigma}_{t_{i-1},t_i} \mathrm{e}^{-(\overline{r}-\underline{r})(t_i - t_{i-1})}, \quad i = 1, \ldots, N+1,$$

and

$$d_i = c_{i+1} = \overline{\sigma}_{t_i,t_{i+1}} \mathrm{e}^{-(\overline{r}-\underline{r})(t_{i+1} - t_i)}, \quad i = 1, \ldots, N,$$

and $\nu_i = \lambda$ for $i = 1, \ldots, N+1$.

(ii) *If $\sigma_{t_0,t_1} > 4$, $\sigma_{t_{N-1},t_N} > 4$, $\sigma_{t_N,t_{N+1}} > 4$, then (A3) and (A4) hold with*

$$\alpha = \frac{\mathrm{e}^{(\overline{r}-\underline{r})(t_{N+1}-t_{N-1})}}{\overline{\sigma}_{t_{N-1},t_N} \overline{\sigma}_{t_N,t_{N+1}}}, \qquad \beta = \frac{\mathrm{e}^{(\overline{r}-\underline{r})(t_1 - t_0)}}{\overline{\sigma}_{t_0,t_1}},$$

and $\nu = \lambda$.

Proof Lemma 3.3 and Eq. (3.2) ensure that for any probability measure μ, $u > v \geq 0$

$$d_{u,v}\mathrm{e}^{\underline{r}(v-u)}\lambda \leq \mu M_{u,v} \leq c_{u,v}\mathrm{e}^{\overline{r}(v-u)}\lambda,$$

where $d_{u,v} := (c_{u,v} - 4/\sigma_{u,v})_+$. Then

$$\overline{\sigma}_{u,v}c_{u,v}\mathrm{e}^{\underline{r}(v-u)}\lambda \leq \mu M_{u,v} \leq c_{u,v}\mathrm{e}^{\overline{r}(v-u)}\lambda, \tag{3.5}$$

recalling that $c_{u,v} \geq 1$. In particular

$$\overline{\sigma}_{u,v}c_{u,v}\mathrm{e}^{\underline{r}(v-u)} \leq \mu(m_{u,v}) \leq c_{u,v}\mathrm{e}^{\overline{r}(v-u)}. \tag{3.6}$$

Combining the two last bounds, we find

$$\mu M_{u,v} \geq \overline{\sigma}_{u,v}c_{u,v}\mathrm{e}^{\underline{r}(v-u)}\lambda \geq \overline{\sigma}_{u,v}\mathrm{e}^{-(\overline{r}-\underline{r})(v-u)}\,\mu(m_{u,v})\lambda$$

and obtain the expected value of c_i, with $u = t_{i-1}$, $v = t_i$.

Moreover for all $w \in (u, v]$ such that $d_{u,w} > 0$, we also have by integration of (3.5)

$$\overline{\sigma}_{u,w}c_{u,w}\mathrm{e}^{\underline{r}(w-u)}\lambda(m_{w,v}) \leq \mu(m_{u,w}) \leq c_{u,w}\mathrm{e}^{\overline{r}(w-u)}\lambda(m_{w,v}).$$

Taking respectively $\mu = \delta_x$ and $\mu = \lambda$ for the second (resp. first) inequality,

$$\overline{\sigma}_{u,w}\mathrm{e}^{-(\overline{r}-\underline{r})(w-u)}m_{u,v}(x) \leq \lambda(m_{u,v}) \tag{3.7}$$

and we get the value of d_i, with $u = t_i$, $w = t_{i+1}$, $v = \tau$.

Let us turn to the proof of (*ii*). Using again (3.7) with now $u = t_N$, $w = t_{N+1}$, $v = \tau$ yields

$$m_{t_N,T}(x) \leq \frac{1}{\overline{\sigma}_{t_N,t_{N+1}}}\mathrm{e}^{-(\overline{r}-\underline{r})(t_{N+1}-t_N)}\lambda(m_{t_N,\tau}).$$

Recalling the expression of c_N from (*i*) provides the expected value of α. Finally, using (3.7) with $u = s$, $w = t_1$, $v = t_N$ yields β. □

Remark that the previous proof could be achieved, without probabilistic notation. Indeed, bounding r, one can build sub and super solutions, which, up to renormalization, satisfy Eq. (3.1) with $r = 0$. It is then enough to use Lemma 3.3 which is only based on the explicit solutions of (3.1) with constant r.

Proof of Theorem 3.1 Consider a sequence $(t_i)_{i=0}^{N+1}$ such that $N \geq 1$, $s = t_0 \leq \cdots \leq t_{N+1} \leq t$ and

$$t_0 - t_1 \leq \rho,\ t_N - t_{N-1} \leq \tau,\ t_{N+1} - t_N \leq \tau \quad \text{and} \quad \text{for } i \in \{0, N-1, N\},\ \sigma_{t_{i+1},t_i} \geq 5.$$

By Lemma 3.4, the following constants

$$c_i = \mathrm{e}^{-g(t_{i-1},t_i)}, \qquad d_i = \mathrm{e}^{-g(t_i,t_{i+1})}$$

are (α, β, λ) admissible coupling constants for M on $[s, t]$, with $\alpha = \gamma_\tau^2$ and $\beta = \gamma_\rho$. Then, optimizing over admissible coupling constants yields

$$C_{\alpha,\beta,\lambda}(s, t) \geq \mathfrak{C}_{\tau,\rho}(s, t),$$

and applying Theorem 2.3 ends the proof. □

3.2 Homogeneous Semigroups and the Renewal Equation

First, we specify the general result in the homogeneous setting, which simplifies the assumptions. Second, we develop an application to the renewal equation.

3.2.1 *Existence of Eigenelements and Speed of Convergence*

In this subsection we consider a semigroup $(M_{s,t})_{0\le s\le t}$ which is homogeneous, meaning that the sets $\mathcal{X}_t = \mathcal{X}$ are time-independent and that there exists a semigroup $(M_t)_{t\ge 0}$ set on $\mathcal{X}$ such that $M_{s,t} = M_{t-s}$ for all $t \ge s \ge 0$. In this case Assumptions (A1), (A2), (A3) and (A4) can be simplified as follows.

First, there exist constants $c > 0$, $r > 0$ and a probability measure ν such that for any $x \in \mathcal{X}$,

$$\delta_x M_r \ge c\, m_r(x)\, \nu. \tag{H1}$$

Second, there exists a constant $d > 0$ such that for any $t \ge 0$,

$$\nu(m_t) \ge d\, \|m_t\|_\infty, \tag{H2}$$

where we recall the notation $m_t = M_t\mathbf{1}$. These assumptions have been obtained in [12] for the study of process conditioned on non-absorption. Let us note that [12] also prove that they are necessary conditions for uniform exponential convergence in total variation distance. Additionally the main result can be strengthened, provided an additional assumption:

$$\text{The function } t \mapsto \|m_t\|_\infty \text{ is locally bounded on } \mathbb{R}_+. \tag{H3}$$

Notice that this last assumption is satisfied by classical semigroups appearing in applications.

Theorem 3.5 *Under Hypotheses* (H1), (H2) *and* (H3), *there exists a unique triplet* $(\gamma, h, \lambda) \in \mathcal{P}(\mathcal{X}) \times \mathcal{B}_b(\mathcal{X}) \times \mathbb{R}$ *such that* $\gamma(h) = 1$ *and for all* $t \ge 0$,

$$\gamma M_t = \mathrm{e}^{\lambda t}\gamma \quad \text{and} \quad M_t h = \mathrm{e}^{\lambda t} h. \tag{3.8}$$

Additionally h is bounded and positive on $\mathcal{X}$ and there exists $C > 0$ such that for all $t \ge 0$ and $\mu \in \mathcal{M}(\mathcal{X})$,

$$\left\| \mathrm{e}^{-\lambda t}\mu M_t - \mu(h)\gamma \right\|_{\mathrm{TV}} \le C\, \|\mu\|_{\mathrm{TV}} (1 - cd)^{t/r}.$$

Theorem 3.5 strictly extends the main result of [11]. For instance their theorems do not apply for the semigroup of Sect. 3.2.2 below. This semigroup cannot be written as a the semigroup of a killed Markov process, even up to exponential normalization.

Remark 3.6 By differentiating (3.8), the triplet (γ, h, λ) is a triplet of eigenelements for the infinitesimal generator of $(M_t)_{t\ge 0}$, that is $\gamma\mathcal{A} = \lambda\gamma$ and $\mathcal{A}h = \lambda h$ where the (unbounded) operator $\mathcal{A}$ is defined by $\mathcal{A} = \lim_{t\to 0} \frac{1}{t}(M_t - I)$.

Remark 3.7 With Lemma 2.5 (*iii*), we also recover (as expected) the statement from [12]:

$$\left\| \frac{\mu M_t}{\mu(m_t)} - \gamma \right\|_{\mathrm{TV}} \le 2(1 - cd)^{t/r}.$$

Proof Assume (H1) and (H2) and consider $(t_i)_{0\le i\le N}$ defined by $t_i = ir$. Then Assumptions (A1)–(A4) hold with, for every $0 \le i \le N$,

$$c_i = c, \qquad d_i = d, \qquad \nu_i = \nu, \qquad \alpha = \frac{1}{cd}, \qquad \beta = \frac{1}{d}.$$

This implies

$$C_{\alpha,\beta,\nu}(0,t) \ge \lfloor (t-r)/r \rfloor \log(1-cd).$$

By Theorem 2.3 applied to M on $[0,t]$, there exist $C > 0$ and $h_0 : \mathcal{X} \to [0,\infty)$ such that for any $\mu, \widetilde{\mu} \in \mathcal{M}(\mathcal{X})$

$$\left\| \mu M_t - \mu(h_0)\nu(m_t)\frac{\widetilde{\mu} M_t}{\widetilde{\mu}(m_t)} \right\|_{\mathrm{TV}} \le C\,\nu(m_t)\|\mu\|_{\mathrm{TV}}(1-cd)^{t/r}. \tag{3.9}$$

Moreover, taking $\mu = \delta_x$,

$$h_0 = \lim_{T\to\infty} \frac{m_T}{\nu(m_T)}$$

for the supremum norm and h_0 is positive, bounded by $1/d$, and $\nu(h_0) = 1$. By the semigroup property we have $m_{t+s}(x) = \delta_x M_t m_s$ from which we deduce that

$$\frac{m_{t+s}(x)}{\nu(m_{t+s})}\frac{\nu M_t m_s}{\nu(m_s)} = \delta_x M_t\left(\frac{m_s}{\nu(m_s)}\right).$$

Letting $s \to \infty$, we get

$$(\nu M_t h_0) h_0 = M_t h_0.$$

This means that h_0 is an eigenvector of M_t associated to the eigenvalue $\nu M_t h_0$. Since the semigroup property yields

$$\nu M_{t+s} h_0 = \nu M_t M_s h_0 = (\nu M_t h_0).(\nu M_s h_0)$$

and $t \mapsto \nu M_t h_0$ is locally bounded (by assumption (H3), because $0 \le \nu M_t h_0 \le \|m_t\|_\infty/d$), we deduce the existence of $\lambda \in \mathbb{R}$ such that

$$\nu M_t h_0 = \mathrm{e}^{\lambda t}\nu(h_0) = \mathrm{e}^{\lambda t}.$$

Let us now show the existence of a left eigenvector γ. Applying (3.9) to $\mu = \nu$ and $\widetilde{\mu} = \nu M_s$ we get that

$$\left\| \frac{\nu M_t}{\nu(m_t)} - \frac{\nu M_{t+s}}{\nu(m_{t+s})} \right\|_{\mathrm{TV}} \le C(1-cd)^{t/r}.$$

This ensures that the family $(\frac{\nu M_t}{\nu(m_t)})_{t\ge 0}$ satisfies the Cauchy property and we deduce the existence of a probability measure γ such that

$$\left\| \frac{\nu M_t}{\nu(m_t)} - \gamma \right\|_{\mathrm{TV}} \le C(1-cd)^{t/r}.$$

Then we use the semigroup property to write that for all $s, t \ge 0$ we have

$$\frac{\nu M_s}{\nu(m_s)} M_t = \nu M_t\left(\frac{m_s}{\nu(m_s)}\right)\frac{\nu M_{s+t}}{\nu(m_{s+t})}$$

and letting s tend to infinity we find

$$\gamma M_t = (\nu M_t h_0)\,\gamma = \mathrm{e}^{\lambda t}\gamma.$$

Now we set $h = h_0/\gamma(h_0)$, so that $\gamma(h) = 1$. Applying (3.9) to $\mu = \widetilde{\mu} = \gamma$ and dividing by $\nu(m_t)$ yields

$$\left|\frac{\mathrm{e}^{\lambda t}}{\nu(m_t)} - \gamma(h_0)\right| \le C(1-cd)^{t/r}.$$

Finally, using (3.9) with $\widetilde{\mu} = \gamma$, we write for $\mu \in \mathcal{M}(\mathcal{X})$ and $t \ge 0$

$$\begin{aligned}\left\|\mathrm{e}^{-\lambda t}\mu M_t - \mu(h)\gamma\right\|_{\mathrm{TV}} &\le \nu(m_t)\mathrm{e}^{-\lambda t}\left(\left\|\frac{\mu M_t}{\nu(m_t)} - \mu(h_0)\gamma\right\|_{\mathrm{TV}} + |\mu(h)|\left|\gamma(h_0) - \frac{e^{\lambda t}}{\nu(m_t)}\right|\right)\\ &\le C'\nu(m_t)\mathrm{e}^{-\lambda t}\|\mu\|_{\mathrm{TV}}(1-cd)^{t/r}.\end{aligned}$$

The conclusion follows from the fact that the function $t \mapsto \nu(m_t)\mathrm{e}^{-\lambda t}$ is bounded. Indeed it is locally bounded due to (H3), and it converges to $1/\gamma(h_0)$ when $t \to +\infty$. □

3.2.2 Example: The Renewal Equation

We consider an age-structured population of proliferating cells which divide at age $a \ge 0$ according to a division rate $b(a)$, giving birth to two daughter cells with age zero. The evolution of the age distribution density u_t is given by the so-called renewal PDE

$$\begin{cases}\partial_t u_t(a) + \partial_a u_t(a) + b(a)u_t(a) = 0, & t, a > 0,\\ u_t(0) = 2\displaystyle\int_0^\infty b(a)u_t(a)\,da, & t > 0.\end{cases} \tag{3.10}$$

This model has been introduced by Sharpe and Lotka [46] in a more general context, namely with a "birth rate" not necessarily equal to twice the "death rate". Since then, it has become a very popular model in population dynamics (see for instance [1, 31, 36, 43, 48, 51]).

