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# Conditions for the Absence of Blowing Up Solutions to Fractional Differential Equations

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**Abstract** When addressing ordinary differential equations in infinite dimensional Banach spaces, an interesting question that arises concerns the existence (or non existence) of blowing up solutions in finite time. In this manuscript we discuss this question for the fractional differential equation $cD_t^\alpha u = f(t,u)$ proving that when $f$ is locally Lipschitz in the second variable, uniformly with respect to the first variable, however does not maps bounded sets into bounded sets, we can construct a maximal local solution that does not "blow up" in finite time.

**Keywords** Caputo derivative · Fractional differential equations · Blow up solutions

## 1 Introduction

To introduce the main aspects of this discussion, consider the following differential equation

$$\begin{cases} u'(t) = f(t,u(t)), & t > t_0, \\ u(t_0) = u_0 \in X, \end{cases} \tag{1}$$

where $X$ denotes a Banach space and $t_0 \in \mathbb{R}$.

Conditions for the existence of solutions to problem (1) were firstly obtained by Peano in [16]. Following his work, several mathematicians proposed improvements to this result. Nonetheless, to our objectives in this manuscript it is worth to present the following version (see [4, 9] for details of the proof).

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**Theorem 1** *Let $X$ be a Banach space, $t_0 \in \mathbb{R}$ and $f : [t_0, \infty) \times X \to X$ a continuous function which is locally Lipschitz in the second variable, uniformly with respect to the first variable (see Theorem 2 for details), and bounded. Then problem (1) has a global solution in the interval $[t_0, \infty)$ or there exists a value $\omega \in (t_0, \infty)$ such that $u : [t_0, \omega) \to X$ is a maximal local solution that satisfies*

$$\limsup_{t \to \omega^-} \|u(t)\|_X = \infty.$$

Clearly the above result establishes conditions such that the solutions of (1) have a dichotomy property concerning its "longtime behavior". Since the need of the Lipschitz condition is classical, it is natural that questions regarding the necessity of the boundedness property of $f$ were raised.

If we focus on finite dimensional spaces, it seems obviously that this hypothesis is dispensable, however in infinite dimensional spaces it plays a fundamental role. Dieudonné in [6] was most likely the first mathematician to address this question. He considered the Banach space

$$X := \left\{ \{x_n\}_{n=1}^{\infty} : x_n \in \mathbb{R} \text{ and } \lim_{n \to \infty} x_n = 0 \right\},$$

with norm $\|\{x_n\}_{n=1}^{\infty}\|_X := \sup_{n \in \mathbb{N}} |x_n|$ and construct a non-bounded and Lipschitz function $f : [0, 1] \times X \to X$ such that (1) posses a local solution that does not admit a continuation and is also bounded.

Following Dieudonné's inspiration, many mathematicians discussed this kind of problem, which in certain sense, is related with the failure of Peano's existence theorem in infinite dimensions. For instance, Deimling improved Dieudonné construction considering more general Banach spaces, as can be seen in [4, 5]; Komornik et al. in [11] addressed the autonomous version of (1), proving that for any infinite dimensional Banach space $X$ and bounded interval $(s, t) \subset \mathbb{R}$, there exists a locally Lipschitz function $f : X \to X$ and $u_0 \in X$ such that the maximal solution of (1) is exactly defined on $(s, t)$ although it remains bounded on $(s, t)$.

On the other hand, it is worth recalling that fractional differential equations are gaining considerable emphasis in the mathematical society and the equivalent question in this context, besides being a very interesting problem, was still unanswered.

Even if this question seems to have an adaptable proof from the standard case of ordinary differential equations, it does not happen. The non-local characteristic carried by the fractional differential operator is very hard to manipulate and new arguments are necessary to obtain such result.

In order to fill this gap, we initially recall some results of the fractional differential equations theory and then we exhibit an example of a locally Lipschitz in the second variable, uniformly with respect to the first variable, function $f$ that does not map bounded sets onto bounded sets and induces (FCP), Fractional Cauchy Problem in Sect. 2, to possess a maximal local solution that is also bounded.

Finally we present the structure of this paper. In Sect. 2 we discuss several important tools concerning the fractional calculus and the respective fractional Cauchy problem, proving the result concerning "blowing up" solutions. In Sect. 3 we construct the aforementioned counter example, discussing also every functional analysis tools used in the process. We left Sect. 4 to address an example where we compute explicitly the result obtained in Theorem 6, which completes the discussion we intend to promote in this paper.

## 2 A Study of Differential Equations with Fractional Caputo Derivative

Let us recall firstly the study of the locally Bochner integrable functions for the Dunford-Schwartz integral with respect to a Banach space (see details in [7]). Hereafter assume that $S \subset \mathbb{R}$ and $X$ is a Banach space.

(i) Denote by $L^1(S, X)$ the set of all measurable functions $x : S \to X$ such that $\|x(t)\|_X$ is integrable. Furthermore, this set equipped with norm

$$\|x(t)\|_{L^1(S,X)} := \int_S \|x(t)\|_X \, dt,$$

is a Banach space.

(ii) Represent by $W^{1,1}(S, X)$ the set of all elements of $L^1(S, X)$ which have weak derivative of order one being in $L^1(S, X)$. This set equipped with norm

$$\|x(t)\|_{W^{1,1}(S,X)} := \big(\|x(t)\|^2_{L^1(S,X)} + \|x'(t)\|^2_{L^1(S,X)}\big)^{1/2},$$

is also a Banach space.

