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Infiltration Equation with Degeneracy on the Boundary

Huashui Zhan[1]

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Abstract This paper is mainly about the infiltration equation

$$u_t = \operatorname{div}\big(a(x)|u|^{\alpha}|\nabla u|^{p-2}\nabla u\big), \quad (x,t)\in\Omega\times(0,T),$$

where $p>1$, $\alpha>0$, $a(x)\in C^1(\overline{\Omega})$, $a(x)\geq 0$ with $a(x)|_{x\in\partial\Omega}=0$. If there is a constant β such that $\int_\Omega a^{-\beta}(x)dx\leq c$, $p>1+\frac{1}{\beta}$, then the weak solution is smooth enough to define the trace on the boundary, the stability of the weak solutions can be proved as usual. Meanwhile, if for any $\beta>\frac{1}{p-1}$, $\int_\Omega a^{-\beta}(x)dxdt=\infty$, then the weak solution lacks the regularity to define the trace on the boundary. The main innovation of this paper is to introduce a new kind of the weak solutions. By these new definitions of the weak solutions, one can study the stability of the weak solutions without any boundary value condition.

Keywords Infiltration equation · Weak solution · Boundary degeneracy · Stability

Mathematics Subject Classification 35K65 · 35K92 · 35K85 · 35R35

1 Introduction

In the study of water infiltration through porous media, Darcy's linear relation

$$V = -K(\theta)\nabla\phi, \tag{1.1}$$

satisfactorily describes the flow conduction provided that the velocities are small. Here V represents the seepage velocity of water, θ is the volumetric moisture content, $K(\theta)$ is the

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✉ H. Zhan
hszhan@jmu.edu.cn; 2012111007@xmut.edu.cn

[1] School of Applied Mathematics, Xiamen University of Technology, Xiamen 361024, China

hydraulic conductivity and ϕ is the total potential, which can be expressed as the sum of a hydrostatic potential $\psi(\theta)$ and a gravitational potential z

$$\phi = \psi(\theta) + z.$$

However, (1.1) fails to describe the flow for large velocities. To get a more accurate description of the flow in this case, several nonlinear versions of (1.1) have been proposed. One of these versions is

$$V^{\alpha} = -K(\theta)\nabla\phi, \tag{1.2}$$

where α is a positive constant. If it is assumed that infiltration takes place in a horizontal column of the medium, then the continuity equation has the form

$$\frac{\partial\theta}{\partial t} + \frac{\partial V}{\partial x} = 0.$$

Then we have

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\big(D(\theta)^{p}|\theta_x|^{p-1}\theta_x\big), \tag{1.3}$$

with $\frac{1}{p} = \alpha$ and $D(\theta) = K(\theta)\psi'(\theta)$.

Considering the flows in fractured media, let ε be the size ratio of the matrix blocks to the whole medium and let the width of the fracture planes and the porous block diameter be in the same order. If the permeability ratio of matrix blocks to fracture planes is of order $\varepsilon^{p_\varepsilon}$, where p_ε is a positive oscillating constant, then the nonlinear Darcy law combined with the continuity equation leads to the following equation

$$\omega^{\varepsilon}u_t^{\varepsilon} - \operatorname{div}\Big(k^{\varepsilon}(x)\big|\nabla u^{\varepsilon}\big|^{p^{\varepsilon}-2}\nabla u^{\varepsilon}\Big) = 0, \tag{1.4}$$

where u^{ε} is the density of the fluid (which is generally denoted as ρ in other references), $\omega^{\varepsilon}, k^{\varepsilon}$ are the porosity and the permeability of the medium.

One can generalize Eqs. (1.3) and (1.4) to the following infiltration equation

$$u_t = \operatorname{div}\big(a(x)|u|^{\alpha}|\nabla u|^{p-2}\nabla u\big), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p > 1$, $a(x) \in C^1(\overline{\Omega})$, $a(x) \geq 0$.

Equation (1.5) also comes from the applications of the other fields such as nonlinear heat conduction, non-Newtonian fluid theory etc. If it is required that $a(x) \geq a^- > 0$, then the equation with the following initial-boundary value conditions

$$u|_{t=0} = u_0(x), \quad x \in \Omega, \tag{1.6}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.7}$$

has been studied thoroughly; one can refer to [1–14] et al. If $a(x) > 0$ and $\alpha > 0$, Eq. (1.5) is degenerate on the boundary by the homogeneous boundary value (1.7), such degeneracy comes from the physics quantity u itself. In this paper, we only assume that $a(x) \geq 0$ with $a(x)|_{x\in\partial\Omega} = 0$. Then Eq. (1.5) is always degenerate on the boundary. Not only the degeneracy comes from the physics quantity u itself, but also comes from the diffusion coefficient $a(x)$ which is affected by the environment.

It is well-known that, if $a(x) \equiv 1$, for the usual infiltration equation

$$u_t = \operatorname{div}\big(|u|^{\alpha}|\nabla u|^{p-2}\nabla u\big), \tag{1.8}$$

we can impose the Dirichlet homogeneous boundary condition (1.7), to prove its well-posedness. So, the degeneracy on the boundary coming from the physics quantity u does not affect the boundary value condition (1.7).

But if $a(x)|_{x\in\partial\Omega} = 0$, the situation may be different. We consider the special case for $\alpha = 0$.

$$u_t = \operatorname{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big), \quad (x,t) \in Q_T. \tag{1.9}$$

Suppose that there is classical solution of Eq. (1.9) for the time being. If u and v are two classical solutions of Eq. (1.9) with the initial values $u(x,0)$ and $v(x,0)$ respectively, then we have

$$\begin{aligned}
&\int_\Omega (u-v)(u-v)_t dx + \int_\Omega a(x)\big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\cdot\nabla(u-v)dx\\
&\quad= \int_{\partial\Omega} a(x)(u-v)\big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\cdot\vec{n}d\Sigma\\
&\quad= 0,
\end{aligned}$$

where $\vec{n}$ is the outer unit normal vector of Ω. Thus,

$$\begin{gathered}
\frac{1}{2}\frac{d}{dt}\int_\Omega (u-v)^2 dx \le 0,\\
\int_\Omega |u(x,t)-v(x,t)|^2 dx \le \int_\Omega |u_0(x)-v_0(x)|^2 dx.
\end{gathered} \tag{1.10}$$

It implies that the classical solutions (if there are) of Eq. (1.9) are controlled by the initial value completely. In other words, the stability of the solutions of Eq. (1.9) is true even if no boundary condition is required. Certainly, since Eq. (1.5) is degenerate on the boundary and may be degenerate or singular at points where $|\nabla u| = 0$, it only has a weak solution generally. So whether the conclusion (1.10) is true or not remains to be verified.