The state space here is $\mathcal{X} = \mathbb{R}_+ = [0,\infty)$. Following [24] we associate to Eq. (3.10) the homogeneous semigroup $(M_t)_{t\ge0}$ defined as follows. For any $f \in \mathcal{B}_b(\mathbb{R}_+)$, we define the family $(M_t f)_{t\ge0} \subset \mathcal{B}_b(\mathbb{R}_+)$ as the unique solution to the equation

$$M_t f(a) = f(a+t)\mathrm{e}^{-\int_0^t b(a+\tau)\,d\tau} + 2\int_0^t \mathrm{e}^{-\int_0^\tau b(a+\tau')\,d\tau'} b(a+\tau)M_{t-\tau}f(0)\,d\tau. \tag{3.11}$$

The proof of the existence and uniqueness of a solution to (3.11) is postponed in Appendix B, Lemma B.1. We also refer to Appendix B for the rigorous definition of μM_t, which provides the unique measure solution to Eq. (3.10) with initial distribution μ. In particular if μ has a density u_0 with respect to the Lebesgue measure, we get that $u_t = \mu M_t$ is the unique L^1 solution to Eq. (3.10) with initial distribution u_0. Appendix B also contains a verification of Assumption 2.1 for the semigroup $(M_t)_{t\ge0}$.

Now we can use Theorem 3.5 to obtain the long time asymptotic behavior of the solutions to Eq. (3.10).

Theorem 3.8 *Assume that b is a non-negative locally bounded function on $\mathbb{R}_+$, and suppose the existence of $a_0 > 0$, $p > 0$, $l \in (p/2, p]$, and $\underline{b} > 0$ for which*

$$\forall k \in \mathbb{N}, \forall a \in [a_0 + kp, a_0 + kp + l], \quad b(a) \ge \underline{b}.$$

Then there exists a unique triplet of eigenelements $(\gamma, h, \lambda) \in \mathcal{P}(\mathbb{R}_+) \times \mathcal{B}_b(\mathbb{R}_+) \times \mathbb{R}_+$ *verifying* $\gamma(h) = 1$ *and*

$$\forall t \geq 0, \quad \gamma M_t = e^{\lambda t}\gamma, \; M_t h = e^{\lambda t} h.$$

Moreover there exist $C > 0$ *and an explicit* $\rho > 0$, *given by* (3.13), *such that for all* $\mu \in \mathcal{M}(\mathbb{R}_+)$ *and all* $t \geq 0$

$$\left\| e^{-\lambda t}\mu M_t - \mu(h)\gamma \right\|_{\mathrm{TV}} \leq C\|\mu\|_{\mathrm{TV}}\, e^{-\rho t}.$$

The convergence of the solutions to a time-independent asymptotic profile multiplied by an exponential function of time, sometimes referred to as *asynchronous exponential growth*, was first conjectured for the renewal equation by Sharpe and Lotka [46] and was then proved by many authors using various methods, see for instance [21, 26–28, 32, 43, 47, 50]. Moreover it is known that the so-called Malthus parameter λ is characterized as the unique real number which satisfies the characteristic equation

$$1 = 2\int_0^\infty b(a)e^{-\int_0^a (\lambda + b(a'))\,da'}\,da,$$

the asymptotic probability measure has an explicit density with respect to the Lebesgue measure

$$\gamma(da) = \kappa\, e^{-\int_0^a (\lambda + b(a'))\,da'}\,da$$

where κ is a normalization constant which ensures that $\gamma \in \mathcal{P}(\mathbb{R}_+)$, and the harmonic function h is explicitly given by

$$h(a) = 2h(0)\int_a^\infty b(a')e^{-\int_a^{a'} (\lambda + b(a''))\,da''}\,da',$$

where $h(0)$ is chosen so that $\gamma(h) = 1$.

The existing results about asynchronous exponential growth for the renewal equation hold for birth and death rates which are not necessarily related by a multiplicative factor, as we assume here. But in these previous works the birth rate is assumed to be bounded or integrable, a condition which is not required in our situation. In Eq. (3.10) if the division rate is unbounded, then the unboundedness of the birth rate $2b$ is “compensated” by the unboundedness of the death rate b. Thus our result is new in the sense that the assumptions on the division rate are very general, but also because it provides an explicit spectral gap in terms of the division rate and a convergence which is valid for measure solutions.

The assumptions on the division rate b include some functions which are not bounded neither from above nor from below by a positive constant when a tends to infinity. The only assumption is that, outside a compact interval, b is larger than a crenel function with period p and a crenel width l. The condition $l > p/2$ is only a technical assumption which simplifies the computations. It can be removed to the price of a larger number of iterations of the Duhamel formula in the proof.

Before proving Theorem 3.8, we define the probability distribution of age of division

$$\Phi(a) := b(a)e^{-\int_0^a b(a')\,da'}$$

and we give a useful property of $m_t = M_t\mathbf{1}$.

Lemma 3.9 *For any* $a \geq 0$ *the function* $t \mapsto m_t(a)$ *is non-decreasing.*

Proof First we check that $m_t(0) \geq 1$ for all $t \geq 0$. By definition $t \mapsto m_t(0)$ is the unique fixed point of

$$\Gamma g(t) = \mathrm{e}^{-\int_0^t b(\tau)\,d\tau} + 2\int_0^t \Phi(\tau) g(t-\tau)\,d\tau$$

and if $g \geq 1$ then for all $t \geq 0$

$$\Gamma g(t) \geq \mathrm{e}^{-\int_0^t b(\tau)\,d\tau} + 2\int_0^t \Phi(\tau)\,d\tau = 2 - \mathrm{e}^{-\int_0^t b(\tau)\,d\tau} \geq 1.$$

So the fixed point necessarily satisfies $m_t(0) \geq 1$, since Γ is a contraction for small times (see Lemma B.1).

In a second step we prove that $t \mapsto m_t(0)$ is non-decreasing. Let $\epsilon > 0$. For all $t \geq 0$ we have by definition of $m_t(0)$

$$\begin{aligned} m_{t+\epsilon}(0) - m_t(0) &= -\int_t^{t+\epsilon} \Phi(\tau)\,d\tau + 2\int_t^{t+\epsilon} \Phi(\tau) m_{t+\epsilon-\tau}(0)\,d\tau \\ &\quad + 2\int_0^t \Phi(\tau)\big(m_{t+\epsilon-\tau}(0) - m_{t-\tau}(0)\big)\,d\tau \\ &= \int_t^{t+\epsilon} \Phi(\tau)\big(2m_{t+\epsilon-\tau}(0) - 1\big)\,d\tau + 2\int_0^t \Phi(\tau)\big(m_{t+\epsilon-\tau}(0) - m_{t-\tau}(0)\big)\,d\tau. \end{aligned}$$

So $t \mapsto m_{t+\epsilon}(0) - m_t(0)$ is the unique fixed point of

$$\Gamma g(t) = f_0(t)\mathrm{e}^{-\int_0^t b(\tau)\,d\tau} + 2\int_0^t \Phi(\tau) g(t-\tau)\,d\tau$$

with $f_0(t) = \int_t^{t+\epsilon} \mathrm{e}^{-\int_t^\tau b(\tau')\,d\tau'} b(\tau)(2m_{t+\epsilon-\tau}(0) - 1)\,d\tau \geq 0$. We deduce from the positivity property in Lemma B.1 that $m_{t+\epsilon}(0) - m_t(0) \geq 0$ for all $t \geq 0$.

The last step consists in extending the result of the second step to $t \mapsto m_t(a)$ for any $a \geq 0$. Let $a \geq 0$ and $\epsilon > 0$. For all $t \geq 0$ we have

$$\begin{aligned} m_{t+\epsilon}(a) - m_t(a) &= \int_t^{t+\epsilon} \mathrm{e}^{-\int_t^\tau b(a+\tau')\,d\tau'} b(a+\tau)\big(2m_{t+\epsilon-\tau}(0) - 1\big)\,d\tau \\ &\quad + 2\int_0^t \mathrm{e}^{-\int_t^\tau b(a+\tau')\,d\tau'} b(a+\tau)\big(m_{t+\epsilon-\tau}(0) - m_{t-\tau}(0)\big)\,d\tau \geq 0. \end{aligned}$$ □

Corollary 3.10 *For all $t, a \geq 0$ we have $m_t(a) \leq 2\,m_t(0)$.*

Proof Starting from the Duhamel formula (3.11) and using Lemma 3.9 we have

$$\begin{aligned} m_t(a) &= \mathrm{e}^{-\int_0^t b(a+\tau)\,d\tau} + 2\int_0^t \mathrm{e}^{-\int_0^s b(a+\tau)\,d\tau} b(a+s) m_{t-s}(0)\,ds \\ &\leq \mathrm{e}^{-\int_0^t b(a+\tau)\,d\tau} + 2m_t(0)\int_0^t \mathrm{e}^{-\int_0^s b(a+\tau)\,d\tau} b(a+s)\,ds \\ &= \mathrm{e}^{-\int_0^t b(a+\tau)\,d\tau} + 2m_t(0)\big[1 - \mathrm{e}^{-\int_0^t b(a+\tau)\,d\tau}\big] \\ &= 2m_t(0) + \mathrm{e}^{-\int_0^t b(a+\tau)\,d\tau}\big(1 - 2m_t(0)\big) \leq 2m_t(0). \end{aligned}$$ □

We are now ready to prove Theorem 3.8.

Proof of Theorem 3.8 We prove that Assumptions (H1), (H2), and (H3) are satisfied by the renewal semigroup and then apply Theorem 3.5.

We start with (H1). For $\alpha > 0$, we define the probability measure ν by

$$\forall f \in C_0(\mathbb{R}_+), \quad \nu(f) := \frac{\int_0^\alpha M_s f(0)\,ds}{\int_0^\alpha m_s(0)\,ds}.$$

We want to prove that for α small enough (to be determined later), there exists a time $t_0 > 0$ and $c > 0$ such that for all $f \geq 0$ and $a \geq 0$,

$$M_{t_0} f(a) \geq c\,\nu(f) m_{t_0}(a). \tag{3.12}$$

Iterating the Duhamel formula (3.11) we have for all $f \geq 0$ and all $t, a \geq 0$,

$$\begin{aligned}
M_t f(a) &= f(a+t)\mathrm{e}^{-\int_0^t b(a+\tau)\,d\tau} + 2\int_0^t \mathrm{e}^{-\int_0^\tau b(a+\tau')\,d\tau'} b(a+\tau) f(t-\tau)\mathrm{e}^{-\int_0^{t-\tau} b(\tau')\,d\tau'}\,d\tau \\
&\quad + 4\int_0^t \mathrm{e}^{-\int_0^\tau b(a+\tau')\,d\tau'} b(a+\tau) \int_0^{t-\tau} \Phi\big(\tau'\big) M_{t-\tau-\tau'} f(0)\,d\tau'\,d\tau \\
&\geq 4\int_0^t \mathrm{e}^{-\int_0^\tau b(a+\tau')\,d\tau'} b(a+\tau) \int_0^{t-\tau} \Phi(t-\tau-s) M_s f(0)\,ds\,d\tau.
\end{aligned}$$

Let $t_0 > 0$, $\alpha \in [0, t_0]$, and $0 \leq t_1 \leq t_2 \leq t_0 - \alpha$. We have

$$M_{t_0} f(a) \geq 4\int_{t_1}^{t_2} \mathrm{e}^{-\int_0^\tau b(a+\tau')\,d\tau'} b(a+\tau) \int_0^\alpha \Phi(t_0-\tau-s) M_s f(0)\,ds\,d\tau.$$

This inequality means that for bounding $M_{t_0} f$ from below we only keep the individuals which: do not divide between times 0 and t_1; divide a first time between t_1 and t_2; do not divide between t_2 and $t_0 - \alpha$; divide a second time between $t_0 - \alpha$ and t_0.

Let us check that we can choose t_0, t_1, t_2 and α such that b and then Φ have a lower bound on $[t_0 - t_2 - \alpha, t_0 - t_1]$ and then give a lower bound for

$$\int_{t_1}^{t_2} \mathrm{e}^{-\int_0^\tau b(a+\tau')\,d\tau'} b(a+\tau)\,d\tau = \mathrm{e}^{-\int_a^{a+t_1} b(\tau)\,d\tau}\big(1 - \mathrm{e}^{-\int_{a+t_1}^{a+t_2} b(\tau)\,d\tau}\big)$$

and obtain (3.12). For that purpose, we define $n = \lfloor a_0/p \rfloor + 1$ the smallest integer such that $np > a_0$. Let $\alpha \in (0, 2l - p)$ and $t_0 = a_0 + np + l$. The choice of t_1 and t_2 depends on whether $a < a_0$ or $a \geq a_0$.

