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Uniqueness Result for Long Range Spatially Segregation Elliptic System

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Abstract We study a class of elliptic competition-diffusion systems of long range segregation models for two and more competing species. We prove the uniqueness result for positive solution of those elliptic and related parabolic systems when the coupling in the right hand side involves a non-local term of integral form.

Moreover, alternate proofs of some known results, such as existence of solutions in the elliptic case and the limiting configuration are given. The free boundary condition in a particular setting is given.

Keywords Spatial segregation · Reaction-diffusion systems · Free boundary problems

Mathematics Subject Classification 35R35 · 92D25 · 35B50

1 Introduction and Problem Setting

One of the important problems in population ecology is modeling of competition and interactions between biological components. To achieve this aim, different models based on reaction-diffusion equations are studied. For spatial segregation, two following models have been studied:

- adjacent segregation: in this model particles interact on contact, and there is a common curve or hyper-surface of separation; free boundary;
- segregation at distance: species interact at a distance from each other. In this model, the annihilation of the coefficient for one component at the point x involves the values of the rest of components in a full neighborhood of the point x.

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The adjacent segregation model and strongly competing systems have been extensively studied from different point of views, we will explain briefly these perspective in the coming section, see [3, 4, 6, 8–10, 12, 14, 15] and references therein. The model describes the steady state of m competing species coexisting in the same area Ω. Let $u_i(x)$ denote the population density of the ith component with the internal dynamic prescribed by $f_i(x, u_i)$. Then, the interaction between components is described by the following system of m differential equations

$$\begin{cases} -\Delta u_i^\varepsilon = f_i(x, u_i^\varepsilon) - \dfrac{1}{\varepsilon} u_i^\varepsilon \displaystyle\sum_{j \neq i} (u_j^\varepsilon)^\beta(x) & \text{in } \Omega, \\ u_i(x) = \phi_i(x) & \text{on } \partial\Omega, \\ i = 1, \dots, m. \end{cases} \tag{1.1}$$

Here ϕ_i are non-negative $C^{1,\alpha}$ functions with disjoint supports that is, $\phi_i \cdot \phi_j = 0$, on the boundary. In the system (1.1), the parameter β can be chosen $\beta = 1$ or 2 which for the case $\beta = 2$, the system is in variational form.

To explain the second model, first we indicate some of the notations that we are dealing with in this paper.

- $\Omega \subset \mathbb{R}^d$, is bounded domain with $C^{1,\alpha}$ boundary;
- $d(x, \partial\Omega)$ denotes the distance of the point x to $\partial\Omega$;
- for a given $D \subset \mathbb{R}^d$, we define $(D)_1 := \{x \in \mathbb{R}^d : d(x, D) \leq 1\}$;
- $(\partial\Omega)_1 := \{x \in \Omega^c : d(x, \partial\Omega) \leq 1\}$;
- $\operatorname{supp} f$: the support of function f;
- $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$;
- $W^+ = \max(W, 0)$ and $W^- = \max(-W, 0)$.

In this work, we consider the following elliptic system studied in [5]:

$$\begin{cases} \Delta u_i^\varepsilon = \dfrac{1}{\varepsilon} u_i^\varepsilon \displaystyle\sum_{j \neq i} H(u_j^\varepsilon)(x) & \text{in } \Omega, \\ u_i^\varepsilon(x) = \phi_i(x) & \text{on } (\partial\Omega)_1, \end{cases} \tag{1.2}$$

where

$$H\left(u_j^\varepsilon\right)(x) = \int_{B_1(x)} u_j^\varepsilon(y)\,dy, \tag{1.3}$$

or

$$H\left(u_j^\varepsilon\right)(x) = \sup_{y \in B_1(x)} u(y). \tag{1.4}$$

Here, the boundary data ϕ_i for $i = 1, \dots, m$ are non-negative, $C^{1,\alpha}$ functions defined on $(\partial\Omega)_1$ with supports at distance, at least one from each other, i.e.,

$$(\operatorname{supp} \phi_i)_1 \cap (\operatorname{supp} \phi_j)^\circ = \emptyset, \quad \text{for } i \neq j.$$

System (1.2) can also be viewed as steady state of the following parabolic system

$$\begin{cases} \dfrac{\partial u_i^\varepsilon}{\partial t} - \Delta u_i^\varepsilon = -\dfrac{1}{\varepsilon} u_i^\varepsilon \displaystyle\sum_{j \neq i} H(u_j^\varepsilon)(x) & \text{in } Q := \Omega \times (0, +\infty), \\ u_i^\varepsilon(x, t) = \phi_i(x, t) & \text{on } (\partial\Omega)_1, \\ u_i^\varepsilon(x, 0) = u_{i,0} & \text{in } \Omega \times \{t = 0\}. \end{cases} \tag{1.5}$$

The main contribution of this work is to provide uniqueness results for system (1.2) (Lemma 3.2) and system (1.5) when H is given by (1.3). Moreover, we provide alternate proof of known results, such as existence of solutions in the elliptic case with right hand side given by (1.3). Also we show that as the competition rate goes to infinity the solution converges, along with suitable sequences, to a spatially long range segregated state (Lemma 4.5), more deep results about properties of limiting configuration can be found in [5].

