

Extended Relativistic Toda Lattice, L-Orthogonal Polynomials and Associated Lax Pair

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Abstract When a measure $\Psi(x)$ on the real line is subjected to the modification $d\Psi^{(t)}(x) = e^{-tx}d\Psi(x)$, then the coefficients of the recurrence relation of the orthogonal polynomials in x with respect to the measure $\Psi^{(t)}(x)$ are known to satisfy the so-called Toda lattice formulas as functions of t . In this paper we consider a modification of the form $e^{-t(px+q/x)}$ of measures or, more generally, of moment functionals, associated with orthogonal L-polynomials and show that the coefficients of the recurrence relation of these L-orthogonal polynomials satisfy what we call an extended relativistic Toda lattice. Most importantly, we also establish the so called Lax pair representation associated with this extended relativistic Toda lattice. These results also cover the (ordinary) relativistic Toda lattice formulations considered in the literature by assuming either $p = 0$ or $q = 0$. However, as far as Lax pair representation is concern, no complete Lax pair representations were established before for the respective relativistic Toda lattice formulations. Some explicit examples of extended relativistic Toda lattice and Langmuir lattice are also presented. As further results, the lattice formulas that follow from the three term recurrence relations associated with kernel polynomials on the unit circle are also established.

Keywords Relativistic Toda lattice · Lax pairs · L-orthogonal polynomials · Kernel polynomials on the unit circle

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1 Introduction

Let Ψ be a positive measure defined on the real line, and let $\{P_n\}_{n=0}^\infty$ be the sequence of monic orthogonal polynomials with respect to Ψ , in the sense that

$$\int_{\mathbb{R}} x^s P_n(x) d\Psi(x) = 0, \quad 0 \leq s \leq n - 1, \quad n \geq 1,$$

where P_n is a polynomial of exact degree n . It is known (see, for example, [5, 14, 34]) that these polynomials satisfy the following three term recurrence relation

$$P_{n+1}(x) = (x - b_{n+1})P_n(x) - a_{n+1}P_{n-1}(x), \quad n \geq 1, \tag{1.1}$$

with $P_0(x) = 1$ and $P_1(x) = x - b_1$. Moreover, for any $n \geq 1$, the coefficients b_n are real and a_{n+1} are positive.

Consider the modified measure $\Psi^{(t)}$, in one time variable t , given by $d\Psi^{(t)}(x) = e^{-tx} d\Psi(x)$, and the associated monic orthogonal polynomials $P_n(x; t)$. As given in [14] (see also [21]) the recursion coefficients $a_{n+1}(t)$ and $b_n(t)$ which appear in the three term recurrence relation (1.1) for $\{P_n(x; t)\}_{n \geq 0}$ satisfy the semi-infinite Toda lattice equations of motion

$$\begin{aligned} \dot{a}_n(t) &= a_n(t)[b_{n-1}(t) - b_n(t)], \\ \dot{b}_n(t) &= a_n(t) - a_{n+1}(t), \end{aligned} \quad n \geq 1, \tag{1.2}$$

with the initial conditions $b_0(t) = 1, a_1(t) = 0, a_n(0) = a_n$ and $b_n(0) = b_n$. Here, we use the usual notation $\dot{f} = \frac{d}{dt} f$.

Toda lattice is a system of particles on the line with exponential interaction of nearest neighbours [33]. Toda was the first to consider such a system for infinitely many particles on the line [35]. The Toda lattice equations (1.2) are obtained from the Newtonian equations of motion (see, for example, [33])

$$\ddot{x}_n = e^{x_{n-1} - x_n} - e^{x_n - x_{n+1}}, \quad n \geq 1, \tag{1.3}$$

when one takes $b_n = \dot{x}_n$ and $a_n = e^{x_{n-1} - x_n}$ for $n \geq 1$.

The present work considers a similar study in the case of the so-called L-orthogonal polynomials. In this respect, let \mathcal{L} be a moment functional defined on the linear space $Span\{1, x^{-1}, x, x^{-2}, \dots\}$ of Laurent polynomials.

Given $p, q \in \mathbb{C}$, we also assume that the moment functional \mathcal{L} is such that $\mathcal{L}[e^{-t(p x + q/x)} x^k]$ exists for all $k \in \mathbb{Z}$ and for all $t \geq 0$, and that

$$\frac{d}{dt} \mathcal{L}[e^{-t(p x + \frac{q}{x})} f(x)] = \mathcal{L}\left[\frac{\partial}{\partial t} (e^{-t(p x + \frac{q}{x})} f(x))\right], \tag{1.4}$$

for any $f(x)$ which is a Laurent polynomial.

Starting with the above \mathcal{L} , we consider the parametric family of moment functionals $\mathcal{L}^{(t)}$, $t \geq 0$, such that

$$\mathcal{L}^{(t)}[x^k] = \mathcal{L}[e^{-t(p x + \frac{q}{x})} x^k] = \nu_k^{(t)}, \quad k = 0, \pm 1, \pm 2, \dots, \quad t \geq 0, \tag{1.5}$$

and assume also that

$$(a) H_n^{(-n)}(t) \neq 0 \quad \text{and} \quad (b) H_{n+1}^{(-n)}(t) \neq 0, \tag{1.6}$$

for $n \geq 0$ and $t \geq 0$, where the associated Hankel determinants $H_n^{(m)}(t)$ are given by $H_0^{(m)}(t) = 1$ and

$$H_n^{(m)}(t) = \begin{vmatrix} v_m^{(t)} & v_{m+1}^{(t)} & \cdots & v_{m+n-1}^{(t)} \\ v_{m+1}^{(t)} & v_{m+2}^{(t)} & \cdots & v_{m+n}^{(t)} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m+n-1}^{(t)} & v_{m+n}^{(t)} & \cdots & v_{m+2n-2}^{(t)} \end{vmatrix}, \quad n, m \in \mathbb{Z}, \quad n \geq 1.$$

We now define, for any fixed $t \geq 0$, the sequence $\{Q_n(x; t)\}_{n \geq 0}$ of polynomials in x by

$$Q_n(x; t) \text{ is a monic polynomial of degree } n \text{ in } x \tag{1.7}$$

$$\mathcal{L}^{(t)}[x^{-n+s} Q_n(x; t)] = 0, \quad s = 0, 1, \dots, n - 1,$$

for $n \geq 1$. It is known (see, for example, [15, 16, 29]) that the existence of these polynomials is assured by the conditions in (1.6). Also when the conditions in (1.6) hold these polynomials are known to satisfy the following three term recurrence relation

$$Q_{n+1}(x; t) = [x - \beta_{n+1}(t)]Q_n(x; t) - \alpha_{n+1}(t)x Q_{n-1}(x; t), \quad n \geq 1, \tag{1.8}$$

with $Q_0(x; t) = 1$, $Q_1(x; t) = x - \beta_1(t)$. The known expressions for the coefficients $\beta_n(t)$ and $\alpha_n(t)$ in terms of the moment functional $\mathcal{L}^{(t)}$ are presented in Sect. 2 of this manuscript.

The polynomials $Q_n(x; 0)$ (to be more general, the polynomials $Q_n(x; t)$ for any fixed t) have been referred to as L-orthogonal polynomials in some previous papers in the subject (see [12] and references therein). Hence, throughout in this manuscript we will refer to the polynomials $Q_n(x; t)$ as L-orthogonal polynomials with respect to the moment functional $\mathcal{L}^{(t)}$ or simply L-orthogonal polynomials. However, it is important to mention that Zhedanov in his many contributions (see, for example, [36]) refers to such polynomials as Laurent biorthogonal polynomials.

The polynomials $Q_n(x; 0)$, when the determinants in (1.6) (for $t = 0$) are positive, have played an important role in the study of the so-called strong Stieltjes moment problem which was introduced in [16].

