Acta Appl Math (2018) 155:9–19
DOI 10.1007/s10440-017-0142-1

Energy Decay of the Solution for a Weak Viscoelastic Equation with a Time-Varying Delay

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Received: 6 November 2016 / Accepted: 14 November 2017 / Published online: 29 November 2017

Abstract In this paper, we consider a weak viscoelastic equation with internal time-varying delay

$$u_{tt}(x,t) - \triangle u(x,t) + \alpha(t)\int_0^t g(t-s)\triangle u(x,s)ds + \mu u_t\big(x, t-\tau(t)\big) = 0$$

in a bounded domain. By introducing suitable energy and Lyapunov functionals, under suitable assumptions, we establish a general decay result for the energy. This work generalizes and improves earlier results in the literature.

Keywords Time-varying delay · Energy decay · Viscoelastic equation

Mathematics Subject Classification (2010) 35B40 · 35L05 · 74Dxx · 93D20

1 Introduction

In this paper, we investigate the following weak viscoelastic equation with a time-varying delay term in the feedback

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) + \alpha(t)\int_0^t g(t-s)\triangle u(x,s)ds + \mu u_t(x,t-\tau(t)) = 0, \\ \quad x \in \Omega, t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega, t \geq 0, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \quad x \in \Omega, \\ u_t(x,t) = f_0(x,t), \quad x \in \Omega,\ t \in [-\tau(0), 0), \end{cases} \tag{1.1}$$

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where Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 2$) with a sufficiently smooth boundary $\partial\Omega$, α and g are positive non-increasing functions defined on $\mathbb{R}^+$, $\tau(t) > 0$ represents the time-varying delay, u_0, u_1, f_0 are given functions belongs to some suitable spaces.

When $\mu = 0$ in the first equation of (1.1), that is in the absence of the delay, problem (1.1) was studied by many authors during the past decades. This type of problem arises in viscoelasticity, especially for the case $\alpha(t) = 1$. We start by mentioning the pioneer works of Dafermos [5, 6], where he discussed a one-dimensional viscoelastic problem, established several existence and asymptotic stability results. For other related works, we refer the readers to [1, 3, 4, 13, 14, 23] and references therein. On the other hand, Messaoudi [17] considered the viscoelastic equation without the delay of the form

$$u_{tt}(x,t) - \Delta u(x,t) + \alpha(t)\int_0^t g(t-s)\Delta u(x,s)ds = 0$$

under suitable conditions on α and g. He obtained general stability by making use of the perturbed energy method.

Introducing the delay term makes the problem different from those considered in the literatures. Time delay arises in many applications depending not only on the present state but also on some past occurrences. The presence of delay may be a source of instability. For example, when $g = 0$, it was shown in [8–10, 18, 19, 27] that an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional control terms have been used. Kirane and Said-Houari [11] considered the problem (1.1) with $\alpha(t) \equiv 1$, an additional linear damping term, $\mu > 0$ and $\tau(t)$ be a constant delay, and established general decay results under some condition.

The case of the time-varying delay in wave equation has also been studied by several authors, see for example [15, 16, 20–22] and the references therein. See also [12, 26] for the case of the transmission problem with delay.

In the works mentioned above, the authors must used the damping term $\mu_1 u_t(x,t)$ to control the delay term in the priori estimate of the solution and the decay estimate of the energy. By the way, in [7, 28], the authors improve earlier results in the literature by making using of the viscoelastic term to control the time constant delay term. Similar results have also been obtained for the problem with infinite memory by Guesmia [10]. Alabau-Boussouira et al. [2] considered the following wave delay equation with past history

$$u_{tt} - \Delta u + \int_0^{+\infty} \mu(s)\Delta u(t-s)ds + k u_t(t-\tau) = 0$$

and established the exponential stability if the coefficient k is sufficiently small by using a perturbation approach for delay problems first introduced in [24]; the result also holds for the anti-damping, i.e. $\tau = 0$ and $k < 0$. In this aspect, it is worth mentioning the work of Pignotti [25]. In this work, the author considered the problem

$$u_{tt} + Au - \int_0^{+\infty} \mu(s)Au(t-s)ds + b(t)u_t(t-\tau) = 0$$

and obtained that asymptotic stability is guaranteed if the delay feedback coefficient belongs to $L^1(0,+\infty)$ and the time intervals where the delay feedback is off are sufficiently large.