The outline of this paper is as follows. In Sect. 2 we provide mathematical background and known results about the systems (1.1) and (1.2). Section 3 deals with existence and uniqueness for systems (1.2) and (1.5) where H is given by (1.3). Section 4 consists analysis of the system (1.2) in the limiting case as ε tends to zero when $m = 2$.

2 Basic Facts

In this section we review some of known results and mathematical background for two systems (1.1) and (1.2). The analysis of the system (1.2) is much more difficult compare with system (1.1). Understanding the properties of the system (1.1) gives some insights in the study of system (1.2). Roughly speaking, the system (1.2) can be reduced to the system (1.1) if in the term

$$H\big(u_j^\varepsilon\big)(x) = \sup_{y\in B_1(x)} u(y),$$

instead of unit ball, we consider the ball with radius r where r tends to zero, then

$$H\big(u_j^\varepsilon\big)(x) = u_j^\varepsilon(x)$$

so it is possible to see some general behavior in the system (1.2).

2.1 Known Results for the First Model

As we already mentioned, the system (1.1) has been studied well. First, the existence of solution for each ε is shown in [6, 14, 15] i.e., for each ε the system (1.1) admits a solution $(u_1^\varepsilon, \dots, u_m^\varepsilon) \in (H^1(\Omega))^m$. Moreover, it is also shown that for each ε the normal derivative of u_i^ε is bounded independent of ε which implies that there exists $(u_1, \dots, u_m) \in (H^1(\Omega))^m$ such that up to subsequences, we have the strong convergence of u_i^ε to u_i in $H^1(\Omega)$, and $u_i \cdot u_j = 0$ for $i \neq j$. For fixed ε uniqueness of elliptic system (1.1) for $f_i \equiv 0$, $\beta = 1$ and parabolic system have been shown in [15]. In [7] for a class of segregation state governed by a variational principle, existence of solutions is shown and also the conditions that provide the uniqueness are given. To see uniqueness result for limiting case when ε tends to zero, see [1, 15]. We refer to [2] to see numerical approximation of the system (1.1) for the limiting case as ε tends to zero.

Another observed result in [6], is that for regular points on the interface separating the support of u_i and u_j the following holds

$$\lim_{\substack{y\to x\\ u_i(y)>0}} \nabla u_i(y) = - \lim_{\substack{y\to x\\ u_j(y)>0}} \nabla u_j(y). \tag{2.1}$$

The limiting solutions of (1.1) share the following properties and belong to class S in below, [6]

$$S=\Big\{U=(u_1,\dots,u_m)\in H^1(\Omega):u_i\geq 0,\ u_i\cdot u_j=0 \text{ if } i\neq j,$$
$$u_i=\phi_i \text{ on } \partial\Omega,\ -\triangle u_i\leq 0,\ -\Delta\Big(u_i-\sum_{j\neq i}u_j\Big)\geq 0\Big\}.$$

Remark 1 In system (1.1) when $\varepsilon\to 0$ the system in variational form, i.e., $\alpha=2$ has same solution as the system with $\alpha=1$.

In the case of two components i.e., $m=2$ the explicit solution can be obtained as following. Note that in this case the difference of two functions, $u_1^\varepsilon-u_2^\varepsilon$, is harmonic for each ε. Let W be the harmonic extension on Ω of the boundary data $\phi_1-\phi_2$. If we set $u_1=W^+$, $u_2=W^-$, then the pair (u_1,u_2) is the limit configuration of any sequences of pairs $(u_1^\varepsilon,u_2^\varepsilon)$, and there exists $C\geq 0$ such that (see [6])

$$\left(\frac{1}{\varepsilon}\right)^{1/6}.\|u_i^\varepsilon-u_i\|_{H_0^1(\Omega)}\leq C\quad \text{as } \varepsilon\to 0. \tag{2.2}$$

Recently in [13], the regularity issues for system of strongly competing Schroödinger equation with nontrivial grouping has been studied. The $C^{0,\alpha}$ estimate that are uniform in competition parameter, also the regularity of free boundary as competition rate tends to infinity, are obtained, we refer to [11] for more related work.