(iii) Finally, $C(S, X)$ denotes the space of the continuous functions $x : S \to X$. When $S$ is a compact set we define the norm

$$\|x(t)\|_{C(S,X)} := \sup_{t \in S} \|x(t)\|_X,$$

which makes $C(S, X)$ a Banach space.

**Definition 1** Let $\alpha \in (0, 1)$, $\tau \in (0, \infty)$ and $h \in L^1(0, \tau; X)$.

(i) The Riemann-Liouville fractional integral of order $\alpha$, is denoted by $J_t^\alpha h(t)$, and is given by

$$J_t^\alpha h(t) := (h * g_\alpha)(t) = \int_0^t g_\alpha(t - r)h(r)\, dr, \qquad \text{a.e. in } [0, \tau],$$

where function $g_\alpha : \mathbb{R} \to \mathbb{R}$ is given by

$$g_\alpha(t) = \begin{cases} t^{\alpha-1}/\Gamma(\alpha), & t > 0, \\ 0, & t \le 0. \end{cases}$$

(ii) If $h * g_{1-\alpha} \in W^{1,1}(0, \tau; X)$, the Riemann-Liouville fractional derivative of order $\alpha$ is given by

$$D_t^\alpha h(t) := D_t^1 J_t^{1-\alpha} h(t) = D_t^1 (h * g_{1-\alpha})(t), \qquad \text{a.e. in } [0, \tau],$$

where $D_t^1 = (d/dt)$.

The above definitions were extensively used in the study of fractional calculus (see for instance [12, 13, 15, 19]) however in this manuscript we discuss only Caputo fractional derivative, which is defined bellow (see for instance [2, 3, 18] for more details).

**Definition 2** Consider real numbers $\alpha \in (0, 1)$, $\tau \in (0, \infty)$ and function $h \in C([0, \tau], X)$ satisfying $h * g_{1-\alpha} \in W^{1,1}(0, \tau; X)$. We define the Caputo fractional derivative of order $\alpha$, which is denoted by $cD_t^\alpha h(t)$, by

$$cD_t^\alpha h(t) := D_t^\alpha \big(h(t) - h(0)\big), \qquad \text{a.e. in } [0, \tau].$$

*Remark 1* It is important to emphasize some facts at this moment.

(i) In [1] the author reproduces the classical proof that ensures, for an integrable function $h \in L^1(0,\tau;X)$, that $D_t^\alpha J_t^\alpha h(t) = h(t)$. Moreover, if it holds that $h * g_{1-\alpha}$ also belongs to $W^{1,1}(0,\tau;X)$, then

$$J_t^\alpha D_t^\alpha h(t) = h(t) - \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \big\{ J_s^{1-\alpha} h(s) \big\}\big|_{s=0}, \quad \text{a.e. in } [0,\tau].$$

It is important to notice that $\{J_s^{1-\alpha} h(s)\}|_{s=0}$ can not even be computable; choose for instance $h(t) = t^{-1/2}$.

(ii) For $h \in C([0,\tau],X)$, it holds that

$$cD_t^\alpha J_t^\alpha h(t) = h(t).$$

Furthermore, if $h * g_{1-\alpha} \in W^{1,1}(0,\tau;X)$ we conclude

$$J_t^\alpha cD_t^\alpha h(t) = h(t) - h(0) - \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \big\{ J_s^{1-\alpha} \big[ h(s) - h(0) \big] \big\}\big|_{s=0},$$

a.e. in $[0,\tau]$. Since $\{J_s^{1-\alpha}[h(s) - h(0)]\}|_{s=0} = 0$, we achieve the equality

$$J_t^\alpha cD_t^\alpha h(t) = h(t) - h(0).$$

(iii) Finally, if $u \in C([0,\tau],X)$ is a function that satisfies $u' \in C([0,\tau],X)$ we obtain $cD_t^\alpha u(t) = J_t^{1-\alpha} u'(t)$ for any $t \in (0,\tau]$. This was the first formal definition of the Caputo fractional derivative.

Once the main tools concerning fractional calculus are introduced, we now formalize the fractional differential equation that we study in this manuscript. The fractional Cauchy problem (FCP) is given by

$$\begin{cases} cD_t^\alpha u(t) = f(t, u(t)), & t > 0, \\ u(0) = u_0 \in X, \end{cases} \tag{FCP}$$

where $\alpha$ is a real number in $(0,1)$, $cD_t^\alpha$ is the Caputo fractional derivative of order $\alpha$ and $f : [0,\infty) \times X \to X$ is a continuous function.

Now it is necessary to define the notion of solution to problem (FCP), which indeed is given by an adaptation of the classical ideas that are applied to ordinary differential equations.

**Definition 3** Assume that $\alpha \in (0,1)$.

(i) We say that a function $u : [0,\infty) \to X$ is a global solution of (FCP) if

$$u \in C^\alpha\big([0,\tau],X\big) := \big\{ u \in C\big([0,\tau],X\big) : cD_t^\alpha u \in C\big([0,\tau],X\big) \big\}$$

for every $\tau > 0$ and satisfies the equations of (FCP).

(ii) If there exists $0 < \tau < \infty$ such that a continuous function $u : [0,\tau] \to X$ belongs to $C^\alpha([0,\tau],X)$ and satisfies (FCP) for $t \in [0,\tau]$, we say that $u$ is a local solution to problem (FCP) on the interval $[0,\tau]$.

*Remark 2* Observe that $C^\alpha([0,\tau],X)$ imbued with norm

$$\|x(t)\|_{C^\alpha([0,\tau],X)} := \sup_{t\in[0,\tau]} \|x(t)\|_X$$

is a Banach space.