If $a(x) = d^{\beta}(x)$, $\beta > 0$, $d(x) = \operatorname{dist}(x, \partial\Omega)$, $\alpha = 0$ in (1.5), we have shown that the usual boundary condition (1.7) is over determined in our previous work [15]. Thus, how to impose the suitable boundary condition to assure the well-posedness of the solutions to Eq. (1.5) is a very interesting problem. Recently, we have done some works on this problem in [16, 17] provided that $\alpha = 0$.

In this paper, firstly, we concern with that when the boundary value condition (1.7) can be imposed. Once we have the boundary value condition (1.7), we can study the stability of the weak solutions as usual. Secondly, since the diffusion coefficient $a(x)$ is degenerate on the boundary, the weak solutions of Eq. (1.5) is not smooth enough to define the trace generally. In such a case, we can not use the boundary value condition (1.7) to study the stability of the weak solutions. In order to solve the problem, we will introduce a new kind of the weak solutions of Eq. (1.5). By these new definitions of the weak solutions, we can study the stability of the weak solutions without any boundary value condition. Moreover, the definitions can be generalized to the other degenerate parabolic equations to study the stability of the weak solutions without any boundary value condition.

2 The Basic Definitions and the Main Results

Definition 2.1 A function $u(x,t)$ is said to be a weak solution of Eq. (1.5) with the initial value (1.6), if

$$u \in L^{\infty}(Q_T), \quad \frac{\partial u}{\partial t} \in L^2(Q_T), \quad a(x)|u|^{\alpha}|\nabla u|^p \in L^1(Q_T), \tag{2.1}$$

and for any function $\varphi \in C_0^1(Q_T)$,

$$\iint_{Q_T} \left(\frac{\partial u}{\partial t}\varphi + a(x)|u|^{\alpha}|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \right) dxdt = 0. \tag{2.2}$$

The initial value (1.6) is satisfied in the sense of

$$\lim_{t\to 0}\int_{\Omega} |u(x,t) - u_0(x)|dx = 0. \tag{2.3}$$

If u satisfies the boundary value condition (1.7) in the sense of the trace, then we say u is a weak solution of the initial-boundary value problem of Eq. (1.5).

In the first place, we will prove the existence of the weak solutions in the sense of Definition 2.1. For simplicity, we can call this kind of solutions as the weak solutions of type I.

Theorem 2.2 *Suppose that* $p > 1$, $\alpha > 0$, $a(x) \in C^1(\overline{\Omega})$, $a(x) > 0$ *when* $x \in \Omega$, $a(x) = 0$ *when* $x \in \partial\Omega$. *If*

$$u_0 \in L^{\infty}(\Omega), \quad a(x)|u_0|^{\alpha}|\nabla u_0|^p \in L^1(\Omega), \tag{2.4}$$

then Eq. (1.5) *with the initial value* (1.6) *has a solution* u *of type I.*

Theorem 2.3 *Besides the conditions in Theorem* 2.2, *if there exists a constant* $\beta > 0$, $p > 1 + \frac{1}{\beta}$, *such that*

$$\int_{\Omega} a^{-\beta}(x)dx \le c, \tag{2.5}$$

then the nonnegative solution of type I with the initial-boundary value conditions (1.6)–(1.7) *is unique.*

Now, we would like to introduce a new kind of the weak solutions of Eq. (1.5). By the new definitions of the weak solutions, the stability of the weak solutions can be researched without any boundary value condition.

Definition 2.4 A nonnegative function $u(x,t)$ is said to be a weak solution of Eq. (1.5) with the initial value (1.6), if u satisfies (2.1), and for any function $\varphi_1 \in C_0^1(Q_T)$, $\varphi_2 \in L^{\infty}(Q_T)$ such that for any given $t \in [0,T)$, $\varphi_2(x,\cdot) \in W_{loc}^{1,p}(\Omega)$, there holds

$$\iint_{Q_T} \left[\frac{\partial u}{\partial t}(\varphi_1\varphi_2) + a(x)|u|^{\alpha}|\nabla u|^{p-2}\nabla u \cdot \nabla(\varphi_1\varphi_2) \right] dxdt = 0. \tag{2.6}$$

The initial value (1.6) is satisfied in the sense of (2.3). If letting $m = 1 + \frac{\alpha}{p-1}$, then (2.6) is equivalent to that

$$\iint_{Q_T} \left[\frac{\partial u}{\partial t}(\varphi_1\varphi_2) + \frac{1}{m^{p-1}} a(x) \left|\nabla u^m\right|^{p-2} \nabla u^m \cdot \nabla(\varphi_1\varphi_2) \right] dxdt = 0. \tag{2.7}$$

We can call this kind of solutions as the weak solutions of type II.