2.2 Long Range Segregated Model

Now, we turn our attention to the second system given by (1.2). System (1.2) is in variational form if

$$H\big(u_j^\varepsilon\big)(x)=\int_{B_1(x)}\big(u_j^\varepsilon(y)\big)^2dy.$$

Remark 2 In system (1.1), the interaction between components is given by the term $u_i(x)u_j(x)$; while in (1.2) components interacting by the nonlocal term $u_i^\varepsilon H(u_j^\varepsilon)(x)$. The analysis and asymptotic behavior of the system (1.1) are more straightforward than system (1.2). For instance, if the number of components $m=2$, then in system (1.1) with $f_i=0$, $\beta=1$, the difference $u_1^\varepsilon-u_2^\varepsilon$ is harmonic for each ε while this is not true for system (1.2).

In [5] rigorous analysis is done to show the following:

- There exist continuous functions $u_1^\varepsilon,\dots,u_m^\varepsilon$ depending on the parameter ε which solve the system (1.2) in viscosity sense.
- As ε tends to zero, there exists a subsequence $u_i^{\varepsilon k}$ converging locally uniformly, to a function u_i, satisfying the properties that the u_i's are locally Lipschitz continuous in Ω and have supports at distance at least one from each other.
- Each function u_i is harmonic on its support. The authors show the semi convexity of the free boundary. For the points belonging to free boundary, there is an exterior tangent ball of radius one at x_0.

- The free boundary set has finite $(n-1)$-dimensional Hausdorff measure and free boundary set is a set of finite perimeter.
- They obtained sharp characterization of the interfaces, i.e., the supports of the limit functions are at distance exactly one from each other.
- Free boundary condition in any dimension for two components is given when H is defined by (1.3).

3 Existence and Uniqueness of the Nonlocal Segregation Model

Consider the following elliptic system

$$\begin{cases} \Delta u_i^\varepsilon = \dfrac{1}{\varepsilon} u_i^\varepsilon \displaystyle\sum_{j\neq i} \int_{B_1(x)} u_j^\varepsilon(y)\, dy & \text{in } \Omega, \\ u_i(x) = \phi_i(x) & \text{on } (\partial\Omega)_1. \end{cases} \tag{3.1}$$

Existence of the solution for system (3.1) has been shown in [5] by Schauder fixed point argument. The aim of this work is to cover the lack of uniqueness for solution of (3.1). We show uniqueness of solution for system (3.1) inspired by the proof of uniqueness for system (1.1) in [15]. Since the proof is constructive it can be used for numerical simulation to approximate the solution of ε problem in (3.1).

Lemma 3.1 *For each $\varepsilon > 0$, there exists a positive solution $(u_1^\varepsilon, \ldots, u_m^\varepsilon)$ of System* (3.1).

Proof To start, consider the harmonic extension u_i^0 given by

$$\begin{cases} \Delta u_i^0 = 0 & \text{in } \Omega, \\ u_i^0 = \phi_i & \text{on } (\Omega)_1 \setminus \Omega. \end{cases} \tag{3.2}$$

Now, given u_i^k consider the solution of the following linear system

$$\begin{cases} \Delta u_i^{k+1} = \dfrac{1}{\varepsilon} u_i^{k+1} \displaystyle\sum_{j\neq i} H(u_j^k)(x) & \text{in } \Omega, \\ u_i^{k+1}(x) = \phi_i(x) & \text{on } (\Omega)_1 \setminus \Omega. \end{cases} \tag{3.3}$$

We show that the following inequalities hold:

$$u_i^0 \geq u_i^2 \cdots \geq u_i^{2k} \geq \cdots \geq u_i^{2k+1} \geq \cdots \geq u_i^3 \geq u_i^1, \quad \text{in } \Omega.$$

Note that since $u_i^0 \geq 0$ then

$$\sum_{j\neq i} \int_{B_1(x)} u_j^0(y) dy \geq 0, \quad x \in \Omega.$$

The boundary conditions $\phi_i(x)$ are non negative so the weak maximum principle implies that $u_i^1 \geq 0$ and consequently

$$u_i^k \geq 0, \quad \text{for } k \geq 1,\ i = 1, \ldots, m.$$

Now we have

$$\begin{cases} \Delta u_i^1 \geq 0 & \text{in } \Omega, \\ u_i^1(x) = u_i^0(x) = \phi_i(x) & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Thus the comparison principle implies that $u_i^1 \leq u_i^0$. To proceed more with induction, assume that

$$u_i^0 \geq u_i^2 \geq \cdots \geq u_i^{2k} \geq u_i^{2k+1} \geq \cdots \geq u_i^3 \geq u_i^1. \tag{3.5}$$