For $a < a_0$ we choose $t_1 = np$ and $t_2 = np + l - \alpha$. We have $b \geq \underline{b}$ on $[t - t_2 - \alpha, t - t_1] = [a_0, a_0 + l]$, so that $\Phi \geq \underline{b}\,\mathrm{e}^{-\int_0^{a_0+l} b(\tau)\,d\tau}$ on $[t - t_2 - \alpha, t - t_1]$, and

$$\int_{t_1}^{t_2} \mathrm{e}^{-\int_0^s b(a+\tau)\,d\tau} b(a+s)\,ds \geq \mathrm{e}^{-\int_0^{a_0+np} b(\tau)\,d\tau}\big(1 - \mathrm{e}^{-\underline{b}(2l-p-\alpha)}\big) > 0.$$

For $a \geq a_0$ we choose $t_1 = 0$ and $t_2 = l - \alpha$. We have $\Phi \geq \underline{b}\,\mathrm{e}^{-\int_0^{A+np+l} b(\tau)\,d\tau} > 0$ on $[t - t_2 - \alpha, t - t_1] = [a_0 + np, a_0 + np + l]$, and

$$\int_0^{t_2} \mathrm{e}^{-\int_0^s b(a+\tau)\,d\tau} b(a+s)\,ds = 1 - \mathrm{e}^{-\int_a^{a+l-\alpha} b(\tau)\,d\tau} \geq 1 - \mathrm{e}^{-\underline{b}(2l-p-\alpha)} > 0.$$

As a consequence, (3.12) is satisfied with

$$c = 4\frac{\int_0^\alpha m_s(0)\,ds}{\|m_{t_0}\|_\infty}\,\underline{b}\left(1-\mathrm{e}^{-\underline{b}(2l-p-\alpha)}\right)\mathrm{e}^{-2\int_0^{a_0+np+l} b(\tau)\,d\tau},$$

which gives Assumption (H1).

Now we turn to (H2). Since we know from Corollary 3.10 that $m_t(0) \geq \frac{1}{2}m_t(a)$ for all $t, a \geq 0$, it suffices to find $d > 0$ such that for all $t \geq 0$,

$$\nu(m_t) \geq 2\,d\,m_t(0).$$

Lemma 3.9 ensures that for all $t \geq 0$,

$$\nu(m_t) = \frac{1}{\int_0^\alpha m_s(0)\,ds}\int_0^\alpha m_{t+s}(0)\,ds \geq \frac{\alpha}{\int_0^\alpha m_s(0)\,ds}m_t(0)$$

and the constant $d = \frac{\alpha}{2\int_0^\alpha m_s(0)\,ds}$ suits.

It remains to check (H3). In that view, we define

$$\mathfrak{b}(a) := \sup_{[0,a]} b$$

and we write for $t \geq s \geq 0$

$$m_s(0) = \mathrm{e}^{-\int_0^s b(\tau)\,d\tau} + 2\int_0^s \mathrm{e}^{-\int_0^{s-\tau} b(\tau')\,d\tau'} b(s-\tau)m_\tau(0)\,d\tau \leq 1 + 2\,\mathfrak{b}(t)\int_0^s m_\tau(0)\,d\tau.$$

Applying the Grönwall's lemma we get $m_s(0) \leq \mathrm{e}^{2\mathfrak{b}(t)s}$ for all $s \in [0,t]$, so $m_t(0) \leq \mathrm{e}^{2\mathfrak{b}(t)t}$, and using Corollary 3.10 we obtain

$$\|m_t\|_\infty \leq 2\,\mathrm{e}^{2\mathfrak{b}(t)t}.$$

Finally we can apply Theorem 3.5 which ensures the exponential convergence with the rate

$$\frac{-\log(1-cd)}{t_0} = \frac{-\log(1-\frac{2\alpha}{\|m_{t_0}\|_\infty}\underline{b}\,(1-\mathrm{e}^{-\underline{b}(2l-p-\alpha)})\,\mathrm{e}^{-2\int_0^{a_0+np+l} b(\tau)\,d\tau})}{a_0+np+l} \geq \rho$$

where

$$\rho = \frac{-\log(1-\alpha\underline{b}\,(1-\mathrm{e}^{-\underline{b}(2l-p-\alpha)})\,\mathrm{e}^{-2\int_0^{2a_0+p+l} b(\tau)\,d\tau-2(2a_0+p+l)\mathfrak{b}(2a_0+p+l)})}{2a_0+p+l} \tag{3.13}$$

and the result follows by choosing $\alpha = l - p/2$. □

3.3 Asymptotically Homogeneous Semigroups and Increasing Maximal Age

In this section, we present a general theorem for semigroups which become homogeneous when time tends to infinity. We then apply this theorem to an age structured population where the state space has a maximal age which increases with time.

3.3.1 Convergence of the Profile and Evolution of the Mass

We consider the situation of a semigroup which becomes homogeneous when time goes to ∞. For the sake of simplicity and in view of our application, we restrict ourselves to the case when the state space is increasing:

$$\forall s < t,\ \mathcal{X}_s \subset \mathcal{X}_t, \quad \mathcal{X} = \bigcup_{s\geq 0} \mathcal{X}_s.$$

We say that a semigroup $(M_{s,t})_{0\leq s\leq t}$ is asymptotically homogeneous if there exists a homogeneous semigroup $(N_t)_{t\geq 0}$ defined on $\mathcal{X}$ and satisfying Assumption 2.1, such that for all $s \geq 0$

$$\lim_{t\to\infty} \sup_{x\in\mathcal{X}_t} \|\delta_x M_{t,t+s} - \delta_x N_s\|_{\mathrm{TV}} = 0 \tag{H'0}$$

In our framework, the assumptions (A1)–(A4) rewrite as follows. There exist $s_0, r > 0$, $c, d > 0$ and some probability measure ν on $\mathcal{X}_{s_0}$ such that for any $t \geq s_0$ and $x \in \mathcal{X}_t$,

$$\delta_x M_{t,t+r} \geq c\, m_{t,t+r}(x)\, \nu, \tag{H'1}$$

and for any $\tau \geq 0$,

$$d\, m_{t,t+\tau}(x) \leq \nu(m_{t,t+\tau}). \tag{H'2}$$

As for the homogeneous case, writing $n_t(x) = \delta_x N_t \mathbf{1}$, for $x \in \mathcal{X}$ and $t \geq 0$, we need that

$$t \mapsto \|n_t\|_\infty \text{ is locally bounded on } \mathbb{R}_+. \tag{H'3}$$

Theorem 3.11 *Let $s \geq 0$. Under Assumptions (H′0), (H′1), (H′2) and (H′3), there exists a probability measure γ on $\mathcal{X}$ and a positive bounded function h_s on $\mathcal{X}_s$ such that*

$$\lim_{t\to\infty} \sup_{\mu\in\mathcal{P}(\mathcal{X}_s)} \left\| \frac{\mu M_{s,t}}{\nu(m_{s,t})} - \mu(h_s)\gamma \right\|_{\mathrm{TV}} = 0.$$

Notice that the TV norm above is the TV norm on the state space $\mathcal{X}$, the measure $\mu M_{s,t} \in \mathcal{M}(\mathcal{X}_t)$ being extended by zero on $\mathcal{X} \setminus \mathcal{X}_t$.

Notice also that in Theorem 3.11 we do not provide a speed of convergence. It could be achieved by taking into account the speed of convergence of M to N.

Proof Following (2.3), we set

$$\delta_x Q^{(t)}_{s,t} = \frac{\delta_x N_{t-s}}{n_{t-s}(x)},$$

for any $x \in \mathcal{X}$, $t \geq s \geq 0$. Fix $x_0 \in \mathcal{X}$. First, using (H′0), for any $u \geq 0$, we obtain

$$\lim_{t\to\infty} m_{t,t+u}(x_0) = n_u(x_0) > 0$$

and then using again (H′0),

$$\lim_{t\to\infty} \left\| \delta_{x_0} P^{(t+u)}_{t,t+u} - \delta_{x_0} Q^{(t+u)}_{t,t+u} \right\|_{\mathrm{TV}} = 0. \tag{3.14}$$

Now we note that (H′1) and (H′2) ensure that (A1)–(A4) are satisfied for any regular subdivision of $[s,t]$ with step r (*i.e.* $t_i - t_{i-1} = r$) and $t_0 \geq s_0$, for the constants

$$c_i = c, \qquad d_i = d, \qquad \nu_i = \nu, \qquad \alpha = 1/(cd), \qquad \beta = 1/d.$$

We apply then Theorem 2.3 to M with $\gamma = \delta_{x_0}$, which ensures that, for every $s \geq 0$,

$$\lim_{t\to\infty} \sup_{\mu\in\mathcal{P}(\mathcal{X}_s)} \left\| \frac{\mu M_{s,t}}{\nu(m_{s,t})} - \mu(h_s)\delta_{x_0} P_{0,t}^{(t)} \right\|_{TV} = 0, \tag{3.15}$$

while Lemma 2.5 (*ii*) guarantees

$$\lim_{u\to\infty} \sup_{t\geq 0, x,y\in\mathcal{X}_t} \left\| \delta_x P_{t,t+u}^{(t+u)} - \delta_y P_{t,t+u}^{(t+u)} \right\|_{TV} = 0. \tag{3.16}$$

Using (H′0) and letting $t \to \infty$ in (H′1), (H′2) and (H′3), we obtain that the semigroup N satisfies (H1), (H2) and (H3). Then by Theorem 3.5, there exists a probability measure γ such that

$$\lim_{u\to\infty} \sup_{x\in\mathcal{X}, t\geq 0} \left\| \delta_x Q_{t,t+u}^{(t+u)} - \gamma \right\|_{TV} = 0.$$

Hence, (3.14) becomes

$$\lim_{u\to\infty} \limsup_{t\to\infty} \left\| \delta_{x_0} P_{t,t+u}^{(t+u)} - \gamma \right\|_{TV} = 0.$$

Using now $\delta_{x_0} P_{0,t+u}^{(t+u)} = (\delta_{x_0} P_{0,t}^{(t+u)}) P_{t,t+u}^{(t+u)}$ and (3.16), we get

$$\lim_{u\to\infty} \sup_{t\geq 0} \left\| \delta_{x_0} P_{0,t+u}^{(t+u)} - \delta_{x_0} P_{t,t+u}^{(t+u)} \right\|_{TV} = 0.$$

Combing the two last bounds yields

$$\begin{aligned} \limsup_{t\to\infty} \left\| \delta_{x_0} P_{0,t}^{(t)} - \gamma \right\|_{TV} &= \lim_{u\to\infty} \sup_{t\geq 0} \left\| \delta_{x_0} P_{0,t+u}^{(t+u)} - \gamma \right\|_{TV} \\ &\leq \lim_{u\to\infty} \sup_{t\geq 0} \left\| \delta_{x_0} P_{0,t+u}^{(t+u)} - \delta_{x_0} P_{t,t+u}^{(t+u)} \right\|_{TV} + \lim_{u\to\infty} \limsup_{t\to\infty} \left\| \delta_{x_0} P_{t,t+u}^{(t+u)} - \gamma \right\|_{TV} = 0. \end{aligned}$$

Plugging this in (3.15) concludes the proof. □

3.3.2 Example: The Renewal Equation with Maximal Age

We consider the renewal equation on a domain bounded by a maximal age which increases along time. When an individual reaches the maximal age, he dies. We denote by a_t the maximal age, which grows from a_0 to a_∞ when $t \to +\infty$. To avoid pathological situations we assume that $t \mapsto t - a_t$ is strictly increasing. Inside the domain $[0, a_t)$ the individuals reproduce with a birth rate $b(a)$ bounded from below by $\underline{b} > 0$ and from above by $\overline{b}$. The partial differential equation which prescribes the evolution of the density $u_{s,t}(a)$ of individuals with age a at time t (starting from a distribution $u_s(a)$ at time $s \in [0,t)$) writes

$$\begin{cases} \partial_t u_{s,t}(a) + \partial_a u_{s,t}(a) = 0, & t > s,\ 0 < a < a_t, \\ u_{s,t}(0) = \displaystyle\int_0^{a_t} b(a) u_{s,t}(a)\, da, & t > s, \\ u_{s,s}(a) = u_s(a), & 0 \leq a < a_s. \end{cases} \tag{3.17}$$

The motivation of such a model comes from [3] which studies a related branching model for the bus paradox problem.

The details of the construction of the associated semigroup are postponed to Appendix C. Here we only give the main steps. For $t \geq s \geq 0$ we define the operator $M_{s,t} : \mathcal{B}_b([0, a_t)) \to \mathcal{B}_b([0, a_s))$ as follows: for any $f \in \mathcal{B}_b([0, a_t))$ the family $(M_{s,t} f)_{t \geq s \geq 0}$ is the unique solution to the equation

$$M_{s,t} f(a) = f(a + t - s) + \int_s^t b_\tau(a + \tau - s) M_{\tau,t} f(0)\, d\tau, \qquad (3.18)$$

where we have denoted $b_t(a) = b(a)\mathbf{1}_{[0,a_t)}(a)$ and f has been extended by 0 beyond a_t. Then for $\mu \in \mathcal{M}(\mathcal{X}_s)$ the measure $\mu M_{s,t}$ is defined on $[0, a_t)$ in such a way that $(\mu M_{s,t})(f) = \mu(M_{s,t} f)$ for all $f \in \mathcal{B}_b([0, a_t))$.

Using Theorem 3.11 we prove the following ergodic result for the semigroup $(M_{s,t})_{t \geq s \geq 0}$. It provides the long time asymptotic behavior of the measure solutions $(\mu M_{s,t})_{t \geq s}$ to (3.17).