Bearing these definitions in mind, we present the classical result that discuss the local existence and uniqueness of a solution to the fractional differential equation (a proof of this theorem can be found in [1, 10]).

**Theorem 2** *Assume that* $\alpha \in (0,1)$, $f : [0,\infty) \times X \to X$ *is a continuous function and* $u_0 \in X$. *If* $f$ *is also a locally Lipschitz in the second variable, uniformly with respect to the first variable, i.e., given* $(t_0, x_0) \in [0,\infty) \times X$ *there exist* $L, r > 0$ (*depending on* $f$, $t_0$ *and* $x_0$) *such that for any* $(t,x), (t,y) \in B_r(t_0,x_0)$ *it holds that*

$$\|f(t,x) - f(t,y)\|_X \leq L\|x-y\|_X,$$

*then there exists* $\tau > 0$ *such that problem* (FCP) *posses a unique local solution* $u$ *in* $[0,\tau]$.

The remainder of this section will be devoted to discuss the continuation of local solutions and global solutions of (FCP). First, it is necessary to introduce some concepts.

**Definition 4** Let $u : [0,\tau] \to X$ be a local solution to (FCP).

(i) If $\tau^* > \tau$ and $u^* : [0,\tau^*] \to X$ is a local solution to (FCP) in $[0,\tau^*]$ such that $u(t) = u^*(t)$ in $[0,\tau]$, then we call $u^*$ a continuation of $u$ over $[0,\tau^*]$.
(ii) Furthermore, if $u : [0,\tau^*) \to X$ is the unique local solution to (FCP) in $[0,\tau]$ for every $\tau \in (0,\tau^*)$ and does not have a continuation, then we call it maximal local solution of (FCP) in $[0,\tau^*)$ (see [17] for more details on maximal solutions).

Now we are able to establish the existence of continuation to a given solution of (FCP).

**Theorem 3** *Let* $\alpha \in (0,1)$, $\tau \in (0,\infty)$ *and* $f : [0,\infty) \times X \to X$ *be as in Theorem* 2. *If* $u : [0,\tau] \to X$ *is the unique local solution to* (FCP) *in* $[0,\tau]$, *then there exists a unique continuation* $u^*$ *of* $u$ *in* $[0,\tau^*]$ *for some value* $\tau^* > \tau$.

Finally, inspired by De Andrade et al. in [3] and based on the results discussed above, for the sake of completeness we state and prove the main theorem of this section.

**Theorem 4** *Let* $\alpha \in (0,1)$, $X$ *be a Banach space and* $f : [0,\infty) \times X \to X$ *a continuous function which is locally Lipschitz in the second variable, uniformly with respect to the first variable and maps bounded sets onto bounded sets. Then problem* (FCP) *has a global solution in the interval* $[0,\infty)$ *or there exists a value* $\omega \in (0,\infty)$ *such that the local solution* $u : [0,\omega) \to X$ *does not admit a continuation and yet satisfies*

$$\limsup_{t\to\omega^-} \|u(t)\|_X = \infty.$$

*Proof* Consider $H \subset \mathbb{R}$, which is given by

$$H := \big\{\tau \in (0,\infty) : \text{ there exists } u_\tau : [0,\tau] \to X \\ \text{unique local solution to (FCP) in } [0,\tau]\big\}. \tag{2}$$

Define $\omega = \sup H$ and consider function $u : [0,\omega) \to X$ which is given by $u(t) = u_\tau(t)$, if $t \in [0,\tau]$. It is not difficult to verify that this function is well defined and is the maximal local solution of (FCP) in $[0,\omega)$.

If $\omega = \infty$, $u$ is a global solution of (FCP). Otherwise, if $\omega < \infty$ we need to prove that

$$\limsup_{t\to\omega^-} \|u(t)\|_X = \infty.$$

The proof is by contradiction. Suppose that there exists $d < \infty$ such that $\|u(t)\|_X \leq d$ for all $t \in [0,\omega)$. Then, since $f$ maps bounded sets onto bounded sets, define

$$M := \sup_{s\in[0,\omega)} \big\|f\big(s,u(s)\big)\big\|_X < \infty$$

and consider $\{t_n\}_n \subset [0,\omega)$ a sequence that converges to $\omega$. Thus, making some computations, we obtain the estimate

$$\big\|u(t_n) - u(t_m)\big\|_X \leq \frac{M^*}{\Gamma(\alpha+1)}\Big[\big|{t_n}^\alpha + |t_m - t_n|^\alpha - t_m^\alpha\big| + |t_m - t_n|^\alpha\Big],$$

for some positive value $M^*$. This ensures that $\{u(t_n)\}_{n=0}^\infty$ is a Cauchy sequence and therefore it has a limit, let us say, $u_\omega \in X$. By extending $u$ over $[0,\omega]$, we conclude that the equality

$$u(t) = u_0 + \int_0^t (t-s)^{\alpha-1} f\big(s,u(s)\big)\,ds,$$

should hold for all $t \in [0,\omega]$. With this, by Theorem 3, we can extend the solution to some bigger interval, which is a contradiction by the definition of $\omega$. Therefore, if $\omega < \infty$ it holds that $\limsup_{t\to\omega^-} \|u(t)\|_X = \infty$. This concludes the proof. □

## 3 Fundamental Structures and the Bounded Maximal Solution

The aim of this section is to recall some fundamental concepts of functional analysis and discuss the existence of a maximal local solution to problem (FCP) which does not "blows up" in finite time, under suitable hypotheses.

For the discussion suggested above to be successfully addressed, we first recall some concepts and notations related to infinite dimensional Banach spaces.