We show that

$$u_i^{2k+2} \geq u_i^{2k+1}. \tag{3.6}$$

By (3.3) and the assumption in (3.5) we have

$$\Delta u_i^{2k+2} \leq \frac{1}{\varepsilon} u_i^{2k+2} \sum_{j \neq i} H\big(u_j^{2k}\big)(x),$$

$$\Delta u_i^{2k+1} = \frac{1}{\varepsilon} u_i^{2k+1} \sum_{j \neq i} H\big(u_j^{2k}\big)(x).$$

Note that u_i^{2k+1} and u_i^{2k+2} have the same boundary value so (3.6) follows from the comparison principle. The same argument using the assumption $u_i^{2k+1} \geq u_i^{2k-1}$ shows that

$$u_i^{2k+2} \leq u_i^{2k}.$$

For the next step, we note that

$$\begin{cases} \Delta u_i^{2k+3} = \dfrac{1}{\varepsilon} u_i^{2k+3} \displaystyle\sum_{j \neq i} H(u_j^{2k+2})(x) & \text{in } \Omega, \\ \Delta u_i^{2k+1} = \dfrac{1}{\varepsilon} u_i^{2k+1} \displaystyle\sum_{j \neq i} H(u_j^{2k})(x) & \text{in } \Omega. \end{cases} \tag{3.7}$$

From previous step we have $u_i^{2k+2} \leq u_i^{2k}$ which implies

$$u_i^{2k+3} \geq u_i^{2k+1}.$$

Now let $\overline{u}_i$ and $\underline{u}_i$ be two families of functions such that

$$u_i^{2k} \to \overline{u}_i \quad \text{uniformly in } \Omega,$$

$$u_i^{2k+1} \to \underline{u}_i \quad \text{uniformly in } \Omega.$$

Taking the limit in (3.3) yields

$$\begin{cases} \Delta \overline{u}_i = \dfrac{1}{\varepsilon} \overline{u}_i \displaystyle\sum_{j \neq i} H(\underline{u}_j)(x) & \text{in } \Omega, \\ \Delta \underline{u}_i = \dfrac{1}{\varepsilon} \underline{u}_i \displaystyle\sum_{j \neq i} H(\overline{u}_j)(x) & \text{in } \Omega. \end{cases} \tag{3.8}$$

The inequality $u_i^{2k+1} \leq u_i^{2k}$ implies that

$$\overline{u}_i \geq \underline{u}_i \quad \text{in } \Omega. \tag{3.9}$$

We will show that, in fact, the equality holds. Since $\overline{u}_i = \underline{u}_i$ on $\partial\Omega$, by (3.9), we have

$$\frac{\partial \overline{u}_i}{\partial n} \leq \frac{\partial \underline{u}_i}{\partial n}, \tag{3.10}$$

where n is the outward normal vector of $\partial\Omega$. Hence

$$\int_\Omega \sum_i \Delta\overline{u}_i(x)\,dx = \int_{\partial\Omega} \sum_i \frac{\partial \overline{u}_i}{\partial n}\,ds \leq \int_{\partial\Omega} \sum_i \frac{\partial \underline{u}_i}{\partial n}\,ds = \int_\Omega \sum_i \Delta\underline{u}_i(x)\,dx. \tag{3.11}$$

Substituting Eq. (3.8) into (3.11), we obtain

$$\int_\Omega \sum_{i,j\, j\neq i} \overline{u}_i(x)\left(\int_{B_1(x)} \underline{u}_j(y)\,dy\right)dx \leq \int_\Omega \sum_{i,j\, j\neq i} \underline{u}_i(x)\left(\int_{B_1(x)} \overline{u}_j(y)\,dy\right)dx. \tag{3.12}$$

Rewriting this, we get a symmetric kernel $K(x, y)$; such that

$$\int_\Omega \int_{\Omega_1} \sum_{i,j\, j\neq i} \overline{u}_i(x)\underline{u}_j(y)K(x, y)\,dy\,dx \leq \int_\Omega \int_{\Omega_1} \sum_{i,j\, j\neq i} \underline{u}_j(x)\overline{u}_i(y)K(x, y)\,dy\,dx, \tag{3.13}$$

where $K(x, y)$ is $\chi_{B_1(0)}(x-y)$ with $\chi_{B_1(0)}$ the characteristic function of the unit ball centered at the origin. Since K is symmetric in x and y,

$$\int_\Omega \int_\Omega \sum_{i,j\, j\neq i} \overline{u}_i(x)\underline{u}_j(y)K(x, y)\,dy\,dx = \int_\Omega \int_\Omega \sum_{i,j\, j\neq i} \underline{u}_j(x)\overline{u}_i(y)K(x, y)\,dy\,dx. \tag{3.14}$$