Theorem 2.5 *Suppose that $\alpha > 0$, $p > 1$, $a(x) \in C^1(\overline{\Omega})$, $a(x) > 0$ when $x \in \Omega$, $a(x) = 0$ when $x \in \partial\Omega$. If $u_0 \geq 0$ satisfies (2.4), then Eq. (1.5) with the initial value (1.6) has a solution u of type II, which satisfies that*

$$u \in L^\infty(Q_T), \quad \frac{\partial u}{\partial t} \in L^2(Q_T), \quad a(x)\left|\nabla u^m\right|^p \in L^1(Q_T). \tag{2.8}$$

Theorem 2.6 *Let u, v be two nonnegative solutions of type II with the initial values u_0, v_0 respectively. If for small enough $\lambda > 0$, $a(x)$ satisfies*

$$\int_{\Omega \setminus \Omega_\lambda} |\nabla a|^p dx \leq c\lambda^{p-1}, \tag{2.9}$$

then

$$\int_\Omega |u(x,t) - v(x,t)| dx \leq \int_\Omega |u_0 - v_0| dx, \tag{2.10}$$

where $\Omega_\lambda = \{x \in \Omega : a(x) > \lambda\}$. Moreover, if $1 < p \leq 2$, the condition (2.9) is unnecessary.

Theorem 2.7 *Let u, v be two nonnegative solutions of type II with the initial values u_0, v_0 respectively. If for small enough $\lambda > 0$, $u(x)$ and $v(x)$ satisfy that*

$$\frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla u^m|^p dx\right)^{\frac{p-1}{p}} \leq c, \qquad \frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla v^m|^p dx\right)^{\frac{p-1}{p}} \leq c, \tag{2.11}$$

then the stability (2.10) is true.

Remark 2.8 In Theorems 2.6–2.7, no boundary value condition is required.

In short, the degeneracy of $|u|^\alpha$ on the boundary does not affects the boundary condition. While the degeneracy of $a(x)$ on the boundary may have far-reaching influence on the boundary condition and adds more difficulties to obtain the stability of the weak solutions. At the last section of the paper, we will give another kind of the weak solutions of Eq. (1.5) with the initial value (1.6), and establish the local stability without any boundary value condition.

3 The Weak Solution

Consider the regularized equation

$$\partial_t u_\varepsilon = \operatorname{div}\left(A_\varepsilon(u_\varepsilon, x, t)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right), \quad (x,t) \in Q_T, \tag{3.1}$$

with the initial boundary conditions (1.6)–(1.7), where

$$A_\varepsilon(u_\varepsilon, x, t) = \big(a(x)+\varepsilon\big)(\varepsilon + |u_\varepsilon|)^{\gamma(p-1)}, \quad \varepsilon > 0, \quad \gamma = \frac{\alpha}{p-1}.$$

Then there is a classical solution u_ε.

Proof of Theorem 2.2 If we choose $\int_0^{u_\varepsilon}(\varepsilon+|s|)^\gamma ds$ as the test function of Eq. (3.1), then

$$\int_\Omega \int_0^{u_\varepsilon(x,t)} (\varepsilon+|s|)^\gamma ds dx + \int_0^t \int_\Omega \big(a(x)+\varepsilon\big)(\varepsilon+|u_\varepsilon|)^{p\gamma}|\nabla u_\varepsilon|^p dxdt$$
$$= \int_\Omega \int_0^{u_\varepsilon(x,0)} (\varepsilon+|s|)^\gamma ds dx.$$

Thus, for any $\Omega_\lambda = \{x \in \Omega : a(x) > \lambda\} \subset \Omega$, since $a(x) \in C^1(\overline{\Omega})$ and is positive in the interior of Ω, we have

$$\begin{aligned} &\int_0^t \int_{\Omega_\lambda} (\varepsilon+|u_\varepsilon|)^{p\gamma}|\nabla u_\varepsilon|^p dxdt \le c(\lambda), \\ &\int_0^t \int_{\Omega_\lambda} \big(|u_\varepsilon|^\gamma |\nabla u_\varepsilon|\big)^p dxdt \le c(\lambda). \end{aligned} \tag{3.2}$$

Now, multiplying (3.1) by $u_{\varepsilon t}$, integrating it over Q_T, similar as the usual infiltration equation, it is not difficult to show that

$$\iint_{Q_T} (u_{\varepsilon t})^2 dxdt + \iint_{Q_T} A_\varepsilon(u_\varepsilon, x, t)\frac{d}{dt}\int_0^{|\nabla u_\varepsilon(x,t)|^2} s^{\frac{p-2}{2}} ds dxdt \leqslant c,$$

by the inequality, we have

$$\iint_{Q_T} (u_{\varepsilon t})^2 dxdt \leqslant c. \tag{3.3}$$

Thus there is a function $u \in L^\infty(Q)$ and a subsequence of $\{u_\varepsilon\}$ (we conserve for this subsequence the same notation u_ε) such that

$$\begin{aligned} &\|u_\varepsilon\|_{\infty, Q_T} \le c, \\ &u_\varepsilon \to u, \quad \text{in } L^s_{loc}(Q_T), \quad (1 < s < \infty), \\ &u_\varepsilon \to u, \quad \text{a.e. in } Q_T, \\ &\partial_t u_\varepsilon \rightharpoonup \partial_t u, \quad \text{weakly in } L^2(Q_T), \\ &(a+\varepsilon)^{\frac{p-1}{p}}(|u_\varepsilon|+\varepsilon)^{p\gamma}|\nabla u_\varepsilon|^{p-2}u_{\varepsilon x_i} \rightharpoonup * \, \xi_i, \quad \text{weakly star in } L^\infty\big(0,\infty; L^{\frac{p}{p-1}}(\Omega)\big), \end{aligned}$$

where $\xi = \{\xi_i : 1 \le i \le N\}$ and every ξ_i is a function in $L^\infty(0,\infty; L^{\frac{p}{p-1}}(\Omega))$, $s = 2$ when $p \ge 2$, $1 < s < \frac{Np}{N-p}$ when $1 < p < 2$. In order to prove the theorem, we only need to prove that

$$\xi_i = a^{\frac{p-1}{p}}|u|^{p\gamma}|\nabla u|^{p-2}u_{x_i}, \quad \text{in } L^\infty\big(0,\infty; L^{\frac{p}{p-1}}(\Omega)\big). \tag{3.4}$$