Theorem 3.12 *Let $s \geq 0$. There exist $\gamma \in \mathcal{P}([0, a_\infty))$, $\nu \in \mathcal{P}([0, a_s))$, and a positive $h_s \in \mathcal{B}_b([0, a_s))$ such that for all $\mu \in \mathcal{M}([0, a_s))$,*

$$\lim_{t \to \infty} \sup_{\mu \in \mathcal{P}(\mathcal{X}_s)} \left\| \frac{\mu M_{s,t}}{\nu(m_{s,t})} - \mu(h_s)\gamma \right\|_{\mathrm{TV}} = 0.$$

Before proving this theorem we start with a useful lemma.

Lemma 3.13 *For any $s \leq t$ and any $f \in C_0([0, a_t))$, we have*

$$\|M_{s,t} f\|_\infty \leq \mathrm{e}^{\overline{b}(t-s)} \|f\|_\infty.$$

Proof By definition of $M_{s,t} f(0)$ we have

$$\left|M_{s,t} f(0)\right| = \left| f(t - s) + \int_s^t b_\tau(\tau - s) M_{\tau,t} f(0)\, d\tau \right| \leq \|f\|_\infty + \overline{b} \int_s^t \left|M_{\tau,t} f(0)\right| d\tau$$

and the Grönwall's lemma gives $|M_{s,t} f(0)| \leq \|f\|_\infty \mathrm{e}^{\overline{b}(t-s)}$. Then for $a \geq 0$ we write

$$\left|M_{s,t} f(a)\right| \leq \|f\|_\infty + \overline{b}\|f\|_\infty \int_s^t \mathrm{e}^{\overline{b}(\tau-s)} d\tau = \mathrm{e}^{\overline{b}(t-s)} \|f\|_\infty. \qquad \square$$

Proof of Theorem 3.12 We start by verifying (H′0) with $\mathcal{X} = [0, a_\infty)$. The homogeneous semigroup N on $\mathcal{M}(\mathcal{X}) \times \mathcal{B}_b(\mathcal{X})$ is defined by the following Duhamel formula

$$N_t f(a) = f(a + t) + \int_0^t b_\infty(a + \tau) N_{t-\tau} f(0)\, d\tau.$$

Existence, uniqueness and Assumption 2.1 can be proved in the same way as for the homogeneous renewal equation (see Appendix B).

Fix $t > 0$, $a \in \mathcal{X}_t = [0, a_t)$, and $f \in C_0([0, a_t))$ such that $\|f\|_\infty \leq 1$. For any $r \geq a_\infty$ we have

$$M_{t,t+r} f(a) = \int_t^{t+r} b_\tau(a + \tau - t) M_{\tau,t+r} f(0)\, d\tau$$

and

$$N_r f(a) = \int_t^{t+r} b_\infty(a + \tau - t) N_{t+r-\tau} f(0)\, d\tau.$$

We start by comparing $M_{t,t+r} f(0)$ and $N_r f(0)$. We have, using Lemma 3.13,

$$\begin{aligned}
&\left|N_r f(0) - M_{t,t+r} f(0)\right| \\
&\le \int_t^{t+r} \left|b_\infty(\tau - t) - b_\tau(\tau - t)\right| \left|N_{t+r-\tau} f(0)\right| d\tau \\
&\quad + \int_t^{t+r} b_\tau(\tau - t) \left|N_{t+r-\tau} f(0) - M_{\tau,t+r} f(0)\right| d\tau \\
&\le \int_t^{t+r} \overline{b}\, \mathbf{1}_{a_\tau \le \tau - t \le a_\infty} \mathrm{e}^{\overline{b}(t+r-\tau)}\, d\tau + \overline{b} \int_t^{t+r} \left|N_{t+r-\tau} f(0) - M_{\tau,t+r} f(0)\right| d\tau \\
&\le \overline{b}\, \mathrm{e}^{\overline{b}r} (a_\infty - a_t) + \overline{b} \int_t^{t+r} \left|N_{t+r-\tau} f(0) - M_{\tau,t+r} f(0)\right| d\tau,
\end{aligned}$$

which gives by Grönwall's lemma

$$\left|N_r f(0) - M_{t,t+r} f(0)\right| \le \overline{b}\, \mathrm{e}^{2\overline{b}r} (a_\infty - a_t).$$

Now we come back to $a \in [0, a_t)$ and we have similarly

$$\begin{aligned}
\left|N_r f(a) - M_{t,t+r} f(a)\right| &\le \overline{b}\mathrm{e}^{\overline{b}r} (a_\infty - a_t) + \overline{b} \int_t^{t+r} \left|N_{t+r-\tau} f(0) - M_{\tau,t+r} f(0)\right| d\tau \\
&\le \overline{b}\, \mathrm{e}^{\overline{b}r} (a_\infty - a_t) + \overline{b}^2 r \mathrm{e}^{2\overline{b}r} (a_\infty - a_t) \\
&\le \max\left(\overline{b}\mathrm{e}^{\overline{b}r}, \left(\overline{b}\mathrm{e}^{\overline{b}r}\right)^2 r\right) (a_\infty - a_t).
\end{aligned}$$

We deduce that for all $r \ge a_\infty$,

$$\sup_{0 \le a < a_t} \|\delta_a M_{t,t+r} - \delta_a N_r\|_{\mathrm{TV}} \le \max\left(\overline{b}\mathrm{e}^{\overline{b}r}, \left(\overline{b}\mathrm{e}^{\overline{b}r}\right)^2 r\right)(a_\infty - a_t) \xrightarrow[t \to +\infty]{} 0.$$

Now we turn to (H′1). Iterating the Duhamel formula (3.18), we get for $f \ge 0$,

$$\begin{aligned}
M_{s,t} f(a) &= f(a + t - s) + \int_s^t b_\tau(a + \tau - s) M_{\tau,t} f(0)\, d\tau \\
&= f(a + t - s) + \int_s^t b_\tau(a + \tau - s) f(t - \tau)\, d\tau \\
&\quad + \int_s^t b_\tau(a + \tau - s) \int_\tau^t b_{\tau'}\left(\tau' - \tau\right) f\left(t - \tau'\right) d\tau'\, d\tau + (\ge 0).
\end{aligned}$$

Thus

$$M_{s,t} f(a) \ge \int_s^t b_\tau(a + \tau - s) \int_\tau^t b_{\tau'}\left(\tau' - \tau\right) f\left(t - \tau'\right) d\tau'\, d\tau.$$

We consider s large enough so that $\Delta := a_\infty - a_s \le a_s/2$. We set $\alpha := \frac{\Delta}{2}$ and take $t-s=2\Delta$. For $s \le \tau \le \tau' \le t$ we have $\tau'-\tau \le t-s = 2\Delta \le a_s \le a'_\tau$ and so $b_{\tau'}(\tau'-\tau) = b(\tau'-\tau) \ge \underline{b}$. We deduce that

$$M_{s,t}f(a) \ge \underline{b} \int_s^t b_\tau(a+\tau-s) \int_\tau^t f(t-\tau')\,d\tau'\,d\tau.$$

We consider separately $a \le a_s - \alpha$ and $a \ge a_s - \alpha$. For $a \le a_s - \alpha$ et $\tau \in [s, s+\alpha]$ we have $a+\tau-s \le a_s \le a_\tau$ and so $b_\tau(a+\tau-s) = b(a+\tau-s) \ge \underline{b}$. Thus we can write

$$\begin{aligned} M_{s,t}f(a) &\ge \underline{b} \int_s^t b_\tau(a+\tau-s) \int_\tau^t f\big(t-\tau'\big)\,d\tau'\,d\tau \\ &\ge \underline{b}^2 \int_s^{s+\alpha} \int_{t-\alpha}^t f\big(t-\tau'\big)\,d\tau'\,d\tau \ge \alpha\underline{b}^2 \int_0^\alpha f(r)\,dr = (\alpha\underline{b})^2 \nu(f), \end{aligned}$$

where $\nu(f) = \frac{1}{\alpha}\int_0^\alpha f(r)\,dr$. Now for $a > a_s - \alpha$ we have that if $\tau \ge t-\alpha$ then $a+\tau-s \ge a_s + \Delta = a_\infty$ and so $b_\tau(a+\tau-s) = 0$. We deduce that

$$\begin{aligned} M_{s,t}f(a) &\ge \underline{b} \int_s^t b_\tau(a+\tau-s) \int_\tau^t f\big(t-\tau'\big)\,d\tau'\,d\tau \\ &\ge \underline{b} \int_s^t b_\tau(a+\tau-s) \int_{t-\alpha}^t f\big(t-\tau'\big)\,d\tau'\,d\tau = \alpha\underline{b}\bigg(\int_s^t b_\tau(a+\tau-s)\,d\tau\bigg)\nu(f). \end{aligned}$$

To conclude that (H′1) is satisfied it remains to compare these lower bounds with $m_{s,t}(a)$. We start from the Duhamel formula

$$m_{s,t}(a) = \mathbf{1}_{a+t-s<a_t} + \int_s^t b_\tau(a+\tau-s) m_{\tau,t}(0)\,d\tau$$

and use Lemma 3.13 which ensures that $m_{\tau,t}(0) \le e^{\overline{b}(t-\tau)}$. If $a \le a_s - \alpha$, we write that

$$m_{s,t}(a) \le 1 + \overline{b} \int_s^t e^{\overline{b}(t-\tau)}\,d\tau = e^{2\overline{b}\Delta}.$$

For $a > a_s - \alpha$ we use that $a+t-s \ge a_s + \Delta = a_\infty \ge a_t$ to write

$$m_{s,t}(a) \le \int_s^t b_\tau(a+\tau-s) m_{\tau,t}(0)\,d\tau \le e^{\overline{b}\Delta} \int_s^t b_\tau(a+\tau-s)\,d\tau.$$

Finally we get

$$M_{s,t}f(a) \ge \min\big(\alpha\underline{b}\,e^{-\overline{b}\Delta}, \big(\alpha\underline{b}\,e^{-\overline{b}\Delta}\big)^2\big)\, m_{s,t}(a)\,\nu(f).$$

For (H′2) we start by comparing $m_{s,t}(a)$ to $m_{s,t}(0)$. We have

$$\begin{aligned} m_{s,t}(a) &= \mathbf{1}_{a+t-s<a_t} + \int_s^t b_\tau(a+\tau-s) m_{\tau,t}(0)\,d\tau \\ &\le \mathbf{1}_{t-s<a_t} + \overline{b} \int_s^t \mathbf{1}_{a+\tau-s\le a_\tau} m_{\tau,t}(0)\,d\tau \le \mathbf{1}_{t-s<a_t} + \overline{b} \int_s^t \mathbf{1}_{\tau-s\le a_\tau} m_{\tau,t}(0)\,d\tau \\ &\le \mathbf{1}_{t-s<a_t} + \frac{\overline{b}}{\underline{b}} \int_s^t b_\tau(\tau-s) m_{\tau,t}(0)\,d\tau \le \frac{\overline{b}}{\underline{b}} m_{s,t}(0). \end{aligned}$$

Then we compare $m_{s,t}(0)$ to $\nu(m_{s,t})$. We consider separately the cases $s \le t \le s+\alpha$ and $t > s+\alpha$. For $s \le t \le s+\alpha$ we have, since $\alpha + t - s \le 2\alpha = \Delta \le a(0) \le a_t$,

$$\int_0^\alpha m_{s,t}(a)\,da \ge \int_0^\alpha \mathbf{1}_{a+t-s<a_t}\,da \ge \alpha$$

and we have already seen that $m_{s,t}(0) \le e^{\overline{b}(t-s)}$, so

$$m_{s,t}(0) \le e^{\alpha\overline{b}}\nu(m_{s,t})$$

and

$$\frac{\underline{b}}{\overline{b}}e^{-\alpha\overline{b}}\|m_{s,t}\|_\infty \le \nu(m_{s,t}).$$

For the case $t > s+\alpha$ we split into two steps. We start by comparing $m_{s,t}(0)$ to $m_{r,t}(0)$ for $s \le r \le s+\alpha$. The semigroup property allows to decompose $m_{s,t}(0) = \delta_0 M_{s,r} m_{r,t}$. Lemma 3.13 ensures that $\|\delta_0 M_{s,r}\|_{\mathrm{TV}} \le e^{\overline{b}(r-s)}$ so, using again the bound $m_{r,t}(a) \le \frac{\overline{b}}{\underline{b}} m_{r,t}(0)$, we get

$$m_{s,t}(0) = \delta_0 M_{s,r} m_{r,t} \le \|\delta_0 M_{s,r}\|_{\mathrm{TV}} \frac{\overline{b}}{\underline{b}} m_{r,t}(0) \le \frac{\overline{b}}{\underline{b}} e^{\alpha\overline{b}} m_{r,t}(0).$$

To conclude we write

$$\begin{aligned}\frac{1}{\alpha}\int_0^\alpha m_{s,t}(a)\,da &\ge \frac{1}{\alpha}\int_0^\alpha \int_s^{s+\alpha} b_\tau(a+\tau-s) m_{\tau,t}(0)\,d\tau\,da \\ &\ge \underline{b}\int_s^{s+\alpha} m_{\tau,t}(0)\,d\tau \ge \frac{\alpha\underline{b}^2}{\overline{b}}e^{-\alpha\overline{b}} m_{s,t}(0).\end{aligned}$$

Finally, we have proved that for all $t > s+\alpha$ we have

$$\frac{\alpha\underline{b}^3}{\overline{b}^2}e^{-\alpha\overline{b}}\|m_{s,t}\|_\infty \le \nu(m_{s,t}).$$

Finally, ($H'3$) comes from similar computations and a generalization of Lemma 3.13 for N. □

3.4 Periodic Semigroups and the Renewal Equation

In this section, we establish the convergence to a periodic profile for periodic semigroups. This generalizes the Floquet theory [22] for periodic matrices. We apply this result to the renewal equation and obtain an explicit exponential rate of convergence. Let us mention that it provides an exponential decay to Floquet eigenelements for a periodic PDE, which up to our knowledge has not been achieved so far.