The remaining part is

$$\begin{aligned} \int_\Omega \int_{\Omega_1\setminus\Omega} \sum_{i,j\, j\neq i} \overline{u}_i(x)\underline{u}_j(y)K(x, y)\,dy\,dx &= \int_\Omega \int_{\Omega_1\setminus\Omega} \sum_{i,j\, j\neq i} \overline{u}_i(x)\phi_j(y)K(x, y)\,dy\,dx \\ &\geq \int_\Omega \int_{\Omega_1\setminus\Omega} \sum_{i,j\, j\neq i} \underline{u}_i(x)\phi_j(y)K(x, y)\,dy\,dx \\ &= \int_\Omega \int_{\Omega_1\setminus\Omega} \sum_{i,j\, j\neq i} \underline{u}_j(x)\overline{u}_i(y)K(x, y)\,dy\,dx. \end{aligned} \tag{3.15}$$

Combining (3.13)–(3.15) we obtain

$$\int_\Omega \int_{\Omega_1\setminus\Omega} \sum_{i,j\, j\neq i} \overline{u}_i(x)\phi_j(y)K(x, y)dydx = \int_\Omega \int_{\Omega_1\setminus\Omega} \sum_{i,j\, j\neq i} \underline{u}_i(x)\phi_j(y)K(x, y)\,dy\,dx. \tag{3.16}$$

Now from (3.16) we obtain

$$\overline{u}_i(x) = \underline{u}_i(x) \quad \text{in } \left\{x \in \Omega : dist(x, \partial\Omega) \leq 1\right\}.$$

This follows from facts that $\overline{u}_i \geq \underline{u}_i$ and non negativity of boundary data ϕ_j and definition of kernel $K(x, y)$. In view of (3.14) and the continuation argument we obtain

$$\overline{u}_i \equiv \underline{u}_i, \quad \text{in } \Omega,$$

which is a solution of (3.1). □

Lemma 3.2 (Uniqueness) *Assume there exists another positive solution* $(w_1, \dots, w_m)$ *of* (3.1), *then*

$$u_i = w_i.$$

Proof We will prove that the following hold:

$$u_i^{2k+1} \le w_i \le u_i^{2k}, \quad \text{for } k \ge 0. \tag{3.17}$$

To begin, we show that

$$w_i \le u_i^0. \tag{3.18}$$

This is a consequence of the fact that w_i satisfies

$$\begin{cases} \Delta w_i \ge 0 & \text{in } \Omega, \\ w_i = u_i^0 & \text{on } \partial\Omega. \end{cases}$$

Next we compare w_i with u_i^1 and we show $w_i \ge u_i^1$. This inequality follows from (3.18) and

$$\begin{cases} \Delta w_i = \dfrac{w_i}{\varepsilon} \displaystyle\sum_{j \ne i} \int_{B_1(x)} w_j & \text{in } \Omega, \\ \Delta u_i^1 = \dfrac{u_i^1}{\varepsilon} \displaystyle\sum_{j \ne i} \int_{B_1(x)} u_j^0 & \text{in } \Omega. \end{cases}$$

Now we proceed by induction and we assume that the claim is true until $2k+1$. This means that we have

$$u_i^{2k+1} \le w_i \le u_i^{2k}.$$

Then we show

$$u_i^{2k+3} \le w_i \le u_i^{2k+2}.$$

Again we can compare the equations in below

$$\begin{cases} \Delta w_i = \dfrac{w_i}{\varepsilon} \displaystyle\sum_{j \ne i} \int_{B_1(x)} w_j & \text{in } \Omega, \\ \Delta u_i^{2k+2} = \dfrac{u_i^{2k+2}}{\varepsilon} \displaystyle\sum_{j \ne i} \int_{B_1(x)} u_j^{2k+1} & \text{in } \Omega. \end{cases}$$

Here we use that $u_j^{2k+1} \le w_j$ which implies that $w_i \le u_i^{2k+2}$. Also we have

$$\begin{cases} \Delta w_i = \dfrac{w_i}{\varepsilon} \displaystyle\sum_{j \ne i} \int_{B_1(x)} w_j & \text{in } \Omega, \\ \Delta u_i^{2k+3} = \dfrac{u_i^{2k+2}}{\varepsilon} \displaystyle\sum_{j \ne i} \int_{B_1(x)} u_j^{2k+2} & \text{in } \Omega. \end{cases}$$

By the last step $u_i^{2k+2} \ge w_i$, which implies

$$u_i^{2k+3} \le w_i.$$

Now taking limit in (3.17) shows that

$$w_i = u_i. \qquad \square$$

As a corollary of Lemmas 3.1 and 3.2 we have the following theorem.

Theorem 3.3 *For each* $\varepsilon > 0$, *there exists a unique positive solution* $(u_1^\varepsilon, \dots, u_m^\varepsilon)$ *of System* (3.1).