Clearly,

$$\iint_{Q_T} (u\varphi_t - \xi \cdot \nabla\varphi)dxdt = 0, \quad \forall\varphi \in C_0^1(Q_T). \tag{3.5}$$

Now, similar as the usual infiltration equation, we can prove that

$$\iint_{Q_T} a^{\frac{p-1}{p}}(x)|u|^{\frac{\alpha(p-1)}{p}}|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dxdt = \iint_{Q_T} \xi \cdot \nabla\varphi dxdt, \quad \forall\varphi \in C_0^1(Q_T), \tag{3.6}$$

we omit the details here. Then (3.4) is true. At the same time, in a similar way as the usual infiltration equation, we can show that the initial value condition (1.6) can be satisfied in the sense of (2.3). The proof is complete. □

4 The Proof of Theorem 2.3

Lemma 4.1 *Let u be a solution of type I with the initial value* (1.6)*. For any constants s, β, satisfying $s > \alpha\beta + 1$, $\frac{1}{\beta} < p - 1$, such that $\int_\Omega a^{-\beta}dx \leq c$, then*

$$\iint_{Q_T} \left|\nabla u^s\right| dxdt \leq c. \tag{4.1}$$

Proof For any constants $s > \alpha\beta + 1$, $\frac{1}{\beta} < p - 1$,

$$\begin{aligned}
\iint_{Q_T} \left|\nabla u^s\right| dxdt &= \iint_{\{(x,t)\in Q_T; a^\beta|u|^{\beta\alpha}|\nabla u| \leqslant 1\}} \left|\nabla u^s\right| dxdt \\
&\quad + \iint_{\{(x,t)\in Q_T; a^\beta|u|^{\beta\alpha}|\nabla u| > 1\}} \left|\nabla u^s\right| dxdt \\
&\leqslant \iint_{Q_T} \left[a^\beta|u|^{\beta\alpha}|\nabla u|\right] a^{-\beta} s|u|^{(s-1-\alpha\beta)} dxdt \\
&\quad + + \iint_{Q_T} \left[a^\beta|u|^{\beta\alpha}|\nabla u|\right]^{\frac{1}{\beta}} s|u|^{(s-1)}|\nabla u| dxdt \\
&\leqslant \iint_{Q_T} a^{-\beta} s|u|^{(s-1-\alpha\beta)} dxdt + \iint_{Q_T} a(x)|u|^\alpha|\nabla u|^{\frac{1}{\beta}+1} s|u|^{(s-1)} dxdt \\
&\leqslant c\iint_{Q_T} a^{-\beta} dxdt + c\iint_{Q_T} a(x)|u|^\alpha\left(1 + |\nabla u|^p\right) dxdt \\
&\leqslant c.
\end{aligned}$$

Then u^s (also u) has trace on the boundary. □

The Proof of Theorem 2.3 By Theorem 2.2 and Lemma 4.1, the existence of the solution of the initial-boundary problem of Eq. (1.5) is clearly. Now, we prove the stability. If u, v are two nonnegative solutions of type I with the same homogeneous boundary value and with the different initial values u_0, v_0 respectively.

By the definition of the weak solution, for all $\varphi \in C_0^1(Q_T)$, we have

$$\int_\Omega \varphi \frac{\partial(u-v)}{\partial t}dx = -\int_\Omega a(x)\big(u^\alpha|\nabla u|^{p-2}\nabla u - v^\alpha|\nabla v|^{p-2}\nabla v\big)\cdot\nabla\varphi dx. \tag{4.2}$$

If we denote that $m = 1 + \frac{\alpha}{p-1}$, then Eq. (4.2) is equivalent to

$$\int_\Omega \varphi \frac{\partial(u-v)}{\partial t}dx = -\frac{1}{m^{p-1}}\int_\Omega a(x)\big(\big|\nabla u^m\big|^{p-2}\nabla u^m - \big|\nabla v^m\big|^{p-2}\nabla v^m\big)\cdot\nabla\varphi dx. \tag{4.3}$$

For small $\eta > 0$, let

$$S_\eta(s) = \int_0^s h_\eta(\tau)d\tau, \qquad h_\eta(s) = \frac{2}{\eta}\left(1-\frac{|s|}{\eta}\right)_+.$$

Obviously $h_\eta(s) \in C(\mathbb{R})$, and

$$h_\eta(s) \geq 0, \quad |sh_\eta(s)| \leq 1, \quad |S_\eta(s)| \leq 1; \quad \lim_{\eta\to 0} S_\eta(s) = \operatorname{sgn} s, \quad \lim_{\eta\to 0} sS'_\eta(s) = 0. \tag{4.4}$$

We can choose $S_\eta(u^m - v^m)$ as the test function in (4.2), then

$$\begin{aligned}&\int_\Omega S_\eta(u-v)\frac{\partial(u-v)}{\partial t}dx\\ &\qquad + \frac{1}{m^{p-1}}\int_\Omega a(x)\big(\big|\nabla u^m\big|^{p-2}\nabla u^m - \big|\nabla v^m\big|^{p-2}\nabla v^m\big)\cdot\nabla\big(u^m - v^m\big)S'_\eta\big(u^m-v^m\big)dx\\ &= 0.\end{aligned} \tag{4.5}$$

Since

$$\begin{aligned}&\lim_{\eta\to 0}\int_\Omega S_\eta\big(u^m - v^m\big)\frac{\partial(u-v)}{\partial t}dx\\ &\quad = \int_\Omega \operatorname{Sign}\big(u^m-v^m\big)\frac{\partial(u-v)}{\partial t}dx = \int_\Omega \operatorname{Sign}(u-v)\frac{\partial(u-v)}{\partial t}dx\\ &\quad = \frac{d}{dt}\|u-v\|_{L^1(\Omega)},\end{aligned} \tag{4.6}$$

and

$$\int_\Omega a(x)\big(\big|\nabla u^m\big|^{p-2}\nabla u - \big|\nabla v^m\big|^{p-2}\nabla v^m\big)\cdot\nabla\big(u^m-v^m\big)S'_\eta\big(u^m-v^m\big)dx \geqslant 0, \tag{4.7}$$

letting $\eta \to 0$ in (4.3), we have

$$\frac{d}{dt}\|u-v\|_{L^1(\Omega)} \leqslant 0.$$

It implies that

$$\int_\Omega \big|u(x,t)-v(x,t)\big|dx \leqslant \int_\Omega |u_0 - v_0|dx, \quad \forall t \in [0,T). \tag{4.8}$$