**Definition 5** Let $X$ be a Banach space. A sequence $\{v_n\}_{n=1}^\infty \subset X$ is called a Schauder basis of $X$, if for every $x \in X$, there exists a unique sequence $\{x_n\}_{n=1}^\infty \subset \mathbb{R}$ such that

$$\lim_{k\to\infty} \left\| x - \sum_{n=1}^k x_n v_n \right\|_X = 0. \tag{3}$$

We write $x = \sum_{n=1}^\infty x_n v_n$ to denote the above limit.

It is worth to emphasize that existence of Schauder basis to general Banach spaces is not a trivial matter. Indeed, as can be found in the literature, it is not true that every Banach space has a Schauder basis (see [8] for details).

Therefore, it is essential to introduce the following result to better adjust our forward computations.

**Theorem 5** *For any infinite dimensional Banach space $X$, there exists an infinite dimensional closed subspace $X_0$ of $X$ with a Schauder Basis $\{v_n\}_{n=0}^{\infty}$. Moreover, we can suppose that $\{v_n\}_{n=1}^{\infty}$ is such that $\|v_n\|_X = 1$, for all $n \in \mathbb{N}$, and that there exists a sequence of linear functionals $\{v_n^*\}_{n=1}^{\infty} \subset X_0^*$ which satisfies $\|v_n^*\|_{X_0^*} = 1$ and also, for any $x \in X_0$,*

$$x = \sum_{n=1}^{\infty} v_n^*(x) v_n. \tag{4}$$

*Proof* It is a classical result. For more details see [14, Theorem I.1.2]. □

At this point we are already prepared to discuss the main ideas proposed by this manuscript. Thus, the remainder of this section is dedicated to prove the following:
**Statement:** There exists an element $u_0 \in X$ and also a continuous and locally Lipschitz function $f : \mathbb{R}^+ \times X \to X$, which does not maps bounded sets into bounded sets, such that

$$\begin{cases} cD_t^{\alpha} u(t) = f(t, u(t)), & t > 0 \\ u(0) = u_0 \in X, \end{cases} \tag{FCP}$$

has a maximal bounded local solution in $[0, 1)$.

A positive answer to the problem above closes any question concerning the adopted hypotheses in Theorem 4. It is worth to stress that the ideas which inspired the proof of this question were given by Dieudonné in [6], Deimling in [4, 5] and Komornik in [11].

To begin this discussion we consider a suitable sequence of functions, which develops a fundamental role forward in the manuscript. More explicitly, for any increasing sequence of positive real numbers $\{t_n\}_{n=1}^{\infty}$ that converges to 1 we consider a sequence of continuously differentiable functions $\{z_n(t)\}_{n=1}^{\infty}$, which satisfies the following properties:

(i) The length of the intervals $[t_n, t_{n+1}]$ decreases when $n$ increases. More specifically,

$$t_n - t_{n-1} > t_{n+1} - t_n,$$

for $n \geq 2$;

(ii) $z_1 \equiv 1$ in $\mathbb{R}$ and for $n \geq 2$

$$z_n(t) := \begin{cases} 1, & t \in [t_n, t_{n+1}], \\ \in (0, 1), & t \in (\frac{t_{n-1}+t_n}{2}, t_n) \cup (t_{n+1}, \frac{t_{n+1}+t_{n+2}}{2}), \\ 0, & \text{otherwise}. \end{cases}$$

Now consider $X$ an infinite dimensional Banach space. As stated in Theorem 5, we assume that $X_0$ denotes an infinite dimensional closed subspace of $X$ (with the topology induced by the topology of $X$) with a Schauder Basis $\{v_n\}_{n=1}^{\infty}$, which satisfies

$$\|v_n\|_X = 1, \quad \forall n \in \mathbb{N}.$$

We also recall that there exists a sequence of linear functionals $\{v_n^*\}_{n=1}^{\infty} \subset X_0^*$ which satisfies

$$\left\|v_n^*\right\|_{X_0^*} = \sup_{\substack{x \in X_0 \\ \|x\|_X \leq 1}} \left|v_n^*(x)\right| = 1, \quad \forall n \in \mathbb{N},$$

and allows us to write every $x \in X_0$ as the sum

$$x = \sum_{n=1}^{\infty} v_n^*(x) v_n.$$

The characteristic function $\chi_I : \mathbb{R} \to \mathbb{R}$ is the function given by

$$\chi_I(t) = \begin{cases} 1, & t \in I, \\ 0, & t \notin I. \end{cases}$$

Based on this last considerations, we prove the following result.

**Lemma 1** *The function* $u : [0, 1) \to X_0\,(\subset X)$ *given by*

$$u(t) := \sum_{n=1}^{\infty} z_n(t) v_n$$

*is continuous and bounded. Moreover, it cannot be extended to* $[0, 1]$ *and if* $\alpha \in (0, 1)$, *it satisfies*

$$cD_t^{\alpha} u(t) = \sum_{n=1}^{\infty} \chi_{[t_n, 1)}(t) cD_t^{\alpha} z_{n+1}(t) v_{n+1}$$

*with* $cD_t^{\alpha} u(t)$ *a continuous function.*

*Proof* The continuity and the boundedness of $u(t)$ follows since $\|z_n(t) v_n\|_X \leq 1$ for every $n \in \mathbb{N}$ and

$$u(t) = \begin{cases} v_1, & \text{for } t \in [0, t_1) \\ v_1 + z_2(t) v_2, & \text{for } t \in [t_1, t_2) \\ v_1 + z_2(t) v_2 + z_3(t) v_3, & \text{for } t \in [t_2, t_3) \\ v_1 + z_2(t) v_2 + z_3(t) v_3 + z_4(t) v_4, & \text{for } t \in [t_3, t_4) \\ v_1 + z_{k-1}(t) v_{k-1} + z_k(t) v_k + z_{k+1}(t) v_{k+1}, & \text{for } t \in [t_k, t_{k+1}) \\ & \text{and } k \geq 4. \end{cases} \tag{5}$$

It is worth mentioning that for each $t \in [0, 1)$ at most four terms in the above sum, which defines $u(t)$, are different from zero.