The same method can be used to construct the unique solution to the parabolic problem (1.5). Indeed, we can proceed as before to construct functions $\overline{u}_i \geq \underline{u}_i$, which satisfy

$$\begin{cases} \Delta \overline{u}_i - \dfrac{\partial \overline{u}_i}{\partial t} = \dfrac{1}{\varepsilon} \overline{u}_i \displaystyle\sum_{j \neq i} H(\underline{u}_j)(x) & \text{in } \Omega, \\ \Delta \underline{u}_i - \dfrac{\partial \underline{u}_i}{\partial t} = \dfrac{1}{\varepsilon} \underline{u}_i \displaystyle\sum_{j \neq i} H(\overline{u}_j)(x) & \text{in } \Omega. \end{cases}$$

Similarly, we still have

$$\frac{\partial \overline{u}_i}{\partial n} \leq \frac{\partial \underline{u}_i}{\partial n}, \quad \text{on } \partial\Omega.$$

Hence for any $T > 0$,

$$\int_0^T \int_{\partial\Omega} \frac{\partial \overline{u}_i}{\partial n} \leq \int_0^T \int_{\partial\Omega} \frac{\partial \underline{u}_i}{\partial n}.$$

Substituting the equation into this, the left hand side equals

$$\int_0^T \int_\Omega \frac{\partial \overline{u}_i}{\partial t} + \frac{1}{\varepsilon} \overline{u}_i \sum_{j \neq i} H(\underline{u}_j) = \int_0^T \int_\Omega \frac{1}{\varepsilon} \overline{u}_i \sum_{j \neq i} H(\underline{u}_j) + \int_\Omega \overline{u}_i(x, T)dx - \int_\Omega u_{i,0}(x)dx,$$

and a similar one holds for the right hand side. By noting that

$$\int_\Omega \overline{u}_i(x, T)dx \geq \int_\Omega \underline{u}_i(x, T)dx,$$

we obtain

$$\int_0^T \int_\Omega \frac{1}{\varepsilon} \overline{u}_i \sum_{j \neq i} H(\underline{u}_j) \leq \int_0^T \int_\Omega \frac{1}{\varepsilon} \underline{u}_i \sum_{j \neq i} H(\overline{u}_j).$$

The rest of the proof is exactly the same as before.

4 Basic Estimates and Asymptotic Behavior as ε Tends to Zero

In this part we study the elliptic systems with highly competitive interaction term. We provide the estimates for the case that competition rate tends to infinity which yields the long range distance of positive components. Although the complete analysis and more results of limiting case can be found in [5], here we simplify some proofs.

For simplicity, we assume that the number of components is $m = 2$ and we consider the following system

$$\begin{cases} \Delta u^{\varepsilon} = \dfrac{u^{\varepsilon}}{\varepsilon} \displaystyle\int_{B_1(x)} v^{\varepsilon}(y)dy & \text{in } \Omega, \\ \Delta v^{\varepsilon} = \dfrac{v^{\varepsilon}}{\varepsilon} \displaystyle\int_{B_1(x)} u^{\varepsilon}(y)dy & \text{in } \Omega, \\ u^{\varepsilon}(x) = \phi(x) & \text{on } (\partial\Omega)_1, \\ v^{\varepsilon}(x) = \varphi(x) & \text{on } (\partial\Omega)_1. \end{cases} \tag{4.1}$$

We use the next Lemma in [5] which states in a strip of size one around the support of a component on the boundary the other components decays to zero exponentially.

Lemma 4.1 *For $\sigma > 0$, let*

$$\overline{\Gamma}^{\sigma} := \left\{\phi(x) > \sigma\right\} \subset \Omega^c.$$

Then on the set $\{x \in \Omega : d(x, \overline{\Gamma}^{\sigma}) \le 1 - r\}$, $0 < r < 1$, we have

$$v^{\varepsilon} \le Ce^{\frac{-c\sigma^{\alpha} r^{\beta}}{\sqrt{\varepsilon}}}.$$

Lemma 4.2 *Assume that the boundary $\partial\Omega$ satisfies an uniform exterior ball condition. Let $(u^{\varepsilon}, v^{\varepsilon})$ be the positive solution of (4.1). There exists a positive constant C independent of ε such that*

$$\sup_{x\in\partial\Omega} \left|\frac{\partial u^{\varepsilon}(x)}{\partial n}\right| \le C,$$

$$\sup_{x\in\partial\Omega} \left|\frac{\partial v^{\varepsilon}(x)}{\partial n}\right| \le C,$$

where n denotes exterior normal to $\partial\Omega$.