With respect to the present objective, it is known that the recursion coefficients $\beta_n(t)$ and $\alpha_n(t)$ (see, for example, [9, 17]) satisfy the equations of motion

$$\dot{\beta}_n = \beta_n \left(\frac{\alpha_{n+1}}{\beta_{n+1}\beta_n} - \frac{\alpha_n}{\beta_n\beta_{n-1}} \right), \quad \dot{\alpha}_n = \alpha_n \left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n} \right), \tag{1.9}$$

for the case where $p = 0$ and $q = 1$, and

$$\dot{\beta}_n = \beta_n(\alpha_n - \alpha_{n+1}), \quad \dot{\alpha}_n = \alpha_n(\alpha_{n-1} + \beta_{n-1} - \alpha_{n+1} - \beta_n), \tag{1.10}$$

for the case where $p = 1$ and $q = 0$, both with $\beta_n(t) \neq 0$, $\alpha_{n+1}(t) \neq 0$ for $n \geq 1$ and $\alpha_1(t) = 0$ (further, $\alpha_{N+1}(t) = 0$ in the finite case, i.e., $n = 1, 2, \dots, N$). The system of equations (1.10) has also been considered in [6, 7], where the authors have showed its connection to Laurent polynomials and T-fractions (or the equivalent M-fractions). Observe that the recurrence formula (1.8) is the three term recurrence relation satisfied by the denominator polynomials of M-fractions.

The above system of equations known nowadays as the *relativistic Toda lattice* was introduced by Ruijsenaars [22] (and studied, for example, in [2, 3, 17, 31–33]). In [17] this system is also considered in the form of the following Newtonian equation of motion

$$\ddot{x}_n = (1 + \epsilon \dot{x}_{n+1})(1 + \epsilon \dot{x}_n) \frac{e^{x_{n+1}-x_n}}{1 + \epsilon^2 e^{x_{n+1}-x_n}} - (1 + \epsilon \dot{x}_n)(1 + \epsilon \dot{x}_{n-1}) \frac{e^{x_n-x_{n-1}}}{1 + \epsilon^2 e^{x_n-x_{n-1}}}, \tag{1.11}$$

where ϵ is a (small) parameter of the model having physical meaning as the inverse speed of light. Notice that the Newtonian equation of motion (1.11) for the relativistic Toda lattice is, actually, a one-parameter perturbation of the Newtonian equation of motion (1.3) for the usual Toda lattice.

In the present manuscript, different from what is considered in [9, 17], we consider the Toda lattice equations that follow from the two directional modification $px + q/x$ in (1.5).

The main purpose of the present manuscript is to show that the recursion coefficients $\beta_n(t)$ and $\alpha_n(t)$ satisfy the lattice equations that we call the *extended relativistic Toda lattice* equations. We also establish the Lax pair formulations for this extended relativistic Toda lattice. The main results of this paper are stated in the following two theorems.

Theorem 1 Consider a moment functional $\mathcal{L}^{(t)}$, defined as in (1.5), and the L -orthogonal polynomials $Q_n(x; t)$, $n \geq 1$, defined in (1.7). The recursion coefficients $\beta_n(t)$ and $\alpha_n(t)$ in the recurrence relation (1.8) satisfy the extended relativistic Toda lattice equations

$$\dot{\beta}_n = p \beta_n (\alpha_n - \alpha_{n+1}) + q \beta_n \left(\frac{\alpha_{n+1}}{\beta_{n+1} \beta_n} - \frac{\alpha_n}{\beta_n \beta_{n-1}} \right) \tag{1.12}$$

and

$$\dot{\alpha}_n = p \alpha_n (\alpha_{n-1} + \beta_{n-1} - \alpha_{n+1} - \beta_n) + q \alpha_n \left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n} \right), \tag{1.13}$$

for $n \geq 1$, with the initial conditions $\beta_0(t) = 1$, $\alpha_0(t) = -1$ and $\alpha_1(t) = 0$. Moreover,

$$\dot{\gamma}_n = p (\alpha_n \gamma_n - \alpha_{n+1} \gamma_{n+1}) + q \left(\frac{\alpha_{n+1}}{\beta_n} - \frac{\alpha_n}{\beta_{n-1}} \right), \quad n \geq 1, \tag{1.14}$$

with $\gamma_n = \alpha_{n+1} + \beta_n$, $n \geq 1$. Here, we have omitted the time variable t for simplification.

The proof of this theorem is given in Sect. 2.

The following interesting result holds with respect to the extended relativistic Toda lattice equations given by Theorem 1.

Theorem 2 Let the infinite Hessenberg matrix $\mathcal{H} = \mathcal{H}(t)$ and the infinite tridiagonal matrix $\mathcal{F} = \mathcal{F}(t)$ be given by

$$\mathcal{H} = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_n & \cdots \\ \alpha_2 & \gamma_2 & \gamma_3 & \cdots & \gamma_n & \cdots \\ 0 & \alpha_3 & \gamma_3 & \cdots & \gamma_n & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & \cdots & 0 & \alpha_n & \gamma_n & \cdots \\ \vdots & & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} f_1 & h_1 & 0 & \cdots & 0 & \cdots \\ e_2 & f_2 & h_2 & \ddots & \vdots & \ddots \\ 0 & e_3 & f_3 & \ddots & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & h_{n-1} & \ddots \\ 0 & \cdots & 0 & e_n & f_n & \ddots \\ \vdots & & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.15}$$

where $\gamma_k = \alpha_{k+1} + \beta_k$,

$$\epsilon_k = -p\alpha_k, \quad \mathfrak{f}_k = p\alpha_k + \frac{q}{\gamma_k - \alpha_{k+1}}, \quad \mathfrak{h}_k = -\frac{q}{\gamma_k - \alpha_{k+1}},$$

for $k \geq 1$. Then the pair $\{\mathcal{H}, \mathcal{F}\}$ is a Lax pair for the extended relativistic Toda lattice equations (1.13) and (1.14) given by Theorem 1. Precisely, $\dot{\mathcal{H}} = [\mathcal{H}, \mathcal{F}]$, where $[\mathcal{H}, \mathcal{F}] = \mathcal{H}\mathcal{F} - \mathcal{F}\mathcal{H}$.

Further information about Lax pairs and the proof of Theorem 2 are given in Sect. 3.

In Sect. 4 we consider the case in which $\mathcal{L}[f] = \int_a^b f(x)d\psi(x)$, where $0 \leq a < b \leq \infty$ and ψ is a strong positive measure defined on $[a, b]$. The word strong, adopted from the work of Jones, Thron and Waadeland [16] on the strong moment problem, is to indicate that the measure has moments of non-negative and negative order. In Sect. 4 we have also given two examples with $pq \neq 0$, where the solutions have very nice representations.

The two parameter extension $p x + q/x$ also permits us to extend the results to the unit circle. If μ is a positive measure on the unit circle $\mathbb{T} = \{x = e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, then with $p = \bar{q}$, the measures $\mu^{(t)}$ given by

$$d\mu^{(t)}(x) = e^{-t(p x + q/x)}d\mu(x),$$

is also positive in \mathbb{T} for any t real. Hence, respectively in Sects. 5 and 6, we have been able to proceed with the study of looking at the recurrence coefficients $\beta_n(t)$ and $\alpha_{n+1}(t)$ that follow from the moment functionals

$$\mathcal{L}^{(t)}[f] = \int_{\mathbb{T}} f(x) e^{-t(\bar{q}x + q/x)}(x - w)d\mu(x), \tag{1.16}$$

where $|w| = 1$ fixed and from the moment functionals

$$\mathcal{L}^{(t)}[f] = \int_{\mathbb{T}} f(x) e^{-t(\bar{q}x + q/x)}x d\mu(x). \tag{1.17}$$

In the case of (1.17) the polynomials $Q_n(x; t)$ are the monic orthogonal polynomials on the unit circle with respect to $\mu^{(t)}$ (also known as Szegő polynomials), and in the case of (1.16) the polynomials $Q_n(x; t)$ are the monic kernel polynomials, say $const K_n^{(t)}(x, w)$, on the unit circle with respect to $\mu^{(t)}$. For a comprehensive collection of information concerning orthogonal polynomials on the unit circle and associated kernel polynomials we refer to [24].