To verify that $u$ cannot be extended in $[0, 1]$ continuously, observe that for $n \geq 2$

$$u\left(\frac{t_n + t_{n+1}}{2}\right) = v_1 + v_n.$$

Thus, define $\sigma_n = (t_n + t_{n+1})/2$ and observe that

$$\left\|u(\sigma_n) - u(\sigma_{n+1})\right\|_X = \|v_n - v_{n+1}\|_X.$$

Since $\{v_n\}_{n=1}^{\infty}$ cannot be convergent, the sequence $u(\frac{t_n+t_{n+1}}{2})$ cannot be a Cauchy sequence, and therefore does not converges. In other words, there is no way to define a value at $t = 1$ such that function $u$ becomes continuous in $[0, 1]$.

By recalling the already mentioned non-local property of the fractional derivative and applying $cD_t^{\alpha}$ in (5) we obtain

$$cD_t^{\alpha}u(t) = \begin{cases} 0, & \text{for } t \in [0, t_1) \\ cD_t^{\alpha}z_2(t)v_2, & \text{for } t \in [t_1, t_2) \\ cD_t^{\alpha}z_2(t)v_2 + cD_t^{\alpha}z_3(t)v_3, & \text{for } t \in [t_2, t_3) \\ cD_t^{\alpha}z_2(t)v_2 + cD_t^{\alpha}z_3(t)v_3 + cD_t^{\alpha}z_4(t)v_4, & \text{for } t \in [t_3, t_4) \\ \sum_{i=2}^{k+1} cD_t^{\alpha}z_i(t)v_i, & \text{for } t \in [t_k, t_{k+1}) \\ & \text{and } k \geq 4. \end{cases}$$

The continuity of $cD_t^{\alpha}u(t)$ follows from the fact that $z_k(t)$ has its first derivative continuous and therefore Remark 1 allows us to conclude that $cD_t^{\alpha}z_k(t) = J_t^{1-\alpha}z_k'(t)$. □

At this point, it remains for us to address the construction of the function $f_{\alpha} : \mathbb{R}^+ \times X \to X$. Thus, consider the following preliminary result.

**Lemma 2** *Let function* $H : [0, \infty) \times X \to \mathbb{R}$ *be given by*

$$H(t, x) = V_1^*(x)\chi_{[0,t_1)}(t) + \sum_{i=1}^{\infty}\left[i^{-2}V_i^*(x)\chi_{[t_i,\infty)}(t) + (i+1)^{-2}V_{i+1}^*(x)\left(\frac{t - t_i}{t_{i+1} - t_i}\right)\chi_{[t_i,t_{i+1})}(t)\right],$$

*where* $V_i^* : X \to \mathbb{R}$ *is an extension of* $v_i^* : X_0 \to \mathbb{R}$ *over* $X$ *and satisfies* $\|V_i^*\|_{X^*} \leq 1$. *Then* $H$ *is a continuous and locally Lipschitz in the second variable, uniformly with respect to the first variable.*

*Proof* Recall that Hahn-Banach theorem allows us to extend our functionals $v_i^*$ to $V_i^*$ defined over $X$ such that $\|V_i^*\|_{X^*} \leq \|v_i^*\|_{X_0^*} = 1$. Then, it is interesting to notice that

$$H(t, x) = \begin{cases} V_1^*(x), & \text{for } t \in [0, t_1), \\ V_1^*(x) + \frac{1}{2^2}V_2^*(x)(\frac{t-t_1}{t_2-t_1}), & \text{for } t \in [t_1, t_2), \\ \sum_{i=1}^{k}\frac{1}{i^2}V_i^*(x) + \frac{1}{(k+1)^2}V_{k+1}^*(x)(\frac{t-t_k}{t_{k+1}-t_k}), & \text{for } t \in [t_k, t_{k+1}) \\ & \text{and } k \geq 2, \\ \sum_{i=1}^{\infty}\frac{1}{i^2}V_i^*(x), & \text{for } t \in [1, \infty). \end{cases}$$

Observe that for $(t, x) \in [0, \infty) \times X$, with $t \neq 1$, the continuity of $H$ follows directly from the above characterization. We verify the continuity of $H$ at $(1, x)$ by definition. Thus, given $\epsilon > 0$, choose $k_0 \in \mathbb{N}$ such that

$$\left(\sum_{i=k+2}^{\infty}\frac{\|x\|_X}{i^2}\right) + \frac{1 + \|x\|_X}{(k+1)^2} < \frac{\epsilon}{2},$$

for any $k \geq k_0$, and let $\delta_0 \in (0, 1)$ be such that $1 - \delta_0 > t_{k_0}$. Then, define

$$\delta := \min\left\{\delta_0, \frac{\epsilon}{2}\left(\sum_{i=1}^{\infty}\frac{1}{i^2}\right)^{-1}\right\}.$$