But, to the best of our knowledge, there is no research on the weak viscoelastic equation (a coefficient $\alpha(t)$ multiplying the memory term) with time-varying delay. Motivated by there results, we investigate the problem (1.1) under suitable assumptions. Our main contribution is an extension of the previous results from [7, 11, 15, 28] to weak viscoelastic equation without the linear damping term. The plan of this paper is as follows. In Sect. 2, we present some notations and assumptions needed for our work. In Sect. 3, we derive a general decay estimate of the energy.

2 Preliminaries and Main Result

We first introduce some notations that will be used in the proof of our results. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms. Throughout this paper, C and C_i are used to denote the generic positive constant. From now on, we shall omit x and t in all functions of x and t if there is no ambiguity.

For the relaxation function g and the potential α, we assumption the following (see [17, 22]):

(G1) $g, \alpha : \mathbb{R}^+ \to R^+$ are nonincreasing differentiable functions satisfying

$$g(0) > 0, \quad \int_0^{+\infty} g(s)ds < \infty, \qquad \alpha(t) > 0, \quad 1 - \alpha(t)\int_0^t g(s)ds \geq l > 0. \quad (2.1)$$

In addition, we assume that there exists a positive constant α_0 such that $\alpha(t) \geq \alpha_0$.

(G2) There exists a nonincreasing differentiable function $\zeta(t) : \mathbb{R}^+ \to R^+$ satisfying

$$\zeta(t) > 0, \quad g'(t) \leq -\zeta(t)g(t) \qquad \text{for } t \geq 0, \quad \lim_{t\to+\infty} \frac{-\alpha'(t)}{\zeta(t)\alpha(t)} = 0. \quad (2.2)$$

For the time-varying delay τ, we assume as in [15, 21, 22] that $\tau \in W^{2,\infty}([0, T])$ for any $T > 0$, and there exist positive constants τ_0, τ_1 and d such that

$$0 < \tau_0 \leq \tau(t) \leq \tau_1 \quad \text{and} \quad \tau'(t) \leq d < 1, \quad \text{for all } t > 0. \quad (2.3)$$

The following lemma is concerned with the global well-posedness of the problem (1.1). By using the classical Faedo-Galerkin method, see, e.g. [15, 17, 28], we can prove the Lemma, and we omit the proof here.

Lemma 2.1 *Let* (2.1)–(2.3) *be satisfied. If the initial data* $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $f_0 \in L^2(\Omega \times (0,1))$ *and any* $T > 0$, *then the problem* (1.1) *has a unique weak solution* $(u, u_t) \in C(0, T; H_0^1(\Omega) \times L^2(\Omega))$ *such that*

$$u \in L^\infty\big(0, T; H_0^1(\Omega)\big), \qquad u_t \in L^\infty\big(0, T; L^2(\Omega)\big).$$

Now, inspired by [15, 17, 21], we define the modified energy functional to the problem (1.1) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \left(1 - \alpha(t)\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}\alpha(t)(g \circ \nabla u)(t)$$
$$+ \frac{\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)} u_t^2(s)ds, \quad (2.4)$$

where ξ, λ are suitable positive constants to be determined later, and

$$(g \circ \nabla u)(t) = \int_\Omega \int_0^t g(t-s)\big|v(t) - v(s)\big|^2 ds dx.$$

First, we fix λ such that

$$\lambda < \min\left\{\frac{1}{\tau_1}\left|\log \frac{|\mu|}{\xi\sqrt{1-d}}\right|, \frac{1}{\tau_1}\log \frac{\alpha_0(1-d)^2 k_1}{2k_2}\right\}, \tag{2.5}$$

with k_1 and k_2 being defined in (3.16).