2 The Proof of Theorem 1

Let $Q_n(x; t) = \sum_{j=0}^n a_{n,j}(t)x^j$, where $a_{n,n}(t) = 1$. From the linear system (1.7) in the coefficients $a_{n,j}(t)$ of $Q_n(x; t)$, one easily finds

$$Q_n(x; t) = \frac{1}{H_n^{(-n)}(t)} \begin{vmatrix} v_{-n}^{(t)} & v_{-n+1}^{(t)} & \cdots & v_0^{(t)} \\ \vdots & \vdots & & \vdots \\ v_{-1}^{(t)} & v_0^{(t)} & \cdots & v_{n-1}^{(t)} \\ 1 & x & \cdots & x^n \end{vmatrix}, \quad n \geq 1,$$

and if $\sigma_{n,-1}(t) = \mathcal{L}^{(t)}[x^{-n-1}Q_n(x; t)]$ and $\sigma_{n,n}(t) = \mathcal{L}^{(t)}[Q_n(x; t)]$ then

$$\sigma_{n,-1}(t) = (-1)^n \frac{H_{n+1}^{(-n-1)}(t)}{H_n^{(-n)}(t)}, \quad \sigma_{n,n}(t) = \frac{H_{n+1}^{(-n)}(t)}{H_n^{(-n)}(t)}, \quad n \geq 1.$$

Also observe that $\sigma_{0,0}(t) = v_0^{(t)}$ and $\sigma_{0,-1}(t) = v_{-1}^{(t)}$.

Clearly, the sequence of polynomials $\{Q_n(x; t)\}_{n \geq 0}$ exists for all $t \geq 0$ because $H_n^{(-n)}(t) \neq 0$ for all $t \geq 0$, which follows from condition (a) of (1.6). Observe that the condition (b) in (1.6) also means that $Q_n(0; t) \neq 0$ for all $n \geq 1$ and for all $t \geq 0$.

With both conditions (a) and (b) in (1.6) the recurrence relation (1.8) holds for any $t \geq 0$, with $\beta_1(t) = \sigma_{0,0}(t)/\sigma_{0,-1}(t)$,

$$\beta_{n+1}(t) = -\alpha_{n+1}(t) \frac{\sigma_{n-1,-1}(t)}{\sigma_{n,-1}(t)} \quad \text{and} \quad \alpha_{n+1}(t) = \frac{\sigma_{n,n}(t)}{\sigma_{n-1,n-1}(t)}, \quad n \geq 1. \tag{2.1}$$

From the recurrence relation (1.8), we directly obtain

$$a_{n,0}(t) = Q_n(0; t) = (-1)^n \beta_n(t)\beta_{n-1}(t) \cdots \beta_2(t)\beta_1(t), \quad n \geq 1 \tag{2.2}$$

and

$$\alpha_{n+1}(t) + \beta_{n+1}(t) = a_{n,n-1}(t) - a_{n+1,n}(t), \quad n \geq 1. \tag{2.3}$$

Notice that using (2.1), we also obtain for $n \geq 0$,

$$\sigma_{n,n}(t) = \alpha_{n+1}(t)\alpha_n(t) \cdots \alpha_2(t)\sigma_{0,0}(t) \tag{2.4}$$

and

$$\sigma_{n,-1}(t) = (-1)^n \frac{\alpha_{n+1}(t)\alpha_n(t) \cdots \alpha_2(t)\sigma_{0,0}(t)}{\beta_{n+1}(t)\beta_n(t) \cdots \beta_2(t)\beta_1(t)} = \frac{\sigma_{n,n}(t)}{\beta_{n+1}(t)a_{n,0}(t)}. \tag{2.5}$$

Lemma 1 *Let us denote $\tau_n(t) = \mathcal{L}^{(t)}[xQ_n(x; t)]$, $n \geq 0$. Then,*

$$\tau_n(t) = \sigma_{n,n}(t) \sum_{k=1}^{n+1} [\alpha_{k+1}(t) + \beta_k(t)] = \sigma_{n,n}(t) \sum_{k=1}^{n+1} \gamma_k(t),$$

for $n \geq 0$, where $\gamma_k(t) = \alpha_{k+1}(t) + \beta_k(t)$, $k = 1, 2, \dots, n + 1$.

Proof The validity of the Lemma for $n = 0$ is easily verified.

From the recurrence relation (1.8), we have

$$xQ_n(x; t) = Q_{n+1}(x; t) + \beta_{n+1}(t)Q_n(x; t) + \alpha_{n+1}(t)xQ_{n-1}(x; t), \quad n \geq 1.$$

Applying the moment function $\mathcal{L}^{(t)}$ in the above equality gives

$$\tau_n(t) = \sigma_{n+1,n+1}(t) + \beta_{n+1}(t)\sigma_{n,n}(t) + \alpha_{n+1}(t)\tau_{n-1}(t), \quad n \geq 1.$$

Consequently, with the observation that $\alpha_2(t) + \beta_1(t) = \mathcal{L}^{(t)}[x]/\mathcal{L}^{(t)}[1]$, we find

$$\frac{\tau_n(t)}{\sigma_{n,n}(t)} = \alpha_{n+2}(t) + \beta_{n+1}(t) + \frac{\tau_{n-1}(t)}{\sigma_{n-1,n-1}(t)} = \sum_{k=1}^{n+1} [\alpha_{k+1}(t) + \beta_k(t)] = \sum_{k=1}^{n+1} \gamma_k(t),$$

for $n \geq 1$, this concludes the proof of the Lemma. □

Since $Q_n(x; t)$ is a polynomial of degree n in the variable x , from (1.7) we have $\mathcal{L}^{(t)}[x^{-n} Q_{n-1}(x; t) Q_n(x; t)] = 0, n \geq 1$. Differentiating this with respect to the variable t and denoting $\frac{\partial}{\partial t} Q_n(x; t) = \dot{Q}_n(x; t)$, we obtain from (1.5),

$$\begin{aligned} &\mathcal{L}^{(t)}[x^{-n} \dot{Q}_{n-1}(x; t) Q_n(x; t)] + \mathcal{L}^{(t)}[x^{-n} Q_{n-1}(x; t) \dot{Q}_n(x; t)] \\ &- \mathcal{L}^{(t)}\left[x^{-n} \left(\mathfrak{p}x + \frac{\mathfrak{q}}{x}\right) Q_{n-1}(x; t) Q_n(x; t)\right] = 0, \quad n \geq 1. \end{aligned} \tag{2.6}$$

However, since $\dot{Q}_n(x; t) = \sum_{j=0}^{n-1} \dot{a}_{n,j}(t)x^j$ is a polynomial of degree at most $n - 1$ in the variable x , again from (1.7), we can see that $\mathcal{L}^{(t)}[x^{-n} \dot{Q}_{n-1}(x; t) Q_n(x; t)] = 0$ and $\mathcal{L}^{(t)}[x^{-n} Q_{n-1}(x; t) \dot{Q}_n(x; t)] = \dot{a}_{n,0}(t)\sigma_{n-1,-1}(t)$, for $n \geq 1$. Furthermore,

$$\mathcal{L}^{(t)}[x^{-n+1} Q_{n-1}(x; t) Q_n(x; t)] = a_{n-1,n-1}(t)\sigma_{n,n}(t) = \sigma_{n,n}(t)$$

and $\mathcal{L}^{(t)}[x^{-n-1} Q_{n-1}(x; t) Q_n(x; t)] = a_{n-1,0}(t)\sigma_{n,-1}(t)$, for $n \geq 1$. By substituting these in (2.6), we then conclude that

$$\dot{a}_{n,0}(t)\sigma_{n-1,-1}(t) = \mathfrak{p} \sigma_{n,n}(t) + \mathfrak{q} a_{n-1,0}(t)\sigma_{n,-1}(t), \quad n \geq 1.$$

Using (2.1), (2.2) and (2.5), we get

$$\sum_{k=1}^n \frac{\dot{\beta}_k(t)}{\beta_k(t)} = -\mathfrak{p} \alpha_{n+1}(t) + \mathfrak{q} \frac{\alpha_{n+1}(t)}{\beta_{n+1}(t)\beta_n(t)}, \quad n \geq 1. \tag{2.7}$$

From this and setting the initial conditions $\beta_0(t) = 1$ and $\alpha_1(t) = 0$, the relation (1.12) of Theorem 1 holds.