3.4.1 Exponential Convergence for Periodic Semigroups

We start by a definition of the so-called Floquet eigenelements.

Definition 3.14 (Periodic semigroup and Floquet eigenelements) We say that a semigroup $(M_{s,t})_{0\le s\le t}$ is periodic with period T if for all $t \ge 0$ we have $\mathcal{X}_{t+T} = \mathcal{X}_t$ and for all $t \ge s \ge 0$,

$$M_{s+T,t+T} = M_{s,t}.$$

We say that $(\lambda_F, \gamma_{s,t}, h_{s,t})_{t\ge s\ge 0}$ is a Floquet family for $(M_{s,t})_{t\ge s\ge 0}$ if for all $t \ge s \ge 0$ the triplet $(\lambda_F, \gamma_{s,t}, h_{s,t}) \in \mathbb{R} \times \mathcal{M}(\mathcal{X}_s) \times \mathcal{B}_b(\mathcal{X}_t)$, for all $s \ge 0$ we have $\gamma_{s,s} \in \mathcal{P}(\mathcal{X}_s)$ and $\gamma_{s,s}(h_{s,s}) = 1$, for all $t \ge s \ge 0$,

$$\gamma_{s+T,t+T} = \gamma_{s,t} = \gamma_{s,t+T} \quad \text{and} \quad h_{s+T,t+T} = h_{s,t} = h_{s,t+T},$$

and

$$\gamma_{s,s} M_{s,t} = \mathrm{e}^{\lambda_F (t-s)} \gamma_{s,t} \quad \text{and} \quad M_{s,t} h_{t,t} = \mathrm{e}^{\lambda_F (t-s)} h_{s,t}.$$

We state the general periodic result, recalling Definition (2.1) of the coupling capacity $C_{\alpha,\beta,\nu}(s,t)$.

Theorem 3.15 *Let $(M_{s,t})_{t\ge s\ge 0}$ be a T-periodic semigroup and let $\alpha, \beta \ge 1$ such that for all $s \ge 0$, there exists $\nu \in \mathcal{P}(\mathcal{X}_s)$ such that $C_{\alpha,\beta,\nu}(s,t) \to +\infty$ when $t \to +\infty$. Assume also that the function $(s,t) \mapsto \|m_{s,t}\|_\infty$ is locally bounded. Then there exists a unique T-periodic Floquet family $(\lambda_F, \gamma_{s,t}, h_{s,t})_{t\ge s\ge 0}$ for $(M_{s,t})_{t\ge s\ge 0}$ and there exist $C, \rho > 0$ such that for all $t \ge s \ge 0$ and all $\mu \in \mathcal{M}(\mathcal{X}_s)$,*

$$\left\| \mathrm{e}^{-\lambda_F (t-s)} \mu M_{s,t} - \mu(h_{s,s}) \gamma_{s,t} \right\|_{\mathrm{TV}} \le C \mathrm{e}^{-\rho(t-s)} \|\mu\|_{\mathrm{TV}}. \tag{3.19}$$

Notice that in general the exponential rate of convergence ρ can be quantified.

Proof We start by the construction of $(\gamma_{s,t})_{t\ge s\ge 0}$. From Theorem 2.3 there exist $C > 0$ and $h_s : \mathcal{X}_s \to (0, \beta]$ such that $\nu(h_s) = 1$ and for any $\mu, \gamma \in \mathcal{M}(\mathcal{X}_s)$ and all $t \ge s$,

$$\left\| \mu M_{s,t} - \mu(h_s) \nu(m_{s,t}) \frac{\gamma M_{s,t}}{\gamma(m_{s,t})} \right\|_{\mathrm{TV}} \le C\, \nu(m_{s,t}) \|\mu\|_{\mathrm{TV}}\, \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}. \tag{3.20}$$

Considering $t = s + kT$ for $k \in \mathbb{N}$, $\mu = \nu$ and $\gamma = \nu M_{s,s+lT}$ for $l \in \mathbb{N}$ and using the periodicity of M, we get

$$\left\| \frac{\nu M_{s,s+kT}}{\nu(m_{s,s+kT})} - \frac{\nu M_{s,s+(k+l)T}}{\nu(m_{s,s+(k+l)T})} \right\|_{\mathrm{TV}} \le C\, \mathrm{e}^{-C_{\alpha,\beta,\nu}(s,s+kT)},$$

which ensures that $\left(\frac{\nu M_{s,s+kT}}{\nu(m_{s,s+kT})}\right)_{k\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{M}(\mathcal{X}_s), \|\cdot\|_{\mathrm{TV}})$. We denote the limit $\gamma_{s,s}$, which belongs to $\mathcal{P}(\mathcal{X}_s)$. Using again (3.20) with $\mu = \nu$ we have that for all $\gamma \in \mathcal{M}_+(\mathcal{X})$,

$$\frac{\gamma M_{s,s+kT}}{\gamma(m_{s,s+kT})} \xrightarrow[k\to\infty]{} \gamma_{s,s}. \tag{3.21}$$

For $f \in \mathcal{B}_b(\mathcal{X}_s)$ we have, using the periodicity of $M_{s,t}$,

$$\gamma_{s,s} M_{s,s+(k+1)T} f = \gamma_{s,s}\, M_{s,s+kT}\, M_{s,s+T} f,$$

which gives

$$\frac{\gamma_{s,s}(m_{s,s+(k+1)T})}{\gamma_{s,s}(m_{s,s+kT})}\frac{\gamma_{s,s}M_{s,s+(k+1)T}f}{\gamma_{s,s}(m_{s,s+(k+1)T})}=\frac{\gamma_{s,s}M_{s,s+kT}}{\gamma_{s,s}(m_{s,s+kT})}M_{s,s+T}f. \tag{3.22}$$

Letting $f=\mathbf{1}$ in (3.22),

$$\Lambda_s=\lim_{k\to\infty}\frac{\gamma_{s,s}(m_{s,s+(k+1)T})}{\gamma_{s,s}(m_{s,s+kT})}=\gamma_{s,s}(m_{s,s+T})$$

and letting $k\to\infty$,

$$\Lambda_s\gamma_{s,s}(f)=\gamma_{s,s}M_{s,s+T}f. \tag{3.23}$$

We check now that Λ_s is independent of s. To do so we start by proving that for any $s'>s$,

$$\gamma_{s',s'}=\frac{\gamma_{s,s}M_{s,s'}}{\gamma_{s,s}(m_{s,s'})}. \tag{3.24}$$

By the semigroup property we have on the one hand, using (3.21) with $\gamma=\gamma_{s,s}M_{s,s'}$,

$$\frac{\gamma_{s,s}M_{s,s'+kT}}{\gamma_{s,s}m_{s,s'+kT}}=\frac{(\gamma_{s,s}M_{s,s'})M_{s',s'+kT}}{(\gamma_{s,s}M_{s,s'})m_{s',s'+kT}}\xrightarrow[k\to\infty]{}\gamma_{s',s'},$$

and on the other hand using (3.23)

$$\frac{\gamma_{s,s}M_{s,s'+kT}}{\gamma_{s,s}m_{s,s'+kT}}=\frac{\gamma_{s,s}M_{s,s+kT}M_{s+kT,s'+kT}}{\gamma_{s,s}M_{s,s+kT}m_{s+kT,s'+kT}}=\frac{\Lambda_s^k\gamma_{s,s}M_{s,s'}}{\Lambda_s^k\gamma_{s,s}(m_{s,s'})}=\frac{\gamma_{s,s}M_{s,s'}}{\gamma_{s,s}(m_{s,s'})}.$$

This proves (3.24) which gives, using again (3.23) and the semigroup property,

$$\Lambda_{s'}\gamma_{s',s'}=\gamma_{s',s'}M_{s',s'+T}=\frac{\gamma_{s,s}M_{s,s'+T}}{\gamma_{s,s}(m_{s,s'})}=\frac{\Lambda_s\gamma_{s,s}M_{s,s'}}{\gamma_{s,s}(m_{s,s'})}.$$

Testing this identity against $\mathbf{1}$ and using that $\gamma_{s,s}$ are probabilities we get that $\Lambda_{s'}=\Lambda_s$, and we denote this constant by Λ. Now we define $\lambda_F=(\log\Lambda)/T$ and for all $t\geq s$,

$$\gamma_{s,t}=\gamma_{s,s}M_{s,t}\mathrm{e}^{-\lambda_F(t-s)}.$$

The definition of λ_F implies that $\gamma_{s,t}=\gamma_{s,t+T}$, and the identity $\gamma_{s+T,s+T}=\gamma_{s,s}$, which is clear on the definition (3.21) of $\gamma_{s,s}$, ensures the periodicity $\gamma_{s+T,t+T}=\gamma_{s,t}$.

We turn to the family $(h_{s,t})_{t\geq s\geq 0}$. Using Remark 2.4 we have that Theorem 2.3 is valid for $\nu=\gamma_{s,s}$ and it ensures the existence of a harmonic function

$$h_{s,s}=\lim_{t\to+\infty}\frac{m_{s,t}}{\gamma_{s,s}(m_{s,t})},$$

which satisfies $\gamma_{s,s}(h_{s,s})=1$. Now we define for $s\leq t$,

$$h_{s,t}=\mathrm{e}^{-\lambda_F(t-s)}M_{s,t}h_{t,t}.$$

It only remains to check that $(h_{s,t})_{t\geq s\geq 0}$ thus defined is T-periodic. By definition of $h_{s,s}$ and using the T-periodicity of $M_{s,t}$ and $\gamma_{s,t}$ we have

$$h_{s,s}=\lim_{k\to\infty}\frac{m_{s,s+kT}}{\gamma_{s,s}(m_{s,s+kT})}=\lim_{k\to\infty}\frac{m_{s+T,s+(k+1)T}}{\gamma_{s+T,s+T}(m_{s+T,s+(k+1)T})}=h_{s+T,s+T}.$$

Then we write

$$h_{s,s}=\lim_{t\to\infty}\frac{m_{s,t+T}}{\gamma_{s,s}(m_{s,t+T})}=\lim_{t\to\infty}\frac{M_{s,s+T}m_{s,t}}{\gamma_{s,s}(M_{s,s+T}m_{s,t})}=\lim_{t\to\infty}\frac{M_{s,s+T}m_{s,t}}{e^{\lambda_F T}\gamma_{s,s}(m_{s,t})}=\mathrm{e}^{-\lambda_F T}M_{s,s+T}h_{s,s},$$

where we have used the dominated convergence theorem for the last equality. And finally

$$h_{s,s+T}=\mathrm{e}^{-\lambda_F T}M_{s,s+T}h_{s+T,s+T}=\mathrm{e}^{-\lambda_F T}M_{s,s+T}h_{s,s}=h_{s,s}.$$

Now we check the convergence (3.19). Applying Theorem 2.3 with $\nu=\gamma=\gamma_{s,s}$ and $s_0=s$, we get that there exists $C>0$ such that for all $t\geq s\geq 0$ and all $\mu\in\mathcal{M}(\mathcal{X}_s)$,

$$\left\|\mu M_{s,t}-\mu(h_{s,s})\mathrm{e}^{\lambda_F(t-s)}\gamma_{s,t}\right\|_{\mathrm{TV}}\leq C\,\mathrm{e}^{\lambda_F(t-s)}\gamma_{s,t}(\mathbf{1})\|\mu\|_{\mathrm{TV}}\,\mathrm{e}^{-C_{\alpha,\beta,\nu}(s,t)}.$$

By periodicity we have $\gamma_{s,t}(\mathbf{1})=\mathrm{e}^{-\lambda_F(t-s)}\gamma_{s,s}(m_{s,t})\leq\mathrm{e}^{|\lambda_F|T}\sup_{0\leq s\leq T,s\leq t\leq s+T}\|m_{s,t}\|_\infty$. Still by periodicity, for all $n\in\mathbb{N}$ and all $t\geq s+nT$, we have $C_{\alpha,\beta,\nu}(s,t)\geq C_{\alpha,\beta,\nu}(0,nT)\lfloor\frac{t-s}{nT}\rfloor$. Using that $C_{\alpha,\beta,\nu}(0,t)\to+\infty$ when $t\to+\infty$ we can choose $n\in\mathbb{N}$ large enough so that $C_{\alpha,\beta,\nu}(0,nT)>0$ and this gives (3.19) with $\rho=C_{\alpha,\beta,\nu}(0,nT)/nT$.