In order give our main theorem, we give the restriction condition on $\zeta(t)$

$$\zeta(t) > b \tag{2.6}$$

where b is a positive constant to be chosen in (3.23).

Our main result reads as follows:

Theorem 2.1 *Let* (G1), (G2) *and* (2.5) *hold. If the coefficient of the time-varying delay satisfies* $|\mu| < a$, *then there exist positive constants* K *and* κ *such that the energy of the problem* (1.1) *satisfies*

$$E(t) \le K e^{-\kappa t}, \quad \forall t \ge t_*, \tag{2.7}$$

where a *is positive constant defined by* (3.21), *which is only dependent on* g_0, l, d.

3 Energy Decay

In this section, we will prove the energy decay result Theorem 2.1 by constructing an appropriate Lyapunov function. First, we have the following lemmas.

Lemma 3.1 *Let* (2.1)–(2.3) *be satisfied. Then for all regular solution of problem* (1.1), *the energy function defined by* (2.4) *satisfies*

$$\begin{aligned} E'(t) \le{}& \left(\frac{|\mu|}{2\sqrt{1-d}} + \frac{\xi}{2}\right)\|u_t\|_2^2 + \left(\frac{|\mu|\sqrt{1-d}}{2} - \frac{\xi}{2}(1-d)e^{-\lambda\tau_1}\right)\big\|u_t\big(t-\tau(t)\big)\big\|_2^2 \\ &+ \frac{1}{2}\alpha(t)\big(g' \circ \nabla u\big)(t) - \frac{1}{2}\alpha'(t)\int_0^t g(s)ds\|\nabla u\|_2^2 \\ &- \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)} u_t^2(s) dx ds. \end{aligned} \tag{3.1}$$

Proof Differentiating (2.4) and using the first equation of (1.1) and then integrating by parts, the assumptions (2.1)–(2.3) and some manipulations as in [16, 21], we obtain

$$\begin{aligned} E'(t) ={}& -\mu\int_\Omega u_t(t)u_t\big(t-\tau(t)\big)dx - \frac{1}{2}\alpha(t)g(t)\|\nabla u\|_2^2 + \frac{1}{2}\alpha(t)\big(g' \circ \nabla u\big)(t) \\ &+ \frac{1}{2}\alpha'(t)(g \circ \nabla u)(t) - \frac{1}{2}\alpha'(t)\left(\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{\xi}{2}\|u_t\|_2^2 \end{aligned}$$

$$
\begin{aligned}
&-\frac{\xi}{2}\big(1-\tau'(t)\big)e^{-\lambda\tau(t)}\int_\Omega u_t^2\big(t-\tau(t)\big)dx-\frac{\lambda\xi}{2}\int_{t-\tau(t)}^t\int_\Omega e^{-\lambda(t-s)}u_t^2(s)dxds\\
\le{}&\frac{1}{2}\alpha(t)\big(g'\circ\nabla u\big)(t)-\frac{1}{2}\alpha(t)g(t)\|\nabla u\|_2^2+\left(\frac{|\mu|}{2\sqrt{1-d}}+\frac{\xi}{2}\right)\|u_t\|_2^2\\
&-\left(\frac{\xi}{2}(1-d)e^{-\lambda\tau_1}-\frac{|\mu|\sqrt{1-d}}{2}\right)\big\|u_t\big(t-\tau(t)\big)\big\|_2^2\\
&-\frac{\lambda\xi}{2}\int_{t-\tau(t)}^t\int_\Omega e^{-\lambda(t-s)}u_t^2(s)dxds\\
&+\frac{1}{2}\alpha'(t)(g\circ\nabla u)(t)-\frac{1}{2}\alpha'(t)\left(\int_0^t g(s)ds\right)\|\nabla u\|_2^2.
\end{aligned}
$$

Noticing (2.5) and the assumption (G1), (3.1) is established. □

Remark 3.1 Since $(\frac{|\mu|}{2\sqrt{1-d}}+\frac{\xi}{2})\|u_t\|_2^2\ge 0$, $-\frac{1}{2}\alpha'(t)(\int_0^t g(s)ds)\|\nabla u\|_2^2\ge 0$, $E'(t)$ may not be non-increasing.