Now from (1.7), we observe that $\mathcal{L}^{(t)}[x^{-n} Q_n^2(x; t)] = \sigma_{n,n}(t)$, for $n \geq 0$. Differentiating $\sigma_{n,n}(t)$ with respect to t and observing that $\mathcal{L}^{(t)}[x^{-n} \dot{Q}_n(x; t) Q_n(x; t)] = 0$, it yields

$$\dot{\sigma}_{n,n}(t) = -\mathcal{L}^{(t)}\left[x^{-n} \left(\mathfrak{p}x + \frac{\mathfrak{q}}{x}\right) Q_n^2(x; t)\right], \quad n \geq 0. \tag{2.8}$$

Observe from (1.7), that $\mathcal{L}^{(t)}[x^{-n-1} Q_n^2(x; t)] = a_{n,0}(t)\sigma_{n,-1}(t)$ for $n \geq 0$ and, from Lemma 1,

$$\mathcal{L}^{(t)}[x^{-n+1} Q_n^2(x; t)] = a_{n,n-1}(t)\sigma_{n,n}(t) + \sigma_{n,n}(t) \sum_{k=1}^{n+1} [\alpha_{k+1}(t) + \beta_k(t)],$$

for $n \geq 0$, where we have taken $a_{0,-1}(t) = 0$. One can verify also from (2.3) that

$$a_{n,n-1}(t) + \sum_{k=1}^{n+1} [\alpha_{k+1}(t) + \beta_k(t)] = \alpha_{n+2}(t) + \beta_{n+1}(t) + \alpha_{n+1}(t), \quad n \geq 0.$$

Hence, using the above results, the equation (2.8) can be given as

$$-p \sigma_{n,n}(t) [\alpha_{n+2}(t) + \beta_{n+1}(t) + \alpha_{n+1}(t)] - q a_{n,0}(t) \sigma_{n,-1}(t) = \dot{\sigma}_{n,n}(t),$$

for $n \geq 0$. Thus, from (2.4) and (2.5), we have

$$-p [\alpha_2(t) + \beta_1(t) + \alpha_1(t)] - q \frac{1}{\beta_1(t)} = \frac{\dot{\sigma}_{0,0}(t)}{\sigma_{0,0}(t)},$$

and, for $n \geq 1$,

$$-p [\alpha_{n+2}(t) + \beta_{n+1}(t) + \alpha_{n+1}(t)] - q \frac{1}{\beta_{n+1}(t)} = \frac{\dot{\sigma}_{0,0}(t)}{\sigma_{0,0}(t)} + \sum_{k=2}^{n+1} \frac{\dot{\alpha}_k(t)}{\alpha_k(t)}.$$

Consequently the relation (1.13) of Theorem 1 holds for $n \geq 2$. On the other hand, since $\beta_0(t) = 1, \alpha_1(t) = 0$ and $\alpha_0(t)$ is arbitrary (but, we set $\alpha_0(t) = -1$) the relation (1.13) clearly holds for $n = 1$.

Finally, since $\gamma_n(t) = \alpha_{n+1}(t) + \beta_n(t)$, for $n \geq 1$ from (1.12) and (1.13), we easily obtain equation (1.14) of Theorem 1.

3 Lax Pairs and the Proof of Theorem 2

As in Nakamura [19], by considering the infinite matrices

$$S = \begin{pmatrix} b_1 & 1 & 0 & \cdots & 0 & \cdots \\ a_2 & b_2 & 1 & \ddots & \vdots & \\ 0 & a_3 & b_3 & \ddots & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & 1 & \ddots \\ 0 & \cdots & 0 & a_n & b_n & \ddots \\ \vdots & & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & \cdots \\ -a_2 & 0 & 0 & & \vdots & \\ 0 & -a_3 & 0 & \ddots & \vdots & \\ \vdots & \ddots & \ddots & \ddots & 0 & \cdots \\ 0 & \cdots & 0 & -a_n & 0 & \ddots \\ \vdots & & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

the Toda lattice equations (1.2) can also be represented in the matrix form

$$\dot{S} = [S, T] = ST - TS. \tag{3.1}$$

The pair $\{S, T\}$ is called a *Lax pair* and (3.1) is called a *Lax representation* for the Toda lattice (1.2). Another Lax pair for the Toda lattice (1.2) can also be found in [11, 35].

For the case of the finite relativistic Toda lattice equations (1.9) and (1.10) (i.e., $n = 1, 2, \dots, N$), in Suris [31, 32] (see also Coussemant et al. [9] for a generalized form of the finite relativistic Toda lattice), by considering the bidiagonal matrices \mathcal{M}_N and \mathcal{P}_N given by

$$\mathcal{M}_N = \begin{pmatrix} \beta_1 & 1 & 0 & \cdots & 0 \\ 0 & \beta_2 & 1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \beta_{N-1} & 1 \\ 0 & \cdots & \cdots & 0 & \beta_N \end{pmatrix}, \quad \mathcal{P}_N = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\alpha_2 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & -\alpha_N & 1 \end{pmatrix},$$

it was shown that (1.9) can be written in the Lax form

$$\begin{cases} \dot{\mathcal{M}}_N = \mathcal{M}_N \mathcal{A}_N - \mathcal{B}_N \mathcal{M}_N, \\ \dot{\mathcal{P}}_N = \mathcal{P}_N \mathcal{A}_N - \mathcal{B}_N \mathcal{P}_N, \end{cases} \tag{3.2}$$

with $\mathcal{A}_N = -(\mathcal{M}_N^{-1} \mathcal{P}_N)_-$ and $\mathcal{B}_N = -(\mathcal{P}_N \mathcal{M}_N^{-1})_-$, where Z_- denotes the strictly lower triangular part of Z . Moreover, by considering $\mathcal{A}_N = -(\mathcal{P}_N^{-1} \mathcal{M}_N)_-$ and $\mathcal{B}_N = -(\mathcal{M}_N \mathcal{P}_N^{-1})_-$, it was also shown that the system (1.10) can also be written in the form (3.2). Notice that in neither case we have a Lax pair of the form (3.1) for the finite relativistic Toda lattice equations (1.9) and (1.10).

The aim of this section is to present a Lax pair of the form (3.1) for the extended relativistic Toda lattice equations (1.13) and (1.14).

Let us first consider the extended relativistic Toda lattice equations of the finite order

$$\dot{\gamma}_n = \mathfrak{p}(\alpha_n \gamma_n - \alpha_{n+1} \gamma_{n+1}) + \mathfrak{q} \left(\frac{\alpha_{n+1}}{\gamma_n - \alpha_{n+1}} - \frac{\alpha_n}{\gamma_{n-1} - \alpha_n} \right), \tag{3.3}$$

and

$$\dot{\alpha}_n = \mathfrak{p} \alpha_n (\alpha_{n-1} + \gamma_{n-1} - \alpha_n - \gamma_n) + \mathfrak{q} \alpha_n \left(\frac{1}{\gamma_{n-1} - \alpha_n} - \frac{1}{\gamma_n - \alpha_{n+1}} \right), \tag{3.4}$$

for $n = 1, 2, \dots, N$, where $\gamma_n(t) = \alpha_{n+1}(t) + \beta_n(t)$, $\beta_0(t) = 1$, $\alpha_0(t) = -1$ and $\alpha_1(t) = 0$, with the additional assumption $\alpha_{N+1}(t) = 0$. Observe that with $\alpha_{N+1}(t) = 0$ the product $\alpha_{N+1}(t) \gamma_{N+1}(t)$ that appears in (3.3) is also zero.