□

5 Without the Boundary Value Condition

Let u be a weak solution of Eq. (1.5) with the initial value (1.6). In general, if for any $\beta > \frac{1}{p-1}$, $\int_\Omega a^{-\beta}(x)dxdt = \infty$, then u lacks the regularity to define the trace on the boundary. So, we are not able to obtain the stability of the weak solutions based on the boundary value condition. Beyond one's imagination, if the weak solutions are of type II, then we can obtain the stability of the weak solutions without any boundary value condition. This is Theorem 2.5.

Proof of Theorem 2.5 First of all, if $u_0 \geq 0$, similar as the usual infiltration equation, we can prove that there is a nonnegative weak solution to Eq. (1.5). Then

$$\iint_{Q_T} \left(\frac{\partial u}{\partial t}\varphi + a(x)|u|^\alpha |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\right)dxdt = 0, \quad \forall \varphi \in C_0^1(Q_T). \tag{5.1}$$

If we denote $\Omega_\varphi = \operatorname{supp}\varphi$, then

$$\int_0^T \int_{\Omega_\varphi} \left[u_t\varphi + a(x)|u|^\alpha |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\right]dxdt = 0. \tag{5.2}$$

Now, for any $\varphi_1 \in C_0^1(Q_T)$, $\varphi_2(x,t) \in W_{loc}^{1,p}(\Omega)$ for any given t, and $|\varphi_2(x,t)| \leq c$, it is clearly that $\varphi_2 \in W^{1,p}(\Omega_{\varphi_1})$. By the fact of that $C^\infty(\Omega_{\varphi_1})$ is dense in $W^{1,p}(\Omega_{\varphi_1})$, by a process of limit, we have

$$\int_0^T \int_{\Omega_{\varphi_1}} \left[u_t(\varphi_1\varphi_2) + a(x)|u|^\alpha |\nabla u|^{p-2}\nabla u \cdot \nabla(\varphi_1\varphi_2)\right]dxdt = 0, \tag{5.3}$$

which implies that

$$\int_0^T \int_{\Omega} \left[u_t(\varphi_1\varphi_2) + a(x)|u|^\alpha |\nabla u|^{p-2}\nabla u \cdot \nabla(\varphi_1\varphi_2)\right]dxdt = 0. \tag{5.4}$$

Therefore u is the weak solution of type II. □

Proof of Theorem 2.6 Denote that $m = 1 + \frac{\alpha}{p-1}$. For any function $\varphi_1 \in C_0^1(Q_T)$, $\varphi_2 \in L^\infty(Q_T)$ such that for any given $t \in [0,T)$, $\varphi_2(x,\cdot) \in W_{loc}^{1,p}(\Omega)$, we have

$$\iint_{Q_T} \left[\frac{\partial(u-v)}{\partial t}(\varphi_1\varphi_2) + \frac{1}{m^{p-1}}a(x)\left(\left|\nabla u^m\right|^{p-2}\nabla u^m - \left|\nabla v^m\right|^{p-2}\nabla v^m\right)\cdot\nabla(\varphi_1\varphi_2)\right]dxdt = 0. \tag{5.5}$$

For a small positive constant $\lambda > 0$, let

$$\phi_\lambda(x) = \begin{cases} 1, & \text{if } x \in \Omega_\lambda, \\ \frac{1}{\lambda}a(x), & x \in \Omega \setminus \Omega_\lambda. \end{cases} \tag{5.6}$$

Here, $\Omega_\lambda = \{x \in \Omega : a(x) > \lambda\}$ as before.

Now, letting $\chi_{[\tau,s]}$ be the characteristic function of $[\tau,s)\subseteq[0,T)$, we choose $\varphi_1=\phi_\lambda(x)\chi_{[\tau,s]}$, $\varphi_2=S_\eta(u^m-v^m)$, and integrate it over Q_T, we have

$$\begin{aligned}
&\int_\tau^s\int_\Omega \phi_\lambda(x)S_\eta\big(u^m-v^m\big)\frac{\partial(u-v)}{\partial t}dxdt\\
&\quad+\frac{1}{m^{p-1}}\int_\tau^s\int_\Omega \phi_\lambda(x)a(x)\big(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^2\nabla v^m\big)\\
&\quad\cdot\nabla\big(u^m-v^m\big)S_\eta'\big(u^m-v^m\big)dxdt\\
&\quad+\frac{1}{m^{p-1}}\int_\tau^s\int_\Omega a(x)\big(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^{p-2}\nabla v^m\big)\cdot\nabla\phi_\lambda(x)S_\eta\big(u^m-v^m\big)\\
&=0,
\end{aligned}\tag{5.7}$$

$$\int_\Omega \phi_\lambda(x)a(x)\big(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^2\nabla v^m\big)\cdot\nabla\big(u^m-v^m\big)S_\eta'\big(u^m-v^m\big)dx\geq 0.\tag{5.8}$$

At the same time,

$$\begin{aligned}
&\left|\int_\Omega a(x)\big(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^2\nabla v^m\big)\cdot\nabla\phi_\lambda(x)S_\eta\big(u^m-v^m\big)dx\right|\\
&\quad\leq\int_{\Omega\setminus\Omega_\lambda}a(x)\Big|\big(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^2\nabla v^m\big)\cdot\nabla\phi_\lambda(x)S_\eta\big(u^m-v^m\big)\Big|dx\\
&\quad\leq\int_{\Omega\setminus\Omega_\lambda}a(x)\Big|\big(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^2\nabla v^m\big)\Big||\nabla\phi_\lambda(x)|dx\\
&\quad\leq\frac{c}{\lambda}\bigg[\int_{\Omega\setminus\Omega_\lambda}a(x)|\nabla u^m|^{p-1}|\nabla a|dx+\int_\tau^s\int_{\Omega\setminus\Omega_\lambda}a(x)|\nabla v^m|^{p-1}|\nabla a|dx\bigg].
\end{aligned}\tag{5.9}$$