If $\|(s, y) - (1, x)\|_{[0,\infty)\times X} < \delta$, we conclude that $s > 1 - \delta_0 > t_{k_0}$ and therefore that $s$ lies in an interval of the form $[t_{k-1}, t_k)$, for some $k \geq k_0$ or $s \geq 1$. In the first situation, we compute

$$\begin{aligned}\big|H(1, x) - H(s, y)\big| &= \left|\sum_{i=1}^{\infty}\frac{1}{i^2}V_i^*(x) - \left[\sum_{i=1}^{k}\frac{1}{i^2}V_i^*(y)\right.\right.\\ &\quad \left.\left. + \frac{1}{(k+1)^2}V_{k+1}^*(y)\left(\frac{s - t_k}{t_{k+1} - t_k}\right)\right]\right|\\ &\leq \left(\sum_{i=k+2}^{\infty}\frac{1}{i^2}\right)\|x\|_X + \left(\sum_{i=1}^{k}\frac{1}{i^2}\right)\|x - y\|_X\\ &\quad + \frac{1}{(k+1)^2}\left\|x - y\frac{s - t_k}{t_{k+1} - t_k}\right\|_X\\ &\leq \left(\sum_{i=k+2}^{\infty}\frac{1}{i^2}\right)\|x\|_X + \left(\sum_{i=1}^{k+1}\frac{1}{i^2}\right)\|x - y\|_X\\ &\quad + \frac{1}{(k+1)^2}\|y\|_X\\ &< \epsilon,\end{aligned}$$

since $\|y\|_X \leq \|y - x\|_X + \|x\|_X \leq 1 + \|x\|_X$ and

$$\left|\left(x - y\frac{s - t_k}{t_{k+1} - t_k}\right)\right| \leq \|x - y\| + \|y\|,$$

while in the second situation we compute

$$\big|H(1, x) - H(s, y)\big| = \left|\sum_{i=1}^{\infty}\frac{1}{i^2}V_i^*(x) - \sum_{i=1}^{\infty}\frac{1}{i^2}V_i^*(y)\right| < \epsilon,$$

proving the continuity.

To conclude that $H$ is locally Lipschitz in the second variable, uniformly with respect to the first variable, choose any point $(\widetilde{t}, \widetilde{x})$ belonging to $[0, \infty) \times X$.

*1st Case*: If $\widetilde{t} \in (0, t_1)$, then consider $2\widetilde{r} := \min\{\widetilde{t}, t_1 - \widetilde{t}\}$ and observe that given pairs $(t, x), (t, y) \in B_{\widetilde{r}}(\widetilde{t}, \widetilde{x})$ we obtain that $t \in (0, t_1)$ and therefore

$$\big|H(t, x) - H(t, y)\big| \leq \left\|V_1^*\right\|_{X^*}\|x - y\|_X.$$

When $\widetilde{t} = 0$, the same inequality holds for any $(t, x), (t, y) \in B_{t_1}(\widetilde{t}, \widetilde{x})$ with $t \geq 0$.

*2nd Case*: If $\widetilde{t} \in (t_k, t_{k+1})$ for some $k \geq 1$, define $2\widetilde{r} := \min\{\widetilde{t} - t_k, t_{k+1} - \widetilde{t}\}$ and notice that for $(t, x), (t, y) \in B_{\widetilde{r}}(\widetilde{t}, \widetilde{x})$ we obtain that $t \in (t_k, t_{k+1})$ what ensures

$$H(t, x) - H(t, y) = \left[\sum_{i=1}^{k} \frac{1}{i^2} V_i^*(x - y)\right] + \frac{1}{(k+1)^2} V_{k+1}^*(x - y)\left(\frac{t - t_k}{t_{k+1} - t_k}\right)$$

therefore,

$$\left|H(t, x) - H(t, y)\right| \leq \left[\sum_{i=1}^{k+1} \frac{1}{i^2} \left\| V_i^* \right\|_{X^*}\right] \|x - y\|_X.$$

*3rd Case*: If $\widetilde{t} = t_k$ for $k \geq 1$, by setting $2\widetilde{r} := \min\{\widetilde{t} - t_{k-1}, t_{k+1} - \widetilde{t}\}$ and following the above computations, we achieve the same conclusion.

*4th Case*: If $\widetilde{t} \in (1, \infty)$ choose $2\widetilde{r} \in (0, \widetilde{t} - 1)$ and observe that for any $(t, x), (t, y) \in B_{\widetilde{r}}(\widetilde{t}, \widetilde{x})$ it holds

$$\left|H(t, x) - H(t, y)\right| \leq \left[\sum_{i=1}^{\infty} \frac{1}{i^2} \left\| V_i^* \right\|_{X^*}\right] \|x - y\|_X \leq \left[\sum_{i=1}^{\infty} \frac{1}{i^2}\right] \|x - y\|_X.$$

*5th Case*: If $\widetilde{t} = 1$, choose $2\widetilde{r} \in (0, 1)$, and the result follows from the above computations, since the image of points $(t, x)$ and $(t, y)$ would be given by an infinite series or a truncated series. This completes the proof. □

**Lemma 3** *Consider $\alpha \in (0, 1)$ and the function $f_\alpha : \mathbb{R}^+ \times X \to X$ given by*

$$f_\alpha(t, x) = \phi\big(H(t, x)\big) cD_t^\alpha u(t),$$

*where $H(t, x)$ is described in Lemma 2 and $\phi : \mathbb{R} \to \mathbb{R}$ is given by*

$$\phi(t) = \min\{t + 1, 1\}.$$

*Then $f_\alpha$ is a continuous and locally Lipschitz in the second variable, uniformly with respect to the first variable.*