Proof We construct barrier functions to control the bound of gradient of u^{ε} and v^{ε} as follows. Firstly, the following inequalities hold

$$-\Delta u^{\varepsilon} \le 0, \qquad -\Delta v^{\varepsilon} \le 0.$$

By the standard sup-sub solution method, we can construct solutions $\overline{u}$ and $\overline{v}$ to the problem

$$\begin{cases} \Delta\overline{u} = 0 & \text{in } \Omega, \\ \Delta\overline{v} = 0 & \text{in } \Omega, \\ \overline{u} = \phi & \text{on } \partial\Omega, \\ \overline{v} = \varphi & \text{on } \partial\Omega. \end{cases}$$

Moreover,

$$u^{\varepsilon} \le \overline{u}, \qquad v^{\varepsilon} \le \overline{v}.$$

Hence

$$\frac{\partial u^{\varepsilon}}{\partial n} \ge \frac{\partial \overline{u}}{\partial n}, \qquad \frac{\partial v^{\varepsilon}}{\partial n} \ge \frac{\partial \overline{v}}{\partial n}. \tag{4.2}$$

Note that such $\overline{u}$ and $\overline{v}$ are independent of ε. At the part where $\phi = 0$, because $u \geq 0$ in Ω, we also have

$$\frac{\partial u^\varepsilon}{\partial n} \leq 0.$$

Combined with (4.2), we get a uniform bound on $\frac{\partial u^\varepsilon}{\partial n}$. It remains to consider the case on $\{\phi > 0\}$. Take an $x_0 \in \partial\Omega$ such that $\phi(x_0) > 0$. By the previous lemma,

$$v^\varepsilon(x) \leq Ce^{-\frac{1}{C\sqrt{\varepsilon}}}, \quad \text{in } B_{\frac{1}{2}}(x_0),$$

where C depends on $\phi(x_0)$. Then in $\Omega \cap B_{\frac{1}{2}}(x_0)$, u^ε satisfies

$$\Delta u^\varepsilon \leq \frac{C}{\varepsilon} e^{-\frac{1}{C\sqrt{\varepsilon}}} u^\varepsilon.$$

From this we can construct a solution w^ε to the problem

$$\begin{cases} \Delta w^\varepsilon = \frac{C}{\varepsilon} e^{-\frac{1}{C\sqrt{\varepsilon}}} w^\varepsilon & \text{in } \Omega \cap B_{\frac{1}{2}}(x_0), \\ w^\varepsilon = u^\varepsilon & \text{on } \partial(\Omega \cap B_{\frac{1}{2}}(x_0)). \end{cases}$$

Moreover,

$$u^\varepsilon \geq w^\varepsilon, \quad \text{in } \Omega \cap B_{\frac{1}{2}}(x_0).$$

Hence

$$\frac{\partial u^\varepsilon}{\partial n} \leq \frac{\partial w^\varepsilon}{\partial n}, \quad \text{on } \partial\Omega \cap B_{\frac{1}{2}}(x_0).$$

Note that

$$0 < \frac{C}{\varepsilon} e^{-\frac{1}{C\sqrt{\varepsilon}}} \leq K,$$

where K is a constant independent of ε. By standard boundary gradient estimates, there exists a constant $C > 0$ independent of ε, such that

$$\frac{\partial w^\varepsilon}{\partial n} \leq C, \quad \text{on } \partial\Omega \cap B_{\frac{1}{4}}(x_0).$$

Take a finite cover of $\partial\Omega \cap \{\phi > 0\}$ using balls $B_{\frac{1}{4}}(x_i)$ with $x_i \in \partial\Omega \cap \{\phi > 0\}$, we see

$$\frac{\partial u^\varepsilon}{\partial n} \leq C, \quad \text{in } \partial\Omega \cap \{\phi > 0\}.$$

Combining this with (4.2) we get a uniform bound on $\frac{\partial u^\varepsilon}{\partial n}$ in the part $\partial\Omega \cap \{\phi > 0\}$. □

Lemma 4.3 *There exist a constant C independent of ε such that if $(u^\varepsilon, v^\varepsilon)$ is a solution of system (4.1) then*

$$\int_\Omega \frac{u^\varepsilon}{\varepsilon}\left(\int_{B_1(x)} v^\varepsilon(y)\,dy\right) dx \leq C,$$

$$\int_\Omega \frac{v^\varepsilon}{\varepsilon}\left(\int_{B_1(x)} u^\varepsilon(y)\,dy\right) dx \leq C.$$

Proof By integrating of the first equation in (4.1) over Ω, we have

$$\int_{\Omega} \frac{u^{\varepsilon}}{\varepsilon}\left(\int_{B_1(x)} v^{\varepsilon}(y)\,dy\right)dx = \int_{\Omega} \Delta u^{\varepsilon}\,dx = \int_{\partial\Omega} \frac{\partial u^{\varepsilon}}{\partial n}\,ds.$$