The uniqueness follows from the convergence. Assume the existence of another Floquet family $(\tilde{\lambda},\tilde{\gamma}_{s,t},\tilde{h}_{s,t})$. Applying (3.19) to $\mu=\tilde{\gamma}_{s,s}$ guarantees that $\tilde{\lambda}_F=\lambda_F$ and $\tilde{\gamma}_{s,t}=\tilde{\gamma}_{s,s}(h_{s,s})\gamma_{s,t}$. Since $\gamma_{s,s}$ and $\tilde{\gamma}_{s,s}$ are both probabilities this implies that $\tilde{\gamma}_{s,s}(h_{s,s})=1$, and $\tilde{\gamma}_{s,t}=\gamma_{s,t}$. Applying again (3.19) but with $\mu=\delta_x$, $t=s+kT$, and the test function $\tilde{h}_{s,s}=\tilde{h}_{s+kT,s+kT}$, we get $\tilde{h}_{s,s}=h_{s,s}$. The equality $\tilde{h}_{s,t}=h_{s,t}$ is then ensured by the identity $\tilde{\lambda}_F=\lambda_F$. □

3.4.2 Example: The Periodic Renewal Equation

We consider the renewal equation with a time-periodic division rate, which is used as a model for circadian rhythms (see [13, 14] and the references therein). More precisely the equation is

$$\begin{cases}\partial_t u_{s,t}(a)+\partial_a u_{s,t}(a)+b(t,a)u_{s,t}(a)=0, & t>s,\ a>0,\\ u_{s,t}(0)=2\displaystyle\int_0^\infty b(t,a)u_{s,t}(a)\,da, & t>s,\\ u_{s,s}(a)=u_s(a), & a\geq 0,\end{cases}\tag{3.25}$$

and we assume that there exists $T>0$ such that $b(t+T,\cdot)=b(t,\cdot)$ for any $t\geq 0$. Additionally b is supposed to be non-negative and globally bounded (in time and age) by a constant $\overline{b}>0$. Similarly as in the homogeneous or asymptotically homogeneous case we associate to Eq. (3.25) a semigroup $(M_{s,t})_{0\leq s\leq t}$ defined on $\mathcal{B}_b(\mathbb{R}_+)$ by the Duhamel formula

$$\begin{aligned}M_{s,t}f(a)=&\ f(a+t-s)\mathrm{e}^{-\int_s^t b(\tau,a+\tau-s)\,d\tau}\\&+2\int_s^t\mathrm{e}^{-\int_s^\tau b(\tau',a+\tau'-s)\,d\tau'}b(\tau,a+\tau-s)M_{\tau,t}f(0)\,d\tau\end{aligned}\tag{3.26}$$

and on $\mathcal{M}(\mathbb{R}_+)$ by setting $(\mu M_{s,t})(f)=\mu(M_{s,t}f)$ for all $f\in\mathcal{B}_b(\mathbb{R}_+)$. As for the homogeneous case, this semigroup is well defined and satisfies Assumption 2.1. Mimicking the proof of Lemma 3.9 we can prove the following monotonicity result.

Lemma 3.16 *For all $s,a\geq 0$, the function $t\mapsto m_{s,t}(a)$ is nondecreasing.*

Notice however that the function $s \mapsto m_{s,t}(a)$ is not nonincreasing in general. Yet, since $(M_{s,t})_{0\le s\le t}$ is T-periodic we have, using Lemma 3.16, $m_{s+T,t}(a) = m_{s,t-T}(a) \le m_{s,t}(a)$.

The aim of the current section is to provide sufficient conditions on b so that we can apply Theorem 3.15. In Theorem 3.17 we give a general such result, and in Theorem 3.18 we optimize the rate of convergence in the case when the division rate depends only on time. Let us point out that in this latter case the mean number of individuals $m_{s,t}(a)$ does not depend on a and satisfies a simple differential equation. Classical Floquet theory [22] then easily applies for proving that it tends to a periodic solution. However it is no longer the case for the age repartition described by the semigroup which remains an infinite-dimensional object. Its asymptotic periodicity cannot be deduced from those of $m_{s,t}$.

The convergence of the solutions of the periodic renewal equation to the Floquet elements has been obtained in [39] by the way of entropy techniques. Here, we provide an explicit exponential rate of convergence.

Theorem 3.17 *Let b be a time-periodic function with period $T > 0$, non-negative and globally bounded by $\overline{b} > 0$. Assume that there exist $A \ge 0$ and $\underline{b} > 0$ such that*

$$\forall t \ge 0,\ \forall a \ge A, \quad b(t,a) \ge \underline{b}.$$

Then there exists a unique Floquet family $(\lambda_F, \gamma_{s,t}, h_{s,t})_{0\le s\le t}$ for the semigroup $(M_{s,t})_{0\le s\le t}$ and a constant $C > 0$ such that for all $t \ge s \ge 0$ and all $\mu \in \mathcal{M}(\mathbb{R}_+)$,

$$\left\| e^{-\lambda_F(t-s)}\mu M_{s,t} - \mu(h_{s,s})\gamma_{s,t} \right\|_{TV} \le C\,\|\mu\|_{TV}\, e^{-\rho(t-s)}$$

where

$$\rho = \frac{-1}{A+2T}\log\left(1 - \frac{2\underline{b}T e^{-\overline{b}(3A+8T)}}{\frac{1}{2\overline{b}T} + \frac{A}{T} + 3 + \frac{1}{1-e^{-\underline{b}T}}}\right).$$

Proof We will use several times the inequality

$$\|m_{s,t}\|_\infty \le e^{2\overline{b}(t-s)},$$

whose proof follows Lemma 3.13. In particular it ensures that the function $(s,t) \mapsto \|m_{s,t}\|_\infty$ is locally bounded.

Now we exhibit constants for the assumptions in Definition 2.2. We start with (A1). Let $s \ge 0$ and define $n := \lfloor \frac{A}{T} \rfloor$, so that $(n+1)T \in (A, A+T]$. From the Duhamel formula (3.26) and using the periodicity of the semigroup we get that for any $f \ge 0$ and any $a \ge 0$

$$\begin{aligned} M_{s,s+(n+2)T} f(a) &\ge 2\int_{s+(n+1)T}^{s+(n+2)T} e^{-\int_s^\tau b(\tau',a+\tau'-s)\,d\tau'} b(\tau,a+\tau-s) M_{\tau,s+(n+1)T} f(0)\,d\tau \\ &\ge 2\underline{b}e^{-\overline{b}(A+2T)} \int_s^{s+T} M_{\tau,s+T} f(0)\,d\tau. \end{aligned}$$

We have $m_{s,s+(n+2)T} \le e^{2\overline{b}(A+2T)}$ and Lemma 3.16 ensures that $\int_s^{s+T} m_{\tau,s+T}(0)\,d\tau \ge T$. So (A1) is satisfied with

$$\nu(f) = \frac{\int_s^{s+T} M_{\tau,s+T} f(0)\,d\tau}{\int_s^{s+T} m_{\tau,s+T}(0)\,d\tau} \quad \text{and} \quad c = 2\underline{b}Te^{-3\overline{b}(A+2T)}.$$

We treat the last three assumptions (A2)–(A4) together by proving that there exists $d > 0$ such that for all $t \geq s$,

$$d\,\|m_{s,t}\|_\infty \leq \nu(m_{s,t}).$$

If $t - s \leq A$, we have $\|m_{s,t}\|_\infty \leq \mathrm{e}^{2\overline{b}A}$, and from Lemma 3.16 $\nu_s(m_{s,t}) \geq 1$, so it remains to treat the case $t - s > A$. Keeping $n = \lfloor \frac{A}{T} \rfloor$ and setting $N = \lfloor \frac{t-s}{T} \rfloor \geq n$, we have from (3.26)

$$\begin{aligned}
m_{s,t}(a) &= \mathrm{e}^{-\int_s^t b(\tau,a+\tau-s)\,d\tau} + 2\int_s^t \mathrm{e}^{-\int_s^\tau b(\tau',a+\tau'-s)\,d\tau'} b(\tau, a+\tau-s) m_{\tau,t}(0)\,d\tau \\
&\leq 1 + 2\overline{b}\sum_{k=0}^{n} \int_{s+kT}^{s+(k+1)T} \mathrm{e}^{-\int_s^\tau b(\tau',a+\tau'-s)\,d\tau'} m_{\tau,t}(0)\,d\tau \\
&\quad + 2\overline{b}\sum_{k=n+1}^{N} \int_{s+kT}^{s+(k+1)T} \mathrm{e}^{-\int_s^\tau b(\tau',a+\tau'-s)\,d\tau'} m_{\tau,t}(0)\,d\tau \\
&\quad + 2\overline{b}\int_{s+NT}^{t} \mathrm{e}^{-\int_s^\tau b(\tau',a+\tau'-s)\,d\tau'} m_{\tau,t}(0)\,d\tau \\
&\leq 1 + 2\overline{b}\sum_{k=0}^{n} \int_s^{s+T} m_{\tau+kT,t}(0)\,d\tau \\
&\quad + 2\overline{b}\sum_{k=n+1}^{N} \int_s^{s+T} \mathrm{e}^{-\int_s^{\tau+kT} b(\tau',a+\tau'-s)\,d\tau'} m_{\tau+kT,t}(0)\,d\tau \\
&\quad + 2\overline{b}\int_s^{t-NT} m_{\tau+NT,t}(0)\,d\tau \\
&\leq 1 + 2\overline{b}(n+1)\int_s^{s+T} m_{\tau,t}(0)\,d\tau + 2\overline{b}\int_s^{t-NT} m_{\tau,t}(0)\,d\tau \\
&\quad + 2\overline{b}\sum_{k=n+1}^{N} \int_s^{s+T} \mathrm{e}^{-\int_{(n+1)T}^{kT} \underline{b}\mathbf{1}_{t_1 \leq u - \lfloor u/T \rfloor \leq t_2}\,du} m_{\tau,t}(0)\,d\tau \\
&\leq \big(1/T + 2\overline{b}(n+2)\big)\int_s^{s+T} m_{\tau,t}(0)\,d\tau + 2\overline{b}\sum_{k=0}^{N-n-1} \mathrm{e}^{-k\underline{b}(t_2-t_1)} \int_s^{s+T} m_{\tau,t}(0)\,d\tau \\
&\leq \left(\frac{1}{T} + 2\left(\frac{A}{T}+3\right)\overline{b} + \frac{2\overline{b}}{1-\mathrm{e}^{-\underline{b}T}}\right)\int_s^{s+T} m_{\tau,t}(0)\,d\tau.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\nu(m_{s,t}) = \nu(m_{s+T,t+T}) &= \frac{1}{\int_s^{s+T} m_{\tau,s+T}(0)\,d\tau}\int_s^{s+T} m_{\tau,t+T}(0)\,d\tau \\
&\geq \frac{1}{\int_s^{s+T} \mathrm{e}^{2\overline{b}(s+T-\tau)}\,d\tau}\int_s^{s+T} m_{\tau,t}(0)\,d\tau \geq 2\overline{b}\mathrm{e}^{-2\overline{b}T}\int_s^{s+T} m_{\tau,t}(0)\,d\tau,
\end{aligned}$$

which gives

$$m_{s,t}(a) \le \left(\frac{1}{2\bar{b}T} + \frac{A}{T} + 3 + \frac{1}{1-\mathrm{e}^{-\underline{b}T}} \right) \mathrm{e}^{2\bar{b}T} \nu_s(m_{s,t})$$

and ends the proof. □

Theorem 3.18 *Assume that $b(t,a) = b(t)$ is a continuous T-periodic function, which is not identically zero. Then there exists a unique Floquet family $(\lambda_F, \gamma_{s,t}, h_{s,t})_{0\le s\le t}$ for the semigroup $(M_{s,t})_{0\le s\le t}$ and a constant $C > 0$ such that for all $t \ge s \ge 0$ and all $\mu \in \mathcal{M}(\mathbb{R}_+)$*

$$\left\| \mathrm{e}^{-\lambda_F(t-s)} \mu M_{s,t} - \mu(h_{s,s})\gamma_{s,t} \right\|_{\mathrm{TV}} \le C \, \|\mu\|_{\mathrm{TV}} \, \mathrm{e}^{-2\int_s^t b(\tau)\,d\tau}.$$

Proof Since b does not depend on a we have the explicit formula

$$m_{s,t}(a) = \mathrm{e}^{\int_s^t b(\tau)\,d\tau}. \tag{3.27}$$

In particular it ensures that the function $(s,t) \mapsto \|m_{s,t}\|_\infty$ is locally bounded.

Now we prove (A1). Let $t \ge s + T$, $k \in \mathbb{N}$, $n = \lfloor \frac{t-s}{T} \rfloor \ge 1$, and $N = (n-1)k + 1$. For all $0 \le i \le N-1$ we set $t_i = s + i\frac{T}{k}$, and $t_N = s + nT$. From the Duhamel formula (3.26) we have for any $1 \le i \le N-1$ and $f \ge 0$

$$M_{t_{i-1},t_i} f(a) \ge 2 \int_{t_{i-1}}^{t_i} \mathrm{e}^{-\int_{t_{i-1}}^{\tau} b(\tau')\,d\tau'} b(\tau) f(t_i - \tau)\,d\tau \ge 2\mathrm{e}^{-\|b\|_\infty \frac{T}{k}} \Big(\min_{[t_{i-1},t_i]} b \Big) \int_0^{\frac{T}{k}} f(\tau)\,d\tau.$$

For $i = N$ we write

$$\begin{aligned} M_{t_{N-1},t_N} f(a) &\ge 2 \int_{s+(n-1)T}^{s+nT} \mathrm{e}^{-\int_{s+(n-1)T}^{\tau} b(\tau')\,d\tau'} b(\tau) f\big(s + (n-1)T - \tau\big)\,d\tau \\ &\ge 2\mathrm{e}^{-\|b\|_\infty T} \int_0^T b(T-\tau) f(\tau)\,d\tau. \end{aligned}$$

So (A1) is verified for $0 \le i \le N-1$ with $\nu_i(f) = \frac{n}{T} \int_0^{\frac{T}{n}} f(\tau)\,d\tau$ and

$$c_i = \frac{2T}{k} \mathrm{e}^{-\|b\|_\infty \frac{T}{k}} \min_{[t_{i-1},t_i]} b$$

and for $i = N$ with $\nu_N(f) = \frac{\int_0^T b(T-\tau) f(\tau)\,d\tau}{\int_0^T b(\tau)\,d\tau}$ and

$$c_N = 2\mathrm{e}^{-\|b\|_\infty T} \int_0^T b(\tau)\,d\tau.$$

From (3.27) we deduce that Assumptions (A2)–(A4) are trivially verified with

$$d_i = 1, \qquad \alpha = \frac{1}{c_N}, \qquad \beta = 1.$$

Thus the coupling capacity satisfies, for all $k \in \mathbb{N}$,

$$C_{\alpha,\beta,\nu}(s,t) \ge -\sum_{i=1}^{N-1} \log\left(1 - \frac{2T}{k} \mathrm{e}^{-\|b\|_\infty \frac{T}{k}} \min_{[t_{i-1},t_i]} b \right).$$

Letting $k \to \infty$ we get

$$C_{\alpha,\beta,\nu}(s,t) \geq 2\int_s^{s+(n-1)T} b(\tau)\,d\tau \geq 2\int_s^t b(\tau)\,d\tau - 2\int_0^T b(\tau)\,d\tau. \qquad \square$$

Remark 3.19 (About optimality) In the constant case $b = const > 0$, we recover the spectral gap $2b$. Indeed, we know that the spectral gap of the operator

$$\mathcal{A}f(a) = f'(a) - bf(a) + 2bf(0)$$

cannot be larger than $2b$ because the dominant Perron eigenvalue is b, and $-b$ belongs to the spectrum of $\mathcal{A}$ (the operator $\mathcal{A} + b$ is not surjective on $C_b(\mathbb{R}_+)$ since all the solutions to the equation $f'(a) = -2bf(0) + \frac{1}{1+a}$ for instance are unbounded).