Now, we define the Lyapunov function

$$L(t)=E(t)+\varepsilon_1\alpha(t)I(t)+\varepsilon_2\alpha(t)K(t), \tag{3.2}$$

where ε_i, $i=1,2$ are two positive real numbers which will be chosen later, and

$$I(t):=\int_\Omega uu_t dx, \tag{3.3}$$

$$K(t):=-\int_\Omega u_t\int_0^t g(t-s)\big(u(t)-u(s)\big)dsdx. \tag{3.4}$$

We can prove that, for sufficiently small $\varepsilon_1,\varepsilon_2$, for any $t\ge 0$, the exist two positive constant β_1,β_2 such that

$$\beta_1 E(t)\le L(t)\le\beta_2 E(t). \tag{3.5}$$

The following estimates hold true.

Lemma 3.2 *Under the assumption* (G1), *there exist two positive constants C_1 and C_2 satisfying*

$$I'(t)\le-\frac{l}{2}\|\nabla u\|_2^2+\|u_t\|_2^2+C_1\big\|u_t\big(t-\tau(t)\big)\big\|_2^2+C_2\alpha(t)(g\circ\nabla u)(t). \tag{3.6}$$

Proof Differentiating and integrating by parts

$$
\begin{aligned}
I'(t)&=\|u_t\|_2^2+\int_\Omega u\left(\triangle u-\alpha(t)\int_0^t g(t-s)\triangle u(s)ds-\mu u_t\big(t-\tau(t)\big)\right)dx\\
&\le\|u_t\|_2^2-l\|\nabla u\|_2^2+\alpha(t)\int_\Omega\nabla u\cdot\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)dsdx\\
&\quad-\mu\int_\Omega uu_t\big(t-\tau(t)\big)dx.
\end{aligned}
$$

Now, Young's and Poincaré's inequalities yields (see [16])

$$\begin{aligned}
&\alpha(t)\int_\Omega \nabla u\cdot\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)dsdx\\
&\quad\le \delta\|\nabla u\|_2^2+\frac{\alpha^2(t)}{4\delta}\int_\Omega\left(\nabla u\cdot\int_0^t g(t-s)\big|\nabla u(s)-\nabla u(t)\big|ds\right)^2dx\\
&\quad\le \delta\|\nabla u\|_2^2++\frac{(1-l)\alpha(t)}{4\delta}(g\circ\nabla u)(t),
\end{aligned}$$

and

$$-\mu\int_\Omega uu_t\big(t-\tau(t)\big)dx\le\delta\|\nabla u\|_2^2+C(\delta)\big\|u_t\big(t-\tau(t)\big)\big\|_2^2.$$

Choosing $\delta>0$ sufficiently small and combining the above estimates, we obtain (3.6). □

Lemma 3.3 *Under the assumption* (G1), *we have the following estimate*

$$\begin{aligned}
K'(t)\le&-\left(\int_0^t g(s)ds-\delta\right)\|u_t\|_2^2+\delta\big(1+2(1-l)^2\big)\|\nabla u\|_2^2\\
&+\left(2\delta\alpha^2(t)+\frac{2+\mu^2C_p^2}{4\delta}\right)\left(\int_0^t g(s)ds\right)(g\circ\nabla u)(t)\\
&+\delta\big\|u_t\big(t-\tau(t)\big)\big\|_2^2dx-\frac{g(0)C_p^2}{4\delta}\big(g'\circ\nabla u\big)(t),
\end{aligned}\tag{3.7}$$

where C_p is the Poincaré constant.