Now Lemma 4.2 give the result. □

Lemma 4.4 *There exists a positive constant C_2 independent of ε such that*

$$\int_{\Omega} |\nabla u^{\varepsilon}|^2\,dx \le C_2,$$

$$\int_{\Omega} |\nabla v^{\varepsilon}|^2\,dx \le C_2.$$

Proof We multiply the differential inequality $-\Delta u_{\varepsilon} \le 0$ by u^{ε} and integration over Ω gives

$$\int_{\Omega} |\nabla u^{\varepsilon}|^2\,dx - \int_{\partial\Omega} u^{\varepsilon}\frac{\partial u^{\varepsilon}}{\partial n}ds \le 0.$$

Now the bound in gradient in Lemma 4.2 give the result. □

Lemma 4.5 *Let u and v be the limiting solution of* (4.1)*. Assume that x_0 is a point in Ω such that $u(x_0) > 0$. Then we have*

$$v \equiv 0 \quad \textit{in } B_1(x_0).$$

Proof By Lemma 4.3 we have

$$\int_{\Omega} u^{\varepsilon}(x)\left(\int_{B_1(x)} v^{\varepsilon}(y)\,dy\right)dx \le C\varepsilon.$$

Let ε tends to zero in the above inequality to get

$$0 \le \int_{\Omega} u(x)\left(\int_{B_1(x)} v(y)\,dy\right)dx \le 0.$$

This implies

$$\int_{\Omega} u(x)\left(\int_{B_1(x)} v(y)\,dy\right)dx = 0,$$

which shows

$$v \equiv 0 \quad \text{in } B_1(x_0).$$ □

Remark 3 Let u and v be the limiting solution of (4.1) as ε tends to zero. Lemma 4.5 shows that the support of u and the support of v are disjoint at distance at least one. In fact in [5] it is shown that u and v are exactly at distance one.

Definition 4.1 The boundaries $\partial\{x \in \Omega : u(x) > 0\}, \partial\{x \in \Omega : v(x) > 0\}$ are called free boundaries.

4.1 Free Boundary Condition in Dimension One

In [5] for any dimension, the free boundary condition for limiting solution is given for

$$H\left(u_j^\varepsilon\right)(x)=\int_{B_1(x)} u_j^\varepsilon(y)dy.$$

The following simple argument gives the free boundary condition in dimension one when H is given by (1.4). Let $d=1$, $\Omega=(-a,a)$, and $a\geq 1$, consider the following system

$$\begin{cases} (u^\varepsilon(x))''=\dfrac{u^\varepsilon(x)}{\varepsilon}\sup\limits_{y\in[x-1,x+1]} v^\varepsilon(y) & \text{in } (-a,a),\\ (v^\varepsilon(x))''=\dfrac{v^\varepsilon(x)}{\varepsilon}\sup\limits_{y\in[x-1,x+1]} u^\varepsilon(y) & \text{in } (-a,a),\\ u^\varepsilon, v^\varepsilon(y)\geq 0 & \text{in } (-a,a),\\ u^\varepsilon(x)=\phi(x) & \text{on } [-a-1,-a],\\ v^\varepsilon(x)=\varphi(x) & \text{on } [a,a+1]. \end{cases} \tag{4.3}$$

It is easy to see that

$$\sup_{y\in[x-1,x+1]} v^\varepsilon(y)=v^\varepsilon(x+1).$$

Also we have that

$$\left(v^\varepsilon(x+1)\right)''=\frac{v^\varepsilon(x+1)}{\varepsilon}\sup_{y\in[x,x+2]} u^\varepsilon(y)=\frac{v^\varepsilon(x+1)}{\varepsilon}u^\varepsilon(x).$$

This shows for every ε,

$$\left(u^\varepsilon(x)-v^\varepsilon(x+1)\right)''=0. \tag{4.4}$$

Let u, v be the limiting points as ε tends to zero. Then u and v satisfy the following system

$$\begin{cases} (u(x)-v(x+1))''=0 & \text{in } (-a,a),\\ (v(x)-u(x-1))''=0 & \text{in } (-a,a),\\ u,v\geq 0 & \text{in } (-a,a),\\ u(-a)=\phi(-a) \quad v(a)=\varphi(a). \end{cases} \tag{4.5}$$

This shows that in (4.5) if x_f be a free boundary point then the following holds, compare with (2.1).

$$u'(x_f)=-v'(x_f+1).$$

5 Conclusion and Further Works

The uniqueness of the solution for a class of elliptic competition-diffusion systems of long range segregation models is shown. Also we show as the competition rate goes to infinity, the solution converges to a spatially long range segregated state satisfying some free boundary problems.

In a forthcoming paper the author will present numerical approximation for the class of elliptic and parabolic competition-diffusion systems of long range segregation models for two and more competing species.

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