For $N \geq 1$, let the $N \times N$ matrices $\mathcal{H}_N = \mathcal{H}_N(t)$, $\mathcal{X}_N = \mathcal{X}_N(t)$ and $\mathcal{Y}_N = \mathcal{Y}_N(t)$ be given by

$$\mathcal{H}_N = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{N-1} & \gamma_N \\ \alpha_2 & \gamma_2 & \gamma_3 & \cdots & \gamma_{N-1} & \gamma_N \\ 0 & \alpha_3 & \gamma_3 & \cdots & \gamma_{N-1} & \gamma_N \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha_{N-1} & \gamma_{N-1} & \gamma_N \\ 0 & \cdots & 0 & 0 & \alpha_N & \gamma_N \end{pmatrix},$$

$$\mathcal{X}_N = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 & 0 \\ -\alpha_2 & \alpha_2 & 0 & \cdots & 0 & 0 \\ 0 & -\alpha_3 & \alpha_3 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & -\alpha_{N-1} & \alpha_{N-1} & 0 \\ 0 & \cdots & 0 & 0 & -\alpha_N & \alpha_N \end{pmatrix}$$

and

$$\mathcal{Y}_N = \begin{pmatrix} \frac{1}{\gamma_1 - \alpha_2} & -\frac{1}{\gamma_1 - \alpha_2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\gamma_2 - \alpha_3} & -\frac{1}{\gamma_2 - \alpha_3} & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \frac{1}{\gamma_{N-2} - \alpha_{N-1}} & -\frac{1}{\gamma_{N-2} - \alpha_{N-1}} & 0 \\ 0 & \cdots & 0 & 0 & \frac{1}{\gamma_{N-1} - \alpha_N} & -\frac{1}{\gamma_{N-1} - \alpha_N} \\ 0 & \cdots & 0 & 0 & 0 & \frac{1}{\gamma_N - \alpha_{N+1}} \end{pmatrix}.$$

The matrix \mathcal{H}_N has already been shown to be interesting in the studies related to L-orthogonal polynomials. From results first appeared in [27] (see also [23, 30] on further studies), the zeros of $Q_N(x; t)$ are exactly the eigenvalues of the Hessenberg matrix $\mathcal{H}_N(t)$.

Now with the above matrices by performing the respective matrix multiplications one easily finds that

$$\dot{\mathcal{H}}_N = \mathfrak{p}(\mathcal{H}_N \mathcal{X}_N - \mathcal{X}_N \mathcal{H}_N) + \mathfrak{q}(\mathcal{H}_N \mathcal{Y}_N - \mathcal{Y}_N \mathcal{H}_N),$$

for any $N \geq 1$. Hence, we can state the following theorem.

Theorem 3 *A Lax representation for the extended relativistic Toda lattice equations of finite order N ($N \geq 1$) given by (3.3) and (3.4) is*

$$\dot{\mathcal{H}}_N = [\mathcal{H}_N, \mathcal{F}_N] = \mathcal{H}_N \mathcal{F}_N - \mathcal{F}_N \mathcal{H}_N,$$

where $\mathcal{F}_N = \mathfrak{p}\mathcal{X}_N + \mathfrak{q}\mathcal{Y}_N$.

To prove Theorem 2 one only needs to let $N \rightarrow \infty$ in Theorem 3.

4 Applications to L-Orthogonal Polynomials on the Positive Real Axis

In this section we consider the case in which the moment functional \mathcal{L} is given by

$$\mathcal{L}[f] = \int_a^b f(x) d\psi(x),$$

where $0 \leq a < b \leq \infty$ and ψ is a strong positive measure defined on $[a, b]$. By the term ‘‘strong’’ we mean that the moments $\mathcal{L}[x^n] = v_n^{(0)}$ exists for all $n \in \mathbb{Z}$. The existence of the moments $\mathcal{L}^{(t)}[x^n] = v_n^{(t)}$, for all $n \in \mathbb{Z}$, depends on the choice of \mathfrak{p} and \mathfrak{q} , especially if $a = 0$ and/or $b = \infty$.

Clearly, the choice $\mathfrak{p} > 0$ and $\mathfrak{q} > 0$, that we will assume throughout in this section, guarantees the existence of all the moments since it is easily verified that

$$v_n^{(t)} = \mathcal{L}^{(t)}[x^n] = \int_a^b x^n e^{-t(\mathfrak{p}x + \frac{\mathfrak{q}}{x})} d\psi(x) \leq e^{-2t\sqrt{\mathfrak{p}\mathfrak{q}}} \int_a^b x^n d\psi(x) = e^{-2t\sqrt{\mathfrak{p}\mathfrak{q}}} v_n^{(0)},$$

for $n = 0, \pm 1, \pm 2, \dots$. With the existence of the moments, the determinant conditions in (1.6) also hold since $e^{-t(\mathfrak{p}x + \mathfrak{q}/x)} d\psi(x)$ leads to a positive measure in $[a, b]$. To be precise, we have $0 < H_n^{(-n)}(t) < \infty$ and $0 < H_{n+1}^{(-n)}(t) < \infty$, for $n \geq 0$. Moreover, since $v_n^{(t)}$ is a bounded and absolutely function of t , also follows that

$$\begin{aligned} \frac{d}{dt} \int_a^b [e^{-t(\mathfrak{p}x + \frac{\mathfrak{q}}{x})} f(x)] d\psi(x) &= \int_a^b \frac{\partial}{\partial t} [e^{-t(\mathfrak{p}x + \frac{\mathfrak{q}}{x})} f(x)] d\psi(x) \\ &= - \int_a^b \left[e^{-t(\mathfrak{p}x + \frac{\mathfrak{q}}{x})} \left[\mathfrak{p}x + \frac{\mathfrak{q}}{x} \right] f(x) \right] d\psi(x), \end{aligned}$$

for any Laurent polynomial f . Thus, the requirement (1.4) for \mathcal{L} .

We can also, without any loss of generality, consider $\mathcal{L}^{(t)}$ in the form

$$\mathcal{L}^{(t)}[f(x)] = \mathcal{L}[f(x)e^{-t(x+\frac{q}{x})}], \tag{4.1}$$

with $q > 0$, by absorbing the positive p into the parameter t .

If we consider the L-orthogonal polynomials $Q_n(x; t)$ with respect to this moment functional then from the positiveness of $e^{-t(x+q/x)}d\psi(x)$ one can state that (see [16]) the coefficients in the associated recurrence relation (1.8) satisfy $\beta_n(t) > 0$ and $\alpha_{n+1}(t) > 0, n \geq 1$.

On the other hand, by Theorem 1 these coefficients satisfy the extended relativistic Toda lattice equations

$$\dot{\beta}_n = \beta_n(\alpha_n - \alpha_{n+1}) + q\beta_n\left(\frac{\alpha_{n+1}}{\beta_{n+1}\beta_n} - \frac{\alpha_n}{\beta_n\beta_{n-1}}\right) \tag{4.2}$$

and

$$\dot{\alpha}_n = \alpha_n(\alpha_{n-1} + \beta_{n-1} - \alpha_{n+1} - \beta_n) + q\alpha_n\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n}\right), \tag{4.3}$$

for $n \geq 1$, with the initial conditions $\beta_0(t) = 1, \alpha_0(t) = -1$ and $\alpha_1(t) = 0$.

We now analyse results corresponding to such $\mathcal{L}^{(t)}$ with two particular examples.

Example 1 For $\delta > 0$, let the moment functional \mathcal{L} be given by $\mathcal{L}[f(x)] = \int_0^\infty f(x) d\psi(x)$, where $d\psi(x) = x^{-\frac{1}{2}}e^{-\delta(x+q/x)}dx$. Then, the moment functional $\mathcal{L}^{(t)}$ defined as in (4.1) satisfies

$$\mathcal{L}^{(t)}[f(x)] = \int_0^\infty f(x) d\psi^{(t)}(x),$$

where $d\psi^{(t)}(x) = x^{-\frac{1}{2}}e^{-(t+\delta)(x+q/x)}dx$.