Proof The proof of this lemma is similar as Lemma 3.4 in [16]. But we do not have the damping term $u_t(t)$ in this paper. We give the sketch of it. Combining (1.1) and (3.4), we obtain

$$\begin{aligned}
K'(t)=&\int_\Omega u_{tt}\int_0^t g(t-s)\big(u(t)-u(s)\big)dsdx\\
&-\int_\Omega u_t\int_0^t g'(t-s)\big(u(t)-u(s)\big)dsdx-\left(\int_0^t g(s)ds\right)\|u_t\|_2^2\\
=&\int_\Omega\nabla u(t)\cdot\left(\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)ds\right)dx\\
&-\alpha(t)\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)ds\right)\cdot\left(\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)ds\right)dx\\
&-\int_\Omega u_t\int_0^t g'(t-s)\big(u(t)-u(s)\big)dsdx-\left(\int_0^t g(s)\right)\|u_t\|_2^2\\
&+\int_\Omega\left(\int_0^t g(t-s)\big(u(t)-u(s)\big)ds\right)\mu u_t\big(t-\tau(t)\big)dx=\sum_{i=1}^5 I_i.
\end{aligned}\tag{3.8}$$

The first to the third terms on the right hand-side of (3.8) can be estimate as in [17], for any $\delta > 0$, that

$$I_1 \le \delta \|\nabla u\|_2^2 + \frac{1}{4\delta}\left(\int_0^t g(s)ds\right)(g \circ \nabla u)(t),$$

$$I_2 \le \left(2\delta\alpha^2(t) + \frac{1}{4\delta}\right)\left(\int_0^t g(s)ds\right)(g \circ \nabla u)(t) + 2\delta(1-l)^2\|\nabla u\|_2^2,$$

$$I_3 \le \delta \|u_t\|_2^2 - \frac{g(0)C_p^2}{4\delta}\big(g' \circ \nabla u\big)(t).$$

Noticing the estimate

$$\int_\Omega \left(\int_0^t g(t-s)\big(u(t)-u(s)\big)ds\right)^2 dx \le C_p^2\left(\int_0^t g(s)ds\right)(g \circ \nabla u)(t),$$

where C_p is the Poincaré constant, we obtain

$$I_5 \le \delta \big\|\nabla u_t\big(t-\tau(t)\big)\big\|_2^2 + \frac{\mu^2 C_p^2}{4\delta}\left(\int_0^t g(s)ds\right)(g \circ \nabla u)(t).$$

Combining the above estimates with (3.8), we get (3.7). □

Now, we are in the position to prove the general decay result.

Proof of Theorem 2.1 Combining (3.1), (3.6), (3.8) and (G1), after a series of computations, for any $t \ge t_0$, we obtain

$$\begin{aligned} L'(t) \le& \left\{\frac{|\mu|}{2\sqrt{1-d}} + \frac{\xi}{2} - \alpha(t)\left(\varepsilon_2 \int_0^t g(s)ds - \varepsilon_2\delta - \varepsilon_1\right)\right\}\|u_t\|_2^2 \\ &+ \left\{\frac{|\mu|\sqrt{1-d}}{2} - \frac{\xi}{2}(1-d)e^{-\lambda\tau_1} + \varepsilon_1 C_1\alpha(t) + \varepsilon_2\delta\alpha(t)\right\}\big\|u_t\big(t-\tau(t)\big)\big\|_2^2 \\ &+ \left\{-\frac{1}{2}\alpha'(t)\int_0^t g(s)ds + \epsilon_2\alpha(t)\delta\big(1+2(1-l)^2\big) - \frac{\varepsilon_1 l}{2}\alpha(t)\right\}\|\nabla u\|_2^2 \\ &+ \left\{\varepsilon_1 C_2\alpha^2(t) + \varepsilon_2\alpha(t)\left(2\delta\alpha^2(t) + \frac{2+\mu^2C_p^2}{4\delta}\right)\left(\int_0^t g(s)ds\right)\right\}(g \circ \nabla u)(t) \\ &+ \alpha(t)\left\{\frac{1}{2} - \frac{\varepsilon_2 g(0)C_p^2}{4\delta}\right\}\big(g' \circ \nabla u\big)(t) + \varepsilon_1\alpha'(t)I(t) + \varepsilon_2\alpha'(t)K(t) \\ &- \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)}u_t(s)dsxds. \end{aligned} \tag{3.9}$$