*Proof* The function $f_\alpha$ is continuous since $\phi(t)$, $cD_t^\alpha u(t)$ and $H(t, x)$ are continuous. As $\phi(t)$ is Lipschitz and $H(t, x)$ is locally Lipschitz in the second variable, uniformly with respect to the first variable, by Lemma 2, it follows from description (limited number of members in the sum) of $cD_t^\alpha u(t)$ in Lemma 1 that $f_\alpha$ is locally Lipschitz in the second variable, uniformly with respect to the first variable. □

*Remark 3* Let us highlight the importance of function $\phi(t)$ on the definition of $f_\alpha(t, x)$ in the above lemma. Observe that the values assumed by $H(t, x)$ can be negative, once it depends on the linear functionals $V_i^*(x)$. Moreover, we can just deduce that $H(t, u(t)) \geq 0$, for $t \in [0, 1)$. It is for this reason that we consider a new function $\phi(t)$ to ensure in the end that $\phi(H(t, u(t))) = 1$ and therefore that $u$ is a solution of (FCP) (see Theorem 6 for details).

Next theorem is the main result of this section, which completely answer the Affirmation.

**Theorem 6** (Sharpness of "blow up" conditions) *Consider* $\alpha \in (0,1)$. *Then there exists* $f_\alpha : \mathbb{R}^+ \times X \to X$ *continuous and locally Lipschitz in the second variable, uniformly with respect to the first variable, which does not map every bounded set into bounded set, such that problem*

$$\begin{cases} cD_t^\alpha u(t) = f_\alpha(t, u(t)), & t > 0 \\ u(0) = v_1 \in X, \end{cases} \tag{FCP}$$

*where* $cD_t^\alpha$ *is the Caputo's fractional derivative and* $v_1$ *is the first element of the Schauder basis defined before in this section, posses a bounded maximal solution* $u : [0,1) \to X$.

*Proof* If we define $f_\alpha$ as in Lemma 3, then it is continuous and locally Lipschitz in the second variable, uniformly with respect to the first variable. Now, it is not difficult to notice that if $u : [0,1) \to X$ is given as in Lemma 1, we deduce

$$H\big(t, u(t)\big) = \begin{cases} V_1^*(u(t)), & \text{for } t \in [0, t_1), \\ V_1^*(u(t)) + \frac{1}{2^2} V_2^*(u(t))\left(\frac{t - t_1}{t_2 - t_1}\right), & \text{for } t \in [t_1, t_2), \\ \sum_{i=1}^k \frac{1}{i^2} V_i^*(u(t)) + \frac{1}{(k+1)^2} V_{k+1}^*(u(t))\left(\frac{t - t_k}{t_{k+1} - t_k}\right), & \text{for } t \in [t_k, t_{k+1}) \\ & \text{and } k \geq 2, \end{cases}$$

which by the linearity of the linear functionals $V_i^*(x)$ and the fact that

$$V_i^*(v_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

allows us to conclude that

$$H\big(t, u(t)\big) = \begin{cases} z_1(t), & \text{for } t \in [0, t_1), \\ z_1(t) + 2^{-2} z_2(t)\left(\frac{t - t_1}{t_2 - t_1}\right), & \text{for } t \in [t_1, t_2), \\ \sum_{i=1}^k i^{-2} z_i(t) + (k+1)^{-2} z_{k+1}(t)\left(\frac{t - t_k}{t_{k+1} - t_k}\right), & \text{for } t \in [t_k, t_{k+1}). \end{cases}$$

Now since $z_k(t) \geq 0$ we obtain $\phi(H(t, u(t))) = 1$.

Thus we achieve that $f_\alpha(t, u(t)) = cD_t^\alpha u(t)$, which means that $u$ is a solution to problem (FCP). Finally $f_\alpha$ does not map bounded sets into bounded sets, since it would contradict Theorem 4 once $u$ is a bounded maximal solution of (FCP). □

## 4 Final Remarks on Theorem 6

In the proof of Theorem 6, to conclude that $f_\alpha(t, x)$ did not map every bounded set into a bounded set, we used an argument based on the conclusions of Theorem 4. In this section, for the completeness of the subject discussed in this paper, we consider a more specific situation where we are able to exhibit the bounded set which is mapped by $f_\alpha(t, x)$ into an unbounded set.

To begin this approach, consider $\{\tau_n\}_{n=1}^\infty$, with $\tau_n \in [(t_{n-1} + t_n)/2, t_n)$, the maximum value of $z_n'(t)$ (which is strictly positive), for each $n \geq 2$.

Fix $\alpha \in (0, 1)$ and now consider the following hypothesis:

$$(\mathrm{P}) \begin{cases} \text{There exists } \{\tau_{n_l}\}_{l=1}^{\infty} \subset \{\tau_n\}_{n=1}^{\infty} \text{ such that} \\ \quad \lim_{l\to\infty} \|cD_t^{\alpha} u(t)|_{t=\tau_{n_l}}\|_X = \infty. \end{cases}$$

Observe that hypothesis (P) is not completely artificial as it seems. Recall initially that the family of functions

$$\left\{z_n'(t) : n \in \mathbb{N}\right\}$$

should posses an increasing property concerning its maximum in the interval $[(t_{n-1} + t_n)/2, t_n)$, once for each higher value of $n$ the length of the interval $[t_{n-1}, t_n)$ should be smaller what implies that the growth of function $z_n'(t)$ should be each time bigger. However proving this assertion to general Banach spaces is a hard task.

To prove the following result, we also need to suppose that $t_1 + t_2 > 1$.