If the condition (2.9) is true

$$\int_{\Omega\setminus\Omega_\lambda}|\nabla a|^p dx\leq c\lambda^{p-1},$$

then

$$\frac{c}{\lambda}\bigg(\int_{\Omega\setminus\Omega_\lambda}a(x)|\nabla a|^p dx\bigg)^{\frac{1}{p}}\leq c.\tag{5.10}$$

By (5.9)–(5.10), using the Hölder inequality,

$$\begin{aligned}
&\left|\int_\Omega a(x)\big(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^2\nabla v^m\big)\cdot\nabla\phi_\lambda(x)S_\eta\big(u^m-v^m\big)dx\right|\\
&\quad\leq\frac{c}{\lambda}\bigg[\int_{\Omega\setminus\Omega_\lambda}a(x)|\nabla u^m|^{p-1}|\nabla a|dx+\int_\tau^s\int_{\Omega\setminus\Omega_\lambda}a(x)|\nabla v^m|^{p-1}|\nabla a|dx\bigg]\\
&\quad\leq\frac{c}{\lambda}\bigg(\int_{\Omega\setminus\Omega_\lambda}a|\nabla a|^p dx\bigg)^{\frac{1}{p}}\bigg(\int_{\Omega\setminus\Omega_\lambda}a(x)|\nabla u^m|^p dx\bigg)^{\frac{p-1}{p}}\\
&\qquad+\frac{c}{\lambda}\bigg(\int_{\Omega\setminus\Omega_\lambda}a(x)|\nabla a|^p dx\bigg)^{\frac{1}{p}}\bigg(\int_{\Omega\setminus\Omega_\lambda}a(x)|\nabla v^m|^p dx\bigg)^{\frac{p-1}{p}}\\
&\quad\leq c\bigg(\int_{\Omega\setminus\Omega_\lambda}a(x)|\nabla u^m|^p dx\bigg)^{\frac{p-1}{p}}+c\bigg(\int_{\Omega\setminus\Omega_\lambda}a(x)|\nabla u^m|^p dx\bigg)^{\frac{p-1}{p}}.
\end{aligned}\tag{5.11}$$

Since

$$\int_\Omega a(x)|\nabla u^m|^{p-1}dx \le cT, \qquad \int_\Omega a(x)|\nabla v^m|^{p-1}dx \le cT,$$

by (5.11), we have

$$\lim_{\lambda\to 0}\left|\int_\Omega a(x)\big(|\nabla u^m|^{p-2}\nabla u^m - |\nabla v^m|^{p-2}\nabla v^m\big)\cdot\nabla\phi_\lambda(x)S_\eta\big(u^m - v^m\big)dx\right| = 0. \tag{5.12}$$

At last,

$$\begin{aligned}
&\lim_{\eta\to 0}\lim_{\lambda\to 0}\int_\tau^s\int_\Omega \phi_\lambda(x)S_\eta\big(u^m - v^m\big)\frac{\partial(u-v)}{\partial t}dxdt\\
&\quad=\lim_{\eta\to 0}\int_\tau^s\lim_{\lambda\to 0}\int_\Omega \phi_\lambda(x)S_\eta(u-v)\frac{\partial(u-v)}{\partial t}dxdt\\
&\quad=\int_\tau^s\frac{d}{dt}\|u-v\|_{L^1(\Omega)}dt.
\end{aligned} \tag{5.13}$$

Now, let $\lambda\to 0$ in (5.7). Then by (5.8), (5.12) and (5.13),

$$\int_\Omega \big|u(x,t) - v(x,t)\big|dx \leqslant \int_\Omega |u_0 - v_0|dx.$$

In particular, if $1 < p \le 2$, since $|\nabla a| \le c$,

$$\frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla a|^p dx\right)^{\frac{1}{p}} \le \lambda^{\frac{1}{p}-1}\left(\int_{\Omega\setminus\Omega_\lambda} dx\right)^{\frac{1}{p}} \le \lambda^{\frac{2}{p}-1} \le c.$$

Then (5.10) is naturally true. Consequently, Theorem 2.6 is proved. □

Proof of Theorem 2.7 As the proof of Theorem 2.6, we have (5.8) and (5.13). If $u(x)$ and $v(x)$ satisfy (2.11)

$$\frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla u^m|^p dx\right)^{\frac{p-1}{p}} \le c, \qquad \frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla u^m|^p dx\right)^{\frac{p-1}{p}} \le c,$$

by (5.9), using the Hölder inequality, we have

$$\begin{aligned}
&\left|\int_\Omega a(x)\big(|\nabla u^m|^{p-2}\nabla u^m - |\nabla v^m|^2\nabla v^m\big)\cdot\nabla\phi_\lambda(x)S_\eta\big(u^m - v^m\big)dx\right|\\
&\quad\le \frac{c}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} a|\nabla a|^p dx\right)^{\frac{1}{p}}\left(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla u^m|^p dx\right)^{\frac{p-1}{p}}\\
&\qquad+\frac{c}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla a|^p dx\right)^{\frac{1}{p}}\left(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla v^m|^p dx\right)^{\frac{p-1}{p}}\\
&\quad\le c\left(\int_{\Omega\setminus\Omega_\lambda} a|\nabla a|^p dx\right)^{\frac{1}{p}} + c\left(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla a|^p dx\right)^{\frac{1}{p}},
\end{aligned}$$

which goes to zero as $\lambda\to 0$ since that $a(x)\in C^1(\overline{\Omega})$. So, as the proof of Theorem 2.6, we know that the stability (2.10) is true. □

6 The Local Stability

Definition 6.1 A function $u(x,t)$ is said to be a weak solution of Eq. (1.5) with the initial value (1.6), if u satisfies (2.1), and for any function $\varphi_1 \in C_0^1(Q_T)$, $\varphi_2 \in L^\infty(Q_T)$ such that for any given $t \in [0,T)$, $\varphi_2(x,\cdot)$ satisfies that

$$\int_\Omega |\varphi_2|^\alpha |\nabla \varphi_2|^p dx < cT, \tag{6.1}$$

we have

$$\iint_{Q_T} \left[\frac{\partial u}{\partial t}(\varphi_1\varphi_2) + a(x)|u|^\alpha |\nabla u|^{p-2}\nabla u \cdot \nabla(\varphi_1\varphi_2)\right] dxdt = 0. \tag{6.2}$$

The initial value (1.6) is satisfied in the sense of (2.3).