By using (3.3), (3.4). Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} &\varepsilon_1\alpha'(t)I(t) + \varepsilon_2\alpha'(t)K(t) \\ &\quad \le -\frac{\varepsilon_1\alpha'(t)C_p^2}{2}\|\nabla u\|_2^2 - \frac{(\varepsilon_1+\varepsilon_2)\alpha'(t)}{2}\|u_t\|_2^2 - \frac{\alpha'(t)C_p^2}{2}\left(\int_0^t g(s)ds\right)(g \circ \nabla u)(t). \end{aligned}$$

Noticing g is positive and non-increasing, we have, for any fixed $t_0>0$, for any $t\geq t_0$, that

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 > 0. \tag{3.10}$$

Hence, (3.9) takes the form

$$\begin{aligned}
&L'(t)\\
&\leq -\alpha(t)\left\{\varepsilon_2 g_0-\varepsilon_2\delta-\varepsilon_1+\frac{-\frac{|\mu|}{2\sqrt{1-d}}-\frac{\xi}{2}+(\varepsilon_1+\varepsilon_2)\alpha'(t)}{2\alpha(t)}\right\}\|u_t\|_2^2\\
&\quad-\left\{\frac{\xi}{2}(1-d)e^{-\lambda\tau_1}-\frac{|\mu|\sqrt{1-d}}{2}-\varepsilon_1C_1\alpha(t)-\varepsilon_2\delta\alpha(t)\right\}\left\|u_t\big(t-\tau(t)\big)\right\|_2^2\\
&\quad-\alpha(t)\left\{\frac{\varepsilon_1 l}{2}-\epsilon_2\delta\big(1+2(1-l)^2\big)+\frac{\alpha'(t)}{2\alpha(t)}\int_0^t g(s)ds+\frac{\varepsilon_1\alpha'(t)C_p^2}{2\alpha(t)}\right\}\|\nabla u\|_2^2\\
&\quad+\alpha(t)\left\{\varepsilon_1C_2\alpha(0)+\varepsilon_2\left(2\delta\alpha^2(0)+\frac{2+\mu^2C_p^2}{4\delta}-\frac{\alpha'(t)C_p^2}{2\alpha(t)}\right)\left(\int_0^t g(s)ds\right)\right\}(g\circ\nabla u)(t)\\
&\quad+\alpha(t)\left\{\frac{1}{2}-\frac{\varepsilon_2 g(0)C_p^2}{4\delta}\right\}\big(g'\circ\nabla u\big)(t)-\frac{\lambda\xi}{2}\int_{t-\tau(t)}^t\int_\Omega e^{-\lambda(t-s)}u_t(s)dsxds.
\end{aligned} \tag{3.11}$$

Now, we should deduce the following system of the inequalities

$$\begin{cases}
\varepsilon_2 g_0-\varepsilon_2\delta-\varepsilon_1+\frac{-\frac{|\mu|}{2\sqrt{1-d}}-\frac{\xi}{2}+(\varepsilon_1+\varepsilon_2)\alpha'(t)}{2\alpha(t)}>0,\\
\frac{\xi}{2}(1-d)e^{-\lambda\tau_1}-\frac{|\mu|\sqrt{1-d}}{2}-\varepsilon_1C_1\alpha(t)-\varepsilon_2\delta\alpha(t)>0,\\
\frac{\varepsilon_1 l}{2}-\epsilon_2\delta(1+2(1-l)^2)+\frac{\alpha'(t)}{2\alpha(t)}\int_0^t g(s)ds+\frac{\varepsilon_1\alpha'(t)C_p^2}{2\alpha(t)}>0,\\
\frac{1}{2}-\frac{\varepsilon_2 g(0)C_p^2}{4\delta}>0\\
\xi,\delta,\varepsilon_1,\varepsilon_2,\lambda>0
\end{cases} \tag{3.12}$$

is solvable only if we add some suitable conditions to μ.

Indeed, we can find solutions of (3.12) according to the following steps.