Considering the L-orthogonal polynomials $Q_n(x; t)$ with respect to this moment functional we find that the coefficients of the associated three term recurrence relation (1.8) satisfy

$$\beta_n(t) = \sqrt{q} \quad \text{and} \quad \alpha_{n+1}(t) = \frac{n}{2(t + \delta)}, \quad n \geq 1. \tag{4.4}$$

This follows from results given in [25] (p. 3139).

Substituting the values (4.4) in the right hand sides of (4.2) and (4.3) we find $\dot{\beta}_n = 0$, for $n \geq 1$, and

$$\dot{\alpha}_n = \alpha_n(\alpha_{n-1} + \beta_{n-1} - \alpha_{n+1} - \beta_n) + q\alpha_n\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n}\right) = -\frac{n-1}{2(t+\delta)^2}, \quad n \geq 2.$$

The values obtained here for $\dot{\beta}_n$ and $\dot{\alpha}_n$ are what we obtain by direct differentiation in (4.4).

It turns out that the measure ψ in Example 1 is such that

$$\frac{d\psi(x)}{\sqrt{x}} = -\frac{d\psi(q/x)}{\sqrt{q/x}}. \tag{4.5}$$

Consequently, the measure $\psi^{(t)}$ in Example 1 also satisfies the symmetric property (4.5).

It is known (see, for example, [25, 28]) that under this symmetric property the coefficients β_n in the three term recurrence (1.8) must satisfy $\beta_n(t) = \sqrt{q}, n \geq 1$.

In general, if we start with any strong measure ψ that satisfies the symmetry (4.5) and then proceed to create the linear functional \mathcal{L} and to build the moment functionals $\mathcal{L}^{(t)}$ as in (4.1), then $\beta_n(t) = \sqrt{q}$, $n \geq 1$. Moreover, for $\alpha_n(t)$ we obtain $\alpha_n(t) > 0$, $n \geq 2$ and from (4.3)

$$\dot{\alpha}_n(t) = \alpha_n(t)[\alpha_{n-1}(t) - \alpha_{n+1}(t)], \quad n \geq 2,$$

with $\alpha_1(t) = 0$, which is known as Langmuir or Volterra lattice (see [21]).

Example 2 For $\delta > 0$, let the moment functional \mathcal{L} be given by $\mathcal{L}[f(x)] = \int_0^\infty f(x) d\tilde{\psi}(x)$, where $d\tilde{\psi}(x) = (x + \sqrt{q})x^{-\frac{3}{2}}e^{-\delta(x+\frac{q}{x})}dx$. Then the moment functional $\mathcal{L}^{(t)}$ defined as in (4.1) satisfies

$$\mathcal{L}^{(t)}[f(x)] = \int_0^\infty f(x) d\tilde{\psi}^{(t)}(x), \tag{4.6}$$

where $d\tilde{\psi}^{(t)}(x) = (x + \sqrt{q})x^{-\frac{3}{2}}e^{-(t+\delta)(x+\frac{q}{x})}dx$.

Let us denote the coefficients in the three term recurrence relation (1.8) with respect to the moment functional $\mathcal{L}^{(t)}$ in (4.6) as $\tilde{\alpha}_n(t)$ and $\tilde{\beta}_n(t)$. The measures $\tilde{\psi}^{(t)}$ can be verified to satisfy the symmetry

$$d\tilde{\psi}^{(t)}(x) = -d\tilde{\psi}^{(t)}(q/x),$$

and further $d\tilde{\psi}^{(t)}(x) = \frac{x+\sqrt{q}}{x}d\psi^{(t)}(x)$, where $\psi^{(t)}$ are the measures given in Example 1. Consequently, using results found in [26], we obtain

$$\tilde{\beta}_n(t) = \frac{l_{n-1}(t)}{l_n(t)}, \quad \tilde{\alpha}_{n+1}(t) = \tilde{\beta}_n(t) [l_n^2(t) - 1], \quad n \geq 1,$$

where $l_n(t) = 1 + \frac{n/[2\sqrt{q}(t+\delta)]}{l_{n-1}(t)+1}$, $n \geq 1$ and $l_0(t) = 1$. These values for $\tilde{\beta}_n$ and $\tilde{\alpha}_n$, together with the values for $\dot{\tilde{\beta}}_n$ and $\dot{\tilde{\alpha}}_n$ obtained from these, can be successively substituted in

$$\begin{aligned} \dot{\tilde{\beta}}_n &= \tilde{\beta}_n(\tilde{\alpha}_n - \tilde{\alpha}_{n+1}) + q\tilde{\beta}_n \left(\frac{\tilde{\alpha}_{n+1}}{\tilde{\beta}_{n+1}\tilde{\beta}_n} - \frac{\tilde{\alpha}_n}{\tilde{\beta}_n\tilde{\beta}_{n-1}} \right), \\ \dot{\tilde{\alpha}}_n &= \tilde{\alpha}_n(\tilde{\alpha}_{n-1} + \tilde{\beta}_{n-1} - \tilde{\alpha}_{n+1} - \tilde{\beta}_n) + q\tilde{\alpha}_n \left(\frac{1}{\tilde{\beta}_{n-1}} - \frac{1}{\tilde{\beta}_n} \right), \end{aligned} \quad n \geq 1,$$

to verify the validity of these extended relativistic Toda lattice equations.

5 Applications to Kernel Polynomials on the Unit Circle

Let μ be a positive measure defined on the unit circle $\mathbb{T} = \{z = e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ and let

$$\mathcal{L}[f] = \int_{\mathbb{T}} f(z)(z-w)d\mu(z) \quad \text{and} \quad \mathcal{L}^{(t)}[f] = \int_{\mathbb{T}} f(z)(z-w)d\mu^{(t)}(z),$$

where

$$d\mu^{(t)}(z) = e^{-t(\bar{q}z+q/z)}d\mu(z) = e^{-2t[\text{Re}(q)\cos\theta+\text{Im}(q)\sin\theta]}d\mu(e^{i\theta}).$$

For convention and also for convenience we have replaced x by z .

Clearly, $\mu^{(t)}$ is a well defined positive measure on the unit circle for any t real. Thus, from now on consider $t \in (-\infty, \infty)$. We denote the n^{th} degree monic orthogonal polynomial and orthonormal polynomial associated with the measure $\mu^{(t)}$ by $\Phi_n(z; t)$ and $\varphi_n(z; t)$, respectively (see [24]). We also denote the associated Verblunsky coefficients by $\alpha_n(t)$. That is, $\alpha_n(t) = -\overline{\Phi_{n+1}(0; t)}$, $n \geq 1$.

Now, with a fixed w such that $|w| = 1$ and fixed t , we consider the L-orthogonal polynomials $Q_n(z; t)$ defined by (1.7) with z in the place of x . The existence of these polynomials and that they satisfy the three term recurrence (1.8), i.e.,

$$Q_{n+1}(z; t) = [z - \beta_{n+1}(t)]Q_n(z; t) - \alpha_{n+1}(t)zQ_{n-1}(z; t), \quad n \geq 1, \tag{5.1}$$

with $Q_0(z; t) = 1$ and $Q_1(z; t) = z - \beta_1(t)$, follow from results given in [8].

These polynomials are actually the monic kernel polynomials (or monic CD kernel as in Simon [24]) with respect to the measure $\mu^{(t)}$. Precisely, we have

$$\overline{\kappa_n \varphi_n(w; t)} Q_n(z; t) = K_n^{(t)}(z, w) = \sum_{j=0}^n \overline{\varphi_j(w; t)} \varphi_j(z; t), \quad n \geq 1,$$

where $\kappa_n^{-2} = \mu_0^{(t)} \prod_{j=0}^{n-1} (1 - |\alpha_j(t)|^2)$ and $\mu_0^{(t)} = \int_{\mathbb{T}} d\mu^{(t)}(z)$. Moreover, for the coefficients $\beta_n(t) = \beta_n(w, t)$ and $\alpha_n(t) = \alpha_n(w, t)$ we also have (see [8] (Thm. 2.1)),

$$\beta_n(t) = -\frac{\rho_n^{(t)}(w)}{\rho_{n-1}^{(t)}(w)}, \quad \alpha_{n+1}(t) = [1 + \rho_n^{(t)}(w)\alpha_{n-1}(t)][1 - \overline{w\rho_n^{(t)}(w)\alpha_n(t)}]w,$$

for $n \geq 1$, where $\rho_n^{(t)}(w) = \Phi_n(w; t)/\Phi_n^*(w; t)$, $n \geq 0$, and $\Phi_n^*(w; t) = w^n \overline{\Phi_n(1/\bar{w}; t)}$ denotes the reciprocal polynomial of $\Phi_n(w; t)$.