Remark 3.20 (Eigenvalue) From the periodicity of $(\gamma_{s,t})_{t\geq s\geq 0}$ and (3.27), we easily get that

$$\lambda_F = \frac{1}{T}\int_0^T b(\tau)\,d\tau.$$

Acknowledgements B.C. and V.B. have received the support of the Chair "Modélisation Mathématique et Biodiversité" of VEOLIA-Ecole Polytechnique-MnHn-FX. The three authors have been supported by ANR projects, funded by the French Ministry of Research: B.C. by ANR PIECE (ANR-12-JS01-0006-01), V.B. by ANR ABIM (ANR-16-CE40-0001) and ANR CADENCE (ANR-16-CE32-0007), and P.G. by ANR KIBORD (ANR-13-BS01-0004).

Appendix A: Branching Models, Absorbed Markov Process and Semigroups

The techniques by coupling used in this paper have been extensively developed in probability, in particular for the study of branching processes and killed process, see the introduction for references. Let us present here informally the probabilistic objects and the interpretation of the auxiliary semigroup.

For that purpose we consider a population of individuals with a trait belonging to the space $\mathcal{X}$. This population can die or give birth to some offsprings with a rate which depends on their trait and independently one from each other (branching property). Moreover the trait may vary in an homogeneous way and without memory (Markov property). Let us also assume that some subspace $\mathcal{S}$ of $\mathcal{X}$ is absorbing, meaning that each individual whose trait reaches this set stop dividing and keeps a constant trait. Writing V_t the set of individuals at time t and $(X_t^i : i \in V_t)$ the set of their traits, the branching and Markov properties and the absorbing property of $\mathcal{S}$ ensure that

$$\delta_x M_{s,t}(f) = \mathbb{E}\bigg(\sum_{i\in V_t} f(X_t^i)1_{X_t^i \notin \mathcal{S}} \,\bigg|\, X_s = \delta_x\bigg)$$

is a semigroup. In general, it is not conservative, since its mass

$$m_{s,t}(x) = \delta_x M_{s,t}\mathbf{1} = \mathbb{E}\big(\#\{i \in V_t : X_t^i \notin \mathcal{S} \mid X_s = \delta_x\}\big)$$

can decrease by absorption in $\mathcal{S}$ or death of individual or created by births. The trait of a typical non-absorbed individual is then given by the auxiliary conservative inhomogeneous semigroup

$$\delta_x P^{(t)}_{u,v} f = \frac{\delta_x M_{u,v}(f m_{v,t})}{m_{u,t}(x)} = \frac{\mathbb{E}(\sum_{i \in V_t} f(X^i_v) 1_{X^i_v \notin \mathcal{S}} \mid X_u = \delta_x)}{\mathbb{E}(\#\{i \in V_t : X^i_t \notin \mathcal{S}\} \mid X_u = \delta_x)} = \mathbb{E}\big(f\big(Y^{(t)}_v\big) \bigm| Y^{(t)}_u = x\big),$$

where X^i_v is the trait of the ancestor of i at time v and $Y^{(t)}$ is the inhomogenous Markov process associated to $P^{(t)}$. Thus, $Y^{(t)}$ is the process describing the dynamics of the trait of a typical individual, which is alive at time t and non-absorbed. Proving that it is ergodic ensures the ergodicity of $\delta_x M_{s,t}\mathbf{1}/m_{s,t}(x)$ as t goes to infinity. In this paper, we make a coupling for that, with Doeblin conditions which ensure exponential uniform ergodicity. Thanks to [12], this Doeblin condition can be rewritten in terms of coupling constants on the original semigroup M.

In homogeneous-time setting, two particular classes of processes have attracted lots of attention. First, if we make $\mathcal{S} = \varnothing$, then X is a branching process and

$$\delta_x M_{s,t}(f) = \mathbb{E}\Bigg(\sum_{i \in V_t} f(X^i_t) \Bigm| X_s = \delta_x\Bigg)$$

is its first moment semigroup which provides the mean number of individuals with a given trait. The auxiliary process describes the dynamical of the trait along the ancestral lineage of an individual chosen uniformly at random, when the population is becoming large. More generally, the genealogical tree of the population can be constructed from this typical lineage, which is called *spine construction*.

Second, if the individuals neither die nor give birth, we get a Markov process in the space trait $\mathcal{X}$ and

$$\delta_x M_{s,t}(f) = \mathbb{E}\big(f(X_t) 1_{X_t \notin \mathcal{S}} \bigm| X_s = x\big)$$

Assume that X_t is eventually absorbed as t goes to infinity a.s. and consider the distribution of the process conditioned on non-absorption:

$$\mathbb{P}_x(X_t \in . \mid X_t \notin \mathcal{S}) = \frac{\delta_x M_{0,t}}{m_{0,t}(x)} = \delta_x P^{(t)}_{0,t}.$$

The ergodic behavior of $P^{(t)}$ and its convergence to a distribution ν yields the convergence of the conditioned distribution (Yaglom limit) to the quasistationary distribution. At fixed time t, $P^{(t)}$ describes the dynamic of the trait for trajectories non-absorbed at time t.

Appendix B: Measure Solutions to the Renewal Equation

We give here the details about the construction of the homogeneous renewal semigroup. It is based on the dual renewal equation

$$\partial_t f_t(a) - \partial_a f_t(a) + b(a) f_t(a) = 2b(a) f_t(0), \quad t, a \geq 0. \tag{B.1}$$

Integrating this equation along the characteristics, we obtain the mild formulation (also called Duhamel formula)

$$f_t(a) = f_0(a+t)\mathrm{e}^{-\int_0^t b(a+\tau)\,d\tau} + 2\int_0^t \mathrm{e}^{-\int_0^\tau b(a+\tau')\,d\tau'} b(a+\tau) f_{t-\tau}(0)\,d\tau. \tag{B.2}$$

The first result is about the well-posedness of this equation in $\mathcal{B}_b(\mathbb{R}_+)$.

Lemma B.1 *For all $f_0 \in \mathcal{B}_b(\mathbb{R}_+)$ there exists a unique family $(f_t)_{t\geq 0} \subset \mathcal{B}_b(\mathbb{R}_+)$ solution to (B.2). Additionally if $f_0 \geq 0$ then $f_t \geq 0$ for all $t \geq 0$.*

Proof First we use the Banach fixed point theorem on a truncated problem. For $T > 0$ and $f_0 \in \mathcal{B}_b(\mathbb{R}_+)$ we define the operator $\Gamma : \mathcal{B}_b([0,T]) \to \mathcal{B}_b([0,T])$ by

$$\Gamma g(t) = f_0(t)\mathrm{e}^{-\int_0^t b(\tau)\,d\tau} + 2\int_0^t \Phi(\tau) g(t-\tau)\,d\tau.$$

We easily have

$$\|\Gamma g_1 - \Gamma g_2\|_\infty \leq 2\int_0^T \Phi(\tau)\,d\tau\, \|g_1 - g_2\|_\infty,$$

so Γ is a contraction if $2\int_0^T \Phi < 1$ and there is a unique fixed point in $\mathcal{B}_b([0,T])$. Additionally since Γ preserves non-negativity when $f_0 \geq 0$, we get that the fixed point g is non-negative when f_0 is non-negative. Since the contraction constant $2\int_0^T \Phi$ is independent of f_0, we can iterate to obtain a unique function $g \in \mathcal{B}_b(\mathbb{R}_+)$ which satisfies

$$g(t) = f_0(t)\mathrm{e}^{-\int_0^t b(\tau)\,d\tau} + 2\int_0^t \Phi(\tau) g(t-\tau)\,d\tau$$

for all $t \geq 0$. Now we set for all $t, a \geq 0$

$$f_t(a) = f_0(a+t)\mathrm{e}^{-\int_0^t b(a+\tau)\,d\tau} + 2\int_0^t \mathrm{e}^{-\int_0^\tau b(a+\tau')\,d\tau'} b(a+\tau) g(t-\tau)\,d\tau,$$

which is a solution to (3.11) since by definition $f_t(0) = \Gamma g(t) = g(t)$. The uniqueness is a direct consequence of the uniqueness of g. The non-negativity follows from the non-negativity of g when $f_0 \geq 0$, and the boundedness is given by the inequality

$$\|f_t\|_\infty \leq \|f_0\|_\infty + 2 \sup_{0\leq s\leq t} \big|g(s)\big|. \qquad \square$$

Lemma B.1 allows to define for all $t \geq 0$ the operator M_t on $\mathcal{B}_b(\mathbb{R}_+)$ by setting $M_t f_0 := f_t$ for all $f_0 \in \mathcal{B}_b(\mathbb{R}_+)$. Then for $\mu \in \mathcal{M}_+(\mathbb{R}_+)$ we define the positive measure μM_t by setting for all Borel set $A \subset \mathbb{R}_+$

$$(\mu M_t)(A) := \mu(M_t \mathbf{1}_A).$$

The axioms of a positive measure are satisfied. First it is clear that $(\mu M_t)(\varnothing) = 0$ and that $(\mu M_t)(A \cup B) = (\mu M_t)(A) + (\mu M_t)(B)$ when A and B are two disjoint Borel sets. The last axiom deserves a bit more attention. Let $(A_n)_{n\geq 0}$ be an increasing sequence of Borel sets and $A = \bigcup_{n\geq 0} A_n$. We want to check that $(\mu M_t)(A) = \lim_{n\to\infty}(\mu M_t)(A_n)$. The sequence $(\mathbf{1}_{A_n})_{n\geq 0}$ is an increasing sequence of Borel functions which converges pointwise to $\mathbf{1}_A$. By positivity of the semigroup, $(M_t \mathbf{1}_{A_n})_{n\geq 0}$ is an increasing sequence of Borel functions bounded by $M_t \mathbf{1}$. Thus this sequence admits a pointwise limit $f_t \in \mathcal{B}_b(\mathbb{R}_+)$ which clearly satisfies the Duhamel formula (3.11) with $f_0 = \mathbf{1}_A$. By uniqueness of the solution to the Duhamel formula we get that $M_t\mathbf{1}_{A_n} \to M_t\mathbf{1}_A$ pointwise. Then by dominated or monotone convergence we deduce that $(\mu M_t)(A) = \mu(M_t\mathbf{1}_A) = \lim_{n\to\infty}\mu(M_t\mathbf{1}_{A_n}) = \lim_{n\to\infty}(\mu M_t)(A_n)$. Finally for a signed measure $\mu \in \mathcal{M}(\mathbb{R}_+)$ we set obviously $\mu M_t := \mu_+ M_t - \mu_- M_t$.

The family $(M_t)_{t\geq 0}$ such defined is a semigroup which satisfies Assumption 2.1. The semigroup property is a consequence of the uniqueness of the solution to the Duhamel formula (3.11). The positivity has been proved in Lemma B.1. For the strong positivity it follows from the Duhamel formula (3.11) that for all $t, a \geq 0$

$$m_t(a) \geq \mathrm{e}^{-\int_0^t b(a+\tau)\,d\tau} > 0.$$

The compatibility $(\mu M_t)(f) = \mu(M_t f)$ follows directly from the definition of μM_t and the definition of Borel functions.

It is claimed in Sect. 3.2.2 that the family $(\mu M_t)_{t\geq 0}$ is a measure solution to the renewal equation. Measure valued solutions to structured population models drew interest in the last few years [9, 10, 24, 28, 29]. They are mainly motivated by biological applications which often require to consider initial distributions which are not densities but measures (Dirac masses for instance). For us it is additionally the suitable framework to apply our ergodic result in Theorem 3.5. We refer to [24] for the proof that the family $(\mu M_t)_{t\geq 0}$ is a measure valued solution to Eq. (3.10) for any $\mu \in \mathcal{M}(\mathbb{R}_+)$. Here we only give a heuristic argument which consists in differentiating the semigroup property $\mu M_t f = \mu M_s M_{t-s} f$ with respect to $s \in [0, t]$. The chain rule gives

$$\partial_s(\mu M_s)M_{t-s}f + \mu M_s \partial_s(M_{t-s}f) = 0$$

and since $M_t f$ is a solution to the dual Eq. (B.1) this gives

$$\partial_s(\mu M_s)M_{t-s}f - \mu M_s \mathcal{A}(M_{t-s}f) = 0$$

where $\mathcal{A}$ is the unbounded operator defined on $C^1(\mathbb{R}_+)$ by $\mathcal{A}f(a) = f'(a) - b(a)f(a) + 2b(a)f(0)$. Taking $s = t$ we get that for all bounded and continuously differentiable function f

$$\partial_t(\mu M_t f) = \mu M_t(\mathcal{A}f),$$

which is a weak formulation of Eq. (3.10).