Step 1. We first take δ sufficiently small such that

$$\begin{aligned}
\delta<\min&\left\{\frac{g_0}{4},\frac{g_0 l}{16[1+2(1-l)^2]},\frac{g_0 l}{2(1+C_1\alpha(0))[1+2(1-l)^2]+1+\alpha(0)},\right.\\
&\left.\frac{g_0\alpha_0 l(1-d)^2}{2[1+2(1-l)^2][\alpha_0(1-d)^2+2C_1\alpha(0)]}\right\},
\end{aligned} \tag{3.13}$$

and

$$\delta<\frac{g_0\alpha_0(1-d)^2}{2[\alpha_0(1-d)^2+2\alpha(0)]}. \tag{3.14}$$

Step 2. Once δ is fixed, we select ε_2 sufficiently small such that

$$\frac{1}{2}-\frac{\varepsilon_2 g(0)C_p^2}{4\delta}>0.$$

Step 3. Then we choose ε_1 satisfying the relation

$$\frac{2\delta[1+2(1-l)^2]}{l} < \frac{g_0}{8} < \frac{\varepsilon_1}{\varepsilon_2} < \min\left\{\frac{g_0-\delta-\alpha(0)\delta}{1+C_1\alpha(0)}, \frac{g_0\alpha_0(1-d)^2}{\alpha_0(1-d)^2+2C_1\alpha(0)}\right\}. \tag{3.15}$$

The choice of ε_1 is possible by the choice of δ in (3.13) and (3.14).

From (3.13) and (3.15), we deduce that

$$\varepsilon_2(g_0-\delta)-\varepsilon_1 > \varepsilon_1 C_1\alpha(0)+\varepsilon_2\delta\alpha(0) > 0. \tag{3.16}$$

Now, let $k_1=\varepsilon_2(g_0-\delta)-\varepsilon_1$ and $k_2=\varepsilon_1 C_1\alpha(0)+\varepsilon_2\delta\alpha(0)$. Thus, (3.16) implies that k_1, k_2 are two positive constants depending on g_0, l.

Step 4. Now, we must ensure that the first to the third inequality in (3.12) hold. That is to say, the following system of inequalities

$$\begin{cases} \frac{\frac{|\mu|}{\sqrt{1-d}}+\xi}{2\alpha_0} - \frac{\alpha'(t)(\varepsilon_1+\varepsilon_2)}{2\alpha(t)} < k_1, \\ \frac{\xi}{2}(1-d)e^{-\lambda\tau_1} - \frac{|\mu|\sqrt{1-d}}{2} > k_2, \\ \frac{\varepsilon_1 l}{2} - \varepsilon_2\delta(1+2(1-l)^2) + \frac{\alpha'(t)}{2\alpha(t)}g_0 + \frac{\varepsilon_1\alpha'(t)C_p^2}{2\alpha(t)} > 0 \end{cases} \tag{3.17}$$

must be solvable.

In fact, by (3.15), the inequality

$$\frac{\varepsilon_1 l}{2} - \epsilon_2\delta\big(1+2(1-l)^2\big) > 0$$

is obtained. Since $\lim_{t\to\infty}\frac{\alpha'(t)}{\alpha(t)}=0$ (which can be deduced from (G2)), we can choose $t_1 \geq t_0$ so that the third term in (3.17) is obtained for all $t \geq t_1$, and the first term in (3.17) becomes

$$\frac{|\mu|}{\sqrt{1-d}} + \xi < k_1\alpha_0. \tag{3.18}$$

Step 5. Moreover, by the choice of δ in (3.14), we can obtain

$$\frac{k_1}{k_2} > \frac{2}{\alpha_0(1-d)^2}, \quad \text{i.e.} \quad \log\frac{\alpha_0(1-d)^2 k_1}{2k_2} > 0. \tag{3.19}$$

Since λ satisfies (2.5), there exists a positive constant ξ such that

$$\frac{2k_2 e^{\lambda\tau_1}}{(\sqrt{1-d})^3} < \xi < \sqrt{1-d}\,k_1\alpha_0. \tag{3.20}$$

Step 6. To ensure the solvability of the system (3.17), we just need to add a condition given by

$$|\mu| < \min\left\{\sqrt{1-d}\,k_1\alpha_0-\xi, \frac{2k_2}{\sqrt{1-d}} - \xi(1-d)e^{-\lambda\tau_1}\right\} := a > 0, \tag{3.21}$$

here a is only dependent on g_0, l, d. Moreover, the condition (3.21) is possible from (3.20).