The monic polynomials $Q_n(z; t)$ can also be considered as the special para-orthogonal polynomials

$$Q_n(z; t) = z\Phi_{n-1}(z; t) - \rho_{n-1}^{(t)}(w)\Phi_{n-1}^*(z; t), \quad n \geq 1.$$

It is well known that quadrature rules on the unit circle are based on the zeros of para-orthogonal polynomials. Para-orthogonal polynomials have also been useful tools in linear prediction and frequency analysis problems (see [10]).

Clearly, by Theorem 1 the coefficients $\beta_n(t)$ and $\alpha_n(t)$ satisfy the extended relativistic Toda lattice equations (1.12) and (1.13), with $\mathfrak{p} = \bar{q}$.

Observe that, in contrast to the results presented in Sect. 4, the coefficients $\beta_n(t)$ and $\alpha_n(t)$ now are complex valued. However, by taking $w = 1$ we can write the three term recurrence (5.1) in a different form which involves only real coefficients.

Let

$$g_n(t) = \frac{1}{2} \frac{|1 - \rho_{n-1}^{(t)}\alpha_{n-1}(t)|^2}{[1 - \text{Re}[\rho_{n-1}^{(t)}\alpha_{n-1}(t)]]}, \quad n \geq 1, \tag{5.2}$$

where $\rho_j^{(t)} = \rho_j^{(t)}(1)$. It is easy to check that all $g_n(t) \in (0, 1)$, hence the terms of the following sequence are all positive

$$\xi_n(t) = \xi_0(t) \prod_{j=1}^n (1 - g_j(t)), \quad n \geq 1, \quad \xi_0(t) := \int_{\mathbb{T}} d\mu^{(t)}(z).$$

With this notation we introduce the normalized CD kernels by $R_n(z; t) := \xi_n(t) K_n^{(t)}(z, 1)$, $n \geq 0$.

It turns out (see [8]) that these kernel polynomials, $R_n(z; t)$, satisfy the following three term recurrence relation

$$R_{n+1}(z; t) = \left[(1 + ic_{n+1}(t))z + (1 - ic_{n+1}(t)) \right] R_n(z; t) - 4d_{n+1}(t)z R_{n-1}(z; t), \tag{5.3}$$

for $n \geq 1$, with $R_0(z; t) = 1$ and $R_1(z; t) = (1 + ic_1(t))z + (1 - ic_1(t))$, where both $\{c_n(t)\}_{n \geq 1}$ and $\{d_{n+1}(t)\}_{n \geq 1}$ are real sequences. In fact,

$$c_n(t) = \frac{\text{Im}(\rho_{n-1}^{(t)} \alpha_{n-1}(t))}{\text{Re}(\rho_{n-1}^{(t)} \alpha_{n-1}(t)) - 1} \in \mathbb{R} \quad \text{and} \quad d_{n+1}(t) = [1 - g_n(t)]g_{n+1}(t), \quad n \geq 1,$$

with $g_n(t)$ given by (5.2). In the standard terminology, this means that $\{d_{n+1}(t)\}_{n \geq 1}$ is a positive chain sequence for any t , and $\{g_{n+1}(t)\}_{n \geq 0}$ is a parameter sequence for $\{d_{n+1}(t)\}_{n \geq 1}$ (for more details on chain sequences see, for example, [5]).

Now, by using Theorem 1, we can state the following.

Theorem 4 *The coefficients $c_n(t)$ and $d_n(t)$ of the three term recurrence relation (5.3) satisfy*

$$\dot{c}_1 = -4 \text{Re}(q) \left[\frac{d_2(c_1 + c_2)}{1 + c_2^2} \right] - 4 \text{Im}(q) \left[\frac{d_2(1 - c_1c_2)}{1 + c_2^2} \right]$$

and, for $n \geq 2$,

$$\begin{aligned} \dot{c}_n &= 4 \text{Re}(q) \left[\frac{d_n(c_n + c_{n-1})}{1 + c_{n-1}^2} - \frac{d_{n+1}(c_n + c_{n+1})}{1 + c_{n+1}^2} \right] \\ &\quad + 4 \text{Im}(q) \left[\frac{d_n(1 - c_n c_{n-1})}{1 + c_{n-1}^2} - \frac{d_{n+1}(1 - c_n c_{n+1})}{1 + c_{n+1}^2} \right], \end{aligned}$$

and

$$\begin{aligned} \dot{d}_n &= 4 \text{Re}(q) \left[\frac{d_n d_{n-1}}{1 + c_{n-2}^2} - \frac{d_n d_{n+1}}{1 + c_{n+1}^2} + \frac{d_n(1 - d_n)(c_{n-1}^2 - c_n^2)}{(1 + c_n^2)(1 + c_{n-1}^2)} \right] \\ &\quad - 4 \text{Im}(q) \left[\frac{d_n d_{n-1} c_{n-2}}{1 + c_{n-2}^2} - \frac{d_n d_{n+1} c_{n+1}}{1 + c_{n+1}^2} + \frac{d_n(1 - d_n)(c_n - c_{n-1})(1 - c_n c_{n-1})}{(1 + c_n^2)(1 + c_{n-1}^2)} \right], \end{aligned}$$

with $c_0(t) = 1$ and $d_1(t) = 0$. Here, we have also omitted the time variable t .

Proof Since, $Q_n(z; t) \prod_{k=1}^n (1 + ic_k(t)) = R_n(z; t)$, $n \geq 1$, the coefficients $\beta_n(t)$, $\alpha_n(t)$, $c_n(t)$ and $d_n(t)$ which appear in the three term recurrence relations (5.1) and (5.3) are such that

$$\beta_n(t) = -\frac{1 - ic_n(t)}{1 + ic_n(t)} \quad \text{and} \quad \alpha_n(t) = \frac{4d_n(t)}{(1 + ic_n(t))(1 - ic_{n-1}(t))}, \tag{5.4}$$

for $n \geq 1$, with $c_0(t) = 1$ and $d_1(t) = 0$. Notice that differentiating $\beta_n(t)$ given by (5.4) with respect to t , we obtain

$$\frac{\dot{\beta}_n(t)}{\beta_n(t)} = -i \frac{2\dot{c}_n(t)}{1 + c_n^2(t)}, \quad n \geq 1. \tag{5.5}$$

Consequently since the coefficients $\beta_n(t)$ and $\alpha_n(t)$ satisfy (1.12), using (5.4) and (5.5), we conclude that, for $n \geq 1$,

$$\begin{aligned} \dot{c}_n(t) = & 2(p + q) \left\{ \frac{d_n(t)[c_n(t) + c_{n-1}(t)]}{1 + c_{n-1}^2(t)} - \frac{d_{n+1}(t)[c_n(t) + c_{n+1}(t)]}{1 + c_{n+1}^2(t)} \right\} \\ & + i 2(p - q) \left\{ \frac{d_n(t)[1 - c_n(t)c_{n-1}(t)]}{1 + c_{n-1}^2(t)} - \frac{d_{n+1}(t)[1 - c_n(t)c_{n+1}(t)]}{1 + c_{n+1}^2(t)} \right\}. \end{aligned}$$

Hence, the expression for $\dot{c}_n(t)$ in Theorem 4 is a consequence of $p = \bar{q}$. In a similar manner, using the value of $\alpha_n(t)$ given by (5.4), one can also prove the expression for $\dot{d}_n(t)$. \square

To obtain a special case of the results presented in Theorem 4, we assume that the measure μ satisfies the symmetry $d\mu(e^{i\theta}) = -d\mu(e^{i(2\pi-\theta)})$ and that q is real. Hence, the measure $\mu^{(t)}$ also satisfies the same symmetry and, as a consequence, $\alpha_{n-1}(t)$ is real and $\rho_n^{(t)}(1) = 1, n \geq 1$. This leads to the following corollary of Theorem 4.