We can call this kind of solutions as the weak solutions of type III.

It is not difficult to prove the existence of the weak solution in the sense of Definition 6.1, we omit the details here.

Theorem 6.2 *Let u, v be two nonnegative solutions of type III with the initial values u_0, v_0 respectively. If*

$$a(x)|\nabla u|^p \le c, \qquad a(x)|\nabla v|^p \le c, \tag{6.3}$$

then there exists a constant $\beta \ge 1$ such that

$$\int_\Omega a^\beta |u(x,t) - v(x,t)|^2 dx \le \int_\Omega a^\beta |u_0 - v_0|^2 dx. \tag{6.4}$$

Corollary 6.3 *For the special case $\alpha = 0$ in Theorem 6.2, even without the condition (6.3), the conclusion (6.4) is still true.*

Proof of Theorem 6.2 Let u, v be two solutions of type III with the initial values $u_0(x)$, $v_0(x)$ respectively. We denote $\varphi_1 = \chi_{[\tau,s]}a^\beta$, $\varphi_2 = (u-v)$, and choose $\varphi_1\varphi_2$ as a test function. Here $\beta \ge 1$ is a constant. Then

$$\begin{aligned}
&\iint_{Q_{\tau s}} (u-v)a^\beta \frac{\partial(u-v)}{\partial t} dxdt \\
&= -\iint_{Q_{\tau s}} a(x)|u|^\alpha \big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\nabla\big[(u-v)a^\beta\big] dxdt \\
&\quad - \iint_{Q_{\tau s}} a(x)\big(|u|^\alpha - |v|^\alpha\big)|\nabla v|^{p-2}\nabla v \nabla\big[(u-v)a^\beta\big] dxdt,
\end{aligned} \tag{6.5}$$

where $Q_{\tau s} = \Omega \times (\tau, s)$.

We have

$$\iint_{Q_{\tau s}} a^{\beta+1}|u|^\alpha \big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\nabla(u-v) dxdt \ge 0, \tag{6.6}$$

and

$$\begin{aligned}&\left|\iint_{Q_{\tau s}}(u-v)a(x)|u|^{\alpha}\big(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\big)\nabla a^{\beta}dxdt\right|\\&\quad\leq\iint_{Q_{\tau s}}|u-v|a(x)|u|^{\alpha}\big(|\nabla u|^{p-1}+|\nabla v|^{p-1}\big)|\nabla a^{\beta}|dxdt\\&\quad\leq c\bigg(\int_{\tau}^{s}\int_{\Omega}a(x)\big(|\nabla u|^{p}+|\nabla v|^{p}\big)dxdt\bigg)^{\frac{p-1}{p}}\cdot\bigg(\int_{\tau}^{s}\int_{\Omega}a^{1+p(\beta-1)}|u-v|^{p}dxdt\bigg)^{\frac{1}{p}}\\&\quad\leq c\bigg(\int_{\tau}^{s}\int_{\Omega}a^{1+p(\beta-1)}|u-v|^{p}dxdt\bigg)^{\frac{1}{p}}.\end{aligned}\tag{6.7}$$

Here, we have used the fact that $|\nabla a|\leq c$. Now, by $\beta\geq 1$, we have

$$\begin{aligned}&\left|\iint_{Q_{\tau s}}(u-v)a(x)|u|^{\alpha}\big(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\big)\nabla a^{\beta}dxdt\right|\\&\quad\leq c\bigg(\int_{\tau}^{s}\int_{\Omega}a^{\beta}|u-v|^{p}dxdt\bigg)^{\frac{1}{p}}.\end{aligned}\tag{6.8}$$

If $p\geq 2$, then

$$\bigg(\int_{\tau}^{s}\int_{\Omega}a^{\beta}|u-v|^{p}dxdt\bigg)^{\frac{1}{p}}\leq c\bigg(\int_{\tau}^{s}\int_{\Omega}a^{\beta}|u-v|^{2}dxdt\bigg)^{\frac{1}{p}}.\tag{6.9}$$

If $1<p<2$, by the Hölder inequality

$$\bigg(\int_{\tau}^{s}\int_{\Omega}a^{\beta}|u-v|^{p}dxdt\bigg)^{\frac{1}{p}}\leq c\bigg(\int_{\tau}^{s}\int_{\Omega}a^{\beta}|u-v|^{2}dxdt\bigg)^{\frac{1}{2}}.\tag{6.10}$$

By (6.8)–(6.10), we have

$$\begin{aligned}&\left|\iint_{Q_{\tau s}}(u-v)a(x)|u|^{\alpha}\big(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\big)\nabla a^{\beta}dxdt\right|\\&\quad\leq c\bigg(\int_{\tau}^{s}\int_{\Omega}a^{\beta}|u-v|^{2}dxdt\bigg)^{\frac{1}{l}},\end{aligned}\tag{6.11}$$

where $l>1$.