Hence, the system (3.12) is solvable. Consequently, there exist two positive constants C_3 and C_4 such that

$$L'(t) \leq -C_3\alpha(t)E(t) + C_4\alpha(t)(g\circ\nabla u)(t), \quad \forall t \geq t_1. \tag{3.22}$$

Multiplying (3.22) by $\zeta(t)$ and using (G2), (3.1) and (3.18), for $t \geq t_1$, we obtain

$$\begin{aligned}\zeta(t)L'(t) &\leq -C_3\zeta(t)\alpha(t)E(t) + C_4\alpha(t)\zeta(t)(g \circ \nabla u)(t)\\ &\leq -C_3\zeta(t)\alpha(t)E(t) - C_4\alpha(t)\big(g' \circ \nabla u\big)(t)\\ &\leq -C_3\zeta(t)\alpha(t)E(t) + C_4\left[-2E'(t) - \alpha'(t)\left(\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + k_1\alpha_0\|u_t\|_2^2\right]\end{aligned}$$

Since $\zeta(t)$ is nonincreasing, we have

$$\begin{aligned}&\big(\zeta(t)L(t) + 2C_4E(t)\big)'\\ &\quad\leq -C_3\zeta(t)\alpha(t)E(t) - C_4\alpha'(t)\left(\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + k_1\alpha_0\|u_t\|_2^2.\end{aligned}$$

Observing from the definition of $E(t)$ and assumption (2.1) that

$$l\|\nabla u\|_2^2 \leq 2E(t),$$

we get

$$\big(\zeta(t)L(t) + 2C_4E(t)\big)' \leq -\zeta(t)\alpha(t)\left(C_3 + \frac{2C_4(1-l)\alpha'(t)}{\alpha(t)\zeta(t)l}\right)E(t) + 2k_1\alpha_0E(t).$$

Since $\lim_{t\to\infty}\frac{-\alpha'(t)}{\zeta(t)\alpha(t)} = 0$, we can choose $t_* \geq t_1$ such that

$$C_5 := C_3 + \frac{2C_4(1-l)\alpha'(t)}{\alpha(t)\zeta(t)l} > 0, \quad \forall t \geq t_*.$$

Hence, if we let

$$b = \frac{2k_1}{C_5}, \tag{3.23}$$

that is $\zeta(t) > b = \frac{2k_1}{C_5} \geq \frac{2k_1\alpha_0}{C_5\alpha(t)}$, we arrive at

$$\big(\zeta(t)L(t) + 2C_4E(t)\big)' \leq -C_5\zeta(t)\alpha(t)E(t) + 2k_1\alpha_0E(t) \leq -C_6E(t), \quad t \geq t_*.$$

Finally, let $\mathcal{L}(t) = \zeta(t)L(t) + 2C_4(t)E(t)$, then we can see that $\mathcal{L}(t)$ is equivalent to $E(t)$. Hence, we arrive at

$$\mathcal{L}'(t) \leq -C_7\mathcal{L}(t), \quad \forall t \geq t_*.$$

Integrating this over (t_*, t), we can deduce that

$$\mathcal{L}(t) \leq \mathcal{L}(t_*)e^{-C_7t}, \quad \forall t \geq t_*.$$

Consequently, the equivalent relations of $E(t)$, $L(t)$ and $\mathcal{L}(t)$ give the desired result (2.7). □

Acknowledgements The authors would like to thank the referees for the careful reading of this paper and for the valuable suggestions to improve the presentation and the style of the paper. This project is supported by Key Scientific Research Foundation of the Higher Education Institutions of Henan Province, China (Grant No. 15A110017) and the Basic Research Foundation of Henan University of Technology, China (No. 2013JCYJ11).

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