**Proposition 1** *If $X$ is a Hilbert space, then property $(P)$ holds.*

*Proof* Let $X$ be a Hilbert space. By definition of $cD_t^{\alpha} u(t)$ we observe

$$\left\|cD_t^{\alpha} u(t)\right|_{t=\tau_n}\Big\|_X^2 = \sum_{k=2}^{n} \left[cD_t^{\alpha} z_k(t)\big|_{t=\tau_n}\right]^2,$$

for each $n \geq 2$.

Now by assuming that property $(P)$ does not hold, there should exists $M > 0$ such that

$$\left\|cD_t^{\alpha} u(t)\right|_{t=\tau_n}\Big\|_X \leq M, \quad \text{for each } n \in \mathbb{N}. \tag{6}$$

Thus, for $n > 2$ and by Remark 1, it holds that

$$\begin{aligned} cD_t^{\alpha} z_k(t)\big|_{t=\tau_n} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{\tau_n} (\tau_n - s)^{-\alpha} z_k'(s)\, ds \\ &= \frac{1}{\Gamma(1-\alpha)} \left[\int_{\frac{t_{k-1}+t_k}{2}}^{t_k} (\tau_n - s)^{-\alpha} z_k'(s)\, ds + \int_{t_{k+1}}^{\frac{t_{k+1}+t_{k+2}}{2}} (\tau_n - s)^{-\alpha} z_k'(s)\, ds\right], \end{aligned} \tag{7}$$

for each $k \in [2, n-1]$. Now by recalling that $z_k'(t) \geq 0$ in $[(t_{k-1} + t_k)/2, t_k]$ and $z_k'(t) \leq 0$ in $[t_{k+1}, (t_{k+1} + t_{k+2})/2]$, we deduce

$$\begin{aligned} &\int_{\frac{t_{k-1}+t_k}{2}}^{t_k} (\tau_n - s)^{-\alpha} z_k'(s)\, ds \\ &\quad \geq \left(\tau_n - \frac{t_{k-1}+t_k}{2}\right)^{-\alpha} \int_{\frac{t_{k-1}+t_k}{2}}^{t_k} z_k'(s)\, ds = \left(\tau_n - \frac{t_{k-1}+t_k}{2}\right)^{-\alpha} \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\int_{t_{k+1}}^{\frac{t_{k+1}+t_{k+2}}{2}} (\tau_n - s)^{-\alpha} z_k'(s)\, ds \\ &\quad \geq \left(\tau_n - \frac{\tau_n}{2}\right)^{-\alpha} \int_{t_{k+1}}^{\frac{t_{k+1}+t_{k+2}}{2}} z_k'(s)\, ds = -\left(\frac{\tau_n}{2}\right)^{-\alpha}. \end{aligned} \tag{9}$$

Therefore, by (6), (7), (8) and (9), we achieve the inequality

$$\frac{1}{\Gamma(1-\alpha)}\left[\left(\tau_n - \frac{t_{k-1}+t_k}{2}\right)^{-\alpha} - \left(\frac{\tau_n}{2}\right)^{-\alpha}\right] \leq \left\| cD_t^\alpha u(t)\big|_{t=\tau_n}\right\|_X \leq M,$$

since $t_{k-1} + t_k \geq t_1 + t_2 > 1$.

Finally by taking the limit when $n \to \infty$ in both sides of the above inequality we obtain

$$\frac{1}{\Gamma(1-\alpha)}\left[\left(1 - \frac{(t_{k-1}+t_k)}{2}\right)^{-\alpha} - \left(\frac{1}{2}\right)^{-\alpha}\right] \leq M,$$

for any $k \geq 2$. However, since $(t_{k-1}+t_k)/2 \to 1$ when $k \to \infty$, the value on the left side of the above inequality would be greater than $M$ for $k$ sufficiently large, which is a contradiction. This completes the proof of the proposition. □

Last result allow us to establish the following theorem.

**Theorem 7** (Sharpness of "blow up" conditions revisited) *Let* $\alpha \in (0, 1)$ *and assume that* $(P)$ *holds in* $X$. *Then there exists* $f_\alpha : \mathbb{R}^+ \times X \to X$ *continuous and locally Lipschitz in the second variable, uniformly with respect to the first variable, which does not map every bounded set into bounded set, such that*

$$\begin{cases} cD_t^\alpha u(t) = f_\alpha(t, u(t)), & t > 0 \\ u(0) = v_1 \in X, \end{cases} \tag{FCP}$$

*where* $cD_t^\alpha$ *is the Caputo's fractional derivative of order* $\alpha$ *and* $v_1$ *is the first element of the Schauder basis defined in Sect.* 3, *posses a bounded maximal solution* $u : [0, 1) \to X$.

*Proof* Here we discuss just the bounded set that is mapped by $f_\alpha$ into an unbounded set. Consider the sequence $\{(s_l, y_l)\}_{l=1}^\infty \subset [0, \infty) \times X$ such that

$$s_l = \tau_{n_l} \quad \text{and} \quad y_l = \sum_{k=1}^{l+1} \left(1/k^2\right) v_k, \quad \text{for } l \geq 1,$$

where $\tau_{n_l}$ is given in hypothesis (P).

It is not difficult to notice that $\{(s_l, y_l)\}_{l=1}^\infty$ is bounded, however

$$\left\| f_\alpha(s_l, y_l)\right\|_X = \left|\phi\big(H(s_l, y_l)\big)\right| \left\| cD_t^\alpha u(t)\big|_{t=\tau_{n_l}}\right\|_X$$

and since

$$\left|\phi\big(H(s_l, y_l)\big)\right| = 1,$$

the sequence $\{f_\alpha(s_l, y_l)\}_{l=1}^\infty$ is unbounded. □

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