At the same time, we have

$$\begin{aligned}&\iint_{Q_{\tau s}}a(x)\big(|u|^{\alpha}-|v|^{\alpha}\big)|\nabla v|^{p-2}\nabla v\nabla\big[(u-v)a^{\beta}\big]dxdt\\&\quad=\iint_{Q_{\tau s}}a(x)\big(|u|^{\alpha}-|v|^{\alpha}\big)|\nabla v|^{p-2}\nabla v\nabla(u-v)a^{\beta}dxdt\\&\qquad+\iint_{Q_{\tau s}}a(x)\big(|u|^{\alpha}-|v|^{\alpha}\big)|\nabla v|^{p-2}\nabla v\nabla a(u-v)\beta a^{\beta-1}dxdt.\end{aligned}\tag{6.12}$$

By the assumption of that $a(x)|\nabla u|^p \le c$, $a(x)|\nabla v|^p \le c$, using the Young inequality, by (6.12), we can show that

$$\iint_{Q_{\tau s}} a(x)\big(|u|^\alpha - |v|^\alpha\big)|\nabla v|^{p-2}\nabla v\nabla\big[(u-v)a^\beta\big]dxdt \\ \le c\left(\int_\tau^s\int_\Omega a^\beta|u-v|^2dxdt\right)^{\frac{1}{k}}, \tag{6.13}$$

where $k > 1$. Clearly,

$$\iint_{Q_{\tau s}} (u-v)a^\beta\frac{\partial(u-v)}{\partial t}dxdt \\ = \int_\Omega a^\beta\big[u(x,s)-v(x,s)\big]^2dx - \int_\Omega a^\beta\big[u(x,\tau)-v(x,\tau)\big]^2dx. \tag{6.14}$$

Now, by (6.5)–(6.14), we have

$$\int_\Omega a^\beta\big[u(x,s)-v(x,s)\big]^2dx - \int_\Omega a^\beta\big[u(x,\tau)-v(x,\tau)\big]^2dx \\ \le c\left(\int_\tau^s\int_\Omega a^\beta|u(x,t)-v(x,t)|^2dxdt\right)^q, \tag{6.15}$$

where $q < 1$. By (6.16), it is not difficult to show that

$$\int_\Omega a^\beta|u(x,s)-v(x,s)|^2dx \le \int_\Omega a^\beta|u(x,\tau)-v(x,\tau)|^2dx. \tag{6.16}$$

Thus, by the arbitrary of τ, we have

$$\int_\Omega a^\beta|u(x,s)-v(x,s)|^2dx \le \int_\Omega a^\beta|u_0-v_0|^2dx. \tag{6.17}$$

The proof is complete. □

Proof of Corollary 6.3 For the special case $\alpha = 0$ in Theorem 6.2, we have

$$\iint_{Q_{\tau s}} (u-v)a^\beta\frac{\partial(u-v)}{\partial t}dxdt \\ = -\iint_{Q_{\tau s}} a(x)\big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\nabla\big[(u-v)a^\beta\big]dxdt. \tag{6.18}$$

Noticing the condition (6.3) is only used to deal with (6.13), which has not appeared now, then without the condition (6.3), we still have the conclusion (6.4). □

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References

1. Gilding, B.G., Peletier, L.A.: The Cauchy problem for an equation in the theory of infiltration. Arch. Ration. Mech. Anal. **61**, 127–140 (1976)
2. Zhao, J., Yuan, H.: The Cauchy problem of some doubly nonlinear degenerate parabolic equations. Chin. Ann. Math., Ser. A **16**(2), 179–194 (1995) (in Chinese)
3. Chen, C., Wang, R.: Global existence and L^{∞} estimates of solution for doubly degenerate parabolic equation. Acta Math. Sin., Ser. A **44**, 1089–1098 (2001) (in Chinese)
4. Lu, G.: Nonlinear degenerate parabolic equations in infiltration through a porous medium. Commun. Nonlinear Sci. Numer. Simul. **3**, 97–100 (1998)
5. Dahlberg, B.E.J., Kenig, C.E.: Nonnegative solutions of the porous medium equation. Commun. Partial Differ. Equ. **9**, 661–678 (1984)
6. Zhao, J.: Singular solutions for a convection diffusion equation with absorption. Math. Acta Sci. **15**, 431–441 (1995)
7. Lions, J.L.: Quelques méthodes de resolution des problèmes aux limites non linear. Gauthier-Villars, Paris (1969)
8. Ohara, Y.: L^{∞} estimates of solutions of some nonlinear degenerate parabolic equations. Nonlinear Anal. TMA **18**, 413–426 (1992)
9. Yuan, J., Lian, Z., Cao, L., Gao, J., Xu, J.: Extinction and positivity for a doubly nonlinear degenerate parabolic equation. Acta Math. Sin. Engl. Ser. **23**, 1751–1756 (2007)
10. Manfredi, J., Vespri, V.: Large time behavior of solutions to a class of doubly nonlinear parabolic equations. Electron. J. Differ. Equ. **1994**(2), 02 (1994)
11. Lee, K., Petrosyan, A., Vázquez, J.L.: Large time geometric properties of solutions of the evolution p-Laplacian equation. J. Differ. Equ. **229**, 389–411 (2006)
12. Zhan, H.: Large time behavior of solutions to a class of doubly nonlinear parabolic equations. Appl. Math. **53**, 521–533 (2008)
13. Zhan, H.: The asymptotic behavior of solutions for a class of doubly degenerate nonlinear parabolic equations. J. Math. Anal. Appl. **370**, 1–10 (2010)
14. Wu, Z., Zhao, J., Yin, J., Li, H.: Nonlinear Diffusion Equations. World Scientific, Singapore (2001)
15. Zhan, H., Yuan, H.: Solutions of a strongly degenerate parabolic equation. J. Jilin Univ. **544**, 671–676 (2016) (in Chinese)
16. Zhan, H.: The stability of the solutions of an equation related to the p-Laplacian with degeneracy on the boundary. Bound. Value Probl. **2016**, 178 (2016). doi:10.1186/s13661-016-0684-6
17. Zhan, H.: On a parabolic equation related to the p-Laplacian. Bound. Value Probl. **2016**, 78 (2016). doi:10.1186/s13661-016-0587-6