Corollary 1 *If μ is such that $d\mu(e^{i\theta}) = -d\mu(e^{i(2\pi-\theta)})$ and q is real then $c_n(t) = 0$ for $n \geq 1$, and $d_n(t)$ satisfy*

$$\dot{d}_n = 4q d_n(d_{n-1} - d_{n+1}), \quad n \geq 2, \tag{5.6}$$

with $d_1(t) = 0$.

Notice that the relation obtained in (5.6) is a type of Langmuir lattice.

Comparing the coefficients in (5.1) when $w = 1$ with the coefficients in (5.3) we have

$$\beta_n(t) = \frac{1 - ic_n(t)}{1 + ic_n(t)}, \quad \alpha_{n+1}(t) = \frac{4d_{n+1}(t)}{[1 + ic_n(t)][1 + ic_{n+1}(t)]}, \quad n \geq 1.$$

We then have with $\gamma_n(t) = \alpha_{n+1}(t) + \beta_n(t), n \geq 1$, the associated Lax pair representation given by Theorem 2. Observe that, in the case of Corollary 1, we only need to replace $\beta_n(t)$ by 1, $\alpha_{n+1}(t)$ by $4d_{n+1}(t)$ in Theorem 2.

6 Applications to Orthogonal Polynomials on the Unit Circle

Let μ be a positive measure defined on the unit circle \mathbb{T} and let

$$\mathcal{L}[f] = \int_{\mathbb{T}} f(z)z d\mu(z) \quad \text{and} \quad \mathcal{L}^{(t)}[f] = \int_{\mathbb{T}} f(z)z d\mu^{(t)}(z),$$

where

$$d\mu^{(t)}(z) = e^{-t(\bar{q}z + q/z)} d\mu(z) = e^{-2t[\text{Re}(q) \cos \theta + \text{Im}(q) \sin \theta]} d\mu(e^{i\theta}).$$

Clearly, in this case the monic L-orthogonal polynomials $Q_n(z; t)$ defined by (1.7) (with z in the place of x) for any t fixed are actually the orthogonal polynomials on the unit circle, $\Phi_n(z; t)$, with respect to the measure $\mu^{(t)}$. Again, we will set the Verblunsky coefficients $\alpha_n(t)$ and the reciprocal polynomials $\Phi_n^*(z; t)$ to be, respectively,

$$\alpha_n(t) = -\overline{\Phi_{n+1}(0; t)} \quad \text{and} \quad \Phi_n^*(z; t) = z^n \overline{\Phi_n(1/\bar{z}; t)}, \quad n \geq 0.$$

It is known (see [24]) that the polynomials $Q_n(z; t) = \Phi_n(z; t)$ satisfy the relations

$$\begin{aligned} Q_n(z; t) &= zQ_{n-1}(z; t) - \overline{a_{n-1}(t)} Q_{n-1}^*(z; t), \\ Q_n(z; t) &= (1 - |a_{n-1}(t)|^2)zQ_{n-1}(z; t) - \overline{a_{n-1}(t)} Q_n^*(z; t), \end{aligned} \quad n \geq 1, \tag{6.1}$$

with $Q_0(z; t) = 1$, $Q_0^*(z; t) = 1$ and $Q_n^*(z; t) = z^n \overline{Q_n(1/\bar{z}; t)}$ for $n \geq 0$.

The existence of those polynomials are guaranteed by the positiveness of the measure $\mu^{(t)}$. That is, with the moments

$$v_k^{(t)} = \mathcal{L}^{(t)}[z^k], \quad k = 0, \pm 1, \pm 2, \dots, \tag{6.2}$$

the associated Hankel determinants satisfy condition (a) of (1.6). However, in general one can not assure condition (b) of (1.6).

It is well known that the sequence of the Verblunsky coefficients $a_n(t)$ associated with the measure $d\mu^{(t)}(z) = e^{t(z+1/z)}d\mu(z)$ satisfies the so-called Schur flow equation

$$\dot{a}_n = (1 - |a_n|^2)(a_{n+1} - a_{n-1}),$$

see, for example, [1, 13, 18]. In [13] a Lax representation for the Schur flow equations was obtained by using the CMV matrices (see [4] for more details about CMV matrices).

In [20] the connection between of the Verblunsky coefficients and the defocusing Ablowitz-Ladik system

$$i \dot{a}_n = a_{n+1} - 2a_n + a_{n-1} - |a_n|^2(a_{n+1} - a_{n-1})$$

was established. Moreover, also in [20], by using the CMV matrices and by defining certain Hamiltonians, a Lax representation for the defocusing Ablowitz-Ladik system was given, with emphasis for the p -periodic Verblunsky coefficients, i.e., when $a_{n+p} = a_p$.

Assumption *Let the moments (6.2) be such that condition (b) of (1.6) also hold.*

With this assumption we also have $a_n(t) \neq 0$, $n \geq 0$. In this case, using (6.1), we can see that the polynomials $Q_n(z; t)$ satisfy the three term recurrence relation

$$Q_{n+1}(z; t) = [z - \beta_{n+1}(t)]Q_n(z; t) - \alpha_{n+1}(t)zQ_{n-1}(z; t), \quad n \geq 1, \tag{6.3}$$

with $Q_0(z; t) = 1$ and $Q_1(z; t) = z - \beta_1(t)$, where $\beta_1(t) = \overline{a_0(t)}$,

$$\beta_{n+1}(t) = -\frac{\overline{a_n(t)}}{a_{n-1}(t)} \quad \text{and} \quad \alpha_{n+1}(t) = \frac{\overline{a_n(t)}}{a_{n-1}(t)}(1 - |a_{n-1}(t)|^2), \quad n \geq 1.$$

Since $\overline{a_n(t)} = (-1)^n \beta_1(t)\beta_2(t) \cdots \beta_{n+1}(t)$, observe that

$$\frac{\overline{\dot{a}_n(t)}}{a_n(t)} = \sum_{j=1}^{n+1} \frac{\dot{\beta}_j(t)}{\beta_j(t)}, \quad n \geq 0.$$

Thus, from (2.7), we find $\overline{\dot{a}_n(t)} = (1 - |a_n(t)|^2)[q \overline{a_{n-1}(t)} - \bar{q} \overline{a_{n+1}(t)}]$, $n \geq 1$.

Hence, we can state the following result.

Theorem 5 Let μ be a positive measure on the unit circle, and let $\{\alpha_n(t)\}_{n=0}^\infty$ be the Verblunsky coefficients associated with the measure $\mu^{(t)}$ given by $d\mu^{(t)}(z) = e^{-t(\bar{q}z + \frac{q}{z})}d\mu(z)$. Assuming that the Verblunsky coefficients are all different from zero we obtain the following

$$\dot{\alpha}_n = (1 - |\alpha_n|^2)(\bar{q}\alpha_{n-1} - q\alpha_{n+1}), \quad n \geq 1. \tag{6.4}$$

Remark 1 The original results associated with Theorem 5 (see [1, 13, 18]) correspond to the case $q = 1$. However, the case $|q| = 1$ also amounts to considering the perturbation $d\tilde{\mu}^{(t)}(z) = d\mu^{(t)}(e^{-i\alpha}z)$, where $q = e^{i\alpha}$.

Remark 2 Because of the approach used in this manuscript, we had to assume that all the Verblunsky coefficients $\alpha_n(t)$ are different from zero to obtain the results of Theorem 5. But, this restriction is not necessary in (6.4).

The system of nonlinear difference differential equations (6.4) also satisfies the Schur flow equation.

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