Appendix C: The Max-Age Semigroup

As for the homogeneous renewal equation, to build a solution to Eq. (3.17) we use a duality approach. We start with the (backward) dual equation

$$\begin{cases} \partial_s f_{s,t}(a) + \partial_a f_{s,t}(a) + b(a) f_{s,t}(0) = 0, & s < t,\ 0 \leq a < a_s, \\ f_{s,t}(a_s) = 0, & s < t, \\ f_{t,t}(a) = f_t(a), & 0 \leq a < a_t. \end{cases} \tag{C.1}$$

Integrating this equation along the characteristics we get the Duhamel formula

$$f_{s,t}(a) = f_t(a + t - s) + \int_s^t b_\tau(a + \tau - s) f_{\tau,t}(0)\,d\tau \tag{C.2}$$

where we have denoted $b_t(a) := b(a)\mathbf{1}_{[0,a_t)}(a)$ and f_t has been extended by 0 beyond a_t.

Lemma C.1 *For all $t > 0$, $f_t \in \mathcal{B}_b([0, a_t))$, and $s \in [0, t]$, there exists a unique $f_{s,t} \in \mathcal{B}_b([0, a_s))$ which satisfies (C.2). Additionally if $f_t \geq 0$ then $f_{s,t} \geq 0$.*

We do not repeat the proof of this result since it follows exactly the strategy of the proof of Lemma B.1. As for the homogeneous renewal equation we define the semigroup $(M_{s,t})_{0\leq s\leq t}$ on $(\mathcal{X}_t)_{t\geq 0} = ([0, a_t))_{t\geq 0}$, first on the right hand side by setting for all $f_t \in \mathcal{B}_b([0, a_t))$

$$M_{s,t} f_t := f_{s,t}$$

where $f_{s,t}$ is the unique solution to Eq. (C.2), and then on the left by setting for all $\mu \in \mathcal{M}([0, a_s))$ and all Borel set $A \subset [0, a_t)$

$$(\mu M_{s,t})(A) = \mu_+(M_{s,t}\mathbf{1}_A) - \mu_-(M_{s,t}\mathbf{1}_A).$$

For any $\mu \in \mathcal{M}([0, a_s))$ the family $(\mu M_{s,t})_{s\leq t}$ is a measure solution to Eq. (3.17). As for the homogeneous case a non rigorous justification is obtained by differentiating the semigroup property $\mu M_{s,t} f = \mu M_{s,r} M_{r,t} f$ with respect to $r \in [s, t]$ and using that $M_{r,t} f$ is solution to (C.1).

The semigroup property for the family $(M_{s,t})_{t\geq s\geq 0}$ is a consequence of the uniqueness of the solution to the Duhamel formula (C.2). We now verify Assumption 2.1. The positivity has been proved in Lemma C.1. For the strong positivity it suffices to check that $m_{s,t}(0) > 0$. Indeed if $m_{s,t}(0) > 0$ for all $0 \leq s \leq t$ the Duhamel formula ensures that for $a < a_s$

$$m_{s,t}(a) = \mathbf{1}_{a+t-s<a_t} + \int_s^t b_\tau(a+\tau-s) m_{\tau,t}(0)\, d\tau \geq \underline{b} \int_s^t \mathbf{1}_{a+\tau-s<a_\tau} m_{\tau,t}(0)\, d\tau > 0.$$

The positivity of $m_{s,t}(0)$ is clear if $t - s < a_t$ since

$$m_{s,t}(0) \geq \mathbf{1}_{t-s<a_t}.$$

Consider now the case $t - s \geq a_t$. The function $r \mapsto m_{r,t}(0)$ is continuous on $[s, t - a_t]$ since for $r \leq t - a_t$ we have

$$m_{r,t}(0) = \int_r^t b_\tau(\tau - r) m_{\tau,t}(0)\, d\tau.$$

Assume by contradiction that there exists $r_0 \in [s, t - a_t]$ such that $m_{r_0,t}(0) = 0$ and $m_{r,t}(0) > 0$ for all $r \in (r_0, t]$. Then the equality above would give for $r = r_0$

$$0 = \int_{r_0}^t b_\tau(\tau - r_0) m_{\tau,t}(0)\, d\tau,$$

which is not possible since the integrand on the right hand side is positive for τ close to r_0. Finally the compatibility condition readily follows from the definition of $\mu M_{s,t}$.

References

1. Arino, O.: A survey of structured cell population dynamics. Acta Biotheor. **43**(1), 3–25 (1995)

2. Bansaye, V.: Ancestral lineages and limit theorems for branching Markov chains in varying environment. J. Theor. Probab. **32**(1), 249–281 (2019)
3. Bansaye, V., Camanes, A.: Queueing for an infinite bus line and aging branching process. Queueing Syst. **88**(1–2), 99–138 (2018)
4. Bansaye, V., Huang, C.: Weak law of large numbers for some Markov chains along non homogeneous genealogies. ESAIM Probab. Stat. **19**, 307–326 (2015)
5. Bansaye, V., Delmas, J.-F., Marsalle, L., Tran, V.C.: Limit theorems for Markov processes indexed by continuous time Galton-Watson trees. Ann. Appl. Probab. **21**(6), 2263–2314 (2011)
6. Bertoin, J.: Random Fragmentation and Coagulation Processes. Cambridge Studies in Advanced Mathematics, vol. 102. Cambridge University Press, Cambridge (2006)
7. Bertoin, J., Watson, A.R.: A probabilistic approach to spectral analysis of growth-fragmentation equations. J. Funct. Anal. **274**(8), 2163–2204 (2018)
8. Birkhoff, G.: Extensions of Jentzsch's theorem. Trans. Am. Math. Soc. **85**, 219–227 (1957)
9. Cañizo, J.A., Carrillo, J.A., Cuadrado, S.: Measure solutions for some models in population dynamics. Acta Appl. Math. **123**, 141–156 (2013)
10. Carrillo, J.A., Colombo, R.M., Gwiazda, P., Ulikowska, A.: Structured populations, cell growth and measure valued balance laws. J. Differ. Equ. **252**(4), 3245–3277 (2012)
11. Champagnat, N., Villemonais, D.: Exponential convergence to quasi-stationary distribution and Q-process. Probab. Theory Relat. Fields **164**(1–2), 243–283 (2016)
12. Champagnat, N., Villemonais, D.: Uniform convergence of penalized time-inhomogeneous Markov processes. ESAIM Probab. Stat. **22**, 129–162 (2018)
13. Clairambault, J., Michel, P., Perthame, B.t.: Circadian rhythm and tumour growth. C. R. Math. Acad. Sci. Paris **342**(1), 17–22 (2006)
14. Clairambault, J., Gaubert, S., Lepoutre, T.: Comparison of Perron and Floquet eigenvalues in age structured cell division cycle models. Math. Model. Nat. Phenom. **4**(3), 183–209 (2009)
15. Cloez, B.: Limit theorems for some branching measure-valued processes. Adv. Appl. Probab. **49**(2), 549–580 (2017)
16. Del Moral, P.: Feynman-Kac Formulae. Genealogical and Interacting Particle Systems with Applications. Probability and Its Applications. Springer, New York (2004)
17. Del Moral, P.: Mean Field Simulation for Monte Carlo Integration. Monographs on Statistics and Applied Probability, vol. 126. CRC Press, Boca Raton (2013)
18. Del Moral, P., Villemonais, D.: Exponential mixing properties for time inhomogeneous diffusion processes with killing. Bernoulli **24**(2), 1010–1032 (2018)
19. Doblin, W.: Éléments d'une théorie générale des chaînes simples constantes de Markoff. Ann. Éc. Norm. **3**(57), 61–111 (1940)
20. Engländer, J., Harris, S.C., Kyprianou, A.E.: Strong law of large numbers for branching diffusions. Ann. Inst. Henri Poincaré Probab. Stat. **46**(1), 279–298 (2010)
21. Feller, W.: On the integral equation of renewal theory. Ann. Math. Stat. **12**, 243–267 (1941)
22. Floquet, G.: Sur les équations différentielles linéaires à coefficients périodiques. Ann. Sci. Éc. Norm. Supér. (2) **12**, 47–88 (1883)
23. Frobenius, G.: Über Matrizen aus nicht negativen Elementen. Berl. Ber. **1912**, 456–477 (1912)
24. Gabriel, P.: Measure solutions to the conservative renewal equation. In: CIMPA School on Mathematical Models in Biology and Medicine. ESAIM Proc. Surveys, vol. 62, pp. 68–78. EDP Sci., Les Ulis (2018)
25. Gaubert, S., Qu, Z.: Dobrushin's ergodicity coefficient for Markov operators on cones. Integral Equ. Oper. Theory **81**(1), 127–150 (2015)
26. Greiner, G.: A typical Perron-Frobenius theorem with applications to an age-dependent population equation. In: Infinite-Dimensional Systems, Retzhof, 1983. Lecture Notes in Math., vol. 1076, pp. 86–100. Springer, Berlin (1984)
27. Gwiazda, P., Perthame, B.: Invariants and exponential rate of convergence to steady state in the renewal equation. Markov Process. Relat. Fields **12**(2), 413–424 (2006)
28. Gwiazda, P., Wiedemann, E.: Generalized entropy method for the renewal equation with measure data. Commun. Math. Sci. **15**(2), 577–586 (2017)
29. Gwiazda, P., Lorenz, T., Marciniak-Czochra, A.: A nonlinear structured population model: Lipschitz continuity of measure-valued solutions with respect to model ingredients. J. Differ. Equ. **248**(11), 2703–2735 (2010)
30. Harris, S.C., Roberts, M.I.: The many-to-few lemma and multiple spines. Ann. Inst. Henri Poincaré Probab. Stat. **53**(1), 226–242 (2017)
31. Iannelli, M.: Mathematical Theory of Age-Structured Population Dynamics. Applied Math. Monographs. CNR, Giardini Editori e Stampatori, Pisa (1995)
32. Khaladi, M., Arino, O.: Estimation of the rate of convergence of semigroups to an asynchronous equilibrium. Semigroup Forum **61**(2), 209–223 (2000)

33. Kreĭn, M.G., Rutman, M.A.: Linear operators leaving invariant a cone in a Banach space. Am. Math. Soc. Transl. **1950**(26), 128 (1950)
34. Marguet, A.: Uniform sampling in a structured branching population. Bernoulli (2019, to appear)
35. Martínez, S., San Martín, J., Villemonais, D.: Existence and uniqueness of a quasistationary distribution for Markov processes with fast return from infinity. J. Appl. Probab. **51**(3), 756–768 (2014)
36. Metz, J.A.J., Diekmann, O. (eds.): The Dynamics of Physiologically Structured Populations. Lecture Notes in Biomathematics, vol. 68. Springer, Berlin (1986)
37. Meyn, S.P., Tweedie, R.L.: Stability of Markovian processes. II. Continuous-time processes and sampled chains. Adv. Appl. Probab. **25**(3), 487–517 (1993)
38. Meyn, S., Tweedie, R.L.: Markov Chains and Stochastic Stability, 2nd edn. Cambridge University Press, Cambridge (2009). With a prologue by Peter W. Glynn
39. Michel, P., Mischler, S., Perthame, B.: General relative entropy inequality: an illustration on growth models. J. Math. Pures Appl. (9) **84**(9), 1235–1260 (2005)
40. Mischler, S., Scher, J.: Spectral analysis of semigroups and growth-fragmentation equations. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **33**(3), 849–898 (2016)
41. Nussbaum, R.D.: Hilbert's projective metric and iterated nonlinear maps. Mem. Am. Math. Soc. **75**(391), iv+137 (1988)
42. Perron, O.: Zur Theorie der Matrices. Math. Ann. **64**(2), 248–263 (1907)
43. Perthame, B.: Transport Equations in Biology. Frontiers in Mathematics. Birkhäuser, Basel (2007)
44. Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, 2nd edn. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 293. Springer, Berlin (1994)
45. Rudin, W.: Real and Complex Analysis, 3rd edn. McGraw-Hill, New York (1987)
46. Sharpe, F.R., Lotka, A.J.: A Problem in Age-Distribution, pp. 97–100. Springer, Berlin, Heidelberg (1977)
47. Song, J., Yu, J.Y., Wang, X.Z., Hu, S.J., Zhao, Z.X., Liu, J.Q., Feng, D.X., Zhu, G.T.: Spectral properties of population operator and asymptotic behaviour of population semigroup. Acta Math. Sci. Engl. Ed. **2**(2), 139–148 (1982)
48. Thieme, H.R.: Mathematics in Population Biology. Princeton Series in Theoretical and Computational Biology. Princeton University Press, Princeton (2003)
49. Villani, C.: Optimal Transport. Old and New. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338. Springer, Berlin (2009)
50. Webb, G.F.: A semigroup proof of the Sharpe-Lotka theorem. In: Infinite-Dimensional Systems, Retzhof, 1983. Lecture Notes in Math., vol. 1076, pp. 254–268. Springer, Berlin (1984)
51. Webb, G.F.: Theory of Nonlinear Age-Dependent Population Dynamics. Monographs and Textbooks in Pure and Applied Mathematics, vol. 89. Dekker, New York (1985)